ISSN 1050-5164 (print)
ISSN 2168-8737 (online)

THE ANNALS *of* APPLIED PROBABILITY

AN OFFICIAL JOURNAL OF THE
INSTITUTE OF MATHEMATICAL STATISTICS

Articles

Vol. 28, No. 4—August 2018

THE ANNALS OF APPLIED PROBABILITY Vol. 28, No. 4, pp. 1977–2634 August 2018

INSTITUTE OF MATHEMATICAL STATISTICS

(Organized September 12, 1935)

The purpose of the Institute is to foster the development and dissemination of the theory and applications of statistics and probability.

The Annals of Applied Probability **[ISSN 1050-5164 (print); ISSN 2168-8737 (online)],** Volume 28, Number 4, August 2018. Published bimonthly by the Institute of Mathematical Statistics, 3163 Somerset Drive, Cleveland, Ohio 44122, USA. Periodicals postage paid at Cleveland, Ohio, and at additional mailing offices.

POSTMASTER: Send address changes to *The Annals of Applied Probability*, Institute of Mathematical Statistics, Dues and Subscriptions Office, 9650 Rockville Pike, Suite L 2310, Bethesda, Maryland 20814-3998, USA.

The Annals of Applied Probability
2018, Vol. 28, No. 4, 1977–2002
https://doi.org/10.1214/17-AAP1289

MIXING TIMES OF RANDOM WALKS ON DYNAMIC CONFIGURATION MODELS

BY LUCA AVENA[*,1], HAKAN GÜLDAŞ[*,1], REMCO VAN DER HOFSTAD[†,1,2] AND FRANK DEN HOLLANDER[*,1,3]

Leiden University[*] and Eindhoven University of Technology[†]

The mixing time of a random walk, with or without backtracking, on a random graph generated according to the configuration model on n vertices, is known to be of order $\log n$. In this paper, we investigate what happens when the random graph becomes *dynamic*, namely, at each unit of time a fraction α_n of the edges is randomly rewired. Under mild conditions on the degree sequence, guaranteeing that the graph is locally tree-like, we show that for every $\varepsilon \in (0,1)$ the ε-mixing time of random walk without backtracking grows like $\sqrt{2\log(1/\varepsilon)/\log(1/(1-\alpha_n))}$ as $n \to \infty$, provided that $\lim_{n\to\infty} \alpha_n(\log n)^2 = \infty$. The latter condition corresponds to a regime of fast enough graph dynamics. Our proof is based on a randomised stopping time argument, in combination with coupling techniques and combinatorial estimates. The stopping time of interest is the first time that the walk moves along an edge that was rewired before, which turns out to be close to a strong stationary time.

REFERENCES

[1] ALDOUS, A. and FILL, J. A. *Reversible Markov Chains and Random Walks on Graphs*. Unpublished manuscript. Available at http://www.stat.berkeley.edu/~aldous/RWG/book.html.

[2] ANGEL, O., VAN DER HOFSTAD, R. and HOLMGREN, C. Limit laws for self-loops and multiple edges in the configuration model. Preprint. Available at arXiv:1603.07172.

[3] BEN-HAMOU, A. and SALEZ, J. (2017). Cutoff for nonbacktracking random walks on sparse random graphs. *Ann. Probab.* **45** 1752–1770. MR3650414

[4] BENDER, E. A. and CANFIELD, E. R. (1978). The asymptotic number of labeled graphs with given degree sequences. *J. Combin. Theory Ser. A* **24** 296–307. MR0505796

[5] BENJAMINI, I., KOZMA, G. and WORMALD, N. (2014). The mixing time of the giant component of a random graph. *Random Structures Algorithms* **45** 383–407. MR3252922

[6] BENJAMINI, I. and MOSSEL, E. (2003). On the mixing time of a simple random walk on the super critical percolation cluster. *Probab. Theory Related Fields* **125** 408–420. MR1967022

[7] BERESTYCKI, N., LUBETZKY, E., PERES, Y. and SLY, A. (2018). Random walks on the random graph. *Ann. Probab.* **46** 456–490. MR3758735

[8] BOLLOBÁS, B. (1980). A probabilistic proof of an asymptotic formula for the number of labelled regular graphs. *European J. Combin.* **1** 311–316. MR0595929

[9] BOLLOBÁS, B. (2001). *Random Graphs*, 2nd ed. *Cambridge Studies in Advanced Mathematics* **73**. Cambridge Univ. Press, Cambridge. MR1864966

MSC2010 subject classifications. Primary 60C05; secondary 37A25, 05C81.

Key words and phrases. Random graph, random walk, mixing time, coupling, dynamic configuration model.

[10] BORDENAVE, C., CAPUTO, P. and SALEZ, J. (2018). Random walk on sparse random digraphs. *Probab. Theory Related Fields* **170** 933–960. MR3773804

[11] DING, J., LUBETZKY, E. and PERES, Y. (2012). Mixing time of near-critical random graphs. *Ann. Probab.* **40** 979–1008. MR2962084

[12] FEDERICO, L. and VAN DER HOFSTAD, R. (2017). Critical window for connectivity in the configuration model. *Combin. Probab. Comput.* **26** 660–680. MR3681976

[13] FOUNTOULAKIS, N. and REED, B. A. (2008). The evolution of the mixing rate of a simple random walk on the giant component of a random graph. *Random Structures Algorithms* **33** 68–86. MR2428978

[14] GREENHILL, C. (2015). The switch Markov chain for sampling irregular graphs (extended abstract). In *Proceedings of the Twenty-Sixth Annual ACM–SIAM Symposium on Discrete Algorithms* 1564–1572. SIAM, Philadelphia, PA. MR3451127

[15] JANSON, S. (2009). The probability that a random multigraph is simple. *Combin. Probab. Comput.* **18** 205–225. MR2497380

[16] JANSON, S. (2014). The probability that a random multigraph is simple. II. *J. Appl. Probab.* **51A** 123–137. MR3317354

[17] LEVIN, D. A., PERES, Y. and WILMER, E. L. (2009). *Markov Chains and Mixing Times*. Amer. Math. Soc., Providence, RI. With a chapter by James G. Propp and David B. Wilson. MR2466937

[18] LUBETZKY, E. and SLY, A. (2010). Cutoff phenomena for random walks on random regular graphs. *Duke Math. J.* **153** 475–510. MR2667423

[19] ŁUCZAK, T. (1992). Sparse random graphs with a given degree sequence. In *Random Graphs, Vol.* 2 (*Poznań*, 1989) 165–182. Wiley, New York. MR1166614

[20] MOLLOY, M. and REED, B. (1995). A critical point for random graphs with a given degree sequence. *Random Structures Algorithms* **6** 161–179. MR1370952

[21] MOLLOY, M. and REED, B. (1998). The size of the giant component of a random graph with a given degree sequence. *Combin. Probab. Comput.* **7** 295–305. MR1664335

[22] NACHMIAS, A. and PERES, Y. (2008). Critical random graphs: Diameter and mixing time. *Ann. Probab.* **36** 1267–1286. MR2435849

[23] PERES, Y., SOUSI, P. and STEIF, J. E. Mixing time for random walk on supercritical dynamical percolation. Preprint. Available at arXiv:1707.07632.

[24] PERES, Y., STAUFFER, A. and STEIF, J. E. (2015). Random walks on dynamical percolation: Mixing times, mean squared displacement and hitting times. *Probab. Theory Related Fields* **162** 487–530. MR3383336

[25] VAN DER HOFSTAD, R. (2017). *Random Graphs and Complex Networks, Vol. I. Cambridge Series in Statistical and Probabilistic Mathematics* **43**. Cambridge Univ. Press, Cambridge. MR3617364

[26] VAN DER HOFSTAD, R. (2018). *Random Graphs and Complex Networks, Vol. II*. Cambridge Univ. Press, Cambridge. To appear.

The Annals of Applied Probability
2018, Vol. 28, No. 4, 2003–2062
https://doi.org/10.1214/17-AAP1313

CONSISTENCY OF MODULARITY CLUSTERING ON RANDOM GEOMETRIC GRAPHS

BY ERIK DAVIS[1] AND SUNDER SETHURAMAN[1]

University of Arizona

Given a graph, the popular "modularity" clustering method specifies a partition of the vertex set as the solution of a certain optimization problem. In this paper, we discuss scaling limits of this method with respect to random geometric graphs constructed from i.i.d. points $\mathcal{X}_n = \{X_1, X_2, \ldots, X_n\}$, distributed according to a probability measure ν supported on a bounded domain $D \subset \mathbb{R}^d$. Among other results, we show, via a Gamma convergence framework, a geometric form of consistency: When the number of clusters, or partitioning sets of $\mathcal{X}_n$ is a priori bounded above, the discrete optimal modularity clusterings converge in a specific sense to a continuum partition of the underlying domain D, characterized as the solution to a "soap bubble" or "Kelvin"-type shape optimization problem.

REFERENCES

[1] ADAMS, R. A. and FOURNIER, J. J. F. (2003). *Sobolev Spaces*, 2nd ed. *Pure and Applied Mathematics* (*Amsterdam*) **140**. Elsevier/Academic Press, Amsterdam. MR2424078

[2] ALBERTI, G. and BELLETTINI, G. (1998). A non-local anisotropic model for phase transitions: Asymptotic behaviour of rescaled energies. *European J. Appl. Math.* **9** 261–284. MR1634336

[3] AMBROSIO, L., FUSCO, N. and PALLARA, D. (2000). *Functions of Bounded Variation and Free Discontinuity Problems*. *Oxford Mathematical Monographs*. The Clarendon Press, Oxford Univ. Press, New York. MR1857292

[4] AMBROSIO, L., GIGLI, N. and SAVARÉ, G. (2005). *Gradient Flows in Metric Spaces and in the Space of Probability Measures*. Birkhäuser, Basel. MR2129498

[5] ANTONIONI, A., EGLOF, M. and TOMASSINI, M. (2013). An energy-based model for spatial social networks. In *Advances in Artificial Life ECAL* 2013 226–231. MIT Press, Cambridge, MA.

[6] ARIAS-CASTRO, E. and PELLETIER, B. (2013). On the convergence of maximum variance unfolding. *J. Mach. Learn. Res.* **14** 1747–1770. MR3104494

[7] ARIAS-CASTRO, E., PELLETIER, B. and PUDLO, P. (2012). The normalized graph cut and Cheeger constant: From discrete to continuous. *Adv. in Appl. Probab.* **44** 907–937. MR3052843

[8] BELKIN, M. and NIYOGI, P. (2003). Laplacian eigenmaps for dimensionality reduction and data representation. *Neural Comput.* **15** 1373–1396.

MSC2010 subject classifications. Primary 60D05; secondary 62G20, 05C82, 49J55, 49J45, 68R10.

Key words and phrases. Modularity, community detection, consistency, random geometric graph, Gamma convergence, Kelvin's problem, scaling limit, shape optimization, optimal transport, total variation, perimeter.

[9] BELKIN, M. and NIYOGI, P. (2008). Towards a theoretical foundation for Laplacian-based manifold methods. *J. Comput. System Sci.* **74** 1289–1308. MR2460286

[10] BETTSTETTER, C. (2002). On the minimum node degree and connectivity of a wireless multi-hop network. In *Proceedings of the 3rd ACM International Symposium on Mobile Ad Hoc Networking & Computing* 80–91. ACM, New York.

[11] BICKEL, P. and CHEN, A. (2009). A nonparametric view of network models and Newman–Girvan and other modularities. *Proc. Natl. Acad. Sci. USA* **106** 21068–21073.

[12] BLONDEL, V., GUILLAUME, J., LAMBIOTTE, R. and LEFEBVRE, E. (2008). Fast unfolding of communities in large networks. *J. Stat. Mech. Theory Exp.* **2008** 10008–10020.

[13] BRAIDES, A. (2002). *Γ-Convergence for Beginners*. *Oxford Lecture Series in Mathematics and Its Applications* **22**. Oxford Univ. Press, Oxford. MR1968440

[14] BRAIDES, A. and GELLI, M. S. (2006). From discrete systems to continuous variational problems: An introduction. In *Topics on Concentration Phenomena and Problems with Multiple Scales*. *Lect. Notes Unione Mat. Ital.* **2** 3–77. Springer, Berlin. MR2267880

[15] BRAIDES, A. and TRUSKINOVSKY, L. (2008). Asymptotic expansions by Γ-convergence. *Contin. Mech. Thermodyn.* **20** 21–62. MR2398821

[16] BRAKKE, K. A. (1992). The surface evolver. *Exp. Math.* **1** 141–165. MR1203871

[17] BRANDES, U., DELLING, D., GAERTLER, M., GÖRKE, R., HOEFER, M., NIKOLOSKI, Z. and WAGNER, D. (2008). On modularity clustering. *IEEE Trans. Knowl. Data Eng.* **20** 172–188.

[18] CAÑETE, A. and RITORÉ, M. (2004). Least-perimeter partitions of the disk into three regions of given areas. *Indiana Univ. Math. J.* **53** 883–904. MR2086704

[19] CLAUSET, A., NEWMAN, M. and MOORE, C. (2004). Finding community structure in very large networks. *Phys. Rev. E* **70** 066111.

[20] COIFMAN, R. R. and LAFON, S. (2006). Diffusion maps. *Appl. Comput. Harmon. Anal.* **21** 5–30. MR2238665

[21] COX, S. J. and FLIKKEMA, E. (2010). The minimal perimeter for N confined deformable bubbles of equal area. *Electron. J. Combin.* **17** Research Paper 45. MR2607331

[22] DAVIS, E. and SETHURAMAN, S. (2017). Consistency of modularity clustering on random geometric graphs. Available at arXiv:1604.03993v1.

[23] DE LA PEÑA, V. H. and MONTGOMERY-SMITH, S. J. (1995). Decoupling inequalities for the tail probabilities of multivariate U-statistics. *Ann. Probab.* **23** 806–816. MR1334173

[24] DHARA, M. and SHUKLA, K. K. (2012). Advanced cost based graph clustering algorithm for random geometric graphs. *Int. J. Comput. Appl.* **60** 20–34.

[25] DÍAZ, J., PETIT, J. and SERNA, M. (2002). A survey of graph layout problems. *ACM Comput. Surv.* **34** 313–356.

[26] DÍ AZ, J., PENROSE, M. D., PETIT, J. and SERNA, M. (2001). Approximating layout problems on random geometric graphs. *J. Algorithms* **39** 78–116. MR1823657

[27] DUDLEY, R. M. (2002). *Real Analysis and Probability*. *Cambridge Studies in Advanced Mathematics* **74**. Cambridge Univ. Press, Cambridge. Revised reprint of the 1989 original. MR1932358

[28] DURRETT, R. (2010). *Probability: Theory and Examples*, 4th ed. *Cambridge Series in Statistical and Probabilistic Mathematics* **31**. Cambridge Univ. Press, Cambridge. MR2722836

[29] EL GAMAL, A., MAMMEN, J., PRABHAKAR, B. and SHAH, D. (2004). Throughput-delay trade-off in wireless networks. In *Twenty-Third Annual Joint Conference Proceedings of the IEEE Computer and Communications Societies*.

[30] FOLLAND, G. B. (2013). *Real Analysis: Modern Techniques and Their Applications*, 2nd ed. Wiley, New York. MR1681462

[31] FORTUNA, M., STOUFFER, D., OLESEN, J., JORDANO, P., MOUILLOT, D., KRASNOV, B., POULIN, R. and BASCOMPTE, J. (2010). Nestedness versus modularity in ecological networks: Two sides of the same coin? *J. Anim. Ecol.* **79** 811–817.

[32] FORTUNATO, S. (2010). Community detection in graphs. *Phys. Rep.* **486** 75–174. MR2580414

[33] FORTUNATO, S. and BARTHÉLEMY, M. (2006). Resolution limit in community detection. *Proc. Natl. Acad. Sci. USA* **104** 36–41.

[34] FRANCESCHETTI, M. and MEESTER, R. (2007). *Random Networks for Communication: From Statistical Physics to Information Systems. Cambridge Series in Statistical and Probabilistic Mathematics* **24**. Cambridge Univ. Press, Cambridge. MR2398551

[35] GARCÍA TRILLOS, N. and SLEPČEV, D. (2016). A variational approach to the consistency of spectral clustering. *Appl. Comput. Harmon. Anal.*

[36] GARCÍA TRILLOS, N., SLEPČEV, D. and VON BRECHT, J. (2016). Estimating perimeter using graph cuts. Available at arXiv:1602.04102.

[37] GARCÍA TRILLOS, N. and SLEPČEV, D. (2015). On the rate of convergence of empirical measures in ∞-transportation distance. *Canad. J. Math.* **67** 1358–1383. MR3415656

[38] GARCÍA TRILLOS, N. and SLEPČEV, D. (2016). Continuum limit of total variation on point clouds. *Arch. Ration. Mech. Anal.* **220** 193–241. MR3458162

[39] GARCÍA TRILLOS, N., SLEPČEV, D., VON BRECHT, J., LAURENT, T. and BRESSON, X. (2016). Consistency of Cheeger and ratio graph cuts. *J. Mach. Learn. Res.* **17** Paper No. 181. MR3567449

[40] GINÉ, E. and KOLTCHINSKII, V. (2006). Empirical graph Laplacian approximation of Laplace–Beltrami operators: Large sample results. In *High Dimensional Probability. Institute of Mathematical Statistics Lecture Notes—Monograph Series* **51** 238–259. IMS, Beachwood, OH. MR2387773

[41] GINÉ, E., LATAŁA, R. and ZINN, J. (2000). Exponential and moment inequalities for U-statistics. In *High Dimensional Probability, II (Seattle, WA*, 1999). *Progress in Probability* **47** 13–38. Birkhäuser, Boston, MA. MR1857312

[42] GOOD, B. H., DE MONTJOYE, Y.-A. and CLAUSET, A. (2010). Performance of modularity maximization in practical contexts. *Phys. Rev. E* (3) **81** 046106. MR2736215

[43] GUIMERA, R. and AMARAL, L. (2005). Functional cartography of complex metabolic networks. *Nature* **433** 895–900.

[44] GUIMERA, R., SALES-PARDO, M. and AMARAL, L. (2004). Modularity from fluctuations in random graphs and complex networks. *Phys. Rev. E* **70** 025101.

[45] GUPTA, P. and KUMAR, P. R. (2000). The capacity of wireless networks. *IEEE Trans. Inform. Theory* **46** 388–404. MR1748976

[46] HAGMANN, P., CAMMOUN, L., GIGANDET, X., MEULI, R., HONEY, C. J., WEDEEN, V. J. and SPORNS, O. (2008). Mapping the structural core of human cerebral cortex. *PLoS Biol.* **6** e159.

[47] HARTIGAN, J. A. (1981). Consistency of single linkage for high-density clusters. *J. Amer. Statist. Assoc.* **76** 388–394. MR0624340

[48] HEIN, M., AUDIBERT, J.-Y. and VON LUXBURG, U. (2005). From graphs to manifolds—Weak and strong pointwise consistency of graph Laplacians. In *Learning Theory. Lecture Notes in Computer Science* **3559** 470–485. Springer, Berlin. MR2203281

[49] HU, H., LAURENT, T., PORTER, M. A. and BERTOZZI, A. L. (2013). A method based on total variation for network modularity optimization using the MBO scheme. *SIAM J. Appl. Math.* **73** 2224–2246. MR3143855

[50] LANCICHINETTI, A. and FORTUNATO, S. (2011). Limits of modularity maximization in community detection. *Phys. Rev. E* **84** 066122.

[51] LE, C. M., LEVINA, E. and VERSHYNIN, R. (2016). Optimization via low-rank approximation for community detection in networks. *Ann. Statist.* **44** 373–400. MR3449772

[52] MEESTER, R. and ROY, R. (1996). *Continuum Percolation. Cambridge Tracts in Mathematics* **119**. Cambridge Univ. Press, Cambridge. MR1409145

[53] MILL, J., TANG, T., KAMINSKY, Z., KHARE, T., YAZDANPANAH, S., BOUCHARD, L., JIA, P., ASSADZADEH, A., FLANAGAN, J., SCHUMACHER, A., WANG, S.-C. and PETRONIS, A. (2008). Epigenomic profiling reveals DNA-methylation changes associated with major psychosis. *Am. J. Hum. Genet.* **82** 696–711.

[54] MORGAN, F. (2009). *Geometric Measure Theory: A Beginner's Guide*, 4th ed. Elsevier/Academic Press, Amsterdam. MR2455580

[55] NEWMAN, M. (2006). Modularity and community structure in networks. *Proc. Natl. Acad. Sci. USA* **103** 8577–8582.

[56] NEWMAN, M. (2013). Spectral methods for community detection and graph partitioning. *Phys. Rev. E* **88** 042822.

[57] NEWMAN, M. and GIRVAN, M. (2004). Finding and evaluating community structure in networks. *Phys. Rev. E* (3) **69** 026113.

[58] NEWMAN, M. E. J. (2006). Finding community structure in networks using the eigenvectors of matrices. *Phys. Rev. E* (3) **74** 036104. MR2282139

[59] OUDET, É. (2011). Approximation of partitions of least perimeter by Γ-convergence: Around Kelvin's conjecture. *Exp. Math.* **20** 260–270. MR2836251

[60] PENROSE, M. (2003). *Random Geometric Graphs. Oxford Studies in Probability* **5**. Oxford Univ. Press, Oxford. MR1986198

[61] POLLARD, D. (1981). Strong consistency of k-means clustering. *Ann. Statist.* **9** 135–140. MR0600539

[62] PORTER, M., MUCHA, P., NEWMAN, M. and WARMBRAND, C. (2005). A network analysis of committees in the US House of Representatives. *Proc. Natl. Acad. Sci. USA* **102** 7057–7062.

[63] PORTER, M. A., ONNELA, J.-P. and MUCHA, P. J. (2009). Communities in networks. *Notices Amer. Math. Soc.* **56** 1082–1097. MR2568495

[64] PRZULJ, N., CORNEIL, D. G. and JURISICA, I. (2004). Modeling interactome: Scale-free or geometric? *Bioinformatics* **20** 3508–3515.

[65] REICHARDT, J. and BORNHOLDT, S. (2006). Statistical mechanics of community detection. *Phys. Rev. E* (3) **74** 016110. MR2276596

[66] ROHE, K., CHATTERJEE, S. and YU, B. (2011). Spectral clustering and the high-dimensional stochastic blockmodel. *Ann. Statist.* **39** 1878–1915. MR2893856

[67] SABIN, M. (1987). Convergence and consistency of fuzzy c-means/ISODATA algorithms. *IEEE Trans. Pattern Anal. Mach. Intell.* **9** 661–668.

[68] SHORACK, G. R. and WELLNER, J. A. (2009). *Empirical Processes with Applications to Statistics. Classics in Applied Mathematics* **59**. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA. Reprint of the 1986 original. MR3396731

[69] SINGER, A. (2006). From graph to manifold Laplacian: The convergence rate. *Appl. Comput. Harmon. Anal.* **21** 128–134. MR2238670

[70] SINGER, A. and WU, H.-T. (2017). Spectral convergence of the connection Laplacian from random samples. *Inf. Inference* **6** 58–123. MR3636868

[71] THORPE, M., THEIL, F., JOHANSEN, A. M. and CADE, N. (2015). Convergence of the k-means minimization problem using Γ-convergence. *SIAM J. Appl. Math.* **75** 2444–2474. MR3422446

[72] TING, D., HUANG, L. and JORDAN, M. I. (2010). An analysis of the convergence of graph Laplacians. In *Proceedings of the 27th International Conference on Machine Learning*.

[73] VAN GENNIP, Y. and BERTOZZI, A. L. (2012). Γ-convergence of graph Ginzburg–Landau functionals. *Adv. Differential Equations* **17** 1115–1180. MR3013414

[74] VAN DER VAART, A. W. (1998). *Asymptotic Statistics. Cambridge Series in Statistical and Probabilistic Mathematics* **3**. Cambridge Univ. Press, Cambridge. MR1652247

[75] VILLANI, C. (2004). *Topics in Optimal Transportation*. American Mathematical Society, Providence, RI.

[76] VILLANI, C. (2009). *Optimal Transport*: *Old and New*. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **338**. Springer, Berlin. MR2459454

[77] VON LUXBURG, U., BELKIN, M. and BOUSQUET, O. (2008). Consistency of spectral clustering. *Ann*. *Statist*. **36** 555–586. MR2396807

[78] WETS, R. J.-B. (1999). Statistical estimation from an optimization viewpoint. *Ann*. *Oper. Res*. **85** 79–101. MR1685588

[79] ZHANG, X. and NEWMAN, M. (2015). Multiway spectral community detection in networks. *Phys*. *Rev. E* **92** 052808.

[80] ZHAO, Y., LEVINA, E. and ZHU, J. (2012). Consistency of community detection in networks under degree-corrected stochastic block models. *Ann*. *Statist*. **40** 2266–2292. MR3059083

The Annals of Applied Probability
2018, Vol. 28, No. 4, 2063–2082
https://doi.org/10.1214/17-AAP1315

CRITICAL PARAMETER OF RANDOM LOOP MODEL ON TREES

BY JAKOB E. BJÖRNBERG[*,†,1] AND DANIEL UELTSCHI[‡]

Chalmers University of Technology[*] and University of Gothenburg[†] and University of Warwick[‡]

We give estimates of the critical parameter for random loop models that are related to quantum spin systems. A special case of the model that we consider is the interchange- or random-stirring process. We consider here the model defined on regular trees of large degrees, which are expected to approximate high spatial dimensions. We find a critical parameter that indeed shares similarity with existing numerical results for the cubic lattice. In the case of the interchange process, our results improve on earlier work by Angel and by Hammond, in that we determine the second-order term of the critical parameter.

REFERENCES

[1] AIZENMAN, M. and NACHTERGAELE, B. (1994). Geometric aspects of quantum spin states. *Comm. Math. Phys.* **164** 17–63. MR1288152

[2] ALON, G. and KOZMA, G. (2013). The probability of long cycles in interchange processes. *Duke Math. J.* **162** 1567–1585. MR3079255

[3] ANGEL, O. (2003). Random infinite permutations and the cyclic time random walk. In *Discrete Random Walks* (*Paris*, 2003). *Discrete Math. Theor. Comput. Sci. Proc.*, *AC* 9–16. Assoc. Discrete Math. Theor. Comput. Sci., Nancy. MR2042369

[4] BARP, A., BARP, E. G., BRIOL, F.-X. and UELTSCHI, D. (2015). A numerical study of the 3D random interchange and random loop models. *J. Phys. A* **48** 345002. MR3376038

[5] BERESTYCKI, N. (2011). Emergence of giant cycles and slowdown transition in random transpositions and k-cycles. *Electron. J. Probab.* **16** 152–173. MR2754801

[6] BERESTYCKI, N. and KOZMA, G. (2015). Cycle structure of the interchange process and representation theory. *Bull. Soc. Math. France* **143** 265–280. MR3351179

[7] BJÖRNBERG, J. E. (2015). Large cycles in random permutation related to the Heisenberg model. *Electron. Commun. Probab.* **20** no. 55. MR3384113

[8] BJÖRNBERG, J. E. (2016). The free energy in a class of quantum spin systems and interchange processes. *J. Math. Phys.* **57** 073303. MR3529570

[9] GLADKICH, A. and PELED, R. (2016). On the cycle structure of Mallows permutations. Available at arXiv:1601.06991.

[10] HAMMOND, A. (2013). Infinite cycles in the random stirring model on trees. *Bull. Inst. Math. Acad. Sin.* (*N.S.*) **8** 85–104. MR3097418

[11] HAMMOND, A. (2015). Sharp phase transition in the random stirring model on trees. *Probab. Theory Related Fields* **161** 429–448. MR3334273

[12] HARRIS, T. E. (1972). Nearest-neighbor Markov interaction processes on multidimensional lattices. *Adv. Math.* **9** 66–89. MR0307392

MSC2010 subject classifications. 60K35, 82B20, 82B26, 82B31.
Key words and phrases. Random loop model, quantum Heisenberg.

[13] KOTECKÝ, R., MIŁOŚ, P. and UELTSCHI, D. (2016). The random interchange process on the hypercube. *Electron. Commun. Probab.* **21** Paper No. 4, 9. MR3485373
[14] MIŁOŚ, P. and ŞENGÜL, B. Existence of a phase transition of the interchange process on the Hamming graph. Preprint. Available at arXiv:1605.03548.
[15] SCHRAMM, O. (2005). Compositions of random transpositions. *Israel J. Math.* **147** 221–243. MR2166362
[16] STARR, S. (2009). Thermodynamic limit for the Mallows model on S_n. *J. Math. Phys.* **50** 095208, 15. MR2566888
[17] TÓTH, B. (1993). Improved lower bound on the thermodynamic pressure of the spin $1/2$ Heisenberg ferromagnet. *Lett. Math. Phys.* **28** 75–84. MR1224836
[18] UELTSCHI, D. (2013). Random loop representations for quantum spin systems. *J. Math. Phys.* **54** 083301. MR3135479

The Annals of Applied Probability
2018, Vol. 28, No. 4, 2083–2104
https://doi.org/10.1214/17-AAP1321

ORDER STATISTICS OF VECTORS WITH DEPENDENT COORDINATES, AND THE KARHUNEN–LOÈVE BASIS

BY ALEXANDER E. LITVAK AND KONSTANTIN TIKHOMIROV

University of Alberta

Let X be an n-dimensional random centered Gaussian vector with independent but not identically distributed coordinates and let T be an orthogonal transformation of $\mathbb{R}^n$. We show that the random vector $Y = T(X)$ satisfies

$$\mathbb{E} \sum_{j=1}^{k} j\text{-}\min_{i \le n} {X_i}^2 \le C \mathbb{E} \sum_{j=1}^{k} j\text{-}\min_{i \le n} {Y_i}^2$$

for all $k \le n$, where "j-min" denotes the jth smallest component of the corresponding vector and $C > 0$ is a universal constant. This resolves (up to a multiplicative constant) an old question of S. Mallat and O. Zeitouni regarding optimality of the Karhunen–Loève basis for the nonlinear signal approximation. As a by-product, we obtain some relations for order statistics of random vectors (not only Gaussian) which are of independent interest.

REFERENCES

[1] ADAMCZAK, R., LATAŁA, R., LITVAK, A. E., PAJOR, A. and TOMCZAK-JAEGERMANN, N. (2014). Tail estimates for norms of sums of log-concave random vectors. *Proc. Lond. Math. Soc.* (3) **108** 600–637. MR3180591

[2] DAVID, H. A. and NAGARAJA, H. N. (2003). *Order Statistics*, 3rd ed. Wiley-Interscience, Hoboken, NJ. MR1994955

[3] GLUSKIN, E. D. (1989). Extremal properties of orthogonal parallelepipeds and their applications to the geometry of Banach spaces. *Mat. Sb.* **64** 85–96.

[4] GORDON, Y., LITVAK, A., SCHÜTT, C. and WERNER, E. (2002). Orlicz norms of sequences of random variables. *Ann. Probab.* **30** 1833–1853. MR1944007

[5] GORDON, Y., LITVAK, A., SCHÜTT, C. and WERNER, E. (2002). Geometry of spaces between polytopes and related zonotopes. *Bull. Sci. Math.* **126** 733–762. MR1941083

[6] GORDON, Y., LITVAK, A., SCHÜTT, C. and WERNER, E. (2005). Minima of sequences of Gaussian random variables. *C. R. Math. Acad. Sci. Paris* **340** 445–448. MR2135327

[7] GORDON, Y., LITVAK, A. E., SCHÜTT, C. and WERNER, E. (2006). On the minimum of several random variables. *Proc. Amer. Math. Soc.* **134** 3665–3675. MR2240681

[8] GORDON, Y., LITVAK, A. E., SCHÜTT, C. and WERNER, E. (2012). Uniform estimates for order statistics and Orlicz functions. *Positivity* **16** 1–28. MR2892571

[9] HARDY, G. H., LITTLEWOOD, J. E. and PÓLYA, G. (1952). *Inequalities*, 2nd ed. The Univ. Press, Cambridge. MR0046395

[10] LATAŁA, R. (2011). Order statistics and concentration of l_r norms for log-concave vectors. *J. Funct. Anal.* **261** 681–696. MR2799576

MSC2010 subject classifications. 62G30, 60E15, 60G15, 60G35, 94A08.

Key words and phrases. Order statistics, Karhunen–Loève basis, nonlinear approximation, INID case.

[11] LECHNER, R., PASSENBRUNNER, M. and PROCHNO, J. (2015). Uniform estimates for averages of order statistics of matrices. *Electron. Commun. Probab.* **20** no. 27. MR3327866

[12] LITVAK, A. E., PAJOR, A. and TOMCZAK-JAEGERMANN, N. (2006). Diameters of sections and coverings of convex bodies. *J. Funct. Anal.* **231** 438–457. MR2195339

[13] LITVAK, A. E. and TOMCZAK-JAEGERMANN, N. (2000). Random aspects of high-dimensional convex bodies. In *Geometric Aspects of Functional Analysis*. *Lecture Notes in Math.* **1745** 169–190. Springer, Berlin. MR1796719

[14] MALLAT, S. (2009). *A Wavelet Tour of Signal Processing: The Sparse Way*, 3rd ed. Elsevier/Academic Press, Amsterdam. MR2479996

[15] MALLAT, S. and ZEITOUNI, O. A conjecture concerning optimality of the Karhunen–Loeve basis in nonlinear reconstruction. Available at arXiv:1109.0489.

[16] MARSHALL, A. W. and PROSCHAN, F. (1965). An inequality for convex functions involving majorization. *J. Math. Anal. Appl.* **12** 87–90. MR0185062

[17] MONTGOMERY-SMITH, S. (2002). Rearrangement invariant norms of symmetric sequence norms of independent sequences of random variables. *Israel J. Math.* **131** 51–60. MR1942301

[18] ŠIDÁK, Z. (1967). Rectangular confidence regions for the means of multivariate normal distributions. *J. Amer. Statist. Assoc.* **62** 626–633. MR0216666

[19] ZEITOUNI, O. (2006). A correlation inequality for nonlinear reconstruction. In *Workshop on the Mathematical Foundations of Learning Theory*, Paris.

The Annals of Applied Probability
2018, Vol. 28, No. 4, 2105–2140
https://doi.org/10.1214/17-AAP1322

BEATING THE OMEGA CLOCK: AN OPTIMAL STOPPING PROBLEM WITH RANDOM TIME-HORIZON UNDER SPECTRALLY NEGATIVE LÉVY MODELS

BY NEOFYTOS RODOSTHENOUS[1] AND HONGZHONG ZHANG

Queen Mary University of London and Columbia University

We study the optimal stopping of an American call option in a random time-horizon under exponential spectrally negative Lévy models. The random time-horizon is modeled as the so-called Omega default clock in insurance, which is the first time when the occupation time of the underlying Lévy process below a level y, exceeds an independent exponential random variable with mean $1/q > 0$. We show that the shape of the value function varies qualitatively with different values of q and y. In particular, we show that for certain values of q and y, some quantitatively different but traditional up-crossing strategies are still optimal, while for other values we may have two disconnected continuation regions, resulting in the optimality of two-sided exit strategies. By deriving the joint distribution of the discounting factor and the underlying process under a random discount rate, we give a complete characterization of all optimal exercising thresholds. Finally, we present an example with a compound Poisson process plus a drifted Brownian motion.

REFERENCES

[1] ALBRECHER, H., GERBER, H. U. and SHIU, E. S. W. (2011). The optimal dividend barrier in the gamma-omega model. *Eur. Actuar. J.* **1** 43–55. MR2843466

[2] ALILI, L. and KYPRIANOU, A. E. (2005). Some remarks on first passage of Lévy processes, the American put and pasting principles. *Ann. Appl. Probab.* **15** 2062–2080. MR2152253

[3] CARR, P. (1998). Randomization and the American put. *Rev. Financ. Stud.* **11** 597–626.

[4] CARR, P. and LINETSKY, V. (2000). The valuation of executive stock options in an intensity-based framework. *Eur. Finance Rev.* **4** 211–230.

[5] CHESNEY, M., JEANBLANC-PICQUÉ, M. and YOR, M. (1997). Brownian excursions and Parisian barrier options. *Adv. in Appl. Probab.* **29** 165–184. MR1432935

[6] CHRISTENSEN, S. and IRLE, A. (2013). American options with guarantee—A class of two-sided stopping problems. *Stat. Risk Model.* **30** 237–254. MR3118459

[7] DAYANIK, S. (2008). Optimal stopping of linear diffusions with random discounting. *Math. Oper. Res.* **33** 645–661. MR2442645

[8] EGAMI, M. and YAMAZAKI, K. (2014). On the continuous and smooth fit principle for optimal stopping problems in spectrally negative Lévy models. *Adv. in Appl. Probab.* **46** 139–167. MR3189052

[9] HALDANE, A. (2010). Patience and finance. In *Remarks at the Oxford China Business Forum, Beijing*. Available at http://www.bankofengland.co.uk/archive/Documents/historicpubs/speeches/2010/speech445.pdf.

MSC2010 subject classifications. Primary 60G40, 60G51; secondary 60G17.
Key words and phrases. Lévy process, optimal stopping, Omega clock, occupation times, random discount rate, impatience.

[10] HART, I. and ROSS, M. (1994). Striking continuity. *Risk* **7** 51–56.

[11] KUZNETSOV, A., KYPRIANOU, A. E. and RIVERO, V. (2012). The theory of scale functions for spectrally negative Lévy processes. In *Lévy Matters II*. *Lecture Notes in Math*. **2061** 97–186. Springer, Heidelberg. MR3014147

[12] KYPRIANOU, A. E. (2006). *Introductory Lectures on Fluctuations of Lévy Processes with Applications*. Springer, Berlin. MR2250061

[13] KYPRIANOU, A. E., PARDO, J. C. and PÉREZ, J. L. (2014). Occupation times of refracted Lévy processes. *J*. *Theoret*. *Probab*. **27** 1292–1315. MR3278941

[14] LANDRIAULT, D., LI, B. and ZHANG, H. (2017). On magnitude, asymptotics and duration of drawdowns for Lévy models. *Bernoulli* **23** 432–458. MR3556778

[15] LEUNG, T., YAMAZAKI, K. and ZHANG, H. (2015). Optimal multiple stopping with negative discount rate and random refraction times under Lévy models. *SIAM J*. *Control Optim*. **53** 2373–2405. MR3379015

[16] LINETSKY, V. (1999). Step options. *Math*. *Finance* **9** 55–96. MR1849356

[17] LOEFFEN, R., CZARNA, I. and PALMOWSKI, Z. (2013). Parisian ruin probability for spectrally negative Lévy processes. *Bernoulli* **19** 599–609. MR3037165

[18] LOEFFEN, R. L. (2015). On obtaining simple identities for overshoots of spectrally negative Lévy processes. Available at https://arxiv.org/abs/1410.5341.

[19] LOEFFEN, R. L., RENAUD, J.-F. and ZHOU, X. (2014). Occupation times of intervals until first passage times for spectrally negative Lévy processes. *Stochastic Process*. *Appl*. **124** 1408–1435. MR3148018

[20] MIURA, R. (1992). A note on look-back options based on order statistics. *Hitotsubashi J*. *Commer. Manag*. **27** 15–28.

[21] MORDECKI, E. (2002). Optimal stopping and perpetual options for Lévy processes. *Finance Stoch*. **6** 473–493. MR1932381

[22] ØKSENDAL, B. and SULEM, A. (2005). *Applied Stochastic Control of Jump Diffusions*. Springer, Berlin. MR2109687

[23] RODOSTHENOUS, N. and ZHANG, H. (2017). Beating the Omega clock: An optimal stopping problem with random time-horizon under spectrally negative Lévy models (Extended version). Available at https://arxiv.org/abs/1706.03724v1.

[24] SURYA, B. A. (2007). An approach for solving perpetual optimal stopping problems driven by Lévy processes. *Stochastics* **79** 337–361. MR2308080

[25] TRABELSI, F. (2011). Asymptotic behavior of random maturity American options. *IAENG Int*. *J*. *Appl*. *Math*. **41** 112–121. MR2841168

[26] ZHANG, H. (2015). Occupation times, drawdowns, and drawups for one-dimensional regular diffusions. *Adv*. *in Appl*. *Probab*. **47** 210–230. MR3327322

The Annals of Applied Probability
2018, Vol. 28, No. 4, 2141–2187
https://doi.org/10.1214/17-AAP1353

MUTATIONS ON A RANDOM BINARY TREE WITH MEASURED BOUNDARY[1]

BY JEAN-JIL DUCHAMPS*,† AND AMAURY LAMBERT*,†

Sorbonne Université and Collège de France*†

Consider a random real tree whose leaf set, or boundary, is endowed with a finite mass measure. Each element of the tree is further given a type, or allele, inherited from the most recent atom of a random point measure (infinitely-many-allele model) on the skeleton of the tree. The partition of the boundary into distinct alleles is the so-called *allelic partition*.

In this paper, we are interested in the infinite trees generated by supercritical, possibly time-inhomogeneous, binary branching processes, and in their boundary, which is the set of particles "coexisting at infinity". We prove that any such tree can be mapped to a random, compact ultrametric tree called the *coalescent point process*, endowed with a "uniform" measure on its boundary which is the limit as $t \to \infty$ of the properly rescaled counting measure of the population at time t.

We prove that the clonal (i.e., carrying the same allele as the root) part of the boundary is a regenerative set that we characterize. We then study the allelic partition of the boundary through the measures of its blocks. We also study the dynamics of the clonal subtree, which is a Markovian increasing tree process as mutations are removed.

REFERENCES

[1] ABRAHAM, R., DELMAS, J.-F. and HOSCHEIT, P. (2013). A note on the Gromov-Hausdorff-Prokhorov distance between (locally) compact metric measure spaces. *Electron. J. Probab.* **18** no. 14, 21. MR3035742

[2] ABRAHAM, R. and SERLET, L. (2002). Poisson snake and fragmentation. *Electron. J. Probab.* **7** no. 17, 15. MR1943890

[3] ALDOUS, D. and PITMAN, J. (1998). The standard additive coalescent. *Ann. Probab.* **26** 1703–1726. MR1675063

[4] BASDEVANT, A.-L. and GOLDSCHMIDT, C. (2008). Asymptotics of the allele frequency spectrum associated with the Bolthausen-Sznitman coalescent. *Electron. J. Probab.* **13** 486–512. MR2386740

[5] BERESTYCKI, J., BERESTYCKI, N. and LIMIC, V. (2014). Asymptotic sampling formulae for Λ-coalescents. *Ann. Inst. Henri Poincaré Probab. Stat.* **50** 715–731. MR3224287

[6] BERTOIN, J. (1999). Subordinators: Examples and applications. In *Lectures on Probability Theory and Statistics* (*Saint-Flour*, 1997). *Lecture Notes in Math.* **1717** 1–91. Springer, Berlin. MR1746300

MSC2010 subject classifications. Primary 05C05, 60J80; secondary 54E45, 60G51, 60G55, 60G57, 60K15, 92D10.

Key words and phrases. Coalescent point process, branching process, random point measure, allelic partition, regenerative set, tree-valued process.

[7] BERTOIN, J. (2009). The structure of the allelic partition of the total population for Galton-Watson processes with neutral mutations. *Ann. Probab.* **37** 1502–1523. MR2546753

[8] CHAMPAGNAT, N. and LAMBERT, A. (2012). Splitting trees with neutral Poissonian mutations I: Small families. *Stochastic Process. Appl.* **122** 1003–1033. MR2891445

[9] CHAMPAGNAT, N. and LAMBERT, A. (2013). Splitting trees with neutral Poissonian mutations II: Largest and oldest families. *Stochastic Process. Appl.* **123** 1368–1414. MR3016227

[10] CHAMPAGNAT, N., LAMBERT, A. and RICHARD, M. (2012). Birth and death processes with neutral mutations. *Int. J. Stoch. Anal.* Art. ID 569081, 20. MR3008829

[11] DELAPORTE, C., ACHAZ, G. and LAMBERT, A. (2016). Mutational pattern of a sample from a critical branching population. *J. Math. Biol.* **73** 627–664. MR3535416

[12] EVANS, S. N. (2008). *Probability and Real Trees. Lecture Notes in Math.* **1920**. Springer, Berlin. Lectures from the 35th Summer School on Probability Theory held in Saint-Flour, July 6–23, 2005. MR2351587

[13] EWENS, W. J. (1972). The sampling theory of selectively neutral alleles. *Theor. Popul. Biol.* **3** 87–112; erratum, ibid. 3 (1972), 240; erratum, ibid. 3 (1972), 376. MR0325177

[14] FREUND, F. (2012). Almost sure asymptotics for the number of types for simple Ξ-coalescents. *Electron. Commun. Probab.* **17** no. 3, 11. MR2872572

[15] FREUND, F. and MÖHLE, M. (2009). On the number of allelic types for samples taken from exchangeable coalescents with mutation. *Adv. in Appl. Probab.* **41** 1082–1101. MR2663237

[16] GRIFFITHS, R. C. and PAKES, A. G. (1988). An infinite-alleles version of the simple branching process. *Adv. in Appl. Probab.* **20** 489–524. MR0955502

[17] KENDALL, D. G. (1948). On the generalized "birth-and-death" process. *Ann. Math. Stat.* **19** 1–15. MR0024091

[18] KINGMAN, J. F. C. (1982). The coalescent. *Stochastic Process. Appl.* **13** 235–248. MR0671034

[19] LAMBERT, A. (2009). The allelic partition for coalescent point processes. *Markov Process. Related Fields* **15** 359–386. MR2554367

[20] LAMBERT, A. and SCHERTZER, E. (2016). Recovering the Brownian coalescent point process from the Kingman coalescent by conditional sampling. Preprint. Available at arXiv:1611.01323.

[21] LAMBERT, A. and URIBE BRAVO, G. (2017). The comb representation of compact ultrametric spaces. *P-Adic Numbers Ultrametric Anal. Appl.* **9** 22–38. MR3607645

[22] MARCHAL, P. (2004). Nested regenerative sets and their associated fragmentation process. In *Mathematics and Computer Science. III*. 461–470. Birkhäuser, Basel. MR2090534

[23] POPOVIC, L. (2004). Asymptotic genealogy of a critical branching process. *Ann. Appl. Probab.* **14** 2120–2148. MR2100386

[24] TAÏB, Z. (1992). *Branching Processes and Neutral Evolution. Lecture Notes in Biomathematics* **93**. Springer, Berlin. MR1176317

The Annals of Applied Probability
2018, Vol. 28, No. 4, 2188–2242
https://doi.org/10.1214/17-AAP1354

N-PLAYER GAMES AND MEAN-FIELD GAMES WITH ABSORPTION

BY LUCIANO CAMPI AND MARKUS FISCHER[1]

London School of Economics and Political Science and University of Padua

We introduce a simple class of mean-field games with absorbing boundary over a finite time horizon. In the corresponding N-player games, the evolution of players' states is described by a system of weakly interacting Itô equations with absorption on first exit from a bounded open set. Once a player exits, her/his contribution is removed from the empirical measure of the system. Players thus interact through a renormalized empirical measure. In the definition of solution to the mean-field game, the renormalization appears in form of a conditional law. We justify our definition of solution in the usual way, that is, by showing that a solution of the mean-field game induces approximate Nash equilibria for the N-player games with approximation error tending to zero as N tends to infinity. This convergence is established provided the diffusion coefficient is nondegenerate. The degenerate case is more delicate and gives rise to counter-examples.

REFERENCES

[1] ALIPRANTIS, C. D. and BORDER, K. C. (2006). *Infinite Dimensional Analysis*: *A Hitchhiker's Guide*, 3rd ed. Springer, Berlin. MR2378491

[2] BENSOUSSAN, A., FREHSE, J. and GRÜN, C. (2014). Stochastic differential games with a varying number of players. *Commun*. *Pure Appl*. *Anal*. **13** 1719–1736. MR3230409

[3] BILLINGSLEY, P. (1999). *Convergence of Probability Measures*, 2nd ed. Wiley, New York. MR1700749

[4] BRIAND, P. and HU, Y. (1998). Stability of BSDEs with random terminal time and homogenization of semilinear elliptic PDEs. *J*. *Funct*. *Anal*. **155** 455–494.

[5] CARMONA, R. and DELARUE, F. (2013). Probabilistic analysis of mean-field games. *SIAM J*. *Control Optim*. **51** 2705–2734.

[6] CARMONA, R., FOUQUE, J.-P. and SUN, L.-H. (2015). Mean field games and systemic risk. *Commun*. *Math*. *Sci*. **13** 911–933.

[7] CARMONA, R. and LACKER, D. (2015). A probabilistic weak formulation of mean field games and applications. *Ann*. *Appl*. *Probab*. **25** 1189–1231. MR3325272

[8] CHAN, P. and SIRCAR, R. (2015). Bertrand and Cournot mean field games. *Appl*. *Math*. *Optim*. **71** 533–569.

[9] CHAN, P. and SIRCAR, R. (2017). Fracking, renewables, and mean field games. *SIAM Review* **59** 588–615.

[10] CVITANIĆ, J., MA, J. and ZHANG, J. (2012). The law of large numbers for self-exciting correlated defaults. *Stochastic Process*. *Appl*. **122** 2781–2810. MR2931341

MSC2010 subject classifications. Primary 60K35, 91A06; secondary 60B10, 93E20.

Key words and phrases. Mean-field game, Nash equilibrium, McKean–Vlasov limit, absorbing boundary, weak convergence, martingale problem, optimal control.

[11] DAI PRA, P., RUNGGALDIER, W. J., SARTORI, E. and TOLOTTI, M. (2009). Large portfolio losses: A dynamic contagion model. *Ann. Appl. Probab.* **19** 347–394. MR2498681

[12] DARLING, R. W. R. and PARDOUX, E. (1997). Backwards SDE with random terminal time and applications to semilinear elliptic PDE. *Ann. Probab.* **25** 1135–1159. MR1457614

[13] DELARUE, F., INGLIS, J., RUBENTHALER, S. and TANRÉ, E. (2015). Global solvability of a networked integrate-and-fire model of McKean–Vlasov type. *Ann. Appl. Probab.* **25** 2096–2133. MR3349003

[14] DELARUE, F., INGLIS, J., RUBENTHALER, S. and TANRÉ, E. (2015). Particle systems with a singular mean-field self-excitation. Application to neuronal networks. *Stochastic Process. Appl.* **125** 2451–2492. MR3322871

[15] EL KAROUI, N., NGUYEN, D. H. and JEANBLANC-PICQUÉ, M. (1987). Compactification methods in the control of degenerate diffusions: Existence of an optimal control. *Stochastics* **20** 169–219.

[16] FISCHER, M. (2017). On the connection between symmetric N-player games and mean field games. *Ann. Appl. Probab.* **27** 757–810. MR3655853

[17] FLEMING, W. H. and RISHEL, R. W. (1975). *Deterministic and Stochastic Optimal Control. Applications of Mathematics* **1**. Springer, New York. MR0454768

[18] FUNAKI, T. (1984). A certain class of diffusion processes associated with nonlinear parabolic equations. *Z. Wahrsch. Verw. Gebiete* **67** 331–348.

[19] GIESECKE, K., SPILIOPOULOS, K. and SOWERS, R. B. (2013). Default clustering in large portfolios: Typical events. *Ann. Appl. Probab.* **23** 348–385. MR3059238

[20] GIESECKE, K., SPILIOPOULOS, K., SOWERS, R. B. and SIRIGNANO, J. A. (2015). Large portfolio asymptotics for loss from default. *Math. Finance* **25** 77–114.

[21] GRABER, P. J. and BENSOUSSAN, A. (2018). Existence and uniqueness of solutions for Bertrand and Cournot mean field games. *Appl. Math. Optim.* **77** 47–71. MR3755719

[22] HAMBLY, B. and LEDGER, S. (2017). A stochastic McKean–Vlasov equation for absorbing diffusions on the half-line. *Ann. Appl. Probab.* **27** 2698–2752. MR3719945

[23] HUANG, M., MALHAMÉ, R. P. and CAINES, P. E. (2006). Large population stochastic dynamic games: Closed-loop McKean–Vlasov systems and the Nash certainty equivalence principle. *Commun. Inf. Syst.* **6** 221–252.

[24] KALLENBERG, O. (2001). *Foundations of Modern Probability. Probability and Its Applications*, 2nd ed. Springer, New York.

[25] KARATZAS, I. and SHREVE, S. E. (1991). *Brownian Motion and Stochastic Calculus*, 2nd ed. *Graduate Texts in Mathematics* **113**. Springer, New York.

[26] KOLOKOLTSOV, V. N., LI, J. and YANG, W. (2011). Mean field games and nonlinear Markov processes. Available at arXiv:1112.3744 [math.PR].

[27] KRYLOV, N. V. (1980). *Controlled Diffusion Processes. Applications of Mathematics* **14**. Springer, New York.

[28] KUSHNER, H. J. (1990). *Weak Convergence Methods and Singularly Perturbed Stochastic Control and Filtering Problems. Systems & Control: Foundations & Applications* **3**. Birkhäuser, Boston.

[29] KUSHNER, H. J. and DUPUIS, P. (2001). *Numerical Methods for Stochastic Control Problems in Continuous Time*, 2nd ed. *Applications of Mathematics* **24**. Springer, New York.

[30] LACKER, D. (2016). A general characterization of the mean field limit for stochastic differential games. *Probab. Theory Related Fields* **165** 581–648. MR3520014

[31] LADYŽENSKAJA, O. A., SOLONNIKOV, V. A. and URAL'CEVA, N. N. (1968). *Linear and Quasi-Linear Equations of Parabolic Type*. Amer. Math. Soc., Providence, RI.

[32] LASRY, J.-M. and LIONS, P.-L. (2006). Jeux à champ moyen. I. Le cas stationnaire. *C. R. Math. Acad. Sci. Paris* **343** 619–625.

[33] LASRY, J.-M. and LIONS, P.-L. (2006). Jeux à champ moyen. II. Horizon fini et contrôle optimal. *C. R. Math. Acad. Sci. Paris* **343** 679–684.

[34] LASRY, J.-M. and LIONS, P.-L. (2007). Mean field games. *Jpn. J. Math.* **2** 229–260. MR2295621

[35] MÉLÉARD, S. (1996). Asymptotic behaviour of some interacting particle systems; McKean–Vlasov and Boltzmann models. In *Probabilistic Models for Nonlinear Partial Differential Equations*. 1*st C.I.M.E* (D. Talay and L. Tubaro, eds.). *Lecture Notes in Math*. **1627** 42–95. Springer, Berlin.

[36] OELSCHLÄGER, K. (1984). A martingale approach to the law of large numbers for weakly interacting stochastic processes. *Ann. Probab.* **12** 458–479. MR0735849

[37] ROGERS, L. C. G. and WILLIAMS, D. (2000). *Diffusions, Markov Processes and Martingales. Volume* 2: *Itô Calculus*. Cambridge Univ. Press, Cambridge.

[38] SPILIOPOULOS, K., SIRIGNANO, J. A. and GIESECKE, K. (2014). Fluctuation analysis for the loss from default. *Stochastic Process. Appl*. **124** 2322–2362. MR3192499

[39] STROOCK, D. W. and VARADHAN, S. R. S. (1979). *Multidimensional Diffusion Processes. Grundlehren der Mathematischen Wissenschaften*. **233**. Springer, Berlin.

[40] SZNITMAN, A.-S. (1991). Topics in propagation of chaos. In *Ecole D'Eté de Probabilités de Saint-Flour XIX*—1989 (P.-L. Hennequin, ed.). *Lecture Notes in Math*. **1464** 165–251. Springer, Berlin.

The Annals of Applied Probability
2018, Vol. 28, No. 4, 2243–2274
https://doi.org/10.1214/17-AAP1355

HYPOELLIPTIC STOCHASTIC FITZHUGH–NAGUMO NEURONAL MODEL: MIXING, UP-CROSSING AND ESTIMATION OF THE SPIKE RATE

BY JOSÉ R. LEÓN[*,†,‡,§,1] AND ADELINE SAMSON[†,2]

Universidad de la República[*], *Université Grenoble Alpes*[†], *INRIA project/team AIRSEA*[‡] *and Universidad Central de Venezuela*[§]

The FitzHugh–Nagumo is a well-known neuronal model that describes the generation of spikes at the intracellular level. We study a stochastic version of the model from a probabilistic point of view. The hypoellipticity is proved, as well as the existence and uniqueness of the stationary distribution. The bi-dimensional stochastic process is β-mixing. The stationary density can be estimated with an adaptive nonparametric estimator. Then we focus on the distribution of the length between successive spikes. Spikes are difficult to define directly from the continuous stochastic process. We study the distribution of the number of up-crossings. We link it to the stationary distribution and propose an estimator of its expectation. We finally prove mathematically that the mean length of inter-up-crossings interval is equal to its up-crossings rate. We illustrate the proposed estimators on a simulation study. Different regimes are explored, with no, few or high generation of spikes.

REFERENCES

[1] AZAÏS, J.-M. and WSCHEBOR, M. (2009). *Level Sets and Extrema of Random Processes and Fields*. Wiley, Hoboken, NJ. MR2478201

[2] BERGLUND, N. and GENTZ, B. (2006). *Noise-Induced Phenomena in Slow–Fast Dynamical Systems*: *A Sample-Paths Approach*. Springer, London. MR2197663

[3] BERGLUND, N. and LANDON, D. (2012). Mixed-mode oscillations and interspike interval statistics in the stochastic FitzHugh-Nagumo model. *Nonlinearity* **25**.

[4] CATTIAUX, P., LEON, J., PINEDA CENTENO, A. and PRIEUR, C. (2017). An overlook on statistical inference issues for stochastic damping Hamiltonian systems under the fluctuation-dissipation condition. *Statistics* **51** 11–29.

[5] COMTE, F., PRIEUR, C. and SAMSON, A. (2017). Adaptive estimation for stochastic damping Hamiltonian systems under partial observation. *Stochastic Process*. *Appl*. **127** 3689–3718. MR3707242

[6] CRAMÉR, H. and LEADBETTER, M. R. (1967). *Stationary and Related Stochastic Processes*. Wiley, New York.

[7] DITLEVSEN, S. and SAMSON, A. (2017). Hypoelliptic diffusion: Discretization, filtering and inference from complete and partial observations. Submitted.

MSC2010 subject classifications. Primary 60J70; secondary 62M05, 60H10, 60J60, 62P10, 35Q62, 37A50.

Key words and phrases. Hypoelliptic diffusion, FitzHugh–Nagumo model, invariant density, nonparametric estimation, up-crossings, pulse rate, spike rate estimation.

[8] DI BERNARDINO, E., LEÓN, J. and TCHUMATCHENKO, T. (2014). Cross-correlations and joint Gaussianity in multivariate level crossing models. *J. Math. Neurosci.* **4** Art. ID 22. MR3197020

[9] DOUKHAN, P. (1994). *Mixing: Properties and Examples. Lecture Notes in Statistics* **85**. Springer, New York. MR1312160

[10] GERSTNER, W. and KISTLER, W. (2002). *Spiking Neuron Models*. Cambridge Univ. Press, Cambridge.

[11] GOLDENSHLUGER, A. and LEPSKI, O. (2011). Bandwidth selection in kernel density estimation: Oracle inequalities and adaptive minimax optimality. *Ann. Statist.* **39** 1608–1632. MR2850214

[12] HODGKIN, A. L. and HUXLEY, A. F. (1952). A quantitative description of ion currents and its applications to conduction and excitation in nerve membranes. *J. Physiol.* **117** 500–544.

[13] KONAKOV, V., MENOZZI, S. and MOLCHANOV, S. (2010). Explicit parametrix and local limit theorems for some degenerate diffusion processes. *Ann. Inst. Henri Poincaré Probab. Stat.* **46** 908–923. MR2744877

[14] KUMAR, P. and NARAYANAN, S. (2006). Solution of Fokker–Planck equation by finite element and finite difference methods for nonlinear systems. *Sādhanā* **31** 445–461.

[15] LACOUR, C., MASSART, P. and RIVOIRARD, V. (2017). Estimator selection: A new method with applications to kernel density estimation. *Sankhya A* **72** 1–38. MR3707423

[16] LINDNER, B., GARCIA-OJALVO, J., NEIMAN, A. and SCHIMANSKY-GEIER, L. (2004). Effects of noise in excitable systems. *Phys. Rep.* **392** 321–424.

[17] LINDNER, B. and SCHIMANSKY-GEIER, L. (1999). Analytical approach to the stochastic FitzHugh–Nagumo system and coherence resonance. *Phys. Rev. E* **60** 7270–7276.

[18] LONGTIN, A. (2000). Effect of noise on the tuning properties of excitable systems. *Chaos Solitons Fractals* **11** 1835–1848.

[19] MATTINGLY, J. C., STUART, A. M. and HIGHAM, D. J. (2002). Ergodicity for SDEs and approximations: Locally Lipschitz vector fields and degenerate noise. *Stochastic Process. Appl.* **101** 185–232. MR1931266

[20] MORRIS, C. and LECAR, H. (1981). Voltage oscillations in the barnacle giant muscle fiber. *Biophys. J.* **35** 193–213.

[21] PAKDAMAN, K., THIEULLEN, M. and WAINRIB, G. (2010). Fluid limit theorems for stochastic hybrid systems with application to neuron models. *Adv. in Appl. Probab.* **42** 761–794. MR2779558

[22] REYNAUD-BOURET, P., RIVOIRARD, V., GRAMMONT, F. and TULEAU-MALOT, C. (2014). Goodness-of-fit tests and nonparametric adaptive estimation for spike train analysis. *J. Math. Neurosci.* **4** Art. ID 3. MR3197017

[23] REYNAUD-BOURET, P., RIVOIRARD, V. and TULEAU-MALOT, C. (2013). Inference of functional connectivity in neurosciences via Hawkes processes. In 1*st IEEE Global Conference on Signal and Information Processing*, *Austin*, *TX*.

[24] SAMSON, A. and THIEULLEN, M. (2012). Contrast estimator for completely or partially observed hypoelliptic diffusion. *Stochastic Process. Appl.* **122** 2521–2552. MR2926166

[25] TUCKWELL, H. C. and RODRIGUEZ, R. (1998). Analytical and simulation results for stochastic Fitzhugh–Nagumo neurons and neural networks. *J. Comput. Neurosci.* **5** 91–113.

[26] WU, L. (2001). Large and moderate deviations and exponential convergence for stochastic damping Hamiltonian systems. *Stochastic Process. Appl.* **91** 205–238. MR1807683

The Annals of Applied Probability
2018, Vol. 28, No. 4, 2275–2308
https://doi.org/10.1214/17-AAP1357

EXCEPTIONAL TIMES OF THE CRITICAL DYNAMICAL ERDŐS–RÉNYI GRAPH[1]

BY MATTHEW I. ROBERTS AND BATI ŞENGÜL

University of Bath and Bank of America Merrill Lynch

In this paper, we introduce a network model which evolves in time, and study its largest connected component. We consider a process of graphs $(G_t : t \in [0, 1])$, where initially we start with a critical Erdős–Rényi graph $\mathrm{ER}(n, 1/n)$, and then evolve forward in time by resampling each edge independently at rate 1. We show that the size of the largest connected component that appears during the time interval $[0, 1]$ is of order $n^{2/3} \log^{1/3} n$ with high probability. This is in contrast to the largest component in the static critical Erdős–Rényi graph, which is of order $n^{2/3}$.

REFERENCES

[1] ADDARIO-BERRY, L., BROUTIN, N. and GOLDSCHMIDT, C. (2012). The continuum limit of critical random graphs. *Probab. Theory Related Fields* **152** 367–406. MR2892951

[2] ALDOUS, D. (1997). Brownian excursions, critical random graphs and the multiplicative coalescent. *Ann. Probab.* **25** 812–854. MR1434128

[3] BENJAMINI, I., HÄGGSTRÖM, O., PERES, Y. and STEIF, J. E. (2003). Which properties of a random sequence are dynamically sensitive? *Ann. Probab.* **31** 1–34. MR1959784

[4] BENJAMINI, I., KALAI, G. and SCHRAMM, O. (1999). Noise sensitivity of Boolean functions and applications to percolation. *Inst. Hautes Études Sci. Publ. Math.* **90** 5–43. MR1813223

[5] BOLLOBÁS, B. (2001). *Random Graphs*, 2nd ed. *Cambridge Studies in Advanced Mathematics* **73**. Cambridge Univ. Press, Cambridge. MR1864966

[6] BROMAN, E. I. and STEIF, J. E. (2006). Dynamical stability of percolation for some interacting particle systems and ϵ-movability. *Ann. Probab.* **34** 539–576. MR2223951

[7] DURRETT, R. (2010). *Random Graph Dynamics*. *Cambridge Series in Statistical and Probabilistic Mathematics* **20**. Cambridge Univ. Press, Cambridge. MR2656427

[8] ERDŐS, P. and RÉNYI, A. (1960). On the evolution of random graphs. *Magy. Tud. Akad. Mat. Kutató Int. Közl.* **5** 17–61. MR0125031

[9] GARBAN, C. (2011). Oded Schramm's contributions to noise sensitivity. In *Selected Works of Oded Schramm. Volume* 1, 2. *Sel. Works Probab. Stat.* 287–350. Springer, New York. MR2883377

[10] GARBAN, C., PETE, G. and SCHRAMM, O. (2010). The Fourier spectrum of critical percolation. *Acta Math.* **205** 19–104.

[11] GARBAN, C. and STEIF, J. E. (2015). *Noise Sensitivity of Boolean Functions and Percolation*. *Institute of Mathematical Statistics Textbooks* **5**. Cambridge Univ. Press, New York. MR3468568

MSC2010 subject classifications. 05C80, 82C20, 60F17.

Key words and phrases. Noise sensitivity, Erdős–Renyi, dynamical random graphs, temporal networks, giant component.

[12] HÄGGSTRÖM, O. and PEMANTLE, R. (1999). On near-critical and dynamical percolation in the tree case. *Random Structures Algorithms* **15** 311–318. MR1716767

[13] HÄGGSTRÖM, O., PERES, Y. and STEIF, J. E. (1997). Dynamical percolation. *Ann. Inst. Henri Poincaré Probab. Stat.* **33** 497–528. MR1465800

[14] HAMMOND, A., PETE, G. and SCHRAMM, O. (2015). Local time on the exceptional set of dynamical percolation and the incipient infinite cluster. *Ann. Probab.* **43** 2949–3005. MR3433575

[15] HOLME, P. and SARAMÄKI, J. (2012). Temporal networks. *Phys. Rep.* **519** 97–125.

[16] JANSON, S., ŁUCZAK, T. and RUCINSKI, A. (2000). *Random Graphs*. Wiley, New York. MR1782847

[17] JONASSON, J. and STEIF, J. E. (2016). Volatility of Boolean functions. *Stochastic Process. Appl.* **126** 2956–2975.

[18] LUBETZKY, E. and STEIF, J. E. (2015). Strong noise sensitivity and random graphs. *Ann. Probab.* **43** 3239–3278. MR3433581

[19] PERES, Y., SCHRAMM, O. and STEIF, J. E. (2009). Dynamical sensitivity of the infinite cluster in critical percolation. *Ann. Inst. Henri Poincaré Probab. Stat.* **45** 491–514. MR2521411

[20] PITTEL, B. (2001). On the largest component of the random graph at a nearcritical stage. *J. Combin. Theory Ser. B* **82** 237–269.

[21] ROBERTS, M. I. (2018). The probability of unusually large components in the near-critical Erdős–Rényi graph. *Adv. Appl. Probab.* **50**. 245–271. MR3781985

[22] ROSSIGNOL, R. Scaling limit of dynamical percolation on the critical Erdős–Rényi random graph. Preprint. Available at arXiv:1710.09101.

[23] SCHRAMM, O. and STEIF, J. E. (2010). Quantitative noise sensitivity and exceptional times for percolation. *Ann. of Math.* (2) **171** 619–672. MR2630053

[24] STEIF, J. E. (2009). A survey of dynamical percolation. In *Fractal Geometry and Stochastics IV. Progress in Probability* **61** 145–174. Birkhäuser, Basel.

[25] TALAGRAND, M. (1994). On Russo's approximate zero–one law. *Ann. Probab.* **22** 1576–1587. MR1303654

[26] VAN DER HOFSTAD, R. (2017). *Random Graphs and Complex Networks. Vol.* 1. Cambridge Univ. Press, Cambridge. MR3617364

The Annals of Applied Probability
2018, Vol. 28, No. 4, 2309–2334
https://doi.org/10.1214/17-AAP1358

WHICH ERGODIC AVERAGES HAVE FINITE ASYMPTOTIC VARIANCE?[1]

BY GEORGE DELIGIANNIDIS* AND ANTHONY LEE†,‡,§

University of Oxford, University of Warwick†, University of Bristol‡ and The Alan Turing Institute§*

We show that the class of L^2 functions for which ergodic averages of a reversible Markov chain have finite asymptotic variance is determined by the class of L^2 functions for which ergodic averages of its associated jump chain have finite asymptotic variance. This allows us to characterize completely which ergodic averages have finite asymptotic variance when the Markov chain is an independence sampler. From a practical perspective, the most important result identifies a simple sufficient condition for all ergodic averages of L^2 functions of the primary variable in a pseudo-marginal Markov chain to have finite asymptotic variance.

REFERENCES

ANDRIEU, C. and ROBERTS, G. O. (2009). The pseudo-marginal approach for efficient Monte Carlo computations. *Ann. Statist.* **37** 697–725. MR2502648

ANDRIEU, C. and VIHOLA, M. (2015). Convergence properties of pseudo-marginal Markov chain Monte Carlo algorithms. *Ann. Appl. Probab.* **25** 1030–1077. MR3313762

ANDRIEU, C. and VIHOLA, M. (2016). Establishing some order amongst exact approximations of MCMCs. *Ann. Appl. Probab.* **26** 2661–2696. MR3563190

BANTERLE, M., GRAZIAN, C., LEE, A. and ROBERT, C. P. (2015). Accelerating Metropolis–Hastings algorithms by delayed acceptance. Available at arXiv:1503.00996.

BEAUMONT, M. A. (2003). Estimation of population growth or decline in genetically monitored populations. *Genetics* **164** 1139–1160.

BEDNORZ, W., ŁATUSZYŃSKI, K. and LATAŁA, R. (2008). A regeneration proof of the central limit theorem for uniformly ergodic Markov chains. *Electron. Commun. Probab.* **13** 85–98.

BORNN, L., PILLAI, N., SMITH, A. and WOODARD, D. (2017). The use of a single pseudo-sample in approximate Bayesian computation. *Stat. Comput.* **27** 583–590.

CARACCIOLO, S., PELISSETTO, A. and SOKAL, A. D. (1990). Nonlocal Monte Carlo algorithm for self-avoiding walks with fixed endpoints. *J. Stat. Phys.* **60** 1–53.

COSTA, O. L. V. (1990). Stationary distributions for piecewise-deterministic Markov processes. *J. Appl. Probab.* **27** 60–73.

COSTA, O. L. V. and DUFOUR, F. (2008). Stability and ergodicity of piecewise deterministic Markov processes. *SIAM J. Control Optim.* **47** 1053–1077.

DOUC, R. and ROBERT, C. P. (2011). A vanilla Rao–Blackwellization of Metropolis–Hastings algorithms. *Ann. Statist.* **39** 261–277. MR2797846

MSC2010 subject classifications. 60J05, 60J22, 60F05, 65C40.
Key words and phrases. Markov chain Monte Carlo, asymptotic variance, jump chain, independent Metropolis–Hastings, pseudo-marginal method.

DOUCET, A., PITT, M. K., DELIGIANNIDIS, G. and KOHN, R. (2015). Efficient implementation of Markov chain Monte Carlo when using an unbiased likelihood estimator. *Biometrika* **102** 295–313.

GEYER, C. J. (1992). Practical Markov chain Monte Carlo. *Statist. Sci.* **7** 473–483.

HÄGGSTRÖM, O. and ROSENTHAL, J. S. (2007). On variance conditions for Markov chain CLTs. *Electron. Commun. Probab.* **12** 454–464.

HASTINGS, W. K. (1970). Monte Carlo sampling methods using Markov chains and their applications. *Biometrika* **57** 97–109.

JARNER, S. F. and HANSEN, E. (2000). Geometric ergodicity of Metropolis algorithms. *Stochastic Process. Appl.* **85** 341–361. MR1731030

JARNER, S. F. and ROBERTS, G. O. (2002). Polynomial convergence rates of Markov chains. *Ann. Appl. Probab.* **12** 224–247. MR1890063

JARNER, S. F. and ROBERTS, G. O. (2007). Convergence of heavy-tailed Monte Carlo Markov chain algorithms. *Scand. J. Stat.* **34** 781–815. MR2396939

KIPNIS, C. and VARADHAN, S. R. S. (1986). Central limit theorem for additive functionals of reversible Markov processes and applications to simple exclusions. *Comm. Math. Phys.* **104** 1–19.

LEE, A. and ŁATUSZYŃSKI, K. (2014). Variance bounding and geometric ergodicity of Markov chain Monte Carlo kernels for approximate Bayesian computation. *Biometrika* **101** 655–671.

LIN, L., LIU, K. F. and SLOAN, J. (2000). A noisy Monte Carlo algorithm. *Phys. Rev. D* **61** 074505.

MENGERSEN, K. L. and TWEEDIE, R. L. (1996). Rates of convergence of the Hastings and Metropolis algorithms. *Ann. Statist.* **24** 101–121. MR1389882

METROPOLIS, N., ROSENBLUTH, A. W., ROSENBLUTH, M. N., TELLER, A. H. and TELLER, E. (1953). Equation of state calculations by fast computing machines. *J. Chem. Phys.* **21** 1087–1092.

MEYN, S. and TWEEDIE, R. L. (2009). *Markov Chains and Stochastic Stability*, 2nd ed. Cambridge Univ. Press, Cambridge. MR2509253

PARDOUX, E. (2008). *Markov Processes and Applications*. Wiley, New York.

PESKUN, P. H. (1973). Optimum Monte-Carlo sampling using Markov chains. *Biometrika* **60** 607–612.

ROBERTS, G. O. and ROSENTHAL, J. S. (2004). General state space Markov chains and MCMC algorithms. *Probab. Surv.* **1** 20–71.

ROBERTS, G. O. and ROSENTHAL, J. S. (2008). Variance bounding Markov chains. *Ann. Appl. Probab.* **18** 1201–1214.

SHERLOCK, C., THIERY, A. H. and LEE, A. (2017). Pseudo-marginal Metropolis–Hastings sampling using averages of unbiased estimators. *Biometrika* **104** 727–734.

SHERLOCK, C., THIERY, A. H., ROBERTS, G. O. and ROSENTHAL, J. S. (2015). On the efficiency of pseudo-marginal random walk Metropolis algorithms. *Ann. Statist.* **43** 238–275. MR3285606

TAVARÉ, S., BALDING, D. J., GRIFFITHS, R. C. and DONNELLY, P. (1997). Inferring coalescence times from DNA sequence data. *Genetics* **145** 505–518.

TIERNEY, L. (1994). Markov chains for exploring posterior distributions. *Ann. Statist.* **22** 1701–1762. MR1329166

TIERNEY, L. (1998). A note on Metropolis–Hastings kernels for general state spaces. *Ann. Appl. Probab.* **8** 1–9. MR1620401

YOSIDA, K. (1980). *Functional Analysis*, 6th ed. Springer, Berlin.

The Annals of Applied Probability
2018, Vol. 28, No. 4, 2335–2369
https://doi.org/10.1214/17-AAP1359

A STOCHASTIC STEFAN-TYPE PROBLEM UNDER FIRST-ORDER BOUNDARY CONDITIONS

BY MARVIN S. MÜLLER[1,2]

ETH Zürich

Moving boundary problems allow to model systems with phase transition at an inner boundary. Motivated by problems in economics and finance, we set up a price-time continuous model for the limit order book and consider a stochastic and nonlinear extension of the classical Stefan-problem in one space dimension. Here, the paths of the moving interface might have unbounded variation, which introduces additional challenges in the analysis. Working on the distribution space, the Itô–Wentzell formula for SPDEs allows to transform these moving boundary problems into partial differential equations on fixed domains. Rewriting the equations into the framework of stochastic evolution equations and stochastic maximal L^p-regularity, we get existence, uniqueness and regularity of local solutions. Moreover, we observe that explosion might take place due to the boundary interaction even when the coefficients of the original problem have linear growths.

REFERENCES

[1] AMANN, H. (2009). *Anisotropic Function Spaces and Maximal Regularity for Parabolic Problems*. *Part* 1: *Function Spaces*. *Jindřich Nečas Center for Mathematical Modeling Lecture Notes* **6**. Matfyzpress, Prague. MR2907677

[2] ANTONOPOULOU, D. C., BLÖMKER, D. and KARALI, G. D. (2015). The sharp interface limit for the stochastic Cahn–Hilliard equation. Available at arXiv:1507.07469.

[3] AUSCHER, P., MCINTOSH, A. and NAHMOD, A. (1997). The square root problem of Kato in one dimension, and first order elliptic systems. *Indiana Univ. Math. J.* **46** 659–695. MR1488331

[4] BARBU, V. and DA PRATO, G. (2002). The two phase stochastic Stefan problem. *Probab. Theory Related Fields* **124** 544–560.

[5] BAYER, C., HORST, U. and QIU, J. (2017). A functional limit theorem for limit order books with state dependent price dynamics. *Ann. Appl. Probab.* **27** 2753–2806. MR3719946

[6] BIAIS, B., HILLION, P. and SPATT, C. (1995). An empirical analysis of the limit order book and the order flow in the Paris bourse. *J. Finance* **50** 1655–1689.

[7] BOUCHAUD, J. P., MÉZARD, M. and POTTERS, M. (2002). Statistical properties of stock order books: Empirical results and models. *Quant. Finance* **2** 251–256.

[8] BRZEŹNIAK, Z. and VERAAR, M. (2012). Is the stochastic parabolicity condition dependent on p and q? *Electron. J. Probab.* **17** no. 56, 24.

[9] CONT, R. (2011). Statistical modeling of high-frequency financial data. *IEEE Signal Processing Magazine* **28** 16–25.

MSC2010 subject classifications. Primary 60H15; secondary 91B70, 91G80.
Key words and phrases. Stochastic partial differential equations, Stefan problem, moving boundary problem, limit order book.

[10] CONT, R., KUKANOV, A. and STOIKOV, S. (2014). The price impact of order book events. *J. Financ. Econom.* **12** 47–88.

[11] CONT, R., STOIKOV, S. and TALREJA, R. (2010). A stochastic model for order book dynamics. *Oper. Res.* **58** 549–563.

[12] DA PRATO, G., KWAPIEŃ, S. and ZABCZYK, J. (1987). Regularity of solutions of linear stochastic equations in Hilbert spaces. *Stochastics* **23** 1–23.

[13] DA PRATO, G. and ZABCZYK, J. (2014). *Stochastic Equations in Infinite Dimensions*, 2nd ed. *Encyclopedia of Mathematics and Its Applications* **152**. Cambridge Univ. Press, Cambridge. MR3236753

[14] DONIER, J., BONART, J., MASTROMATTEO, I. and BOUCHAUD, J.-P. (2015). A fully consistent, minimal model for non-linear market impact. *Quant. Finance* **15** 1109–1121. MR3354126

[15] GOULD, M. D., PORTER, M. A., WILLIAMS, S., MCDONALD, M., FENN, D. J. and HOWISON, S. D. (2013). Limit order books. *Quant. Finance* **13** 1709–1742.

[16] GRISVARD, P. (1967). Caractérisation de quelques espaces d'interpolation. *Arch. Ration. Mech. Anal.* **25** 40–63. MR0213864

[17] HAASE, M. (2006). *The Functional Calculus for Sectorial Operators. Operator Theory: Advances and Applications* **169**. Birkhäuser, Basel.

[18] HUANG, W., LEHALLE, C.-A. and ROSENBAUM, M. (2015). Simulating and analyzing order book data: The queue-reactive model. *J. Amer. Statist. Assoc.* **110** 107–122. MR3338490

[19] JACOD, J. and SHIRYAEV, A. N. (2003). *Limit Theorems for Stochastic Processes*, 2nd ed. *Grundlehren der Mathematischen Wissenschaften* **288**. Springer, Berlin.

[20] KELLER-RESSEL, M. and MÜLLER, M. S. (2016). A Stefan-type stochastic moving boundary problem. *Stoch. Partial Differ. Equ. Anal. Computat.* **4** no. 4, 746–790. MR3554430

[21] KIM, K., MUELLER, C. and SOWERS, R. B. (2010). A stochastic moving boundary value problem. *Illinois J. Math.* **54** 927–962.

[22] KIM, K., ZHENG, Z. and SOWERS, R. B. (2012). A stochastic Stefan problem. *J. Theoret. Probab.* **25** 1040–1080.

[23] KIRILENKO, A. A., KYLE, A. S., SAMADI, M. and TUZUN, T. (2011). The flash crash: The impact of high frequency trading on an electronic market. *Social Science Research Network Working Paper Series*.

[24] KRYLOV, N. V. (1999). An analytic approach to SPDEs. In *Stochastic Partial Differential Equations: Six Perspectives. Math. Surveys Monogr.* **64** 185–242. Amer. Math. Soc., Providence, RI. MR1661766

[25] KRYLOV, N. V. (2011). On the Itô–Wentzell formula for distribution-valued processes and related topics. *Probab. Theory Related Fields* **150** 295–319. MR2800911

[26] LASRY, J.-M. and LIONS, P.-L. (2007). Mean field games. *Jpn. J. Math.* **2** 229–260. MR2295621

[27] LIONS, J. L. and MAGENES, E. (1972). *Non-homogeneous Boundary Value Problems and Applications*—1. Springer, Berlin.

[28] LIPTON, A., PESAVENTO, U. and SOTIROPOULOS, M. G. (2014). Trading strategies via book imbalance. *Risk*.

[29] LUNARDI, A. (1995). *Analytic Semigroups and Optimal Regularity in Parabolic Problems*. Birkhäuser, Basel.

[30] LUNARDI, A. (2004). An introduction to parabolic moving boundary problems. In *Functional Analytic Methods for Evolution Equations* (M. Iannelli, R. Nagel and S. Piazzera, eds.). *Lecture Notes in Mathematics* **1855** 371–399. Springer, Berlin Heidelberg.

[31] LUNARDI, A. (2009). *Interpolation Theory*, 2nd ed. Edizioni della Normale, Pisa.

[32] MARKOWICH, P. A., TEICHMANN, J. and WOLFRAM, M. T. (2016). Parabolic free boundary price formation models under market size fluctuations. *Multiscale Model. Simul.* **14** 1211–1237.

[33] MASTROMATTEO, I., TÓTH, B. and BOUCHAUD, J.-P. (2014). Anomalous impact in reaction-diffusion financial models. *Phys. Rev. Lett.* **113** 268701.

[34] MUELLER, M. S. (2016). Semilinear stochastic moving boundary problems. Doctoral thesis, TU Dresden.

[35] SMITH, E., FARMER, J. D., GILLEMOT, L. and KRISHNAMURTHY, S. (2003). Statistical theory of the continuous double auction. *Quant. Finance* **3** 481–514.

[36] STEFAN, J. (1888). Über die Theorie der Eisbildung, insbesondere über die Eisbildung im Polarmeere. Wien. Ber. XCVIII, Abt. 2a (965–983).

[37] TRIEBEL, H. (1978). *Interpolation Theory, Function Spaces, Differential Operators. North-Holland Mathematical Library* **18**. North-Holland, Amsterdam.

[38] VAN NEERVEN, J., VERAAR, M. and WEIS, L. (2012). Maximal L^p-regularity for stochastic evolution equations. *SIAM J. Math. Anal.* **44** 1372–1414.

[39] VAN NEERVEN, J., VERAAR, M. and WEIS, L. (2012). Stochastic maximal L^p-regularity. *Ann. Probab.* **40** 788–812. MR2952092

[40] ZHENG, Z. (2012). Stochastic Stefan problems: Existence, uniqueness and modeling of market limit orders. Ph.D. thesis, Graduate College of the Univ. Illinois at Urbana-Champaign.

The Annals of Applied Probability
2018, Vol. 28, No. 4, 2370–2416
https://doi.org/10.1214/17-AAP1360

STOCHASTIC APPROXIMATION OF QUASI-STATIONARY DISTRIBUTIONS ON COMPACT SPACES AND APPLICATIONS

BY MICHEL BENAIM[1], BERTRAND CLOEZ AND FABIEN PANLOUP

Institut de mathématiques, Université de Montpellier and Université d'Angers

As a continuation of a recent paper, dealing with finite Markov chains, this paper proposes and analyzes a recursive algorithm for the approximation of the quasi-stationary distribution of a general Markov chain living on a compact metric space killed in finite time. The idea is to run the process until extinction and then to bring it back to life at a position randomly chosen according to the (possibly weighted) empirical occupation measure of its past positions. General conditions are given ensuring the convergence of this measure to the quasi-stationary distribution of the chain. We then apply this method to the numerical approximation of the quasi-stationary distribution of a diffusion process killed on the boundary of a compact set. Finally, the sharpness of the assumptions is illustrated through the study of the algorithm in a nonirreducible setting.

REFERENCES

[1] ALDOUS, D., FLANNERY, B. and PALACIOS, J. (1988). Two applications of urn processes: The fringe analysis of search trees and the simulation of quasi-stationary distributions of Markov chains. *Probab. Engrg. Inform. Sci.* **2** 293–307.

[2] BANSAYE, V., MÉLÉARD, S. and RICHARD, M. (2016). Speed of coming down from infinity for birth-and-death processes. *Adv. in Appl. Probab.* **48** 1183–1210. MR3595771

[3] BASS, R. F. (1998). *Diffusions and Elliptic Operators*. Springer, New York. MR1483890

[4] BEN-ARI, I. and PINSKY, R. G. (2007). Spectral analysis of a family of second-order elliptic operators with nonlocal boundary condition indexed by a probability measure. *J. Funct. Anal.* **251** 122–140. MR2353702

[5] BENAÏM, M. (1999). Dynamics of stochastic approximation algorithms. In *Séminaire de Probabilités, XXXIII. Lecture Notes in Math.* **1709** 1–68. Springer, Berlin. MR1767993

[6] BENAÏM, M. and CLOEZ, B. (2015). A stochastic approximation approach to quasi-stationary distributions on finite spaces. *Electron. Commun. Probab.* **20** no. 37. MR3352332

[7] BENAÏM, M. and HIRSCH, M. W. (1996). Asymptotic pseudotrajectories and chain recurrent flows, with applications. *J. Dynam. Differential Equations* **8** 141–176. MR1388167

[8] BENAÏM, M., LEDOUX, M. and RAIMOND, O. (2002). Self-interacting diffusions. *Probab. Theory Related Fields* **122** 1–41. MR1883716

[9] BERGLUND, N. and LANDON, D. (2012). Mixed-mode oscillations and interspike interval statistics in the stochastic FitzHugh–Nagumo model. *Nonlinearity* **25** 2303–2335. MR2946187

MSC2010 subject classifications. Primary 65C20, 60B12, 60J10; secondary 34F05, 60J20, 60J60.

Key words and phrases. Quasi-stationary distributions, stochastic approximation, reinforced random walks, random perturbations of dynamical systems, extinction rate, Euler scheme.

[10] BIENIEK, M., BURDZY, K. and FINCH, S. (2012). Non-extinction of a Fleming–Viot particle model. *Probab. Theory Related Fields* **153** 293–332. MR2925576

[11] BIENIEK, M., BURDZY, K. and PAL, S. (2012). Extinction of Fleming–Viot-type particle systems with strong drift. *Electron. J. Probab.* **17** no. 11. MR2878790

[12] BLANCHET, J., GLYNN, P. and ZHENG, S. (2016). Analysis of a stochastic approximation algorithm for computing quasi-stationary distributions. *Adv. in Appl. Probab.* **48** 792–811. MR3568892

[13] BOULEAU, N. and LÉPINGLE, D. (1994). *Numerical Methods for Stochastic Processes*. Wiley, New York. MR1274043

[14] BURDZY, K., HOŁYST, R. and MARCH, P. (2000). A Fleming–Viot particle representation of the Dirichlet Laplacian. *Comm. Math. Phys.* **214** 679–703. MR1800866

[15] CATTIAUX, P., COLLET, P., LAMBERT, A., MARTÍNEZ, S., MÉLÉARD, S. and SAN MARTÍN, J. (2009). Quasi-stationary distributions and diffusion models in population dynamics. *Ann. Probab.* **37** 1926–1969. MR2561437

[16] CHAMPAGNAT, N. and VILLEMONAIS, D. (2016). Exponential convergence to quasi-stationary distribution and Q-process. *Probab. Theory Related Fields* **164** 243–283. MR3449390

[17] CLOEZ, B. and THAI, M.-N. (2016). Fleming–Viot processes: Two explicit examples. *ALEA Lat. Am. J. Probab. Math. Stat.* **13** 337–356.

[18] CLOEZ, B. and THAI, M.-N. (2016). Quantitative results for the Fleming–Viot particle system and quasi-stationary distributions in discrete space. *Stochastic Process. Appl.* **126** 680–702. MR3452809

[19] COLLET, P., MARTÍNEZ, S., MÉLÉARD, S. and SAN MARTÍN, J. (2011). Quasi-stationary distributions for structured birth and death processes with mutations. *Probab. Theory Related Fields* **151** 191–231. MR2834717

[20] DEL MORAL, P. and GUIONNET, A. (1999). On the stability of measure valued processes with applications to filtering. *C. R. Acad. Sci. Paris, Sér. I Math.* **329** 429–434. MR1710091

[21] DEL MORAL, P. and MICLO, L. (2000). A Moran particle system approximation of Feynman–Kac formulae. *Stochastic Process. Appl.* **86** 193–216. MR1741805

[22] DUDLEY, R. M. (2002). *Real Analysis and Probability*. *Cambridge Studies in Advanced Mathematics* **74**. Cambridge Univ. Press, Cambridge. Revised reprint of the 1989 original. MR1932358

[23] DUFLO, M. (2000). *Random Iterative Models*. Springer, Berlin.

[24] ETHIER, S. N. and KURTZ, T. G. (1986). *Markov Processes*: *Characterization and Convergence*. Wiley, New York.

[25] FERRARI, P. A., KESTEN, H., MARTINEZ, S. and PICCO, P. (1995). Existence of quasi-stationary distributions. A renewal dynamical approach. *Ann. Probab.* **23** 501–521. MR1334159

[26] FERRARI, P. A. and MARIĆ, N. (2007). Quasi stationary distributions and Fleming–Viot processes in countable spaces. *Electron. J. Probab.* **12** 684–702. MR2318407

[27] GOBET, E. (2000). Weak approximation of killed diffusion using Euler schemes. *Stochastic Process. Appl.* **87** 167–197. MR1757112

[28] GONG, G. L., QIAN, M. P. and ZHAO, Z. X. (1988). Killed diffusions and their conditioning. *Probab. Theory Related Fields* **80** 151–167. MR970476

[29] GRIGORESCU, I. and KANG, M. (2012). Immortal particle for a catalytic branching process. *Probab. Theory Related Fields* **153** 333–361. MR2925577

[30] LAMBERTON, D. and PAGÈS, G. (2008). A penalized bandit algorithm. *Electron. J. Probab.* **13** 341–373. MR2386736

[31] LAMBERTON, D., PAGÈS, G. and TARRÈS, P. (2004). When can the two-armed bandit algorithm be trusted? *Ann. Appl. Probab.* **14** 1424–1454. MR2071429

[32] LEMAIRE, V. (2007). An adaptive scheme for the approximation of dissipative systems. *Stochastic Process. Appl.* **117** 1491–1518. MR2353037

[33] LEMAIRE, V. and MENOZZI, S. (2010). On some non asymptotic bounds for the Euler scheme. *Electron. J. Probab.* **15** 1645–1681. MR2735377

[34] MÉLÉARD, S. and VILLEMONAIS, D. (2012). Quasi-stationary distributions and population processes. *Probab. Surv.* **9** 340–410. MR2994898

[35] MEYN, S. P. and TWEEDIE, R. L. (1993). Stability of Markovian processes. III. Foster–Lyapunov criteria for continuous-time processes. *Adv. in Appl. Probab.* **25** 518–548. MR1234295

[36] OÇAFRAIN, W. and VILLEMONAIS, D. (2016). Non-failable approximation method for conditioned distributions. Preprint. Available at arXiv:1606.08978.

[37] PANLOUP, F. (2008). Recursive computation of the invariant measure of a stochastic differential equation driven by a Lévy process. *Ann. Appl. Probab.* **18** 379–426. MR2398761

[38] PEMANTLE, R. (2007). A survey of random processes with reinforcement. *Probab. Surv.* **4** 1–79. MR2282181

[39] PINSKY, R. G. (1995). *Positive Harmonic Functions and Diffusion*. *Cambridge Studies in Advanced Mathematics* **45**. Cambridge Univ. Press, Cambridge. MR1326606

[40] POLLOCK, M., FEARNHEAD, P., JOHANSEN, A. M. and ROBERTS, G. O. (2016). The scalable Langevin exact algorithm: Bayesian inference for big data. Preprint. Available at arXiv:1609.03436.

[41] RACHEV, S. T., KLEBANOV, L. B., STOYANOV, S. V. and FABOZZI, F. J. (2013). *The Methods of Distances in the Theory of Probability and Statistics*. Springer, New York. MR3024835

[42] VILLEMONAIS, D. (2011). Interacting particle systems and Yaglom limit approximation of diffusions with unbounded drift. *Electron. J. Probab.* **16** 1663–1692. MR2835250

[43] VILLEMONAIS, D. (2014). General approximation method for the distribution of Markov processes conditioned not to be killed. *ESAIM Probab. Stat.* **18** 441–467. MR3333998

The Annals of Applied Probability
2018, Vol. 28, No. 4, 2417–2450
https://doi.org/10.1214/17-AAP1361

GLIVENKO–CANTELLI THEORY, ORNSTEIN–WEISS QUASI-TILINGS, AND UNIFORM ERGODIC THEOREMS FOR DISTRIBUTION-VALUED FIELDS OVER AMENABLE GROUPS

BY CHRISTOPH SCHUMACHER*, FABIAN SCHWARZENBERGER† AND IVAN VESELIĆ*

TU Dortmund and HTW Dresden†*

We consider random fields indexed by finite subsets of an amenable discrete group, taking values in the Banach-space of bounded right-continuous functions. The field is assumed to be equivariant, local, coordinate-wise monotone and almost additive, with finite range dependence. Using the theory of quasi-tilings we prove an uniform ergodic theorem, more precisely, that averages along a Følner sequence converge uniformly to a limiting function. Moreover, we give explicit error estimates for the approximation in the sup norm.

REFERENCES

[1] ADACHI, T. (1993). A note on the Følner condition for amenability. *Nagoya Math*. *J*. **131** 67–74. MR1238633

[2] ALIPRANTIS, C. D. and BORDER, K. C. (2006). *Infinite Dimensional Analysis*: *A Hitchhiker's Guide*, 3rd ed. Springer, Berlin. MR2378491

[3] DEHARDT, J. (1971). Generalizations of the Glivenko–Cantelli theorem. *Ann*. *Math*. *Stat*. **42** 2050–2055.

[4] KLENKE, A. (2008). *Probability Theory*. *Number* 223 *in Universitext*, 1st ed. Springer, London.

[5] KRIEGER, F. (2009). Le lemme d'Ornstein–Weiss d'aprés Gromov. *Dyn*., *Ergod*. *Theory Geom*. **54** 99–112.

[6] LENZ, D., MÜLLER, P. and VESELIĆ, I. (2008). Uniform existence of the integrated density of states for models on $\mathbb{Z}^d$. *Positivity* **12** 571–589. MR2448749

[7] LENZ, D., SCHWARZENBERGER, F. and VESELIĆ, I. (2010). A Banach space-valued ergodic theorem and the uniform approximation of the integrated density of states. *Geom*. *Dedicata* **150** 1–34.

[8] LENZ, D. and STOLLMANN, P. (2005). An ergodic theorem for Delone dynamical systems and existence of the integrated density of states. *J*. *Anal*. *Math*. **97** 1–24. MR2274971

[9] LENZ, D. and VESELIĆ, I. (2009). Hamiltonians on discrete structures: Jumps of the integrated density of states and uniform convergence. *Math*. *Z*. **263** 813–835.

[10] ORNSTEIN, D. and WEISS, B. (1987). Entropy and isomorphism theorems for actions of amenable groups. *J*. *Anal*. *Math*. **48** 1–141.

[11] POGORZELSKI, F. and SCHWARZENBERGER, F. (2016). A Banach space-valued ergodic theorem for amenable groups and applications. *J*. *Anal*. *Math*. **130** 19–69.

[12] SCHUMACHER, C., SCHWARZENBERGER, F. and VESELIĆ, I. (2017). A Glivenko–Cantelli theorem for almost additive functions on lattices. *Stochastic Process*. *Appl*. **127** 179–208.

MSC2010 subject classifications. 60F99, 60B12, 62E20, 60K35.
Key words and phrases. Følner sequence, amenable group, quasi-tilings, Glivenko–Cantelli theory, Uniform convergence, Empirical measures.

[13] SCHWARZENBERGER, F. (2013). The integrated density of states for operators on groups. Ph.D. thesis, Technische Universität Chemnitz. Available at http://nbn-resolving.de/urn:nbn:de:bsz:ch1-qucosa-123241.

[14] TAO, T. (2015). Failure of the L^1 pointwise and maximal ergodic theorems for the free group. *Forum Math*. *Sigma* **3** 19.

[15] VESELIĆ, I. (2005). Spectral analysis of percolation Hamiltonians. *Math*. *Ann*. **331** 841–865.

[16] WEISS, B. (2001). Monotileable amenable groups. In *Topology*, *Ergodic Theory*, *Real Algebraic Geometry*. *Amer. Math*. *Soc*. *Transl*. *Ser.* 2 **202** 257–262. Amer. Math. Soc., Providence, RI. MR1819193

[17] WRIGHT, F. T. (1981). The empirical discrepancy over lower layers and a related law of large numbers. *Ann*. *Probab.* **9** 323–329. MR0606996

The Annals of Applied Probability
2018, Vol. 28, No. 4, 2451–2500
https://doi.org/10.1214/17-AAP1363

POLYNOMIAL JUMP-DIFFUSIONS ON THE UNIT SIMPLEX

BY CHRISTA CUCHIERO*, MARTIN LARSSON†,1 AND SARA SVALUTO-FERRO†,1

Vienna University and ETH Zurich†*

Polynomial jump-diffusions constitute a class of tractable stochastic models with wide applicability in areas such as mathematical finance and population genetics. We provide a full parameterization of polynomial jump-diffusions on the unit simplex under natural structural hypotheses on the jumps. As a stepping stone, we characterize well-posedness of the martingale problem for polynomial operators on general compact state spaces.

REFERENCES

ACKERER, D. and FILIPOVIĆ, D. (2016). Linear credit risk models. Available at http://ssrn.com/abstract=2782455.

ACKERER, D., FILIPOVIĆ, D. and PULIDO, S. (2016). The Jacobi stochastic volatility model. Available at http://ssrn.com/abstract=2782486.

BERTSEKAS, D. P. (1995). *Nonlinear Programming*. Athena Scientific, Nashua, NH.

BIAGINI, S. and ZHANG, Y. (2016). Polynomial diffusion models for life insurance liabilities. *Insurance Math. Econom.* **71** 114–129.

BÖTTCHER, B., SCHILLING, R. and WANG, J. (2013). *Lévy Matters. III. Lévy-Type Processes: Construction, Approximation and Sample Path Properties. Lecture Notes in Math.* **2099**. Springer, Cham. MR3156646

COURRÈGE, P. (1965). Sur la forme intégro-différentielle des opérateurs de C_k^∞ dans C satisfaisant au principe du maximum. In *Séminaire Brelot-Choquet-Deny* (*Théorie du Potentiel*) **10**.

CUCHIERO, C. (2017). Polynomial processes in stochastic portfolio theory. ArXiv e-prints. Available at http://arxiv.org/abs/1705.03647.

CUCHIERO, C., KELLER-RESSEL, M. and TEICHMANN, J. (2012). Polynomial processes and their applications to mathematical finance. *Finance Stoch.* **16** 711–740.

CUCHIERO, C., GELLERT, K., GUIRICICH, M., PLATTS, A., SOOKDEO, S. and TEICHMANN, J. (2016). Polynomial models for market weights at work in stochastic portfolio theory: Theory, tractable estimation, calibration and implementation. Working paper.

DAWSON, D. A. and LI, Z. (2006). Skew convolution semigroups and affine Markov processes. *Ann. Probab.* **34** 1103–1142. MR2243880

DELBAEN, F. and SHIRAKAWA, H. (2002). An interest rate model with upper and lower bounds. *Asia-Pac. Financ. Mark.* **9** 191–209.

DUFFIE, D. (2004). *Credit Risk Modeling with Affine Processes*. Scuola Normale Superiore, Classe di Scienze, Pisa. MR2142955

EPSTEIN, C. L. and MAZZEO, R. (2013). Degenerate diffusion operators arising in population biology. In *Number* **185**. Princeton Univ. Press.

MSC2010 subject classifications. 60J25, 60H30.

Key words and phrases. Polynomial processes, unit simplex, stochastic models with jumps, Wright–Fisher diffusion, stochastic invariance.

ETHERIDGE, A. (2011). *Some Mathematical Models from Population Genetics*. *Lectures from the* 39*th Probability Summer School Held in Saint-Flour*, 2009, *École d'Été de Probabilités de Saint-Flour*. [*Saint-Flour Probability Summer School*]. *Lecture Notes in Math*. **2012**. Springer, Heidelberg. MR2759587

ETHIER, S. N. and KURTZ, T. G. (2005). *Markov Processes*: *Characterization and Convergence*. Wiley, New York. MR0838085

FERNHOLZ, R. (2002). *Stochastic Portfolio Theory*. Springer, New York.

FERNHOLZ, R. and KARATZAS, I. (2005). Relative arbitrage in volatility-stabilized markets. *Ann. Finance* **1** 149–177.

FERNHOLZ, R. and KARATZAS, I. (2009). Stochastic portfolio theory: An overview. *Handb. Numer. Anal*. **15** 89–167.

FILIPOVIĆ, D., GOURIER, E. and MANCINI, L. (2016). Quadratic variance swap models. *J. Financ. Econ*. **119** 44–68.

FILIPOVIĆ, D. and LARSSON, M. (2016). Polynomial diffusions and applications in finance. *Finance Stoch*. **20** 931–972. MR3551857

FILIPOVIĆ, D. and LARSSON, M. (2017). Polynomial jump-diffusion models. arXiv:1711.08043.

FILIPOVIĆ, D., LARSSON, M. and TROLLE, A. (2017). Linear-rational term structure models. *J. Finance* **72** 655–704.

FILIPOVIĆ, D. and TROLLE, A. B. (2013). The term structure of interbank risk. *J. Financ. Econ*. **109** 707–733.

GALLARDO, L. and YOR, M. (2006). Some remarkable properties of the Dunkl martingales. In *In Memoriam Paul-André Meyer*: *Séminaire de Probabilités XXXIX*. *Lecture Notes in Math*. **1874** 337–356. Springer, Berlin. MR2276903

GOURIEROUX, C. and JASIAK, J. (2006). Multivariate Jacobi process with application to smooth transitions. *J. Econometrics* **131** 475–505.

HAVILAND, E. K. (1935). On the Momentum Problem for Distribution Functions in More than One Dimension. *Amer. J. Math*. **57** 562–568. MR1507095

HAVILAND, E. K. (1936). On the momentum problem for distribution functions in more than one dimension. ii. *Amer. J. Math*. **58** 164–168.

HOH, W. (1998). Pseudo differential operators generating Markov processes. Univ. Bielefeld. Habilitationsschrift.

JACOD, J. and SHIRYAEV, A. N. (2003). *Limit Theorems for Stochastic Processes*, 2nd ed. Springer.

JARROW, R. and TURNBULL, S. (1998). A unified approach for pricing contingent claims on multiple term structures. *Rev. Quant. Finance Account*. **10** 5–19.

JEANBLANC, M., YOR, M. and CHESNEY, M. (2009). *Mathematical Methods for Financial Markets*. Springer Finance. Springer, London.

KRABICHLER, T. and TEICHMANN, J. (2017). The Jarrow & Turnbull setting revisited. Working paper.

KRÜHNER, P. and LARSSON, M. (2017). Affine processes with compact state space. arXiv:1706.07579.

LARSSON, M. and PULIDO, S. (2017). Polynomial diffusions on compact quadric sets. *Stochastic Process. Appl*. **127** 901–926. MR3605715

MARSHALL, M. (2008). *Positive Polynomials and Sums of Squares*. Amer. Math. Soc, Providence, RI.

ZHENG, H. (2006). Interaction of credit and liquidity risks: Modelling and valuation. *J. Bank. Financ*. **30** 391–407.

The Annals of Applied Probability
2018, Vol. 28, No. 4, 2501–2543
https://doi.org/10.1214/17-AAP1365

RAPID MIXING OF GEODESIC WALKS ON MANIFOLDS WITH POSITIVE CURVATURE

BY OREN MANGOUBI[1] AND AARON SMITH[2]

École Polytechnique Fédérale de Lausanne (EPFL) and University of Ottawa

We introduce a Markov chain for sampling from the uniform distribution on a Riemannian manifold $\mathcal{M}$, which we call the *geodesic walk*. We prove that the mixing time of this walk on any manifold with positive sectional curvature $C_x(u, v)$ bounded both above and below by $0 < \mathfrak{m}_2 \le C_x(u, v) \le \mathfrak{M}_2 < \infty$ is $\mathcal{O}^*(\frac{\mathfrak{M}_2}{\mathfrak{m}_2})$. In particular, this bound on the mixing time does not depend explicitly on the dimension of the manifold. In the special case that $\mathcal{M}$ is the boundary of a convex body, we give an explicit and computationally tractable algorithm for approximating the exact geodesic walk. As a consequence, we obtain an algorithm for sampling uniformly from the surface of a convex body that has running time bounded solely in terms of the curvature of the body.

REFERENCES

[1] ABRESCH, U. and MEYER, W. T. (1997). Injectivity radius estimates and sphere theorems. In *Comparison Geometry* (*Berkeley*, *CA*, 1993–94). *Math*. *Sci*. *Res*. *Inst*. *Publ*. **30** 1–47. Cambridge Univ. Press, Cambridge. MR1452866

[2] ALEXANDROV, A. D. (2005). *Convex Polyhedra*. Springer, Berlin. MR2127379

[3] BÉLISLE, C. J. P., ROMEIJN, H. E. and SMITH, R. L. (1993). Hit-and-run algorithms for generating multivariate distributions. *Math*. *Oper. Res*. **18** 255–266. MR1250117

[4] BELKIN, M., NARAYANAN, H. and NIYOGI, P. (2006). Heat flow and a faster algorithm to compute the surface area of a convex body. In *Foundations of Computer Science*, 2006. *FOCS'*06. 47*th Annual IEEE Symposium on* 47–56. IEEE, New York.

[5] BONAHON, F. and SÖZEN, Y. (2003). Variation of the Liouville measure of a hyperbolic surface. *Ergodic Theory Dynam*. *Systems* **23** 729–758.

[6] BORNN, L., SHEPHARD, N. and SOLGI, R. (2015). Moment conditions and Bayesian nonparametrics. Preprint. Available at arXiv:1507.08645.

[7] CHEEGER, J. and EBIN, D. G. (2008). *Comparison Theorems in Riemannian Geometry* **365**. Am. Math. Soc., Providence.

[8] CHOW, B. and KNOPF, D. (2004). *The Ricci Flow*: *An Introduction* **110**. American Mathematical Society, Providence.

[9] COMETS, F., POPOV, S., SCHÜTZ, G. M. and VACHKOVSKAIA, M. (2009). Billiards in a general domain with random reflections. *Arch*. *Ration*. *Mech*. *Anal*. **191** 497–537.

[10] DAI, X. and WEI, G. (1995). A comparison-estimate of Toponogov type for Ricci curvature. *Math*. *Ann*. **303** 297–306.

MSC2010 subject classifications. Primary 60J05; secondary 68W20, 60J20, 53D25.

Key words and phrases. Markov chain Monte Carlo (MCMC), Hamiltonian Monte Carlo (HMC), manifolds, convex bodies, positive curvature, momentum.

[11] DIACONIS, P., HOLMES, S. and SHAHSHAHANI, M. (2013). Sampling from a manifold. In *Advances in Modern Statistical Theory and Applications*: *A Festschrift in Honor of Morris L. Eaton*. *Inst*. *Math*. *Stat*. (*IMS*) *Collect*. **10** 102–125. IMS, Beachwood, OH. MR3586941

[12] DIEKER, A. B. and VEMPALA, S. S. (2015). Stochastic billiards for sampling from the boundary of a convex set. *Math*. *Oper*. *Res*. **40** 888–901.

[13] DOBRUŠIN, R. L. (1970). Definition of a system of random variables by means of conditional distributions. *Teor. Verojatnost*. *i Primenen*. **15** 469–497. MR0298716

[14] DURMUS, A. and MOULINES, E. (2016). Sampling from strongly log-concave distributions with the unadjusted Langevin algorithm. Preprint. Available at arXiv:1605.01559.

[15] DURMUS, A. and MOULINES, E. (2017). Nonasymptotic convergence analysis for the unadjusted Langevin algorithm. *Ann*. *Appl*. *Probab*. **27** 1551–1587. MR3678479

[16] ESCHENBURG, J.-H. (1994). *Comparison Theorems in Riemannian Geometry*. Dipartimento di matematica, Univ. delli studi di Trento.

[17] GUDMUNDSSON, S. (2008). *An Introduction to Riemannian Geometry*. Lund Univ.

[18] HOFFMAN, M. D. and GELMAN, A. (2014). The no-U-turn sampler: Adaptively setting path lengths in Hamiltonian Monte Carlo. *J*. *Mach*. *Learn*. *Res*. **15** 1593–1623.

[19] HOPF, H. and RINOW, W. (1931). Über den Begriff der vollständigen differentialgeometrischen Fläche. *Comment*. *Math*. *Helv*. **3** 209–225.

[20] JOULIN, A. and OLLIVIER, Y. (2010). Curvature, concentration and error estimates for Markov chain Monte Carlo. *Ann*. *Probab*. **38** 2418–2442. MR2683634

[21] KLINGENBERG, W. P. (1995). *Riemannian Geometry*, 2nd ed. de Gruyter. Berlin.

[22] LEBEAU, G. and MICHEL, L. (2010). Semi-classical analysis of a random walk on a manifold. *Ann*. *Probab*. **38** 277–315. MR2599200

[23] LEE, J. M. (2006). *Riemannian Manifolds*: *An Introduction to Curvature* **176**. Springer, Berlin.

[24] LEE, Y. T. and VEMPALA, S. S. (2017). Geodesic walks in polytopes. In *STOC*'17—*Proceedings of the* 49*th Annual ACM SIGACT Symposium on Theory of Computing* 927–940. ACM, New York. MR3678240

[25] LOVÁSZ, L. and VEMPALA, S. (2007). The geometry of logconcave functions and sampling algorithms. *Random Structures Algorithms* **30** 307–358.

[26] MANGOUBI, O. Sampling the interior of a convex body from its boundary. *In preparation*.

[27] MANGOUBI, O. and SMITH, A. (2017). Rapid mixing of hamiltonian Monte Carlo on strongly log-concave distributions. Preprint. Available at arXiv:1708.07114.

[28] MYERS, S. B. (1941). Riemannian manifolds with positive mean curvature. *Duke Math*. *J*. **8** 401–404. MR0004518

[29] NARAYANAN, H. and NIYOGI, P. (2008). Sampling hypersurfaces through diffusion. In *Approximation*, *Randomization and Combinatorial Optimization*. *Algorithms and Techniques* 535–548. Springer, Berlin.

[30] OLLIVIER, Y. A visual introduction to Riemannian curvatures and some discrete generalizations. *Analysis and Geometry of Metric Measure Spaces*: *Lecture Notes of the* 50*th Séminaire de Mathématiques Supérieures* (*SMS*), *Montréal* 197–219.

[31] OLLIVIER, Y. (2009). Ricci curvature of Markov chains on metric spaces. *J*. *Funct*. *Anal*. **256** 810–864.

[32] ÖZTIRELI, A. C., ALEXA, M. and GROSS, M. (2010). Spectral sampling of manifolds. *ACM Transactions on Graphics* (*TOG*) **29** 168.

[33] PAULIN, D. (2015). Concentration inequalities for Markov chains by Marton couplings and spectral methods. *Electron*. *J*. *Probab*. **20**.

[34] PETERSEN, P. (2006). *Riemannian Geometry*, 2nd ed. *Graduate Texts in Mathematics* **171**. Springer, New York. MR2243772

[35] RAUCH, H. E. (1951). A contribution to differential geometry in the large. *Ann*. *of Math*. (2) **54** 38–55. MR0042765

[36] RHEE, C-H., ZHOU, E. and QIU, P. (2014). An iterative algorithm for sampling from manifolds. In *Proceedings of the* 2014 *Winter Simulation Conference* 574–585. IEEE Press, New York.

[37] SEILER, C., RUBINSTEIN-SALZEDO, S. and HOLMES, S. (2014). Positive curvature and Hamiltonian Monte Carlo. In *Advances in Neural Information Processing Systems* 586–594.

[38] VEMPALA, S. (2005). Geometric random walks: A survey. *Combinatorial and Computational Geometry* **52** 2.

[39] VILLANI, C. (2008). *Optimal Transport*: *Old and New* **338**. Springer, Berlin.

[40] WALCZAK, P. G. (1994). Existence of smooth invariant measures for geodesic flows of foliations of Riemannian manifolds. *Proc. Amer. Math. Soc.* **120** 903–906.

[41] ZHU, S. (1997). The comparison geometry of Ricci curvature. In *Comparison Geometry* (*Berkeley*, *CA*, 1993–94). *Math. Sci. Res. Inst. Publ.* **30** 221–262. Cambridge Univ. Press, Cambridge. MR1452876

[42] http://math.stackexchange.com/users/8508/robert israel. Surface area of a convex set less than that of its enclosing sphere? Mathematics stack exchange. Available at http://math.stackexchange.com/q/58556 (version: 2011-08-19).

The Annals of Applied Probability
2018, Vol. 28, No. 4, 2544–2591
https://doi.org/10.1214/17-AAP1366

CONVERGENCE AND QUALITATIVE PROPERTIES OF MODIFIED EXPLICIT SCHEMES FOR BSDES WITH POLYNOMIAL GROWTH

BY ARNAUD LIONNET*,[1], GONCALO DOS REIS†,[2] AND LUKASZ SZPRUCH†

*INRIA Paris—Ecole Nationale des Ponts et Chaussées** and *University of Edinburgh*†

The theory of Forward–Backward Stochastic Differential Equations (FBSDEs) paves a way to probabilistic numerical methods for nonlinear parabolic PDEs. The majority of the results on the numerical methods for FBSDEs relies on the global Lipschitz assumption, which is not satisfied for a number of important cases such as the Fisher–KPP or the FitzHugh–Nagumo equations. Furthermore, it has been shown in [*Ann. Appl. Probab.* **25** (2015) 2563–2625] that for BSDEs with monotone drivers having polynomial growth in the primary variable y, only the (sufficiently) implicit schemes converge. But these require an additional computational effort compared to explicit schemes.

This article develops a general framework that allows the analysis, in a systematic fashion, of the integrability properties, convergence and qualitative properties (e.g., comparison theorem) for whole families of modified explicit schemes. The framework yields the convergence of some modified explicit scheme with the same rate as implicit schemes and with the computational cost of the standard explicit scheme.

To illustrate our theory, we present several classes of easily implementable modified explicit schemes that can computationally outperform the implicit one and preserve the qualitative properties of the solution to the BSDE. These classes fit into our developed framework and are tested in computational experiments.

REFERENCES

[1] BENDER, C. and DENK, R. (2007). A forward scheme for backward SDEs. *Stochastic Process. Appl.* **117** 1793–1812. MR2437729

[2] BOUCHARD, B. and TOUZI, N. (2004). Discrete-time approximation and Monte-Carlo simulation of backward stochastic differential equations. *Stochastic Process. Appl.* **111** 175–206. MR2056536

[3] BRIAND, P., DELYON, B. and MÉMIN, J. (2002). On the robustness of backward stochastic differential equations. *Stochastic Process. Appl.* **97** 229–253. MR1875334

[4] BRIAND, P. and LABART, C. (2014). Simulation of BSDEs by Wiener chaos expansion. *Ann. Appl. Probab.* **24** 1129–1171.

[5] CHASSAGNEUX, J.-F. An introduction to the numerical approximation of BSDEs. In 2*nd Summer School of the Euro-Mediterranean Research Center for Mathematics and Its Applications* (*EMRCMA*).

MSC2010 subject classifications. Primary 65C30, 60H35; secondary 60H30.

Key words and phrases. FBSDE, monotone driver, polynomial growth, time discretization, modified explicit schemes, nonexplosion, numerical stability.

[6] CHASSAGNEUX, J.-F., JACQUIER, A. and MIHAYLOV, I. (2016). An explicit Euler scheme with strong rate of convergence for financial SDEs with non-Lipschitz coefficients. *SIAM J. Financial Math.* **7** 993–1021. MR3582409

[7] CHASSAGNEUX, J.-F. and RICHOU, A. (2015). Numerical stability analysis of the Euler scheme for BSDEs. *SIAM J. Numer. Anal.* **53** 1172–1193. Available at arXiv:1407.0887.

[8] CHASSAGNEUX, J.-F. and RICHOU, A. (2016). Numerical simulation of quadratic BSDEs. *Ann. Appl. Probab.* **26** 262–304. MR3449318

[9] CHERIDITO, P. and STADJE, M. (2013). BSΔEs and BSDEs with non-Lipschitz drivers: Comparison, convergence and robustness. *Bernoulli* **19** 1047–1085. Available at arXiv:1002.0175v1.

[10] CRISAN, D. and MANOLARAKIS, K. (2014). Second order discretization of backward SDEs and simulation with the cubature method *Ann. Appl. Probab.* **24** 652–678. MR3178494

[11] EL KAROUI, N., PENG, S. and QUENEZ, M. C. (1997). Backward stochastic differential equations in finance. *Math. Finance* **7** 1–71.

[12] GOBET, E. and TURKEDJIEV, P. (2016). Approximation of backward stochastic differential equations using Malliavin weights and least-squares regression. *Bernoulli* **22** 530–562.

[13] HENRY, D. (1981). *Geometric Theory of Semilinear Parabolic Equations. Lecture Notes in Math.* **840**. Springer, Berlin.

[14] HUTZENTHALER, M. and JENTZEN, A. (2014). On a perturbation theory and on strong convergence rates for stochastic ordinary and partial differential equations with non-globally monotone coefficients. Preprint. Available at arXiv:1401.0295.

[15] HUTZENTHALER, M. and JENTZEN, A. (2015). *Numerical Approximations of Stochastic Differential Equations with Non-globally Lipschitz Continuous Coefficients. Memoirs of the American Mathematical Society* **236**. Amer. Math. Soc., Providence, RI.

[16] HUTZENTHALER, M., JENTZEN, A. and KLOEDEN, P. E. (2011). Strong and weak divergence in finite time of Euler's method for stochastic differential equations with non-globally Lipschitz continuous coefficients. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **467** 1563–1576.

[17] HUTZENTHALER, M., JENTZEN, A. and KLOEDEN, P. E. (2012). Strong convergence of an explicit numerical method for SDEs with nonglobally Lipschitz continuous coefficients. *Ann. Appl. Probab.* **22** 1611–1641. MR2985171

[18] HUTZENTHALER, M., JENTZEN, A. and WANG, X. (2013). Exponential integrability properties of numerical approximation processes for nonlinear stochastic differential equations. Preprint. Available at arXiv:1309.7657.

[19] LIONNET, A., DOS REIS, G. and SZPRUCH, L. (2015). Time discretization of FBSDE with polynomial growth drivers and reaction–diffusion PDEs. *Ann. Appl. Probab.* **25** 2563–2625. MR3375884

[20] ROTHE, F. (1984). *Global Solutions of Reaction–Diffusion Systems. Lecture Notes in Math.* **1072**. Springer, Berlin.

[21] SZPRUCH, L. and ZHANG, X. (2013). V-integrability, asymptotic stability and comparison theorem of explicit numerical schemes for SDEs. Preprint. Available at arXiv:1310.0785.

[22] ZHANG, J. (2004). A numerical scheme for BSDEs. *Ann. Appl. Probab.* **14** 459–488. MR2023027

The Annals of Applied Probability
2018, Vol. 28, No. 4, 2592–2634
https://doi.org/10.1214/17-AAP1370

UNIFORM CONTROL OF LOCAL TIMES OF SPECTRALLY POSITIVE STABLE PROCESSES[1]

BY NOAH FORMAN *, SOUMIK PAL*, DOUGLAS RIZZOLO† AND MATTHIAS WINKEL‡

University of Washington, University of Delaware† and University of Oxford‡*

We establish two results about local times of spectrally positive stable processes. The first is a general approximation result, uniform in space and on compact time intervals, in a model where each jump of the stable process may be marked by a random path. The second gives moment control on the Hölder constant of the local times, uniformly across a compact spatial interval and in certain random time intervals. For the latter, we introduce the notion of a Lévy process restricted to a compact interval, which is a variation of Lambert's Lévy process confined in a finite interval and of Pistorius' doubly reflected process. We use the results of this paper to exhibit a class of path-continuous branching processes of Crump–Mode–Jagers-type with continuum genealogical structure. A further motivation for this study lies in the construction of diffusion processes in spaces of interval partitions and $\mathbb{R}$-trees, which we explore in forthcoming articles. In that context, local times correspond to branch lengths.

REFERENCES

[1] BARLOW, M. T. (1988). Necessary and sufficient conditions for the continuity of local time of Lévy processes. *Ann. Probab.* **16** 1389–1427. MR0958195

[2] BARLOW, M. T., PERKINS, E. A. and TAYLOR, S. J. (1986). The behaviour and construction of local times for Lévy processes. In *Seminar on Stochastic Processes*, 1984 (*Evanston, Ill.*, 1984). *Progr. Probab. Statist.* **9** 23–54. Birkhäuser, Boston, MA. MR0896720

[3] BARLOW, M. T., PERKINS, E. A. and TAYLOR, S. J. (1986). Two uniform intrinsic constructions for the local time of a class of Lévy processes. *Illinois J. Math.* **30** 19–65. MR0822383

[4] BERTOIN, J. (1996). *Lévy Processes*. *Cambridge Tracts in Mathematics* **121**. Cambridge Univ. Press, Cambridge. MR1406564

[5] BERTOIN, J. (1996). On the first exit time of a completely asymmetric stable process from a finite interval. *Bull. Lond. Math. Soc.* **28** 514–520. MR1396154

[6] BERTOIN, J. (1997). Exponential decay and ergodicity of completely asymmetric Lévy processes in a finite interval. *Ann. Appl. Probab.* **7** 156–169. MR1428754

[7] BIHARI, I. (1956). A generalization of a lemma of Bellman and its application to uniqueness problems of differential equations. *Acta Math. Hungar.* **7** 81–94.

[8] BLUMENTHAL, R. M. and GETOOR, R. K. (1968). *Markov Processes and Potential Theory*. *Pure and Applied Mathematics* **29** Academic Press, New York. MR0264757

MSC2010 subject classifications. 60G51, 60G52, 60J55, 60J80.

Key words and phrases. Stable process, squared Bessel processes, CMJ process, local time approximation, excursion theory, restricted Lévy process, Hölder continuity.

[9] BOYLAN, E. S. (1964). Local times for a class of Markoff processes. *Illinois J. Math.* **8** 19–39. MR0158434

[10] CABALLERO, M. E. and CHAUMONT, L. (2006). Conditioned stable Lévy processes and the Lamperti representation. *J. Appl. Probab.* **43** 967–983.

[11] CHACON, R. V., LE JAN, Y., PERKINS, E. and TAYLOR, S. J. (1981). Generalised arc length for Brownian motion and Lévy processes. *Z. Wahrsch. Verw. Gebiete* **57** 197–211. MR0626815

[12] CSÖRGŐ, M. and RÉVÉSZ, P. (1986). Mesure du voisinage and occupation density. *Probab. Theory Related Fields* **73** 211–226. MR0855223

[13] DELAPORTE, C. (2013). Lévy processes with marked jumps II: Application to a population model with mutations at birth. Preprint. Available at arXiv:1305.6491 [math.PR].

[14] DELAPORTE, C. (2015). Lévy processes with marked jumps I: Limit theorems. *J. Theoret. Probab.* **28** 1468–1499. MR3422939

[15] DEMBO, A. and ZEITOUNI, O. (2010). *Large Deviations Techniques and Applications*. *Stochastic Modelling and Applied Probability* **38**. Springer, Berlin. Corrected reprint of the second (1998) edition. MR2571413

[16] DONEY, R. A. (2007). *Fluctuation Theory for Lévy Processes*. *Lecture Notes in Math.* **1897**. Springer, Berlin. MR2320889

[17] ETHIER, S. N. and KURTZ, T. G. (1981). The infinitely-many-neutral-alleles diffusion model. *Adv. in Appl. Probab.* **13** 429–452. MR0615945

[18] ETHIER, S. N. and KURTZ, T. G. (1986). *Markov Processes. Characterization and Convergence*. Wiley, New York. MR0838085

[19] FORMAN, N., PAL, S., RIZZOLO, D. and WINKEL, M. (2017). Diffusions on a space of interval partitions with Poisson-Dirichlet stationary distributions. Available at arXiv:1609.06706v2 [math.PR].

[20] GETOOR, R. K. and SHARPE, M. J. (1981). Two results on dual excursions. In *Seminar on Stochastic Processes*, 1981 (*Evanston*, *Ill.*, 1981). *Progr. Prob. Statist.* **1** 31–52. Birkhäuser, Boston, MA. MR0647780

[21] GNEDIN, A., PITMAN, J. and YOR, M. (2006). Asymptotic laws for compositions derived from transformed subordinators. *Ann. Probab.* **34** 468–492. MR2223948

[22] GÖING-JAESCHKE, A. and YOR, M. (2003). A survey and some generalisations of Bessel processes. *Bernoulli* **9** 313–349.

[23] HUTZENTHALER, M., JENTZEN, A. and NOLL, M. (2014). Strong convergence rates and temporal regularity for Cox–Ingersoll–Ross processes and Bessel processes with accessible boundaries. ArXiv preprint. Available at arXiv:1403.6385.

[24] JAGERS, P. (1974). Convergence of general branching processes and functionals thereof. *J. Appl. Probab.* 471–478.

[25] JAGERS, P. (1975). *Branching Processes with Biological Applications*. Wiley, New York. MR0488341

[26] KALLENBERG, O. (2002). *Foundations of Modern Probability*, 2nd ed. Springer, New York. MR1876169

[27] KHOSHNEVISAN, D. (1994). Exact rates of convergence to Brownian local time. *Ann. Probab.* **22** 1295–1330. MR1303646

[28] KYPRIANOU, A. E. (2006). *Introductory Lectures on Fluctuations of Lévy Processes with Applications*. Springer, Berlin. MR2250061

[29] KYPRIANOU, A. E., PARDO, J. C. and WATSON, A. R. (2014). Hitting distributions of α-stable processes via path censoring and self-similarity. *Ann. Probab.* **42** 398–430. MR3161489

[30] LAMBERT, A. (2000). Completely asymmetric Lévy processes confined in a finite interval. *Ann. Inst. Henri Poincaré Probab. Stat.* **36** 251–274. MR1751660

[31] LAMBERT, A. (2010). The contour of splitting trees is a Lévy process. *Ann. Probab.* **38** 348–395. MR2599603

[32] LAMBERT, A. and SIMATOS, F. (2013). Asymptotic behavior of local times of compound Poisson processes with drift in the infinite variance case. *J. Theoret. Probab.* **28** 41–91.

[33] LAMBERT, A. and URIBE BRAVO, G. (2016). Totally ordered measured trees and splitting trees with infinite variation. ArXiv preprint. Available at arXiv:1607.02114.

[34] PARDO, J. C., PÉREZ, J. L. and RIVERO, V. M. (2015). The excursion measure away from zero for spectrally negative Lévy processes. ArXiv preprint. Available at arXiv:1507.05225.

[35] PERKINS, E. (1981). A global intrinsic characterization of Brownian local time. *Ann. Probab.* **9** 800–817. MR0628874

[36] PERMAN, M. (1996). An excursion approach to Ray–Knight theorems for perturbed Brownian motion. *Stochastic Process. Appl.* **63** 67–74.

[37] PETROV, L. A. (2009). A two-parameter family of infinite-dimensional diffusions on the Kingman simplex. *Funktsional. Anal. i Prilozhen.* **43** 45–66. MR2596654

[38] PISTORIUS, M. R. (2003). On doubly reflected completely asymmetric Lévy processes. *Stochastic Process. Appl.* **107** 131–143. MR1995924

[39] PISTORIUS, M. R. (2004). On exit and ergodicity of the spectrally one-sided Lévy process reflected at its infimum. *J. Theoret. Probab.* **17** 183–220. MR2054585

[40] PITMAN, J. (2006). *Combinatorial Stochastic Processes. Lecture Notes in Math.* **1875**. Springer, Berlin. MR2245368

[41] PITMAN, J. and YOR, M. (1982). A decomposition of Bessel bridges. *Z. Wahrsch. Verw. Gebiete* **59** 425–457. MR0656509

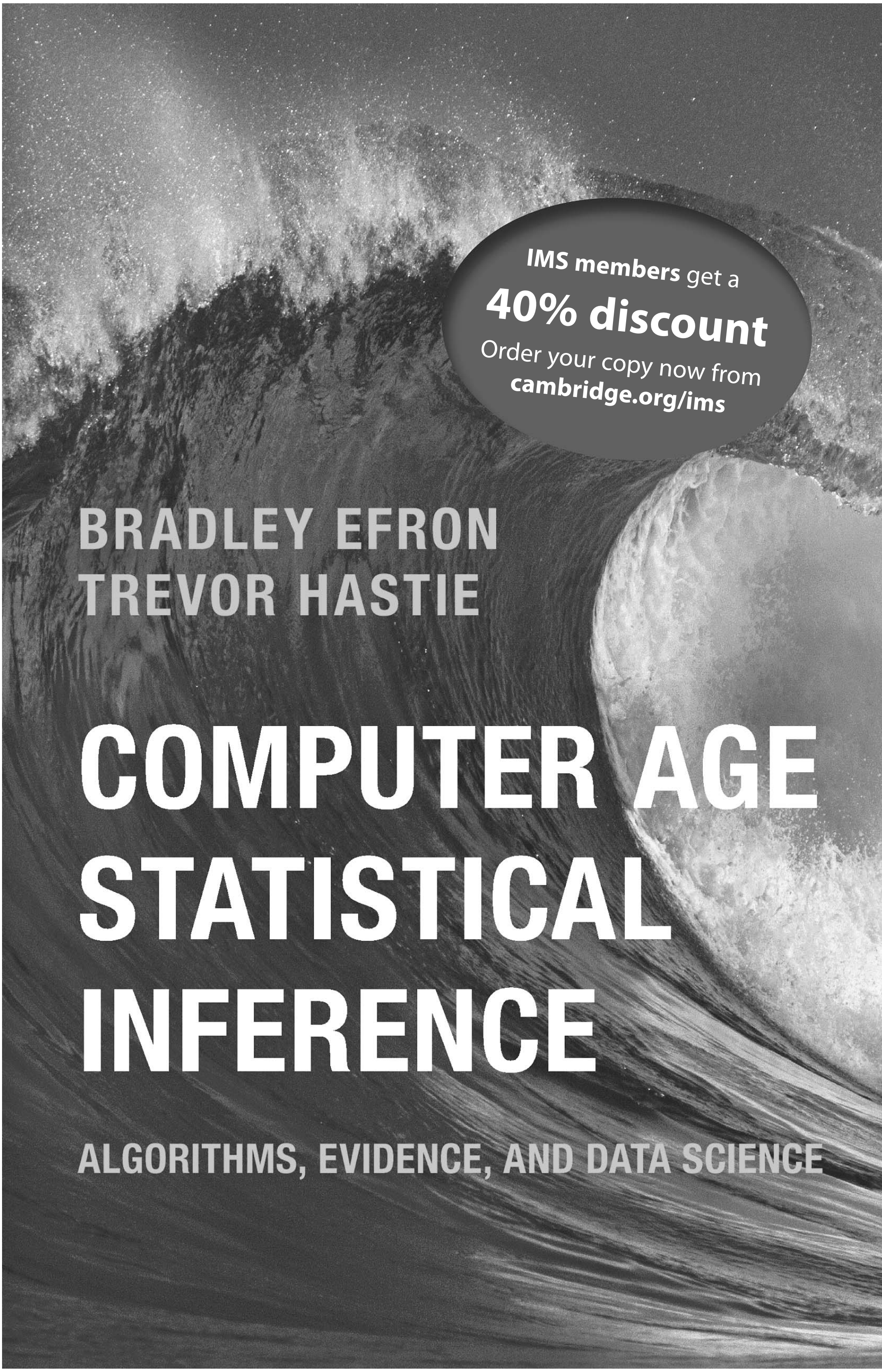
[42] PITMAN, J. and YOR, M. (1997). The two-parameter Poisson–Dirichlet distribution derived from a stable subordinator. *Ann. Probab.* **25** 855–900. MR1434129

[43] REVUZ, D. and YOR, M. (1999). *Continuous Martingales and Brownian Motion*, 3rd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **293**. Springer, Berlin. MR1725357

[44] ROGERS, L. C. G. and WILLIAMS, D. (2000). *Diffusions, Markov Processes, and Martingales. Vol. 2. Itô Calculus*, Cambridge Univ. Press, Cambridge. MR1780932

[45] RUGGIERO, M., WALKER, S. G. and FAVARO, S. (2013). Alpha-diversity processes and normalized inverse-Gaussian diffusions. *Ann. Appl. Probab.* **23** 386–425. MR3059239

[46] SIMON, T. (2011). Hitting densities for spectrally positive stable processes. *Stochastics* **83** 203–214.


IMS members get a
40% discount
Order your copy now from
cambridge.org/ims
BRADLEY EFRON
TREVOR HASTIE
COMPUTER AGE
STATISTICAL
INFERENCE
ALGORITHMS, EVIDENCE, AND DATA SCIENCE