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# ON METEORS, EARTHWORMS AND WIMPS

BY SARA BILLEY[1], KRZYSZTOF BURDZY[1], SOUMIK PAL[1] AND BRUCE E. SAGAN

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We study a model of mass redistribution on a finite graph. We address the questions of convergence to equilibrium and the rate of convergence. We present theorems on the distribution of empty sites and the distribution of mass at a fixed vertex. These distributions are related to random permutations with certain peak sets.

## REFERENCES

[1] ALDOUS, D. and FILL, J. (2014). *Reversible Markov Chains and Random Walks on Graphs*. Book in preparation. Available at http://www.stat.berkeley.edu/~aldous/RWG/book.html.
[2] ALDOUS, D. J. (1991). Meeting times for independent Markov chains. *Stochastic Process. Appl.* **38** 185–193. MR1119980
[3] BILLEY, S., BURDZY, K. and SAGAN, B. E. (2013). Permutations with given peak set. *J. Integer Seq.* **16** Article 13.6.1, 18. MR3083179
[4] BILLINGSLEY, P. (1968). *Convergence of Probability Measures*. Wiley, New York. MR0233396
[5] BÓNA, M. (2007). The copies of any permutation pattern are asymptotically normal. Available at arXiv:0712.2792.
[6] BURDZY, K. (2013). Meteor process on $\mathbb{Z}^d$. Available at arXiv:1312.6865.
[7] BURDZY, K., CHEN, Z.-Q. and PAL, S. (2013). Brownian earthworm. *Ann. Probab.* **41** 4002–4049. MR3161468
[8] CAPUTO, P., LIGGETT, T. M. and RICHTHAMMER, T. (2010). Proof of Aldous' spectral gap conjecture. *J. Amer. Math. Soc.* **23** 831–851. MR2629990
[9] CHAN, O.-Y. and PRAŁAT, P. (2012). Chipping away at the edges: How long does it take? *J. Comb.* **3** 101–121. MR2975324
[10] CHAO, C.-C. (1997). A note on applications of the martingale central limit theorem to random permutations. *Random Structures Algorithms* **10** 323–332. MR1606222
[11] CHEN, L. H. Y., GOLDSTEIN, L. and SHAO, Q.-M. (2011). *Normal Approximation by Stein's Method*. Springer, Heidelberg. MR2732624
[12] CONGER, M. and VISWANATH, D. (2007). Normal approximations for descents and inversions of permutations of multisets. *J. Theoret. Probab.* **20** 309–325. MR2324533
[13] CRANE, H. (2014). The cut-and-paste process. *Ann. Probab.* **42** 1952–1979. MR3262496
[14] CRANE, H. and LALLEY, S. P. (2013). Convergence rates of Markov chains on spaces of partitions. *Electron. J. Probab.* **18** 1–23. MR3078020
[15] DE HAAN, L. and FERREIRA, A. (2006). *Extreme Value Theory*: *An Introduction*. Springer, New York. MR2234156

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[16] DIACONIS, P. and FREEDMAN, D. (1999). Iterated random functions. *SIAM Rev.* **41** 45–76. MR1669737

[17] DYSON, F. J. (1962). A Brownian-motion model for the eigenvalues of a random matrix. *J. Math. Phys.* **3** 1191–1198. MR0148397

[18] EHRENBORG, R. and MAHAJAN, S. (1998). Maximizing the descent statistic. *Ann. Comb.* **2** 111–129. MR1682923

[19] FERRARI, P. A. and FONTES, L. R. G. (1998). Fluctuations of a surface submitted to a random average process. *Electron. J. Probab.* **3** 34 pp. (electronic). MR1624854

[20] FEY-DEN BOER, A., MEESTER, R., QUANT, C. and REDIG, F. (2008). A probabilistic approach to Zhang's sandpile model. *Comm. Math. Phys.* **280** 351–388. MR2395474

[21] FURSTENBERG, H. and KESTEN, H. (1960). Products of random matrices. *Ann. Math. Statist.* **31** 457–469. MR0121828

[22] HAIRER, M., MATTINGLY, J. C. and SCHEUTZOW, M. (2011). Asymptotic coupling and a general form of Harris' theorem with applications to stochastic delay equations. *Probab. Theory Related Fields* **149** 223–259. MR2773030

[23] HOUGH, J. B., KRISHNAPUR, M., PERES, Y. and VIRÁG, B. (2009). *Zeros of Gaussian Analytic Functions and Determinantal Point Processes*. *University Lecture Series* **51**. Amer. Math. Soc., Providence, RI. MR2552864

[24] HOWITT, C. and WARREN, J. (2009). Consistent families of Brownian motions and stochastic flows of kernels. *Ann. Probab.* **37** 1237–1272. MR2546745

[25] KASRAOUI, A. (2012). The most frequent peak set of a random permutation. Available at arXiv:1210.5869.

[26] KERMACK, W. O. and MCKENDRICK, A. G. (1937). Some distributions associated with a randomly arranged set of numbers. *Proc. Roy. Soc. Edinburgh* **57** 332–376.

[27] KERMACK, W. O. and MCKENDRICK, A. G. (1937). Tests for randomness in a series of numerical observations. *Proc. Roy. Soc. Edinburgh* **57** 228–240.

[28] LEVIN, D. A., PERES, Y. and WILMER, E. L. (2009). *Markov Chains and Mixing Times*. Amer. Math. Soc., Providence, RI. MR2466937

[29] WOLFRAM RESEARCH (2010). *Mathematica*. *Version* 8.0. Wolfram Research, Champaign, IL.

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# DEGREE DISTRIBUTION OF SHORTEST PATH TREES AND BIAS OF NETWORK SAMPLING ALGORITHMS

BY SHANKAR BHAMIDI[1], JESSE GOODMAN[2], REMCO VAN DER HOFSTAD[3] AND JÚLIA KOMJÁTHY[3]

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In this article, we explicitly derive the limiting degree distribution of the shortest path tree from a single source on various random network models with edge weights. We determine the asymptotics of the degree distribution for large degrees of this tree and compare it to the degree distribution of the original graph. We perform this analysis for the complete graph with edge weights that are powers of exponential random variables (weak disorder in the stochastic mean-field model of distance), as well as on the configuration model with edge-weights drawn according to any continuous distribution. In the latter, the focus is on settings where the degrees obey a power law, and we show that the shortest path tree again obeys a power law with the *same* degree power-law exponent. We also consider random $r$-regular graphs for large $r$, and show that the degree distribution of the shortest path tree is closely related to the shortest path tree for the stochastic mean-field model of distance. We use our results to shed light on an empirically observed bias in network sampling methods.

This is part of a general program initiated in previous works by Bhamidi, van der Hofstad and Hooghiemstra [*Ann. Appl. Probab.* **20** (2010) 1907–1965], [*Combin. Probab. Comput.* **20** (2011) 683–707], [*Adv. in Appl. Probab.* **42** (2010) 706–738] of analyzing the effect of attaching random edge lengths on the geometry of random network models.

## REFERENCES

[1] ACHLIOPTAS, D., CLAUSET, A., KEMPE, D. and MOORE, C. (2005). On the bias of traceroute sampling. In *STOC*'05, *May* 22–24, 2005. ACM, Baltimore, MD.

[2] ALDOUS, D. (1992). Asymptotics in the random assignment problem. *Probab. Theory Related Fields* **93** 507–534. MR1183889

[3] ALDOUS, D. and STEELE, J. M. (2004). The objective method: Probabilistic combinatorial optimization and local weak convergence. In *Probability on Discrete Structures*. *Encyclopaedia Math*. *Sci*. **110** 1–72. Springer, Berlin. MR2023650

[4] ATHREYA, K. B. and NEY, P. E. (2004). *Branching Processes*. Dover, Mineola, NY. MR2047480

[5] BARABÁSI, A.-L. and ALBERT, R. (1999). Emergence of scaling in random networks. *Science* **286** 509–512. MR2091634

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*Key words and phrases.* Flows, random graph, random network, first passage percolation, hopcount, Bellman–Harris processes, stable-age distribution, bias, network algorithms, power law, mean-field model of distance, weak disorder.

[6] BHAMIDI, S. and VAN DER HOFSTAD, R. (2012). Weak disorder asymptotics in the stochastic mean-field model of distance. *Ann. Appl. Probab.* **22** 29–69. MR2932542

[7] BHAMIDI, S., VAN DER HOFSTAD, R. and HOOGHIEMSTRA, G. (2010). Extreme value theory, Poisson–Dirichlet distributions, and first passage percolation on random networks. *Adv. in Appl. Probab.* **42** 706–738. MR2779556

[8] BHAMIDI, S., VAN DER HOFSTAD, R. and HOOGHIEMSTRA, G. (2010). First passage percolation on random graphs with finite mean degrees. *Ann. Appl. Probab.* **20** 1907–1965. MR2724425

[9] BHAMIDI, S., VAN DER HOFSTAD, R. and HOOGHIEMSTRA, G. (2011). First passage percolation on the Erdős–Rényi random graph. *Combin. Probab. Comput.* **20** 683–707. MR2825584

[10] BHAMIDI, S., VAN DER HOFSTAD, R. and HOOGHIEMSTRA, G. (2012). Universality for first passage percolation on sparse random graphs. Available at arXiv:1210.6839.

[11] BOLLOBÁS, B. (2001). *Random Graphs*, 2nd ed. *Cambridge Studies in Advanced Mathematics* **73**. Cambridge Univ. Press, Cambridge. MR1864966

[12] BOLLOBÁS, B. and RIORDAN, O. (2004). The diameter of a scale-free random graph. *Combinatorica* **24** 5–34. MR2057681

[13] BOLLOBÁS, B., RIORDAN, O., SPENCER, J. and TUSNÁDY, G. (2001). The degree sequence of a scale-free random graph process. *Random Structures Algorithms* **18** 279–290. MR1824277

[14] BRAUNSTEIN, L. A., BULDYREV, S. V., COHEN, R., HAVLIN, S. and STANLEY, H. E. (2003). Optimal paths in disordered complex networks. *Phys. Rev. Lett.* **91** 168701.

[15] BROIDO, A. and CLAFFY, K. (2001). Internet topology: Connectivity of IP graphs. In *SPIE International Symposium on Convergence of IT and Communication*, 172–187. Denver, CO.

[16] CHUNG, F. and LU, L. (2006). *Complex Graphs and Networks*. *CBMS Regional Conference Series in Mathematics* **107**. Published for the Conference Board of the Mathematical Sciences, Washington, DC. MR2248695

[17] DOROGOVTSEV, S. N. and MENDES, J. F. F. (2003). *Evolution of Networks*. Oxford Univ. Press, Oxford. MR1993912

[18] DURRETT, R. (2007). *Random Graph Dynamics*. Cambridge Univ. Press, Cambridge. MR2271734

[19] ECKHOFF, M., GOODMAN, J., VAN DER HOFSTAD, R. and NARDI, F. R. (2013). Short paths for first passage percolation on the complete graph. *J. Stat. Phys.* **151** 1056–1088. MR3063496

[20] EMBRECHTS, P., KLÜPPELBERG, C. and MIKOSCH, T. (1997). *Modelling Extremal Events: For Insurance and Finance*. *Applications of Mathematics* (*New York*) **33**. Springer, Berlin. MR1458613

[21] FALOUTSOS, M., FALOUTSOS, P. and FALOUTSOS, C. (1999). On power-law relationships of the internet topology. *SIGCOMM Comput. Commun. Rev.* **29** 251–262.

[22] GOVINDAN, R. and TANGMUNARUNKIT, H. (2000). Heuristics for internet map discovery. In *INFOCOM 2000. Nineteenth Annual Joint Conference of the IEEE Computer and Communications Societies. Proceedings. IEEE* **3** 1371–1380. IEEE, New York.

[23] GREY, D. R. (1973/74). Explosiveness of age-dependent branching processes. *Z. Wahrsch. Verw. Gebiete* **28** 129–137. MR0359040

[24] JAGERS, P. (1975). *Branching Processes with Biological Applications*. Wiley, London. MR0488341

[25] JAGERS, P. and NERMAN, O. (1984). The growth and composition of branching populations. *Adv. in Appl. Probab.* **16** 221–259. MR0742953

[26] JANSON, S. (1999). One, two and three times $\log n/n$ for paths in a complete graph with random weights. *Combin. Probab. Comput.* **8** 347–361. MR1723648

[27] JANSON, S. (2009). The probability that a random multigraph is simple. *Combin. Probab. Comput.* **18** 205–225. MR2497380

[28] LAKHINA, A., BYERS, J., CROVELLA, M. and XIE, P. (2003). Sampling biases in ip topology measurements. In *INFOCOM. Twenty-Second Annual Joint Conference of the IEEE Computer and Communications. IEEE Societies* **1** 332–341. IEEE, New York.

[29] LEVINSON, N. (1960). Limiting theorems for age-dependent branching processes. *Illinois J. Math.* **4** 100–118. MR0112191

[30] MOLLOY, M. and REED, B. (1995). A critical point for random graphs with a given degree sequence. In *Proceedings of the Sixth International Seminar on Random Graphs and Probabilistic Methods in Combinatorics and Computer Science, "Random Graphs '93"* (*Poznań*, 1993) **6** 161–179. MR1370952

[31] NEWMAN, M., BARABÁSI, A.-L. and WATTS, D. J., eds. (2006). *The Structure and Dynamics of Networks*. Princeton Univ. Press, Princeton, NJ. MR2352222

[32] NEWMAN, M. E. J. (2003). The structure and function of complex networks. *SIAM Rev.* **45** 167–256 (electronic). MR2010377

[33] PANSIOT, J. J. and GRAD, D. (1998). On routes and multicast trees in the internet. *ACM SIGCOMM Computer Communication Review* **28** 41–50.

[34] RESNICK, S. I. (1986). Point processes, regular variation and weak convergence. *Adv. in Appl. Probab.* **18** 66–138. MR0827332

[35] SALEZ, J. (2013). Joint distribution of distances in large random regular networks. *J. Appl. Probab.* **50** 861–870. MR3102519

[36] VAN DER HOFSTAD, R. (2014). Random graphs and complex networks. Lecture notes in preparation. Available at http://www.win.tue.nl/~rhofstad/.

[37] VAN MIEGHEM, P. (2009). *Performance Analysis of Communications Networks and Systems*. Cambridge Univ. Press, Cambridge.

[38] WÄSTLUND, J. (2006). Random assignment and shortest path problems. In *Fourth Colloquium on Mathematics and Computer Science Algorithms, Trees, Combinatorics and Probabilities. Discrete Math. Theor. Comput. Sci. Proc., AG* 31–38. Assoc. Discrete Math. Theor. Comput. Sci., Nancy. MR2509621

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# STRICT LOCAL MARTINGALES AND BUBBLES

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This paper deals with asset price bubbles modeled by strict local martingales. With any strict local martingale, one can associate a new measure, which is studied in detail in the first part of the paper. In the second part, we determine the "default term" apparent in risk-neutral option prices if the underlying stock exhibits a bubble modeled by a strict local martingale. Results for certain path dependent options and last passage time formulas are given.

## REFERENCES

[1] BAYRAKTAR, E., KARDARAS, C. and XING, H. (2012). Strict local martingale deflators and valuing American call-type options. *Finance Stoch*. **16** 275–291. MR2903625

[2] CARR, P., FISHER, T. and RUF, J. (2014). On the hedging of options on exploding exchange rates. *Finance Stoch*. **18** 115–144. MR3146489

[3] CHERIDITO, P., NIKEGHBALI, A. and PLATEN, E. (2012). Processes of class sigma, last passage times, and drawdowns. *SIAM J. Financial Math*. **3** 280–303. MR2968035

[4] CHYBIRYAKOV, O. (2007). Itô's integrated formula for strict local martingales with jumps. In *Séminaire de Probabilités XL. Lecture Notes in Math*. **1899** 375–388. Springer, Berlin. MR2409017

[5] COX, A. M. G. and HOBSON, D. G. (2005). Local martingales, bubbles and option prices. *Finance Stoch*. **9** 477–492. MR2213778

[6] DELBAEN, F. and SCHACHERMAYER, W. (1995). Arbitrage possibilities in Bessel processes and their relations to local martingales. *Probab. Theory Related Fields* **102** 357–366. MR1339738

[7] DELBAEN, F. and SCHACHERMAYER, W. (1998). The fundamental theorem of asset pricing for unbounded stochastic processes. *Math. Ann*. **312** 215–250. MR1671792

[8] DELBAEN, F. and SHIRAKAWA, H. (2002). No arbitrage condition for positive diffusion price processes. *Asia-Pac. Financ. Mark*. **9** 159–168.

[9] DELLACHERIE, C. (1969). Ensembles aléatoires. I. In *Séminaire de Probabilités*, *III* (*Univ. Strasbourg*, 1967/68) 97–114. Springer, Berlin. MR0258120

[10] ELWORTHY, K. D., LI, X. M. and YOR, M. (1997). On the tails of the supremum and the quadratic variation of strictly local martingales. In *Séminaire de Probabilités*, *XXXI*. *Lecture Notes in Math*. **1655** 113–125. Springer, Berlin. MR1478722

[11] ELWORTHY, K. D., LI, X.-M. and YOR, M. (1999). The importance of strictly local martingales; applications to radial Ornstein–Uhlenbeck processes. *Probab. Theory Related Fields* **115** 325–355. MR1725406

[12] ERŠOV, M. P. (1975). Extension of measures and stochastic equations. *Theory Probab. Appl*. **19** 431–444.

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*Key words and phrases.* Strict local martingales, bubbles.

[13] FERNHOLZ, E. R. and KARATZAS, I. (2009). Stochastic portfolio theory: An overview. In *Special Volume*: *Mathematical Modeling and Numerical Methods in Finance* (P. G. Ciarlet et al., eds). *Handbook of Numerical Analysis* **XV** 89–167. Elsevier, Oxford.

[14] FERNHOLZ, D. and KARATZAS, I. (2010). On optimal arbitrage. *Ann*. *Appl*. *Probab*. **20** 1179–1204. MR2676936

[15] FÖLLMER, H. (1972). The exit measure of a supermartingale. *Z*. *Wahrsch*. *Verw*. *Gebiete* **21** 154–166. MR0309184

[16] FÖLLMER, H. and PROTTER, P. (2011). Local martingales and filtration shrinkage. *ESAIM Probab*. *Stat*. **15** S25–S38. MR2817343

[17] HULLEY, H. (2010). The economic plausibility of strict local martingales in financial modelling. In *Contemporary Quantitative Finance* 53–75. Springer, Berlin. MR2732840

[18] JACOD, J. (1979). *Calcul Stochastique et Problèmes de Martingales*. *Lecture Notes in Math*. **714**. Springer, Berlin. MR0542115

[19] JACOD, J. and SHIRYAEV, A. N. (2003). *Limit Theorems for Stochastic Processes*, 2nd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **288**. Springer, Berlin. MR1943877

[20] JARROW, R. A. and PROTTER, P. (2009). Forward and futures prices with bubbles. *Int*. *J*. *Theor*. *Appl*. *Finance* **12** 901–924. MR2574496

[21] JARROW, R. A., PROTTER, P. and SHIMBO, K. (2007). Asset price bubbles in complete markets. In *Advances in Mathematical Finance*. *Appl*. *Numer*. *Harmon*. *Anal*. 97–121. Birkhäuser, Boston, MA. MR2359365

[22] JARROW, R. A., PROTTER, P. and SHIMBO, K. (2010). Asset price bubbles in incomplete markets. *Math*. *Finance* **20** 145–185. MR2650245

[23] KREHER, D. and NIKEGHBALI, A. (2013). A new kind of augmentation of filtrations suitable for a change of probability measure by a strict local martingale. Preprint, available at arXiv:1108.4243v2.

[24] MADAN, D., ROYNETTE, B. and YOR, M. (2008a). From Black–Scholes formula, to local times and last passage times for certain submartingales. Available at hal.archives-ouvertes.fr/hal-00261868.

[25] MADAN, D., ROYNETTE, B. and YOR, M. (2008b). Option prices as probabilities. *Finance Res*. *Lett*. **5** 79–87.

[26] MADAN, D. B. and YOR, M. (2006). Ito's integrated formula for strict local martingales. In *In Memoriam Paul-André Meyer*: *Séminaire de Probabilités XXXIX*. *Lecture Notes in Math*. **1874** 157–170. Springer, Berlin. MR2276895

[27] MEYER, P. A. (1972). La mesure de H. Föllmer en théorie des surmartingales. In *Séminaire de Probabilités*, *VI* (*Univ. Strasbourg*, *Année Universitaire* 1970–1971; *Journées Probabilistes de Strasbourg*, 1971) 118–129. *Lecture Notes in Math*. **258**. Springer, Berlin. MR0368131

[28] MIJATOVIĆ, A. and URUSOV, M. (2012). On the martingale property of certain local martingales. *Probab*. *Theory Related Fields* **152** 1–30. MR2875751

[29] NAJNUDEL, J. and NIKEGHBALI, A. (2011). A new kind of augmentation of filtrations. *ESAIM Probab*. *Stat*. **15** S39–S57. MR2817344

[30] PAL, S. and PROTTER, P. (2010). Analysis of continuous strict local martingales via $h$-transforms. *Stochastic Process*. *Appl*. **120** 1424–1443. MR2653260

[31] PARTHASARATHY, K. R. (1967). *Probability Measures on Metric Spaces*. *Probability and Mathematical Statistics* **3**. Academic Press, New York. MR0226684

[32] PLATEN, E. and HEATH, D. (2006). *A Benchmark Approach to Quantitative Finance*. Springer, Berlin. MR2267213

[33] PROFETA, C., ROYNETTE, B. and YOR, M. (2010). *Option Prices as Probabilities*. *A New Look at Generalized Black–Scholes Formulae*. Springer, Berlin. MR2582990

[34] PROTTER, PH. E. (2013). Strict local martingales with jumps. Preprint, available at arXiv:1307.2436v2.
[35] REVUZ, D. and YOR, M. (1999). *Continuous Martingales and Brownian Motion*, 3rd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **293**. Springer, Berlin. MR1725357
[36] RUF, J. (2013). Hedging under arbitrage. *Math*. *Finance* **23** 297–317. MR3034079
[37] YEN, J.-Y. and YOR, M. (2011). Call option prices based on Bessel processes. *Methodol*. *Comput*. *Appl*. *Probab*. **13** 329–347. MR2788861

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# ON GERBER–SHIU FUNCTIONS AND OPTIMAL DIVIDEND DISTRIBUTION FOR A LÉVY RISK PROCESS IN THE PRESENCE OF A PENALTY FUNCTION

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This paper concerns an optimal dividend distribution problem for an insurance company whose risk process evolves as a spectrally negative Lévy process (in the absence of dividend payments). The management of the company is assumed to control timing and size of dividend payments. The objective is to maximize the sum of the expected cumulative discounted dividend payments received until the moment of ruin and a penalty payment at the moment of ruin, which is an increasing function of the size of the shortfall at ruin; in addition, there may be a fixed cost for taking out dividends. A complete solution is presented to the corresponding stochastic control problem. It is established that the value-function is the unique *stochastic solution* and the pointwise smallest *stochastic supersolution* of the associated HJB equation. Furthermore, a necessary and sufficient condition is identified for optimality of a single dividend-band strategy, in terms of a particular Gerber–Shiu function. A number of concrete examples are analyzed.

## REFERENCES

[1] ALBRECHER, H. and THONHAUSER, S. (2008). Optimal dividend strategies for a risk process under force of interest. *Insurance Math. Econom.* **43** 134–149. MR2442038

[2] ALVAREZ, L. H. R. and VIRTANEN, J. (2006). A class of solvable stochastic dividend optimization problems: On the general impact of flexibility on valuation. *Econom. Theory* **28** 373–398. MR2217337

[3] ALVAREZ, O. and TOURIN, A. (1996). Viscosity solutions of nonlinear integro-differential equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **13** 293–317. MR1395674

[4] AVRAM, F., KYPRIANOU, A. E. and PISTORIUS, M. R. (2004). Exit problems for spectrally negative Lévy processes and applications to (Canadized) Russian options. *Ann. Appl. Probab.* **14** 215–238. MR2023021

[5] AVRAM, F., PALMOWSKI, Z. and PISTORIUS, M. (2010). On optimal dividend distribution for a Cramér–Lundberg process with exponential jumps in the presence of a linear Gerber–Shiu penalty function. In *The Pyrenees International Workshop and Summer School on Statistics, Probability and Operations Research—SPO* 2009. *Monogr. Mat. García Galdeano* **36** 69–77. Prensas Univ. Zaragoza, Zaragoza. MR2856705

[6] AVRAM, F., PALMOWSKI, Z. and PISTORIUS, M. R. (2007). On the optimal dividend problem for a spectrally negative Lévy process. *Ann. Appl. Probab.* **17** 156–180. MR2292583

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[7] AZCUE, P. and MULER, N. (2005). Optimal reinsurance and dividend distribution policies in the Cramér–Lundberg model. *Math*. *Finance* **15** 261–308. MR2132192

[8] AZCUE, P. and MULER, N. (2012). Optimal dividend policies for compound Poisson processes: The case of bounded dividend rates. *Insurance Math*. *Econom*. **51** 26–42. MR2909138

[9] BARDI, M. and CAPUZZO-DOLCETTA, I. (1997). *Optimal Control and Viscosity Solutions of Hamilton–Jacobi–Bellman Equations*. Birkhäuser, Boston, MA. MR1484411

[10] BAYRAKTAR, E. and SÎRBU, M. (2012). Stochastic Perron's method and verification without smoothness using viscosity comparison: The linear case. *Proc*. *Amer*. *Math*. *Soc*. **140** 3645–3654. MR2929032

[11] BENTH, F. E., KARLSEN, K. H. and REIKVAM, K. (2001). Optimal portfolio selection with consumption and nonlinear integro-differential equations with gradient constraint: A viscosity solution approach. *Finance Stoch*. **5** 275–303. MR1849422

[12] BERTOIN, J. (1996). *Lévy Processes*. *Cambridge Tracts in Mathematics* **121**. Cambridge Univ. Press, Cambridge. MR1406564

[13] BIFFIS, E. and KYPRIANOU, A. E. (2010). A note on scale functions and the time value of ruin for Lévy insurance risk processes. *Insurance Math*. *Econom*. **46** 85–91. MR2586158

[14] CAI, J., FENG, R. and WILLMOT, G. E. (2009). On the expectation of total discounted operating costs up to default and its applications. *Adv. in Appl*. *Probab.* **41** 495–522. MR2541187

[15] DE FINETTI, B. (1957). Su un'impostazione alternativa dell teoria colletiva del rischio. *Transactions of the XV International Congress of Actuaries* **2** 433–443.

[16] DICKSON, D. C. M. and WATERS, H. R. (2004). Some optimal dividends problems. *Astin Bull*. **34** 49–74. MR2055537

[17] FLEMING, W. H. and SONER, H. M. (1993). *Controlled Markov Processes and Viscosity Solutions*. *Applications of Mathematics* (*New York*) **25**. Springer, New York. MR1199811

[18] GERBER, H. U. (1969). Entscheidungskriterien für den Zusammengesetzten Poisson Prozess. *Mitteilungen der Vereinigung Schweizerischer Versicherungsmathematiker* **69** 185–228.

[19] GERBER, H. U. (1972). Games of economic survival with discrete- and continuous-income processes. *Oper. Res*. **20** 37–45.

[20] GERBER, H. U. (1979). *An Introduction to Mathematical Risk Theory*. *S.S. Heubner Foundation Monograph Series* **8**. S.S. Huebner Foundation for Insurance Education, Philadelphia, PA. MR0579350

[21] GERBER, H. U., LIN, X. S. and YANG, H. (2006). A note on the dividends-penalty identity and the optimal dividend barrier. *Astin Bull*. **36** 489–503. MR2312676

[22] HALLIN, M. (1979). Band strategies: The random walk of reserves. *Blätter der DGVFM* **14** 321–236.

[23] KULENKO, N. and SCHMIDLI, H. (2008). Optimal dividend strategies in a Cramér–Lundberg model with capital injections. *Insurance Math*. *Econom*. **43** 270–278. MR2456621

[24] KUZNETSOV, A., KYPRIANOU, A. E. and RIVERO, V. (2012). The theory of scale functions for spectrally negative Lévy processes. In *Lévy Matters II*. *Lecture Notes in Math*. **2061** 97–186. Springer, Heidelberg. MR3014147

[25] KYPRIANOU, A. E. (2006). *Introductory Lectures on Fluctuations of Lévy Processes with Applications*. Springer, Berlin. MR2250061

[26] KYPRIANOU, A. E., RIVERO, V. and SONG, R. (2010). Convexity and smoothness of scale functions and de Finetti's control problem. *J*. *Theoret*. *Probab*. **23** 547–564. MR2644875

[27] LOEFFEN, R. L. (2008). On optimality of the barrier strategy in de Finetti's dividend problem for spectrally negative Lévy processes. *Ann*. *Appl*. *Probab*. **18** 1669–1680. MR2462544

[28] LOEFFEN, R. L. (2009). An optimal dividends problem with transaction costs for spectrally negative Lévy processes. *Insurance Math*. *Econom*. **45** 41–48. MR2549863

[29] LOEFFEN, R. L. and RENAUD, J.-F. (2010). De Finetti's optimal dividends problem with an affine penalty function at ruin. *Insurance Math. Econom.* **46** 98–108. MR2586160
[30] NEVEU, J. (1975). *Discrete-parameter Martingales*, Revised ed. North-Holland, Amsterdam. MR0402915
[31] NGUYEN-NGOC, L. and YOR, M. (2005). Some martingales associated to reflected Lévy processes. In *Séminaire de Probabilités XXXVIII. Lecture Notes in Math.* **1857** 42–69. Springer, Berlin. MR2126966
[32] PESKIR, G. and SHIRYAEV, A. (2006). *Optimal Stopping and Free-Boundary Problems*. Birkhauser, Basel.
[33] PHAM, H. (1998). Optimal stopping of controlled jump diffusion processes: A viscosity solution approach. *J. Math. Systems Estim. Control* **8** 1–27 pp. (electronic). MR1650147
[34] PISTORIUS, M. R. (2004). On exit and ergodicity of the spectrally one-sided Lévy process reflected at its infimum. *J. Theoret. Probab.* **17** 183–220. MR2054585
[35] SATO, K.-I. (1999). *Lévy Processes and Infinitely Divisible Distributions*. *Cambridge Studies in Advanced Mathematics* **68**. Cambridge Univ. Press, Cambridge. MR1739520
[36] SAYAH, A. (1991). Equations d'Hamilton–Jacobi du premier ordre avec termes intégro-différentiels, I; II. *Comm. Partial Differential Equations* **16** 1057–1074, 1075–1093.
[37] SCHMIDLI, H. (2008). *Stochastic Control in Insurance*. Springer, London. MR2371646
[38] SHREVE, S. E., LEHOCZKY, J. P. and GAVER, D. P. (1984). Optimal consumption for general diffusions with absorbing and reflecting barriers. *SIAM J. Control Optim.* **22** 55–75. MR0728672
[39] SONER, H. M. (1986). Optimal control with state-space constraint. I, II. *SIAM J. Control Optim.* **24** 552–561, 1110–1122. MR0838056
[40] STROOCK, D. and VARADHAN, S. R. S. (1972). On degenerate elliptic-parabolic operators of second order and their associated diffusions. *Comm. Pure Appl. Math.* **25** 651–713. MR0387812
[41] THONHAUSER, S. and ALBRECHER, H. (2007). Dividend maximization under consideration of the time value of ruin. *Insurance Math. Econom.* **41** 163–184. MR2324572
[42] THONHAUSER, S. and ALBRECHER, H. (2011). Optimal dividend strategies for a compound Poisson process under transaction costs and power utility. *Stoch. Models* **27** 120–140. MR2769360
[43] ZAJIC, T. (2000). Optimal dividend payout under compound Poisson income. *J. Optim. Theory Appl.* **104** 195–213. MR1741397

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# CRITICAL POPULATION AND ERROR THRESHOLD ON THE SHARP PEAK LANDSCAPE FOR THE WRIGHT–FISHER MODEL

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We pursue the task of developing a finite population counterpart to Eigen's model. We consider the classical Wright–Fisher model describing the evolution of a population of size $m$ of chromosomes of length $\ell$ over an alphabet of cardinality $\kappa$. The mutation probability per locus is $q$. The replication rate is $\sigma > 1$ for the master sequence and 1 for the other sequences. We study the equilibrium distribution of the process in the regime where

$$\ell \to +\infty, \qquad m \to +\infty, \qquad q \to 0,$$
$$\ell q \to a \in ]0, +\infty[, \qquad \frac{m}{\ell} \to \alpha \in [0, +\infty].$$

We obtain an equation $\alpha\psi(a) = \ln\kappa$ in the parameter space $(a, \alpha)$ separating the regime where the equilibrium population is totally random from the regime where a quasispecies is formed. We observe the existence of a critical population size necessary for a quasispecies to emerge, and we recover the finite population counterpart of the error threshold. The result is the twin brother of the corresponding result for the Moran model. The proof is more complex, and it relies on the Freidlin–Wentzell theory of random perturbations of dynamical systems.

## REFERENCES

[1] ALVES, D. and FONTANARI, J. F. (1998). Error threshold in finite populations. *Phys. Rev. E* **57** 7008–7013.

[2] CERF, R. (2010). Critical control of a genetic algorithm. Preprint. Available at arXiv:1005.3390.

[3] CERF, R. (2015). *Critical Population and Error Threshold on the Sharp Peak Landscape for a Moran Model*. *Mem. Amer. Math. Soc.* **233**. Amer. Math. Soc., Providence, RI.

[4] DARDEN, T. (1983). Asymptotics of fixation under heterosis: A large deviation approach. Unpublished manuscript.

[5] DEMETRIUS, L., SCHUSTER, P. and SIGMUND, K. (1985). Polynucleotide evolution and branching processes. *Bull. Math. Biol.* **47** 239–262. MR0803564

[6] DIXIT, N. M., SRIVASTAVA, P. and VISHNOI, N. K. (2012). A finite population model of molecular evolution: Theory and computation. *J. Comput. Biol.* **19** 1176–1202. MR2990752

[7] EIGEN, M. (1971). Self-organization of matter and the evolution of biological macromolecules. *Naturwissenschaften* **58** 465–523.

[8] ELLIS, R. S. (2006). *Entropy, Large Deviations, and Statistical Mechanics*. Springer, Berlin. Reprint of the 1985 original. MR2189669

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[9] FELLER, W. (1968). *An Introduction to Probability Theory and Its Applications*. *Vol*. *I*, 3rd ed. Wiley, New York. MR0228020

[10] FREIDLIN, M. I. and WENTZELL, A. D. (1998). *Random Perturbations of Dynamical Systems*, 2nd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **260**. Springer, New York. Translated from the 1979 Russian original by Joseph Szücs. MR1652127

[11] GILLESPIE, D. T. (1976). A general method for numerically simulating the stochastic time evolution of coupled chemical reactions. *J*. *Comput*. *Phys*. **22** 403–434. MR0503370

[12] GRIMMETT, G. (1999). *Percolation*, 2nd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **321**. Springer, Berlin. MR1707339

[13] HARRIS, T. E. (1977). A correlation inequality for Markov processes in partially ordered state spaces. *Ann*. *Probab*. **5** 451–454. MR0433650

[14] KIFER, Y. (1988). *Random Perturbations of Dynamical Systems*. *Progress in Probability and Statistics* **16**. Birkhäuser, Boston, MA. MR1015933

[15] KIFER, Y. (1990). A discrete-time version of the Wentzell–Freidlin theory. *Ann*. *Probab*. **18** 1676–1692. MR1071818

[16] LIGGETT, T. M. (2005). *Interacting Particle Systems*. Springer, Berlin. Reprint of the 1985 original. MR2108619

[17] MCCASKILL, J. (1984). A stochastic theory of macromolecular evolution. *Biol*. *Cybernet*. **50** 63–73.

[18] MEZIĆ, I. (1997). FKG inequalities in cellular automata and coupled map lattices. *Phys*. *D* **103** 491–504. MR1464259

[19] MORROW, G. J. and SAWYER, S. (1989). Large deviation results for a class of Markov chains arising from population genetics. *Ann*. *Probab*. **17** 1124–1146. MR1009448

[20] MUSSO, F. (2011). A stochastic version of the Eigen model. *Bull*. *Math*. *Biol*. **73** 151–180. MR2770281

[21] NOWAK, M. A. and SCHUSTER, P. (1989). Error thresholds of replication in finite populations. Mutation frequencies and the onset of Muller's ratchet. *J*. *Theoret*. *Biol*. **137** 375–395.

[22] OCHOA, G. (2001). Error thresholds and optimal mutation rates in genetic algorithms. Ph.D. thesis, Univ. Sussex, Brighton.

[23] PARK, J.-M., MUÑOZ, E. and DEEM, M. W. (2010). Quasispecies theory for finite populations. *Phys*. *Rev*. *E* **81** 011902.

[24] SAAKIAN, D. B., DEEM, M. W. and HU, C.-K. (2012). Finite population size effects in quasispecies models with single-peak fitness landscape. *Europhys*. *Lett*. **98** 18001.

[25] WEINBERGER, E. D. (1987). A stochastic generalization of Eigen's theory of natural selection. Ph.D. dissertation, The Courant Institute of Mathematical Sciences, New York Univ., New York.

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# BRANCHING RANDOM WALKS AND MULTI-TYPE CONTACT-PROCESSES ON THE PERCOLATION CLUSTER OF $\mathbb{Z}^d$

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In this paper we prove that, under the assumption of quasi-transitivity, if a branching random walk on $\mathbb{Z}^d$ survives locally (at arbitrarily large times there are individuals alive at the origin), then so does the same process when restricted to the infinite percolation cluster $\mathcal{C}_\infty$ of a supercritical Bernoulli percolation. When no more than $k$ individuals per site are allowed, we obtain the $k$-type contact process, which can be derived from the branching random walk by killing all particles that are born at a site where already $k$ individuals are present. We prove that local survival of the branching random walk on $\mathbb{Z}^d$ also implies that for $k$ sufficiently large the associated $k$-type contact process survives on $\mathcal{C}_\infty$. This implies that the strong critical parameters of the branching random walk on $\mathbb{Z}^d$ and on $\mathcal{C}_\infty$ coincide and that their common value is the limit of the sequence of strong critical parameters of the associated $k$-type contact processes. These results are extended to a family of restrained branching random walks, that is, branching random walks where the success of the reproduction trials decreases with the size of the population in the target site.

## REFERENCES

[1] BARLOW, M. T. (2004). Random walks on supercritical percolation clusters. *Ann. Probab.* **32** 3024–3084. MR2094438

[2] BARLOW, M. T., PERES, Y. and SOUSI, P. (2012). Collisions of random walks. *Ann. Inst. Henri Poincaré Probab. Stat.* **48** 922–946. MR3052399

[3] BELHADJI, L., BERTACCHI, D. and ZUCCA, F. (2010). A self-regulating and patch subdivided population. *Adv. in Appl. Probab.* **42** 899–912. MR2779564

[4] BELHADJI, L. and LANCHIER, N. (2006). Individual versus cluster recoveries within a spatially structured population. *Ann. Appl. Probab.* **16** 403–422. MR2209347

[5] BERTACCHI, D. and BORRELLO, D. (2011). The small world effect on the coalescing time of random walks. *Stochastic Process. Appl.* **121** 925–956. MR2775102

[6] BERTACCHI, D., LANCHIER, N. and ZUCCA, F. (2011). Contact and voter processes on the infinite percolation cluster as models of host-symbiont interactions. *Ann. Appl. Probab.* **21** 1215–1252. MR2857447

[7] BERTACCHI, D., POSTA, G. and ZUCCA, F. (2007). Ecological equilibrium for restrained branching random walks. *Ann. Appl. Probab.* **17** 1117–1137. MR2344301

[8] BERTACCHI, D. and ZUCCA, F. (2003). Uniform asymptotic estimates of transition probabilities on combs. *J. Aust. Math. Soc.* **75** 325–353. MR2015321

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[9] BERTACCHI, D. and ZUCCA, F. (2008). Critical behaviors and critical values of branching random walks on multigraphs. *J. Appl. Probab.* **45** 481–497. MR2426846

[10] BERTACCHI, D. and ZUCCA, F. (2009). Characterization of critical values of branching random walks on weighted graphs through infinite-type branching processes. *J. Stat. Phys.* **134** 53–65. MR2489494

[11] BERTACCHI, D. and ZUCCA, F. (2009). Approximating critical parameters of branching random walks. *J. Appl. Probab.* **46** 463–478. MR2535826

[12] BERTACCHI, D. and ZUCCA, F. (2014). Strong local survival of branching random walks is not monotone. *Adv. in Appl. Probab.* **46** 400–421. MR3215539

[13] COMETS, F., MENSHIKOV, M. V. and POPOV, S. Y. (1998). One-dimensional branching random walk in a random environment: A classification. *Markov Process. Related Fields* **4** 465–477. MR1677053

[14] COURONNÉ, O. and MESSIKH, R. J. (2004). Surface order large deviations for 2D FK-percolation and Potts models. *Stochastic Process. Appl.* **113** 81–99. MR2078538

[15] DEN HOLLANDER, F., MENSHIKOV, M. V. and POPOV, S. Y. (1999). A note on transience versus recurrence for a branching random walk in random environment. *J. Stat. Phys.* **95** 587–614. MR1700867

[16] DURRETT, R. (1984). Oriented percolation in two dimensions. *Ann. Probab.* **12** 999–1040. MR0757768

[17] DURRETT, R. and JUNG, P. (2007). Two phase transitions for the contact process on small worlds. *Stochastic Process. Appl.* **117** 1910–1927. MR2437735

[18] GANTERT, N., MÜLLER, S., POPOV, S. and VACHKOVSKAIA, M. (2010). Survival of branching random walks in random environment. *J. Theoret. Probab.* **23** 1002–1014. MR2735734

[19] GRIMMETT, G. (1999). *Percolation*, 2nd ed. *Grundlehren der Mathematischen Wissenschaften* **321**. Springer, Berlin. MR1707339

[20] HUETER, I. and LALLEY, S. P. (2000). Anisotropic branching random walks on homogeneous trees. *Probab. Theory Related Fields* **116** 57–88. MR1736590

[21] LIGGETT, T. M. (1996). Branching random walks and contact processes on homogeneous trees. *Probab. Theory Related Fields* **106** 495–519. MR1421990

[22] LIGGETT, T. M. (1999). Branching random walks on finite trees. In *Perplexing Problems in Probability. Progress in Probability* **44** 315–330. Birkhäuser, Boston, MA. MR1703138

[23] LIGGETT, T. M. and SPITZER, F. (1981). Ergodic theorems for coupled random walks and other systems with locally interacting components. *Z. Wahrsch. Verw. Gebiete* **56** 443–468. MR0621659

[24] MACHADO, F. P. and POPOV, S. Y. (2000). One-dimensional branching random walks in a Markovian random environment. *J. Appl. Probab.* **37** 1157–1163. MR1808881

[25] MACHADO, F. P. and POPOV, S. Y. (2003). Branching random walk in random environment on trees. *Stochastic Process. Appl.* **106** 95–106. MR1983045

[26] MADRAS, N. and SCHINAZI, R. (1992). Branching random walks on trees. *Stochastic Process. Appl.* **42** 255–267. MR1176500

[27] MÜLLER, S. (2014). Interacting growth processes and invariant percolation. *Ann. Appl. Probab.* **25** 268–286. MR3297773

[28] MÜLLER, S. (2008). A criterion for transience of multidimensional branching random walk in random environment. *Electron. J. Probab.* **13** 1189–1202. MR2430704

[29] PEMANTLE, R. (1992). The contact process on trees. *Ann. Probab.* **20** 2089–2116. MR1188054

[30] PEMANTLE, R. and STACEY, A. M. (2001). The branching random walk and contact process on Galton–Watson and nonhomogeneous trees. *Ann. Probab.* **29** 1563–1590. MR1880232

[31] PISZTORA, A. (1996). Surface order large deviations for Ising, Potts and percolation models. *Probab. Theory Related Fields* **104** 427–466. MR1384040

[32] SENETA, E. (2006). *Non-negative Matrices and Markov Chains*. Springer, New York. MR2209438

[33] STACEY, A. (2003). Branching random walks on quasi-transitive graphs. *Combin*. *Probab. Comput*. **12** 345–358. MR1988981

[34] ZUCCA, F. (2011). Survival, extinction and approximation of discrete-time branching random walks. *J*. *Stat*. *Phys*. **142** 726–753. MR2773785

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# JIGSAW PERCOLATION: WHAT SOCIAL NETWORKS CAN COLLABORATIVELY SOLVE A PUZZLE?

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We introduce a new kind of percolation on finite graphs called *jigsaw percolation*. This model attempts to capture networks of people who innovate by merging ideas and who solve problems by piecing together solutions. Each person in a social network has a unique piece of a jigsaw puzzle. Acquainted people with compatible puzzle pieces merge their puzzle pieces. More generally, groups of people with merged puzzle pieces merge if the groups know one another and have a pair of compatible puzzle pieces. The social network solves the puzzle if it eventually merges all the puzzle pieces. For an Erdős–Rényi social network with $n$ vertices and edge probability $p_n$, we define the critical value $p_c(n)$ for a connected puzzle graph to be the $p_n$ for which the chance of solving the puzzle equals $1/2$. We prove that for the $n$-cycle (ring) puzzle, $p_c(n) = \Theta(1/\log n)$, and for an arbitrary connected puzzle graph with bounded maximum degree, $p_c(n) = O(1/\log n)$ and $\omega(1/n^b)$ for any $b > 0$. Surprisingly, with probability tending to 1 as the network size increases to infinity, social networks with a power-law degree distribution cannot solve any bounded-degree puzzle. This model suggests a mechanism for recent empirical claims that innovation increases with social density, and it might begin to show what social networks stifle creativity and what networks collectively innovate.

## REFERENCES

[1] AIZENMAN, M. and LEBOWITZ, J. L. (1988). Metastability effects in bootstrap percolation. *J. Phys. A* **21** 3801–3813. MR0968311
[2] BALL, P. (2014). Crowd-sourcing: Strength in numbers. *Nature* **506** 422–423.
[3] BARABÁSI, A. L., JEONG, H., NÉDA, Z., RAVASZ, E., SCHUBERT, A. and VICSEK, T. (2002). Evolution of the social network of scientific collaborations. *Phys. A* **311** 590–614. MR1943379
[4] BETTENCOURT, L., LOBO, J., HELBING, D., KÜHNERT, C. and WEST, G. B. (2007). Growth, innovation, scaling, and the pace of life in cities. *Proc. Natl. Acad. Sci. USA* **104** 7301.
[5] BETTENCOURT, L. M. A., CINTRÓN-ARIAS, A., KAISER, D. I. and CASTILLO-CHÁVEZ, C. (2006). The power of a good idea: Quantitative modeling of the spread of ideas from epidemiological models. *Phys. A* **364** 513–536.
[6] BETTENCOURT, L. M. A., KAISER, D. I. and KAUR, J. (2009). Scientific discovery and topological transitions in collaboration networks. *J. Informetr.* **3** 210–221.

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[7] BETTENCOURT, L. M. A., KAISER, D. I., KAUR, J., CASTILLO-CHÁVEZ, C. and WOJICK, D. E. (2008). Population modeling of the emergence and development of scientific fields. *Scientometrics* **75** 495–518.

[8] BETTENCOURT, L. M. A., LOBO, J., STRUMSKY, D. and WEST, G. B. (2010). Urban scaling and its deviations: Revealing the structure of wealth, innovation and crime across cities. *PLoS ONE* **5** e13541.

[9] CHAI, S. and FLEMING, L. (2011). Emergence of Breakthroughs. In *DIME-DRUID ACADEMY Winter Conference* 1–47. DRUID-DIME Academy, Aalborg, Denmark.

[10] CHEN, C., CHEN, Y., HOROWITZ, M., HOU, H., LIU, Z. and PELLEGRINO, D. (2009). Towards an explanatory and computational theory of scientific discovery. *J. Informetr.* **3** 191–209.

[11] CHUNG, F. and LU, L. (2002). The average distances in random graphs with given expected degrees. *Proc. Natl. Acad. Sci. USA* **99** 15879–15882 (electronic). MR1944974

[12] CIOINSIGHT (2004). *Web extra: Who's on first*?, CIOinsight (2004), 1–2. Available at http://www.cioinsight.com/c/a/Past-News/Web-Extra-Whos-on-First/.

[13] COPPERSMITH, D., GAMARNIK, D. and SVIRIDENKO, M. (2002). The diameter of a long-range percolation graph. *Random Structures Algorithms* **21** 1–13. MR1913075

[14] COWAN, R. and JONARD, N. (2007). Structural holes, innovation and the distribution of ideas. *J. Econ. Interac. Coord.* **2** 93–110.

[15] D'SOUZA, R. M. and MITZENMACHER, M. (2010). Local cluster aggregation models of explosive percolation. *Phys. Rev. Lett.* **104** 195702.

[16] DUCH, J., WAITZMAN, J. S. and AMARAL, L. A. N. (2010). Quantifying the performance of individual players in a team activity. *PLoS ONE* **5** e10937.

[17] ERDŐS, P. and RÉNYI, A. (1961). On the strength of connectedness of a random graph. *Acta Math. Acad. Sci. Hung.* **12** 261–267. MR0130187

[18] GERSTEIN, M. and DOUGLAS, S. M. (2007). RNAi development. *PLoS Comput. Biol.* **3** e80.

[19] GOWERS, T. and NIELSEN, M. (2009). Massively collaborative mathematics. *Nature* **461** 879–881.

[20] GRAVNER, J. and HOLROYD, A. E. (2008). Slow convergence in bootstrap percolation. *Ann. Appl. Probab.* **18** 909–928. MR2418233

[21] GRAVNER, J. and SIVAKOFF, D. (2013). Nucleation scaling in jigsaw percolation. Preprint. Available at arXiv:1310.2194.

[22] GRIMMETT, G. (1999). *Percolation*, 2nd ed. Springer, Berlin. MR1707339

[23] HOLROYD, A. E. (2003). Sharp metastability threshold for two-dimensional bootstrap percolation. *Probab. Theory Related Fields* **125** 195–224. MR1961342

[24] INTRONE, J., LAUBACHER, R., OLSON, G. and MALONE, T. (2011). The Climate CoLab: Large scale model-based collaborative planning. In *International Conference on Collaboration Technologies and Systems* (*CTS*), *May* 2011 40–47. MIT Center for Collective Intelligence, Cambridge, MA.

[25] JOHNSON, S. (2010). *Where Good Ideas Come from*: *The Natural History of Innovation*. Riverhead Hardcover, New York.

[26] LAKHANI, K. R., GARVIN, D. A. and LONSTEIN, E. (2010). TopCoder (A): Developing software through crowdsourcing. Harvard Business School General Management Unit 610–032 1–18.

[27] LAMBIOTTE, R. and PANZARASA, P. (2009). Communities, knowledge creation, and information diffusion. *J. Informetr.* **3** 180–190.

[28] LILJEROS, F., EDLING, C. R., AMARAL, L. A. N., STANLEY, H. E. and ABERG, Y. (2001). The web of human sexual contacts. *Nature* **411** 907–908.

[29] MOLLOY, M. and REED, B. (1995). A critical point for random graphs with a given degree sequence. *Random Structures Algorithms* **6** 161–180.

[30] MOORE, K. and NEELY, P. (2011). From social networks to collaboration networks: The next evolution of social media for business. *Forbes* September 15 1–3.

[31] NEWMAN, M. (2001). Scientific collaboration networks. I and II. *Phys*. *Rev*. *E* **64** 016131, 016132.

[32] NEWMAN, M. E. J. (2001). The structure of scientific collaboration networks. *Proc*. *Natl*. *Acad*. *Sci*. *USA* **98** 404–409 (electronic). MR1812610

[33] REDNER, S. (1998). How popular is your paper? An empirical study of the citation distribution. *Eur. Phys*. *J*. *B* **4** 131–134.

[34] SCHLÄPFER, M., BETTENCOURT, L. M. A., RASCHKE, M., CLAXTON, R., SMOREDA, Z., WEST, G. B. and RATTI, C. (2014). The scaling of human interactions with city size. *Journal of The Royal Society Interface* **11** 98.

[35] SCHULMAN, L. S. (1983). Ong range percolation in one dimension. *J*. *Phys*. *A* **16** L639–L641.

[36] SLIVKEN, E. (2013). Jigsaw percolation of Erdös–Rènyi random graphs. Preprint. Available at http://www.math.washington.edu/~slivken/jigsawER.pdf.

[37] SOOD, V., MATHIEU, M., SHREIM, A., GRASSBERGER, P. and PACZUSKI, M. (2010). Interacting branching process as a simple model of innovation. *Phys*. *Rev. Lett*. **105** 178701.

[38] UZZI, B. (2008). A social network's changing statistical properties and the quality of human innovation. *J*. *Phys*. *A* **41** 224023, 12. MR2453835

[39] UZZI, B. and SPIRO, J. (2005). Collaboration and creativity: The small world problem. *Am*. *J*. *Sociol*. **111** 447–504.

[40] VAN DEN ESKER, H., VAN DER HOFSTAD, R., HOOGHIEMSTRA, G. and ZNAMENSKI, D. (2005). Distances in random graphs with infinite mean degrees. *Extremes* **8** 111–141. MR2275914

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# AN INTEGRAL EQUATION FOR ROOT'S BARRIER AND THE GENERATION OF BROWNIAN INCREMENTS

BY PAUL GASSIAT[1], ALEKSANDAR MIJATOVIĆ[2] AND HARALD OBERHAUSER[1,3]

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We derive a nonlinear integral equation to calculate Root's solution of the Skorokhod embedding problem for atom-free target measures. We then use this to efficiently generate bounded time–space increments of Brownian motion and give a parabolic version of Muller's classic "Random walk over spheres" algorithm.

## REFERENCES

[1] ANKIRCHNER, S., HOBSON, D. and STRACK, P. (2014). Finite, integrable and bounded time embeddings for diffusions. *Bernoulli* **21** 1067–1088. MR3338657

[2] ANKIRCHNER, S. and STRACK, P. (2011). Skorokhod embeddings in bounded time. *Stoch. Dyn.* **11** 215–226. MR2836522

[3] BEIGLBOECK, M. and HUESMANN, M. (2013). Optimal transport and Skorokhod embedding. Preprint. Available at arXiv:1307.3656.

[4] CHACON, R. V. (1977). Potential processes. *Trans. Amer. Math. Soc.* **226** 39–58. MR0501374

[5] CHEN, X. and CHADAM, J. (2007). A mathematical analysis of the optimal exercise boundary for American put options. *SIAM J. Math. Anal.* **38** 1613–1641 (electronic). MR2286022

[6] COSTANTINI, C., GOBET, E. and EL KAROUI, N. (2006). Boundary sensitivities for diffusion processes in time dependent domains. *Appl. Math. Optim.* **54** 159–187. MR2239532

[7] COX, A. M. G. and WANG, J. (2011). Root's barrier: Construction, optimality and applications to variance options. Preprint. Available at arXiv:1104.3583.

[8] COX, A. M. G. and WANG, J. (2013). Optimal robust bounds for variance options. *Ann. Appl. Probab.* **23** 859–894.

[9] DEACONU, M. and HERRMANN, S. (2013). Hitting time for Bessel processes—Walk on moving spheres algorithm (WoMS). *Ann. Appl. Probab.* **23** 2259–2289. MR3127935

[10] DUPIRE, B. (2005). Arbitrage bounds for volatility derivatives as free boundary problem. Presentation, KTH, Stockholm.

[11] FELLER, W. (2008). *An Introduction to Probability Theory and Its Applications*. Wiley, New York.

[12] FRIEDMAN, A. and KINDERLEHRER, D. (1974/1975). A one phase Stefan problem. *Indiana Univ. Math. J.* **24** 1005–1035. MR0385326

[13] GALICHON, A., HENRY-LABORDÈRE, P. and TOUZI, N. (2014). A stochastic control approach to no-arbitrage bounds given marginals, with an application to lookback options. *Ann. Appl. Probab.* **24** 312–336. MR3161649

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[14] HOBSON, D. (2011). The Skorokhod embedding problem and model-independent bounds for option prices. In *Paris–Princeton Lectures on Mathematical Finance* 2010. *Lecture Notes in Math*. **2003** 267–318. Springer, Berlin. MR2762363

[15] HOBSON, D. G. (1998). Robust hedging of the lookback option. *Finance Stoch*. **2** 329–347.

[16] HUNTER, J. D. (2007). Matplotlib: A 2D graphics environment. *Comput*. *Sci*. *Eng*. **9** 90–95.

[17] ISHII, H. (1995). On the equivalence of two notions of weak solutions, viscosity solutions and distribution solutions. *Funkcial*. *Ekvac*. **38** 101–120. MR1341739

[18] JONES, E., OLIPHANT, T., PETERSON, P. et al. (2001). SciPy: Open source scientific tools for Python. Available at http://www.scipy.org/.

[19] LIEBERMAN, G. M. (1996). *Second Order Parabolic Differential Equations*. World Scientific, River Edge, NJ. MR1465184

[20] LINZ, P. (1985). *Analytical and Numerical Methods for Volterra Equations*. *SIAM Studies in Applied Mathematics* **7**. SIAM, Philadelphia, PA. MR0796318

[21] LOYNES, R. M. (1970). Stopping times on Brownian motion: Some properties of Root's construction. *Z. Wahrsch*. *Verw*. *Gebiete* **16** 211–218. MR0292170

[22] MCCONNELL, T. R. (1991). The two-sided Stefan problem with a spatially dependent latent heat. *Trans*. *Amer. Math*. *Soc*. **326** 669–699. MR1008699

[23] MCKEAN, H. P. JR. (1965). Appendix: A free boundary problem for the heat equation arising from a problem in mathematical economics. *Ind*. *Manage*. *Rev*. **6** 32–39.

[24] MILSTEIN, G. N. (1997). Weak approximation of a diffusion process in a bounded domain. *Stoch*. *Stoch*. *Rep*. **62** 147–200. MR1489185

[25] MILSTEIN, G. N. and TRETYAKOV, M. V. (1999). Simulation of a space–time bounded diffusion. *Ann*. *Appl*. *Probab*. **9** 732–779. MR1722281

[26] MONROE, I. (1972). On embedding right continuous martingales in Brownian motion. *Ann*. *Math*. *Statist*. **43** 1293–1311. MR0343354

[27] MULLER, M. E. (1956). Some continuous Monte Carlo methods for the Dirichlet problem. *Ann*. *Math*. *Statist*. **27** 569–589. MR0088786

[28] OBERHAUSER, H. and DOS REIS, G. (2013). Root's barrier, viscosity solutions of obstacle problems and reflected FBSDEs. Preprint. Available at arXiv:1301.3798.

[29] OBŁÓJ, J. (2004). The Skorokhod embedding problem and its offspring. *Probab*. *Surv*. **1** 321–390. MR2068476

[30] PESKIR, G. (2005). On the American option problem. *Math*. *Finance* **15** 169–181. MR2116800

[31] ROOT, D. H. (1969). The existence of certain stopping times on Brownian motion. *Ann*. *Math*. *Statist*. **40** 715–718. MR0238394

[32] ROST, H. (1976). Skorokhod stopping times of minimal variance. In *Séminaire de Probabilités*, *X* (*Première Partie*, *Univ*. *Strasbourg*, *Strasbourg*, *Année Universitaire* 1974/1975). *Lecture Notes in Math*. **511** 194–208. Springer, Berlin. MR0445600

[33] SKOROHOD, A. V. (1961). *Issledovaniya po Teorii Sluchainykh Protsessov* (*Stokhasticheskie Differentsialnye Uravneniya i Predelnye Teoremy Dlya Protsessov Markova*). Izdat. Kiev. Univ., Kiev. MR0185619

[34] ZEIN, S., LEJAY, A. and DEACONU, M. (2010). An efficient algorithm to simulate a Brownian motion over irregular domains. *Commun*. *Comput*. *Phys*. **8** 901–916. MR2673769

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# HEDGING, ARBITRAGE AND OPTIMALITY WITH SUPERLINEAR FRICTIONS

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In a continuous-time model with multiple assets described by càdlàg processes, this paper characterizes superhedging prices, absence of arbitrage, and utility maximizing strategies, under general frictions that make execution prices arbitrarily unfavorable for high trading intensity. Such frictions induce a duality between feasible trading strategies and shadow execution prices with a martingale measure. Utility maximizing strategies exist even if arbitrage is present, because it is not scalable at will.

## REFERENCES

ALMGREN, R. and CHRISS, N. (2001). Optimal execution of portfolio transactions. *Journal of Risk* **3** 5–40.

ASTIC, F. and TOUZI, N. (2007). No arbitrage conditions and liquidity. *J. Math. Econom.* **43** 692–708. MR2331745

BALDER, E. J. (1989). Infinite-dimensional extension of a theorem of Komlós. *Probab. Theory Related Fields* **81** 185–188. MR0982652

BANK, P. and KRAMKOV, D. (2011a). A model for a large investor trading at market indifference prices. I: Single-period case. Preprint. Available at arXiv:1110.3224.

BANK, P. and KRAMKOV, D. (2011b). A model for a large investor trading at market indifference prices. II: Continuous-time case. Preprint. Available at arXiv:1110.3229.

BERTSIMAS, D. and LO, A. (1998). Optimal control of execution costs. *Journal of Financial Markets* **1** 1–50.

BLACK, F. (1986). Noise. *J. Finance* **41** 529–543.

BOUCHARD, B. and NGUYEN HUU, A. (2013). No marginal arbitrage of the second kind for high production regimes in discrete time production-investment models with proportional transaction costs. *Math. Finance* **23** 366–386. MR3034082

BOUCHARD, B. and TAFLIN, E. (2013). No-arbitrage of second kind in countable markets with proportional transaction costs. *Ann. Appl. Probab.* **23** 427–454. MR3059265

ÇETIN, U., JARROW, R. A. and PROTTER, P. (2004). Liquidity risk and arbitrage pricing theory. *Finance Stoch.* **8** 311–341. MR2213255

ÇETIN, U. and ROGERS, L. C. G. (2007). Modeling liquidity effects in discrete time. *Math. Finance* **17** 15–29. MR2281790

ÇETIN, U., SONER, H. M. and TOUZI, N. (2010). Option hedging for small investors under liquidity costs. *Finance Stoch.* **14** 317–341. MR2670416

CONT, R., KUKANOV, A. and STOIKOV, S. (2014). The price impact of order book events. *Journal of Financial Econometrics* **12** 47–88.

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DELBAEN, F. and SCHACHERMAYER, W. (2006). *The Mathematics of Arbitrage*. Springer, Berlin. MR2200584

DELLACHERIE, C. and MEYER, P.-A. (1978). *Probabilities and Potential*. *North-Holland Mathematics Studies* **29**. North-Holland, Amsterdam. MR0521810

DELLACHERIE, C. and MEYER, P.-A. (1982). *Probabilities and Potential*. *B*: *Theory of Martingales*. *North-Holland Mathematics Studies* **72**. North-Holland, Amsterdam. MR0745449

DENIS, E. and KABANOV, Y. (2012). Consistent price systems and arbitrage opportunities of the second kind in models with transaction costs. *Finance Stoch*. **16** 135–154. MR2872651

DERMODY, J. C. and ROCKAFELLAR, R. T. (1991). Cash stream valuation in the face of transaction costs and taxes. *Math*. *Finance* **1** 31–54.

DERMODY, J. C. and ROCKAFELLAR, R. T. (1995). Tax basis and nonlinearity in cash stream valuation. *Math*. *Finance* **5** 97–119.

DOLINSKY, Y. and SONER, H. M. (2013). Duality and convergence for binomial markets with friction. *Finance Stoch*. **17** 447–475. MR3066984

DUFOUR, A. and ENGLE, R. F. (2000). Time and the price impact of a trade. *J. Finance* **55** 2467–2498.

FERNHOLZ, R., KARATZAS, I. and KARDARAS, C. (2005). Diversity and relative arbitrage in equity markets. *Finance Stoch*. **9** 1–27. MR2210925

GARLEANU, N. and PEDERSEN, L. (2013). Dynamic trading with predictable returns and transaction costs. *J. Finance* **68** 2309–2340.

GUASONI, P., LÉPINETTE, E. and RÁSONYI, M. (2012). The fundamental theorem of asset pricing under transaction costs. *Finance Stoch*. **16** 741–777. MR2972240

GUASONI, P., RÁSONYI, M. and SCHACHERMAYER, W. (2008). Consistent price systems and face-lifting pricing under transaction costs. *Ann*. *Appl*. *Probab.* **18** 491–520. MR2398764

INGERSOLL, J. E. (1987). *Theory of Financial Decision Making*. Rowman & Littlefield, Lanham, MD.

JOUINI, E. and KALLAL, H. (1995). Martingales and arbitrage in securities markets with transaction costs. *J*. *Econom*. *Theory* **66** 178–197. MR1338025

KABANOV, YU. M. and KRAMKOV, D. O. (1994). Large financial markets: Asymptotic arbitrage and contiguity. *Teor. Veroyatn*. *Primen*. **39** 222–229. MR1348197

KABANOV, Y. and SAFARIAN, M. (2009). *Markets with Transaction Costs*: *Mathematical Theory*. Springer, Berlin. MR2589621

KALLSEN, J. and MUHLE-KARBE, J. (2010). On using shadow prices in portfolio optimization with transaction costs. *Ann*. *Appl*. *Probab.* **20** 1341–1358. MR2676941

KARATZAS, I. and KARDARAS, C. (2007). The numéraire portfolio in semimartingale financial models. *Finance Stoch*. **11** 447–493. MR2335830

KOMLÓS, J. (1967). A generalization of a problem of Steinhaus. *Acta Math*. *Acad*. *Sci*. *Hungar.* **18** 217–229. MR0210177

KYLE, A. (1985). Continuous auctions and insider trading. *Econometrica* **29** 1315–1335.

MARIS, F. and SAYIT, H. (2012). Consistent price systems in multiasset markets. *Int*. *J*. *Stoch*. *Anal*. Art. ID 687376, 14. MR2964813

PENNANEN, T. (2011a). Superhedging in illiquid markets. *Math*. *Finance* **21** 519–540. MR2830432

PENNANEN, T. (2011b). Arbitrage and deflators in illiquid markets. *Finance Stoch*. **15** 57–83. MR2764443

PENNANEN, T. (2014). Optimal investment and contingent claim valuation in illiquid markets. *Finance Stoch.* To appear.

PENNANEN, T. and PENNER, I. (2010). Hedging of claims with physical delivery under convex transaction costs. *SIAM J*. *Financial Math*. **1** 158–178. MR2658577

RÁSONYI, M. (2009). Arbitrage under transaction costs revisited. In *Optimality and Risk—Modern Trends in Mathematical Finance* 211–225. Springer, Berlin. MR2648605

ROGERS, L. C. G. (1997). Arbitrage with fractional Brownian motion. *Math*. *Finance* **7** 95–105. MR1434408

ROGERS, L. C. G. and SINGH, S. (2010). The cost of illiquidity and its effects on hedging. *Math*. *Finance* **20** 597–615. MR2731409

SCHACHERMAYER, W. (2004). The fundamental theorem of asset pricing under proportional transaction costs in finite discrete time. *Math*. *Finance* **14** 19–48. MR2030834

SCHIED, A. and SCHÖNEBORN, T. (2009). Risk aversion and the dynamics of optimal liquidation strategies in illiquid markets. *Finance Stoch*. **13** 181–204. MR2482051

V. WEIZSÄCKER, H. (2004). Can one drop $L^1$-boundedness in Komlós's subsequence theorem? *Amer. Math*. *Monthly* **111** 900–903. MR2104694

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# GLOBAL SOLVABILITY OF A NETWORKED INTEGRATE-AND-FIRE MODEL OF MCKEAN–VLASOV TYPE[1]

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We here investigate the well-posedness of a networked integrate-and-fire model describing an infinite population of neurons which interact with one another through their common statistical distribution. The interaction is of the self-excitatory type as, at any time, the potential of a neuron increases when some of the others fire: precisely, the kick it receives is proportional to the instantaneous proportion of firing neurons at the same time. From a mathematical point of view, the coefficient of proportionality, denoted by $\alpha$, is of great importance as the resulting system is known to blow-up for large values of $\alpha$. In the current paper, we focus on the complementary regime and prove that existence and uniqueness hold for all time when $\alpha$ is small enough.

## REFERENCES

BRUNEL, N. (2000). Dynamics of sparsely connected networks of excitatory and inhibitory spiking neurons. *J. Comput. Neurosci.* **8** 183–208.

BRUNEL, N. and HAKIM, V. (1999). Fast global oscillations in networks of integrate-and-fire neurons with low firing rates. *Neural Comput.* **11** 1621–1671.

CÁCERES, M. J., CARRILLO, J. A. and PERTHAME, B. (2011). Analysis of nonlinear noisy integrate & fire neuron models: Blow-up and steady states. *J. Math. Neurosci.* **1** Art. 7, 33. MR2853216

CARRILLO, J. A., GONZÁLEZ, M. D. M., GUALDANI, M. P. and SCHONBEK, M. E. (2013). Classical solutions for a nonlinear Fokker–Planck equation arising in computational neuroscience. *Comm. Partial Differential Equations* **38** 385–409. MR3019444

DELARUE, F. and MENOZZI, S. (2010). Density estimates for a random noise propagating through a chain of differential equations. *J. Funct. Anal.* **259** 1577–1630. MR2659772

DELARUE, F., INGLIS, J., RUBENTHALER, S. and TANRÉ, E. (2013). First hitting times for general non-homogeneous 1d diffusion processes: Density estimates in small time. Technical report. Available at http://hal.archives-ouvertes.fr/hal-00870991.

FRIEDMAN, A. (1964). *Partial Differential Equations of Parabolic Type*. Prentice Hall International, Englewood Cliffs, NJ. MR0181836

GARRONI, M. G. and MENALDI, J. L. (1992). *Green Functions for Second Order Parabolic Integro-Differential Problems*. Longman, Harlow, UK. MR1202037

JOLIVET, R., LEWIS, T. J. and GERSTNER, W. (2004). Generalized integrate-and-fire models of neuronal activity approximate spike trains of a detailed model to a high degree of accuracy. *J. Neurophysiol.* **92** 959–976.

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*Key words and phrases.* McKean nonlinear diffusion process, renewal process, first hitting time density estimates, integrate-and-fire network, nonhomogeneous diffusion process, neuroscience.

KISTLER, W., GERSTNER, W. and VAN HEMMEN, J. L. (1997). Reduction of the Hodgkin–Huxley equations to a single-variable threshold model. *Neural Comput*. **5** 1015–1045.

KRYLOV, N. V. (1980). *Controlled Diffusion Processes*. *Applications of Mathematics* **14**. Springer, New York. Translated from the Russian by A. B. Aries. MR0601776

KRYLOV, N. V. and SAFONOV, M. V. (1979). An estimate for the probability of a diffusion process hitting a set of positive measure. *Dokl*. *Akad*. *Nauk SSSR* **245** 18–20. MR0525227

LEWIS, T. J. and RINZEL, J. (2003). Dynamics of spiking neurons connected by both inhibitory and electrical coupling. *J*. *Comput*. *Neurosci*. **14** 283–309.

LIEBERMAN, G. M. (1996). *Second Order Parabolic Differential Equations*. World Scientific, River Edge, NJ. MR1465184

OSTOJIC, S., BRUNEL, N. and HAKIM, V. (2009). Synchronization properties of networks of electrically coupled neurons in the presence of noise and heterogeneities. *J*. *Comput*. *Neurosci*. **26** 369–392. MR2505036

PESKIR, G. and SHIRYAEV, A. (2006). *Optimal Stopping and Free-Boundary Problems*. Birkhäuser, Basel. MR2256030

RENART, A., BRUNEL, N. and WANG, X.-J. (2004). Mean-field theory of irregularly spiking neuronal populations and working memory in recurrent cortical networks. In *Computational Neuroscience* 431–490. Chapman & Hall, Boca Raton, FL. MR2029670

SACERDOTE, L. and GIRAUDO, M. T. (2013). Stochastic integrate and fire models: A review on mathematical methods and their applications. In *Stochastic Biomathematical Models*. *Lecture Notes in Math*. **2058** 99–148. Springer, Heidelberg. MR3051031

STROOCK, D. W. and VARADHAN, S. R. S. (1979). *Multidimensional Diffusion Processes*. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **233**. Springer, Berlin. MR0532498

SZNITMAN, A.-S. (1991). Topics in propagation of chaos. In *École d'Été de Probabilités de Saint–Flour XIX*—1989. *Lecture Notes in Math*. **1464** 165–251. Springer, Berlin. MR1108185

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# RANDOMIZED AND BACKWARD SDE REPRESENTATION FOR OPTIMAL CONTROL OF NON-MARKOVIAN SDES

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We study optimal stochastic control problems for non-Markovian stochastic differential equations (SDEs) where the drift, diffusion coefficients and gain functionals are path-dependent, and importantly we do not make any ellipticity assumptions on the SDE. We develop a control randomization approach and prove that the value function can be reformulated under a family of dominated measures on an enlarged filtered probability space. This value function is then characterized by a backward SDE with nonpositive jumps under a single probability measure, which can be viewed as a path-dependent version of the Hamilton–Jacobi–Bellman equation, and an extension to $G$-expectation.

## REFERENCES

[1] BERTSEKAS, D. P. and SHREVE, S. E. (1978). *Stochastic Optimal Control*: *The Discrete Time Case*. *Mathematics in Science and Engineering* **139**. Academic Press, New York. MR0511544

[2] BOUCHARD, B. (2009). A stochastic target formulation for optimal switching problems in finite horizon. *Stochastics* **81** 171–197. MR2571686

[3] CHERIDITO, P., SONER, H. M., TOUZI, N. and VICTOIR, N. (2007). Second-order backward stochastic differential equations and fully nonlinear parabolic PDEs. *Comm*. *Pure Appl*. *Math*. **60** 1081–1110. MR2319056

[4] CRANDALL, M. G., ISHII, H. and LIONS, P.-L. (1992). User's guide to viscosity solutions of second order partial differential equations. *Bull*. *Amer*. *Math*. *Soc*. (*N.S.*) **27** 1–67. MR1118699

[5] DENIS, L., HU, M. and PENG, S. (2011). Function spaces and capacity related to a sublinear expectation: Application to $G$-Brownian motion paths. *Potential Anal*. **34** 139–161. MR2754968

[6] EKREN, I., KELLER, C., TOUZI, N. and ZHANG, J. (2014). On viscosity solutions of path dependent PDEs. *Ann*. *Probab*. **42** 204–236. MR3161485

[7] EL KAROUI, N. (1981). Les aspects probabilistes du contrôle stochastique. In *Ninth Saint Flour Probability Summer School*—1979 (*Saint Flour*, 1979). *Lecture Notes in Math*. **876** 73–238. Springer, Berlin. MR0637471

[8] EL KAROUI, N., NGUYEN, D. H. and JEANBLANC-PICQUÉ, M. (1988). Existence of an optimal Markovian filter for the control under partial observations. *SIAM J. Control Optim*. **26** 1025–1061. MR0957652

[9] EL KAROUI, N. and TAN, X. (2013). Capacities measurable selection and dynamic programming Part II: Application in stochastic control problems. Preprint. Available at arXiv:1310.3364.

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[10] FABBRI, G., GOZZI, F. and ŚWIĘCH, A. (2015). *Stochastic Optimal Control in Infinite Dimensions: Dynamic Programming and HJB Equations*. Book in preparation.

[11] FLEMING, W. H. (1977). Generalized solutions in optimal stochastic control. In *Differential Games and Control Theory*, *II* (*Proc*. 2*nd Conf*., *Univ. Rhode Island*, *Kingston*, *R.I*., 1976) *Lecture Notes in Pure and Appl*. *Math*. **30** 147–165. Dekker, New York. MR0688675

[12] FLEMING, W. H. and SONER, H. M. (2006). *Controlled Markov Processes and Viscosity Solutions*, 2nd ed. *Stochastic Modelling and Applied Probability* **25**. Springer, New York. MR2179357

[13] HE, S. W., WANG, J. G. and YAN, J. A. (1992). *Semimartingale Theory and Stochastic Calculus*. CRC Press, Boca Raton, FL. MR1219534

[14] JACOD, J. (1975). Multivariate point processes: Predictable projection, Radon–Nikodým derivatives, representation of martingales. *Z. Wahrsch*. *Verw*. *Gebiete* **31** 235–253. MR0380978

[15] KHARROUBI, I., LANGRENÉ, N. and PHAM, H. (2014). Discrete time approximation of fully nonlinear HJB equations via BSDEs with nonpositive jumps. *Ann. Appl. Probab.* To appear. Available at arXiv:1311.4505.

[16] KHARROUBI, I., MA, J., PHAM, H. and ZHANG, J. (2010). Backward SDEs with constrained jumps and quasi-variational inequalities. *Ann*. *Probab*. **38** 794–840. MR2642892

[17] KHARROUBI, I. and PHAM, H. (2014). Feynman–Kac representation for Hamilton–Jacobi–Bellman IPDE. *Ann. Probab.* To appear.

[18] KRYLOV, N. V. (1980). *Controlled Diffusion Processes*. *Applications of Mathematics* **14**. Springer, Berlin. MR0601776

[19] NUTZ, M. (2012). A quasi-sure approach to the control of non-Markovian stochastic differential equations. *Electron*. *J*. *Probab*. **17** 23. MR2900464

[20] NUTZ, M. and VAN HANDEL, R. (2013). Constructing sublinear expectations on path space. *Stochastic Process*. *Appl*. **123** 3100–3121. MR3062438

[21] PARDOUX, É. and PENG, S. (1992). Backward stochastic differential equations and quasilinear parabolic partial differential equations. In *Stochastic Partial Differential Equations and Their Applications* (*Charlotte*, *NC*, 1991) (B. Rozovskii and R. Sowers, eds.). *Lecture Notes in Control and Inform*. *Sci*. **176** 200–217. Springer, Berlin. MR1176785

[22] PENG, S. (2007). $G$-expectation, $G$-Brownian motion and related stochastic calculus of Itô type. In *Stochastic Analysis and Applications*. *Abel Symp*. **2** 541–567. Springer, Berlin. MR2397805

[23] PENG, S. (2011). Note on viscosity solutions of path-dependent PDE and $G$-martingales. Preprint. Available at arXiv:1106.1144v1.

[24] PHAM, H. (2009). *Continuous-Time Stochastic Control and Optimization with Financial Applications*. *Stochastic Modelling and Applied Probability* **61**. Springer, Berlin. MR2533355

[25] PROTTER, P. and SHIMBO, K. (2008). No arbitrage and general semimartingales. In *Markov Processes and Related Topics*: *A Festschrift for Thomas G*. *Kurtz*. *Inst*. *Math*. *Stat*. *Collect*. **4** 267–283. IMS, Beachwood, OH. MR2574236

[26] ROGERS, L. C. G. and WILLIAMS, D. (2000). *Diffusions*, *Markov Processes*, *and Martingales*: *Itô Calculus*, *Vol*. 2. Cambridge Univ. Press, Cambridge. MR1780932

[27] SONER, H. M., TOUZI, N. and ZHANG, J. (2012). Wellposedness of second order backward SDEs. *Probab. Theory Related Fields* **153** 149–190. MR2925572

[28] TANG, S. and ZHANG, F. (2013). Path-dependent optimal stochastic control and viscosity solution of associated Bellman equations. Preprint. Available at arxiv:1210.2078.

[29] TANG, S. J. and LI, X. J. (1994). Necessary conditions for optimal control of stochastic systems with random jumps. *SIAM J*. *Control Optim*. **32** 1447–1475. MR1288257

[30] YONG, J. and ZHOU, X. Y. (1999). *Stochastic Controls*: *Hamiltonian Systems and HJB Equations*. *Applications of Mathematics* (*New York*) **43**. Springer, New York. MR1696772

[31] ZABCZYK, J. (1996). *Chance and Decision*: *Stochastic Control in Discrete Time*. Scuola Normale Superiore, Pisa. MR1678432

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# ON WELL-POSEDNESS OF FORWARD–BACKWARD SDES—A UNIFIED APPROACH

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In this paper, we study the well-posedness of the Forward–Backward Stochastic Differential Equations (FBSDE) in a general non-Markovian framework. The main purpose is to find a unified scheme which combines all existing methodology in the literature, and to address some fundamental long-standing problems for non-Markovian FBSDEs. An important device is a *decoupling random field* that is *regular* (uniformly Lipschitz in its spatial variable). We show that the regulariy of such decoupling field is closely related to the bounded solution to an associated *characteristic BSDE*, a backward stochastic Riccati-type equation with superlinear growth in both components $Y$ and $Z$. We establish various sufficient conditions for the well-posedness of an ODE that dominates the characteristic BSDE, which leads to the existence of the desired regular decoupling random field, whence the solvability of the original FBSDE. A synthetic analysis of the solvability is given, as a "User's Guide," for a large class of FBSDEs that are not covered by the existing methods. Some of them have important implications in applications.

## REFERENCES

[1] ANTONELLI, F. (1993). Backward–forward stochastic differential equations. *Ann. Appl. Probab.* **3** 777–793. MR1233625

[2] BRIAND, P. and HU, Y. (2006). BSDE with quadratic growth and unbounded terminal value. *Probab. Theory Related Fields* **136** 604–618. MR2257138

[3] CVITANIĆ, J. and MA, J. (1996). Hedging options for a large investor and forward–backward SDE's. *Ann. Appl. Probab.* **6** 370–398. MR1398050

[4] CVITANIĆ, J. and ZHANG, J. (2005). The steepest descent method for forward–backward SDEs. *Electron. J. Probab.* **10** 1468–1495 (electronic). MR2191636

[5] DELARUE, F. (2002). On the existence and uniqueness of solutions to FBSDEs in a non-degenerate case. *Stochastic Process. Appl.* **99** 209–286. MR1901154

[6] HU, Y., MA, J. and YONG, J. (2002). On semi-linear degenerate backward stochastic partial differential equations. *Probab. Theory Related Fields* **123** 381–411. MR1918539

[7] HU, Y. and PENG, S. (1995). Solution of forward–backward stochastic differential equations. *Probab. Theory Related Fields* **103** 273–283. MR1355060

[8] KARATZAS, I. and SHREVE, S. E. (1991). *Brownian Motion and Stochastic Calculus*, 2nd ed. Springer, New York. MR1121940

[9] KAZAMAKI, N. and SEKIGUCHI, T. (1979). On the transformation of some classes of martingales by a change of law. *Tohoku Math. J.* **31** 261–279. MR0547641

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[10] KOBYLANSKI, M. (2000). Backward stochastic differential equations and partial differential equations with quadratic growth. *Ann. Probab.* **28** 558–602. MR1782267

[11] KOHLMANN, M. and TANG, S. (2002). Global adapted solution of one-dimensional backward stochastic Riccati equations, with application to the mean-variance hedging. *Stochastic Process. Appl.* **97** 255–288. MR1875335

[12] MA, J., PROTTER, P. and YONG, J. M. (1994). Solving forward–backward stochastic differential equations explicitly—a four step scheme. *Probab. Theory Related Fields* **98** 339–359. MR1262970

[13] MA, J., YIN, H. and ZHANG, J. (2012). On non-Markovian forward–backward SDEs and backward stochastic PDEs. *Stochastic Process. Appl.* **122** 3980–4004. MR2971722

[14] MA, J. and YONG, J. (1997). Adapted solution of a degenerate backward SPDE, with applications. *Stochastic Process. Appl.* **70** 59–84. MR1472959

[15] MA, J. and YONG, J. (1999). On linear, degenerate backward stochastic partial differential equations. *Probab. Theory Related Fields* **113** 135–170. MR1676768

[16] MA, J. and YONG, J. (1999). *Forward–Backward Stochastic Differential Equations and Their Applications. Lecture Notes in Math.* **1702**. Springer, Berlin. MR1704232

[17] PARDOUX, E. and TANG, S. (1999). Forward–backward stochastic differential equations and quasilinear parabolic PDEs. *Probab. Theory Related Fields* **114** 123–150. MR1701517

[18] PENG, S. and WU, Z. (1999). Fully coupled forward–backward stochastic differential equations and applications to optimal control. *SIAM J. Control Optim.* **37** 825–843. MR1675098

[19] TANG, S. (2003). General linear quadratic optimal stochastic control problems with random coefficients: Linear stochastic Hamilton systems and backward stochastic Riccati equations. *SIAM J. Control Optim.* **42** 53–75 (electronic). MR1982735

[20] WU, Z. (1999). The comparison theorem of FBSDE. *Statist. Probab. Lett.* **44** 1–6. MR1706370

[21] WU, Z. (2003). Fully coupled FBSDE with Brownian motion and Poisson process in stopping time duration. *J. Aust. Math. Soc.* **74** 249–266. MR1957882

[22] WU, Z. and XU, M. (2009). Comparison theorems for forward backward SDEs. *Statist. Probab. Lett.* **79** 426–435. MR2494634

[23] WU, Z. and YU, Z. (2014). Probabilistic interpretation for a system of quasilinear parabolic partial differential equation combined with algebra equations. *Stochastic Process. Appl.* **124** 3921–3947.

[24] YONG, J. (1997). Finding adapted solutions of forward–backward stochastic differential equations: Method of continuation. *Probab. Theory Related Fields* **107** 537–572. MR1440146

[25] YONG, J. (2006). Linear forward–backward stochastic differential equations with random coefficients. *Probab. Theory Related Fields* **135** 53–83. MR2214151

[26] YONG, J. (2010). Forward–backward stochastic differential equations with mixed initial-terminal conditions. *Trans. Amer. Math. Soc.* **362** 1047–1096. MR2551515

[27] YU, Z. (2013). Equivalent cost functionals and stochastic linear quadratic optimal control problems. *ESAIM Control Optim. Calc. Var.* **19** 78–90. MR3023061

[28] ZHANG, J. (2006). The wellposedness of FBSDEs. *Discrete Contin. Dyn. Syst. Ser. B* **6** 927–940 (electronic). MR2223916

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# THE VERTEX-CUT-TREE OF GALTON–WATSON TREES CONVERGING TO A STABLE TREE

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We consider a fragmentation of discrete trees where the internal vertices are deleted independently at a rate proportional to their degree. Informally, the associated cut-tree represents the genealogy of the nested connected components created by this process. We essentially work in the setting of Galton–Watson trees with offspring distribution belonging to the domain of attraction of a stable law of index $\alpha \in (1, 2)$. Our main result is that, for a sequence of such trees $\mathcal{T}_n$ conditioned to have size $n$, the corresponding rescaled cut-trees converge in distribution to the stable tree of index $\alpha$, in the sense induced by the Gromov–Prokhorov topology. This gives an analogue of a result obtained by Bertoin and Miermont in the case of Galton–Watson trees with finite variance.

## REFERENCES

[1] ABRAHAM, R. and DELMAS, J. F. (2013). Beta-coalescents and stable Galton–Watson trees. Unpublished manuscript.

[2] ABRAHAM, R. and DELMAS, J.-F. (2013). The forest associated with the record process on a Lévy tree. *Stochastic Process. Appl.* **123** 3497–3517. MR3071387

[3] ABRAHAM, R., DELMAS, J.-F. and VOISIN, G. (2010). Pruning a Lévy continuum random tree. *Electron. J. Probab.* **15** 1429–1473. MR2727317

[4] ADDARIO-BERRY, L., BROUTIN, N. and HOLMGREN, C. (2014). Cutting down trees with a Markov chainsaw. *Ann. Appl. Probab.* **24** 2297–2339. MR3262504

[5] ALDOUS, D. (1993). The continuum random tree. III. *Ann. Probab.* **21** 248–289. MR1207226

[6] ALDOUS, D. and PITMAN, J. (1998). The standard additive coalescent. *Ann. Probab.* **26** 1703–1726. MR1675063

[7] BERTOIN, J. (1998). *Lévy Processes*. Cambridge Univ. Press, Cambridge.

[8] BERTOIN, J. (2000). A fragmentation process connected to Brownian motion. *Probab. Theory Related Fields* **117** 289–301. MR1771665

[9] BERTOIN, J. (2002). Self-similar fragmentations. *Ann. Inst. Henri Poincaré Probab. Stat.* **38** 319–340. MR1899456

[10] BERTOIN, J. (2006). *Random Fragmentation and Coagulation Processes*. *Cambridge Studies in Advanced Mathematics* **102**. Cambridge Univ. Press, Cambridge. MR2253162

[11] BERTOIN, J. (2012). Fires on trees. *Ann. Inst. Henri Poincaré Probab. Stat.* **48** 909–921. MR3052398

[12] BERTOIN, J. and MIERMONT, G. (2013). The cut-tree of large Galton–Watson trees and the Brownian CRT. *Ann. Appl. Probab.* **23** 1469–1493. MR3098439

[13] BINGHAM, N. H., GOLDIE, C. M. and TEUGELS, J. L. (1987). *Regular Variation*. Cambridge Univ. Press, Cambridge. MR0898871

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[14] DONEY, R. A. (1997). One-sided local large deviation and renewal theorems in the case of infinite mean. *Probab. Theory Related Fields* **107** 451–465. MR1440141

[15] DROSSEL, B. and SCHWABL, F. (1992). Self-organized critical forest fire model. *Phys. Rev. Lett.* **69** 1629–1632.

[16] DUQUESNE, T. (2003). A limit theorem for the contour process of conditioned Galton–Watson trees. *Ann. Probab.* **31** 996–1027. MR1964956

[17] DUQUESNE, T. and LE GALL, J.-F. (2002). Random trees, Lévy processes and spatial branching processes. *Astérisque* **281** vi+147. MR1954248

[18] DUQUESNE, T. and LE GALL, J.-F. (2005). Probabilistic and fractal aspects of Lévy trees. *Probab. Theory Related Fields* **131** 553–603. MR2147221

[19] GREVEN, A., PFAFFELHUBER, P. and WINTER, A. (2009). Convergence in distribution of random metric measure spaces ($\Lambda$-coalescent measure trees). *Probab. Theory Related Fields* **145** 285–322. MR2520129

[20] IBRAGIMOV, I. A. and LINNIK, YU. V. (1971). *Independant and Stationnary Sequences of Random Variables*. Wolters-Noordhoff, Groningen.

[21] JANSON, S. (2006). Random cutting and records in deterministic and random trees. *Random Structures Algorithms* **29** 139–179. MR2245498

[22] LYONS, R., PEMANTLE, R. and PERES, Y. (1995). Conceptual proofs of $L \log L$ criteria for mean behavior of branching processes. *Ann. Probab.* **23** 1125–1138. MR1349164

[23] MEIR, A. and MOON, J. W. (1970). Cutting down random trees. *J. Aust. Math. Soc.* **11** 313–324. MR0284370

[24] MIERMONT, G. (2003). Self-similar fragmentations derived from the stable tree. I. Splitting at heights. *Probab. Theory Related Fields* **127** 423–454. MR2018924

[25] MIERMONT, G. (2005). Self-similar fragmentations derived from the stable tree. II. Splitting at nodes. *Probab. Theory Related Fields* **131** 341–375. MR2123249

[26] PANHOLZER, A. (2006). Cutting down very simple trees. *Quaest. Math.* **29** 211–227. MR2233368

[27] PITMAN, J. (2006). *Combinatorial Stochastic Processes*. Springer, Berlin. MR2245368

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# OPTIMAL SCALING FOR THE TRANSIENT PHASE OF THE RANDOM WALK METROPOLIS ALGORITHM: THE MEAN-FIELD LIMIT[1]

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We consider the random walk Metropolis algorithm on $\mathbb{R}^n$ with Gaussian proposals, and when the target probability measure is the $n$-fold product of a one-dimensional law. In the limit $n \to \infty$, it is well known (see [*Ann. Appl. Probab.* **7** (1997) 110–120]) that, when the variance of the proposal scales inversely proportional to the dimension $n$ whereas time is accelerated by the factor $n$, a diffusive limit is obtained for each component of the Markov chain if this chain starts at equilibrium. This paper extends this result when the initial distribution is not the target probability measure. Remarking that the interaction between the components of the chain due to the common acceptance/rejection of the proposed moves is of mean-field type, we obtain a propagation of chaos result under the same scaling as in the stationary case. This proves that, in terms of the dimension $n$, the same scaling holds for the transient phase of the Metropolis–Hastings algorithm as near stationarity. The diffusive and mean-field limit of each component is a diffusion process nonlinear in the sense of McKean. This opens the route to new investigations of the optimal choice for the variance of the proposal distribution in order to accelerate convergence to equilibrium (see [Optimal scaling for the transient phase of Metropolis–Hastings algorithms: The longtime behavior *Bernoulli* (2014) To appear]).

## REFERENCES

[1] BÉDARD, M. (2007). Weak convergence of Metropolis algorithms for non-i.i.d. target distributions. *Ann. Appl. Probab.* **17** 1222–1244. MR2344305

[2] BÉDARD, M. (2008). Optimal acceptance rates for Metropolis algorithms: Moving beyond 0.234. *Stochastic Process. Appl.* **118** 2198–2222. MR2474348

[3] BÉDARD, M., DOUC, R. and MOULINES, E. (2012). Scaling analysis of multiple-try MCMC methods. *Stochastic Process. Appl.* **122** 758–786. MR2891436

[4] BÉDARD, M., DOUC, R. and MOULINES, E. (2014). Scaling analysis of delayed rejection MCMC methods. *Methodol. Comput. Appl. Probab.* **16** 811–838. MR3270597

[5] BESKOS, A., PILLAI, N., ROBERTS, G., SANZ-SERNA, J.-M. and STUART, A. (2013). Optimal tuning of the hybrid Monte Carlo algorithm. *Bernoulli* **19** 1501–1534. MR3129023

[6] BESKOS, A., ROBERTS, G. and STUART, A. (2009). Optimal scalings for local Metropolis–Hastings chains on nonproduct targets in high dimensions. *Ann. Appl. Probab.* **19** 863–898. MR2537193

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[7] BREYER, L. A., PICCIONI, M. and SCARLATTI, S. (2004). Optimal scaling of MaLa for nonlinear regression. *Ann. Appl. Probab.* **14** 1479–1505. MR2071431

[8] BREYER, L. A. and ROBERTS, G. O. (2000). From Metropolis to diffusions: Gibbs states and optimal scaling. *Stochastic Process. Appl.* **90** 181–206. MR1794535

[9] CHRISTENSEN, O. F., ROBERTS, G. O. and ROSENTHAL, J. S. (2005). Scaling limits for the transient phase of local Metropolis–Hastings algorithms. *J. R. Stat. Soc. Ser. B Stat. Methodol.* **67** 253–268. MR2137324

[10] ETHIER, S. N. and KURTZ, T. G. (1986). *Markov Processes: Characterization and Convergence*. Wiley, New York. MR0838085

[11] HASTINGS, W. K. (1970). Monte Carlo sampling methods using Markov chains and their applications. *Biometrika* **57** 97–109.

[12] JOURDAIN, B., LELIÈVRE, T. and MIASOJEDOW, B. (2014). Optimal scaling for the transient phase of Metropolis–Hastings algorithms: The longtime behavior. *Bernoulli* **20** 1930–1978. MR3263094

[13] KARATZAS, I. and SHREVE, S. E. (1988). *Brownian Motion and Stochastic Calculus*, 2nd ed. Springer, New York. MR0917065

[14] MATTINGLY, J. C., PILLAI, N. S. and STUART, A. M. (2012). Diffusion limits of the random walk Metropolis algorithm in high dimensions. *Ann. Appl. Probab.* **22** 881–930. MR2977981

[15] METROPOLIS, N., ROSENBLUTH, A., ROSENBLUTH, M., TELLER, A. and TELLER, E. (1953). Equation of state calculations by fast computing machines. *J. Chem. Phys.* **21** 1087–1092.

[16] NEAL, P. and ROBERTS, G. (2011). Optimal scaling of random walk Metropolis algorithms with non-Gaussian proposals. *Methodol. Comput. Appl. Probab.* **13** 583–601. MR2822397

[17] NEAL, P., ROBERTS, G. and YUEN, W. K. (2012). Optimal scaling of random walk Metropolis algorithms with discontinuous target densities. *Ann. Appl. Probab.* **22** 1880–1927. MR3025684

[18] PILLAI, N. S., STUART, A. M. and THIÉRY, A. H. (2012). Optimal scaling and diffusion limits for the Langevin algorithm in high dimensions. *Ann. Appl. Probab.* **22** 2320–2356. MR3024970

[19] PILLAI, N. S., STUART, A. M. and THIÉRY, A. H. (2014). Noisy gradient flow from a random walk in Hilbert space. *Stoch. PDE: Anal. Comput.* **2** 196–232. MR3249584

[20] RACHEV, S. T. and RÜSCHENDORF, L. (1998). *Mass Transportation Problems: Theory. Probability and Its Applications (New York)* **1**. Springer, New York. MR1619170

[21] ROBERTS, G. O., GELMAN, A. and GILKS, W. R. (1997). Weak convergence and optimal scaling of random walk Metropolis algorithms. *Ann. Appl. Probab.* **7** 110–120. MR1428751

[22] ROBERTS, G. O. and ROSENTHAL, J. S. (1998). Optimal scaling of discrete approximations to Langevin diffusions. *J. R. Stat. Soc. Ser. B Stat. Methodol.* **60** 255–268. MR1625691

[23] ROBERTS, G. O. and ROSENTHAL, J. S. (2001). Optimal scaling for various Metropolis–Hastings algorithms. *Statist. Sci.* **16** 351–367. MR1888450

[24] SZNITMAN, A.-S. (1991). Topics in propagation of chaos. In *École D'Été de Probabilités de Saint-Flour XIX—1989. Lecture Notes in Math.* **1464** 165–251. Springer, Berlin. MR1108185

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# DISCRETE TIME APPROXIMATION OF FULLY NONLINEAR HJB EQUATIONS VIA BSDES WITH NONPOSITIVE JUMPS

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We propose a new probabilistic numerical scheme for fully nonlinear equation of Hamilton–Jacobi–Bellman (HJB) type associated to stochastic control problem, which is based on the Feynman–Kac representation in [Kharroubi and Pham (2014)] by means of control randomization and backward stochastic differential equation with nonpositive jumps. We study a discrete time approximation for the minimal solution to this class of BSDE when the time step goes to zero, which provides both an approximation for the value function and for an optimal control in feedback form. We obtained a convergence rate without any ellipticity condition on the controlled diffusion coefficient. An explicit implementable scheme based on Monte Carlo simulations and empirical regressions, associated error analysis and numerical experiments are performed in the companion paper [*Monte Carlo Methods Appl.* **20** (2014) 145–165].

## REFERENCES

[1] BALLY, V. and PAGÈS, G. (2003). Error analysis of the optimal quantization algorithm for obstacle problems. *Stochastic Process. Appl.* **106** 1–40. MR1983041

[2] BARLES, G., BUCKDAHN, R. and PARDOUX, E. (1997). Backward stochastic differential equations and integral-partial differential equations. *Stochastics Stochastics Rep.* **60** 57–83. MR1436432

[3] BARLES, G. and JAKOBSEN, E. R. (2007). Error bounds for monotone approximation schemes for parabolic Hamilton–Jacobi–Bellman equations. *Math. Comp.* **76** 1861–1893 (electronic). MR2336272

[4] BONNANS, J. F. and ZIDANI, H. (2003). Consistency of generalized finite difference schemes for the stochastic HJB equation. *SIAM J. Numer. Anal.* **41** 1008–1021. MR2005192

[5] BOUCHARD, B. and CHASSAGNEUX, J.-F. (2008). Discrete-time approximation for continuously and discretely reflected BSDEs. *Stochastic Process. Appl.* **118** 2269–2293. MR2474351

[6] BOUCHARD, B. and ELIE, R. (2008). Discrete-time approximation of decoupled forward-backward SDE with jumps. *Stochastic Process. Appl.* **118** 53–75. MR2376252

[7] BOUCHARD, B. and TOUZI, N. (2004). Discrete-time approximation and Monte-Carlo simulation of backward stochastic differential equations. *Stochastic Process. Appl.* **111** 175–206. MR2056536

[8] CRANDALL, M. G., ISHII, H. and LIONS, P.-L. (1992). User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc.* (*N.S.*) **27** 1–67. MR1118699

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[9] FAHIM, A., TOUZI, N. and WARIN, X. (2011). A probabilistic numerical method for fully nonlinear parabolic PDEs. *Ann*. *Appl*. *Probab.* **21** 1322–1364. MR2857450

[10] FLEMING, W. H. and SONER, H. M. (2006). *Controlled Markov Processes and Viscosity Solutions*, 2nd ed. *Stochastic Modelling and Applied Probability* **25**. Springer, New York. MR2179357

[11] GUO, W., ZHANG, J. and ZHUO, J. (2006). A monotone scheme for high dimensional fully nonlinear PDE. Preprint.

[12] KHARROUBI, I., LANGRENÉ, N. and PHAM, H. (2014). A numerical algorithm for fully nonlinear HJB equations: An approach by control randomization. *Monte Carlo Methods Appl*. **20** 145–165. MR3213592

[13] KHARROUBI, I. and PHAM, H. (2014). Feynman–Kac representation for Hamilton–Jacobi–Bellman IPDE. *Ann. Probab.* To appear.

[14] KLOEDEN, P. E. and PLATEN, E. (1992). *Numerical Solution of Stochastic Differential Equations*. *Applications of Mathematics* (*New York*) **23**. Springer, Berlin. MR1214374

[15] KRYLOV, N. V. (1999). Approximating value functions for controlled degenerate diffusion processes by using piece-wise constant policies. *Electron*. *J*. *Probab.* **4** 1–19. MR1668597

[16] KRYLOV, N. V. (2000). On the rate of convergence of finite-difference approximations for Bellman's equations with variable coefficients. *Probab. Theory Related Fields* **117** 1–16. MR1759507

[17] KUSHNER, H. J. and DUPUIS, P. G. (1992). *Numerical Methods for Stochastic Control Problems in Continuous Time*. *Applications of Mathematics* (*New York*) **24**. Springer, New York. MR1217486

[18] LEMOR, J.-P., GOBET, E. and WARIN, X. (2006). Rate of convergence of an empirical regression method for solving generalized backward stochastic differential equations. *Bernoulli* **12** 889–916. MR2265667

[19] PHAM, H. (2009). *Continuous-Time Stochastic Control and Optimization with Financial Applications*. *Stochastic Modelling and Applied Probability* **61**. Springer, Berlin. MR2533355

[20] POSSAMAÏ, D. and TAN, X. (2013). Weak approximation of second order BSDEs. Preprint.

[21] SONER, H. M., TOUZI, N. and ZHANG, J. (2012). Wellposedness of second order backward SDEs. *Probab. Theory Related Fields* **153** 149–190. MR2925572

[22] TAN, X. (2013). A splitting method for fully nonlinear degenerate parabolic PDEs. *Electron*. *J. Probab.* **18** 24. MR3035743

[23] ZHANG, J. (2004). A numerical scheme for BSDEs. *Ann*. *Appl*. *Probab.* **14** 459–488. MR2023027

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# LIMIT SHAPES FOR GROWING EXTREME CHARACTERS OF $U(\infty)$

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We prove the existence of a limit shape and give its explicit description for certain probability distribution on signatures (or highest weights for unitary groups). The distributions have representation theoretic origin—they encode decomposition on irreducible characters of the restrictions of certain extreme characters of the infinite-dimensional unitary group $U(\infty)$ to growing finite-dimensional unitary subgroups $U(N)$. The characters of $U(\infty)$ are allowed to depend on $N$. In a special case, this describes the hydrodynamic behavior for a family of random growth models in $(2+1)$-dimensions with varied initial conditions.

## REFERENCES

[1] ANDERSON, G. W., GUIONNET, A. and ZEITOUNI, O. (2010). *An Introduction to Random Matrices*. *Cambridge Studies in Advanced Mathematics* **118**. Cambridge Univ. Press, Cambridge. MR2760897

[2] BIANE, P. (1995). Representations of unitary groups and free convolution. *Publ. Res. Inst. Math. Sci.* **31** 63–79. MR1317523

[3] BIANE, P. (2001). Approximate factorization and concentration for characters of symmetric groups. *Int. Math. Res. Not. IMRN* **4** 179–192. MR1813797

[4] BORODIN, A. and FERRARI, P. L. (2014). Anisotropic growth of random surfaces in $2+1$ dimensions. *Comm. Math. Phys.* **325** 603–684. MR3148098

[5] BORODIN, A. and GORIN, V. (2012). Lectures on integrable probability. Preprint. Available at arXiv:1212.3351.

[6] BORODIN, A. and KUAN, J. (2008). Asymptotics of Plancherel measures for the infinite-dimensional unitary group. *Adv. Math.* **219** 894–931. MR2442056

[7] BORODIN, A. and OLSHANSKI, G. (2012). The boundary of the Gelfand–Tsetlin graph: A new approach. *Adv. Math.* **230** 1738–1779. MR2927353

[8] BORODIN, A. M. and BUFETOV, A. I. (2013). Plancherel representations of $U(\infty)$ and correlated Gaussian Free Fields. Preprint. Available at arXiv:1301.0511.

[9] BUFETOV, A. and GORIN, V. (2013). Representations of classical Lie groups and quantized free convolution. Preprint.

[10] CHEN, W. (2012). Discrete-time anisotropic KPZ growth of random surface in $2+1$ dimensions. Technical report for an MIT summer undergraduate research project.

[11] EDREI, A. (1953). On the generation function of a doubly infinite, totally positive sequence. *Trans. Amer. Math. Soc.* **74** 367–383. MR0053989

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[12] IVANOV, V. and OLSHANSKI, G. (2002). Kerov's central limit theorem for the Plancherel measure on Young diagrams. In *Symmetric Functions* 2001: *Surveys of Developments and Perspectives*. *NATO Sci. Ser. II Math. Phys. Chem.* **74** 93–151. Kluwer Academic, Dordrecht. MR2059361

[13] KEROV, S. (1993). Gaussian limit for the Plancherel measure of the symmetric group. *C. R. Acad. Sci. Paris Sér. I Math.* **316** 303–308. MR1204294

[14] KEROV, S. and OLSHANSKI, G. (1994). Polynomial functions on the set of Young diagrams. *C. R. Acad. Sci. Paris Sér. I Math.* **319** 121–126. MR1288389

[15] KREWERAS, G. (1972). Sur les partitions non croisées d'un cycle. *Discrete Math.* **1** 333–350. MR0309747

[16] LOGAN, B. F. and SHEPP, L. A. (1977). A variational problem for random Young tableaux. *Adv. Math.* **26** 206–222. MR1417317

[17] MACDONALD, I. G. (1995). *Symmetric Functions and Hall Polynomials*, 2nd ed. Oxford Univ. Press, New York. MR1354144

[18] NICA, A. and SPEICHER, R. (2006). *Lectures on the Combinatorics of Free Probability*. *London Mathematical Society Lecture Note Series* **335**. Cambridge Univ. Press, Cambridge. MR2266879

[19] OKOUNKOV, A. (2002). Symmetric functions and random partitions. In *Symmetric Functions* 2001: *Surveys of Developments and Perspectives*. *NATO Sci. Ser. II Math. Phys. Chem.* **74** 223–252. Kluwer Academic, Dordrecht. MR2059364

[20] OKOUNKOV, A. and OLSHANSKI, G. (1998). Asymptotics of Jack polynomials as the number of variables goes to infinity. *Int. Math. Res. Not. IMRN* **13** 641–682. MR1636541

[21] OKUNKOV, A. and OLSHANSKI, G. (1997). Shifted Schur functions. *Algebra i Analiz* **9** 73–146. MR1468548

[22] OLSHANSKI, G. (2003). The problem of harmonic analysis on the infinite-dimensional unitary group. *J. Funct. Anal.* **205** 464–524. MR2018416

[23] OLSHANSKI, G. I. (1990). Unitary representations of infinite-dimensional pairs $(G, K)$ and the formalism of R. Howe. In *Representation of Lie Groups and Related Topics*. (A. M. Vershik and D. P. Zhelobenko, eds.). *Adv. Stud. Contemp. Math.* **7** 269–463. Gordon and Breach, New York. MR1104279

[24] SAGAN, B. E. (2001). *The Symmetric Group*: *Representations*, *Combinatorial Algorithms*, *and Symmetric Functions*, 2nd ed. *Graduate Texts in Mathematics* **203**. Springer, New York. MR1824028

[25] STANLEY, R. P. (1999). *Enumerative Combinatorics*. *Vol.* 2. *Cambridge Studies in Advanced Mathematics* **62**. Cambridge Univ. Press, Cambridge. MR1676282

[26] VERSHIK, A. M. and KEROV, S. V. (1977). Asymptotic behavior of the Plancherel measure of the symmetric group and the limit form of Young tableaux. *Dokl. Akad. Nauk SSSR* **233** 1024–1027. MR0480398

[27] VERSHIK, A. M. and KEROV, S. V. (1981). Asymptotic theory of the characters of a symmetric group. *Funktsional. Anal. i Prilozhen.* **15** 15–27, 96. MR0639197

[28] VERSHIK, A. M. and KEROV, S. V. (1982). Characters and factor-representations of the infinite unitary group. *Dokl. Akad. Nauk SSSR* **267** 272–276. MR0681202

[29] VOICULESCU, D. (1976). Représentations factorielles de type $\mathrm{II}_1$ de $U(\infty)$. *J. Math. Pures Appl.* (9) **55** 1–20. MR0442153

[30] WEYL, H. (1997). *The Classical Groups*: *Their Invariants and Representations*. Princeton Univ. Press, Princeton, NJ. MR1488158

[31] ZHELOBENKO, D. P. (1970). *Kompaktnye Gruppy Li i Ikh Predstavleniya*. Nauka, Moscow. English translation: Transl. Math. Monographs 40, Amer. Math. Soc., Providence, RI, 1973. MR0473097