ISSN 0091-1798 (print)
ISSN 2168-894X (online)

THE ANNALS *of* PROBABILITY

AN OFFICIAL JOURNAL OF THE
INSTITUTE OF MATHEMATICAL STATISTICS

Articles

Vol. 43, No. 5—September 2015

The Annals of Probability
2015, Vol. 43, No. 5, 2205–2249
DOI: 10.1214/14-AOP931

BASIC PROPERTIES OF CRITICAL LOGNORMAL MULTIPLICATIVE CHAOS

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We study one-dimensional exact scaling lognormal multiplicative chaos measures at criticality. Our main results are the determination of the exact asymptotics of the right tail of the distribution of the total mass of the measure, and an almost sure upper bound for the modulus of continuity of the cumulative distribution function of the measure. We also find an almost sure lower bound for the increments of the measure almost everywhere with respect to the measure itself, strong enough to show that the measure is supported on a set of Hausdorff dimension 0.

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MSC2010 subject classifications. Primary 60G57; secondary 60G18, 83C45.
Key words and phrases. Multiplicative chaos, critical temperature.

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The Annals of Probability
2015, Vol. 43, No. 5, 2250–2281
DOI: 10.1214/14-AOP932

A COUNTEREXAMPLE TO THE CANTELLI CONJECTURE THROUGH THE SKOROKHOD EMBEDDING PROBLEM

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In this paper, we construct a counterexample to a question by Cantelli, asking whether there exists a nonconstant positive measurable function φ such that for i.i.d. r.v. X, Y of law $\mathcal{N}(0,1)$, the r.v. $X + \varphi(X) \cdot Y$ is also Gaussian.

This construction is made by finding an unusual solution to the Skorokhod embedding problem (showing that the corresponding Brownian transport, contrary to the Root barrier, is not unique). To find it, we establish some sufficient conditions for the continuity of the Root barrier function.

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MSC2010 subject classifications. Primary 60G40; secondary 60J65.

Key words and phrases. Cantelli conjecture, Skorokhod embedding, Root barrier, Gaussian variable, Stefan problem, mass transport, Brownian motion.

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The Annals of Probability
2015, Vol. 43, No. 5, 2282–2331
DOI: 10.1214/14-AOP933

RATIOS OF PARTITION FUNCTIONS FOR THE LOG-GAMMA POLYMER

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We introduce a random walk in random environment associated to an underlying directed polymer model in $1 + 1$ dimensions. This walk is the positive temperature counterpart of the competition interface of percolation and arises as the limit of quenched polymer measures. We prove this limit for the exactly solvable log-gamma polymer, as a consequence of almost sure limits of ratios of partition functions. These limits of ratios give the Busemann functions of the log-gamma polymer, and furnish centered cocycles that solve a variational formula for the limiting free energy. Limits of ratios of point-to-point and point-to-line partition functions manifest a duality between tilt and velocity that comes from quenched large deviations under polymer measures. In the log-gamma case, we identify a family of ergodic invariant distributions for the random walk in random environment.

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MSC2010 subject classifications. 60K35, 60K37.

Key words and phrases. Busemann function, competition interface, convex duality, directed polymer, geodesic, Kardar–Parisi–Zhang universality, large deviations, log-gamma polymer, random environment, random walk in random environment, variational formula.

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The Annals of Probability
2015, Vol. 43, No. 5, 2332–2373
DOI: 10.1214/14-AOP934

DISORDER, ENTROPY AND HARMONIC FUNCTIONS

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We study harmonic functions on random environments with particular emphasis on the case of the infinite cluster of supercritical percolation on $\mathbb{Z}^d$. We prove that the vector space of harmonic functions growing at most linearly is $(d+1)$-dimensional almost surely. Further, there are no nonconstant sublinear harmonic functions (thus implying the uniqueness of the corrector). A main ingredient of the proof is a quantitative, annealed version of the Avez entropy argument. This also provides bounds on the derivative of the heat kernel, simplifying and generalizing existing results. The argument applies to many different environments; even reversibility is not necessary.

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MSC2010 subject classifications. Primary 60K37; secondary 31A05, 82B43, 37A35, 60B15, 60J10, 20P05.

Key words and phrases. Harmonic functions, percolation, random walk in random environment, stationary graphs, entropy, Avez, Kaimanovich–Vershik, corrector, IIC, UIPQ, planar map, anomalous diffusion.

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The Annals of Probability
2015, Vol. 43, No. 5, 2374–2404
DOI: 10.1214/14-AOP938

MARTIN BOUNDARY OF RANDOM WALKS WITH UNBOUNDED JUMPS IN HYPERBOLIC GROUPS

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Given a probability measure on a finitely generated group, its Martin boundary is a natural way to compactify the group using the Green function of the corresponding random walk. For finitely supported measures in hyperbolic groups, it is known since the work of Ancona and Gouëzel–Lalley that the Martin boundary coincides with the geometric boundary. The goal of this paper is to weaken the finite support assumption. We first show that, in any nonamenable group, there exist probability measures with exponential tails giving rise to pathological Martin boundaries. Then, for probability measures with superexponential tails in hyperbolic groups, we show that the Martin boundary coincides with the geometric boundary by extending Ancona's inequalities. We also deduce asymptotics of transition probabilities for symmetric measures with superexponential tails.

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MSC2010 subject classifications. 31C35, 60J50, 60B99.

Key words and phrases. Random walk, hyperbolic group, Martin boundary, Gromov boundary, infinite range, local limit theorem.

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The Annals of Probability
2015, Vol. 43, No. 5, 2405–2457
DOI: 10.1214/14-AOP939

RANDOMLY TRAPPED RANDOM WALKS

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We introduce a general model of trapping for random walks on graphs. We give the possible scaling limits of these *Randomly Trapped Random Walks* on $\mathbb{Z}$. These scaling limits include the well-known fractional kinetics process, the Fontes–Isopi–Newman singular diffusion as well as a new broad class we call *spatially subordinated Brownian motions*. We give sufficient conditions for convergence and illustrate these on two important examples.

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MSC2010 subject classifications. Primary 60K37, 60G52; secondary 60F17.
Key words and phrases. Bouchaud trap model, random walk, scaling limit, percolation.

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The Annals of Probability
2015, Vol. 43, No. 5, 2458–2480
DOI: 10.1214/14-AOP940

A LOWER BOUND ON THE TWO-ARMS EXPONENT FOR CRITICAL PERCOLATION ON THE LATTICE

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We consider the standard site percolation model on the d-dimensional lattice. A direct consequence of the proof of the uniqueness of the infinite cluster of Aizenman, Kesten and Newman [*Comm. Math. Phys.* **111** (1987) 505–531] is that the two-arms exponent is larger than or equal to $1/2$. We improve slightly this lower bound in any dimension $d \geq 2$. Next, starting only with the hypothesis that $\theta(p) > 0$, without using the slab technology, we derive a quantitative estimate establishing long-range order in a finite box.

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MSC2010 subject classifications. Primary 60K35; secondary 82B43.
Key words and phrases. Critical percolation, arms exponent.

The Annals of Probability
2015, Vol. 43, No. 5, 2481–2510
DOI: 10.1214/14-AOP941

EMBEDDING LAWS IN DIFFUSIONS BY FUNCTIONS OF TIME

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We present a constructive probabilistic proof of the fact that if $B = (B_t)_{t \ge 0}$ is standard Brownian motion started at 0, and μ is a given probability measure on $\mathbb{R}$ such that $\mu(\{0\}) = 0$, then there exists a unique left-continuous increasing function $b : (0, \infty) \to \mathbb{R} \cup \{+\infty\}$ and a unique left-continuous decreasing function $c : (0, \infty) \to \mathbb{R} \cup \{-\infty\}$ such that B stopped at $\tau_{b,c} = \inf\{t > 0 \mid B_t \ge b(t) \text{ or } B_t \le c(t)\}$ has the law μ. The method of proof relies upon weak convergence arguments arising from Helly's selection theorem and makes use of the Lévy metric which appears to be novel in the context of embedding theorems. We show that $\tau_{b,c}$ is minimal in the sense of Monroe so that the stopped process $B^{\tau_{b,c}} = (B_{t \wedge \tau_{b,c}})_{t \ge 0}$ satisfies natural uniform integrability conditions expressed in terms of μ. We also show that $\tau_{b,c}$ has the smallest truncated expectation among all stopping times that embed μ into B. The main results extend from standard Brownian motion to all recurrent diffusion processes on the real line.

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MSC2010 subject classifications. Primary 60G40, 60J65; secondary 60F05, 60J60.

Key words and phrases. Skorokhod embedding, Brownian motion, diffusion process, Markov process, Helly's selection theorem, weak convergence, Lévy metric, reversed barrier, minimal stopping time.

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The Annals of Probability
2015, Vol. 43, No. 5, 2511–2544
DOI: 10.1214/14-AOP942

EXACT ROSENTHAL-TYPE BOUNDS

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It is shown that, for any given $p \geq 5$, $A > 0$ and $B > 0$, the exact upper bound on $\mathsf{E}|\sum X_i|^p$ over all independent zero-mean random variables (r.v.'s) $X_1, \dots, X_n$ such that $\sum \mathsf{E}X_i^2 = B$ and $\sum \mathsf{E}|X_i|^p = A$ equals $c^p \mathsf{E}|\Pi_\lambda - \lambda|^p$, where $(\lambda, c) \in (0, \infty)^2$ is the unique solution to the system of equations $c^p \lambda = A$ and $c^2 \lambda = B$, and Π_λ is a Poisson r.v. with mean λ. In fact, a more general result is obtained, as well as other related ones. As a tool used in the proof, a calculus of variations of moments of infinitely divisible distributions with respect to variations of the Lévy characteristics is developed.

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MSC2010 subject classifications. Primary 60E15; secondary 60E07.

Key words and phrases. Rosenthal inequality, bounds on moments, sums of independent random variables, probability inequalities, calculus of variations, infinitely divisible distributions, Lévy characteristics.

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The Annals of Probability
2015, Vol. 43, No. 5, 2545–2610
DOI: 10.1214/14-AOP944

SPINES, SKELETONS AND THE STRONG LAW OF LARGE NUMBERS FOR SUPERDIFFUSIONS

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Consider a supercritical superdiffusion $(X_t)_{t\geq 0}$ on a domain $D \subseteq \mathbb{R}^d$ with branching mechanism

$$(x, z) \mapsto -\beta(x)z + \alpha(x)z^2 + \int_{(0,\infty)} (e^{-zy} - 1 + zy)\Pi(x, dy).$$

The skeleton decomposition provides a pathwise description of the process in terms of immigration along a branching particle diffusion. We use this decomposition to derive the strong law of large numbers (SLLN) for a wide class of superdiffusions from the corresponding result for branching particle diffusions. That is, we show that for suitable test functions f and starting measures μ,

$$\frac{\langle f, X_t\rangle}{P_\mu[\langle f, X_t\rangle]} \to W_\infty \qquad P_\mu\text{-almost surely as } t \to \infty,$$

where W_∞ is a finite, non-deterministic random variable characterized as a martingale limit. Our method is based on skeleton and spine techniques and offers structural insights into the driving force behind the SLLN for superdiffusions. The result covers many of the key examples of interest and, in particular, proves a conjecture by Fleischmann and Swart [*Stochastic Process. Appl.* **106** (2003) 141–165] for the super-Wright–Fisher diffusion.

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MSC2010 subject classifications. Primary 60J68; secondary 60J80, 60F15.

Key words and phrases. Superdiffusion, measure-valued diffusion, skeleton decomposition, spine decomposition, strong law of large numbers, additive and multiplicative martingales, almost sure limit theorem.

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The Annals of Probability
2015, Vol. 43, No. 5, 2611–2646
DOI: 10.1214/14-AOP945

REGENERATIVE TREE GROWTH: MARKOVIAN EMBEDDING OF FRAGMENTERS, BIFURCATORS, AND BEAD SPLITTING PROCESSES[1]

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Some, but not all processes of the form $M_t = \exp(-\xi_t)$ for a pure-jump subordinator ξ with Laplace exponent Φ arise as residual mass processes of particle 1 (tagged particle) in Bertoin's partition-valued exchangeable fragmentation processes. We introduce the notion of a *Markovian embedding* of $M = (M_t, t \geq 0)$ in a fragmentation process, and we show that for each Φ, there is a unique (in distribution) binary fragmentation process in which M has a Markovian embedding. The identification of the Laplace exponent Φ^* of its tagged particle process M^* gives rise to a symmetrisation operation $\Phi \mapsto \Phi^*$, which we investigate in a general study of pairs (M, M^*) that coincide up to a random time and then evolve independently. We call M a *fragmenter* and (M, M^*) a *bifurcator*.

For $\alpha > 0$, we equip the interval $R_1 = [0, \int_0^\infty M_t^\alpha \, dt]$ with a purely atomic probability measure μ_1, which captures the jump sizes of M suitably placed on R_1. We study binary tree growth processes that in the nth step sample an atom ("bead") from μ_n and build (R_{n+1}, μ_{n+1}) by replacing the atom by a rescaled independent copy of (R_1, μ_1) that we tie to the position of the atom. We show that any such *bead splitting process* $((R_n, \mu_n), n \geq 1)$ converges almost surely to an α-self-similar continuum random tree of Haas and Miermont, in the Gromov–Hausdorff–Prohorov sense. This generalises Aldous's line-breaking construction of the Brownian continuum random tree.

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MSC2010 subject classifications. 60J80.
Key words and phrases. Fragmentation, self-similar tree, continuum random tree, $\mathbb{R}$-tree, weighted $\mathbb{R}$-tree.

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The Annals of Probability
2015, Vol. 43, No. 5, 2647–2700
DOI: 10.1214/14-AOP946

FORWARD–BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS AND CONTROLLED MCKEAN–VLASOV DYNAMICS

BY RENÉ CARMONA[1] AND FRANÇOIS DELARUE

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The purpose of this paper is to provide a detailed probabilistic analysis of the optimal control of nonlinear stochastic dynamical systems of McKean–Vlasov type. Motivated by the recent interest in mean-field games, we highlight the connection and the differences between the two sets of problems. We prove a new version of the stochastic maximum principle and give sufficient conditions for existence of an optimal control. We also provide examples for which our sufficient conditions for existence of an optimal solution are satisfied. Finally we show that our solution to the control problem provides approximate equilibria for large stochastic controlled systems with mean-field interactions when subject to a common policy.

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MSC2010 subject classifications. Primary 93E20; secondary 60H10, 60K35.
Key words and phrases. Stochastic control, McKean–Vlasov diffusion, stochastic Pontryagin principle, mean-field interaction, mean-field forward–backward stochastic differential equation.

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The Annals of Probability
2015, Vol. 43, No. 5, 2701–2728
DOI: 10.1214/14-AOP947

THE RANGE OF TREE-INDEXED RANDOM WALK IN LOW DIMENSIONS

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We study the range R_n of a random walk on the d-dimensional lattice $\mathbb{Z}^d$ indexed by a random tree with n vertices. Under the assumption that the random walk is centered and has finite fourth moments, we prove in dimension $d \leq 3$ that $n^{-d/4} R_n$ converges in distribution to the Lebesgue measure of the support of the integrated super-Brownian excursion (ISE). An auxiliary result shows that the suitably rescaled local times of the tree-indexed random walk converge in distribution to the density process of ISE. We obtain similar results for the range of critical branching random walk in $\mathbb{Z}^d$, $d \leq 3$. As an intermediate estimate, we get exact asymptotics for the probability that a critical branching random walk starting with a single particle at the origin hits a distant point. The results of the present article complement those derived in higher dimensions in our earlier work.

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MSC2010 subject classifications. Primary 60G50, 60J80; secondary 60G57.

Key words and phrases. Tree-indexed random walk, range, ISE, branching random walk, super-Brownian motion, hitting probability.

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The Annals of Probability
2015, Vol. 43, No. 5, 2729–2762
DOI: 10.1214/14-AOP949

EXPECTED SIGNATURE OF BROWNIAN MOTION UP TO THE FIRST EXIT TIME FROM A BOUNDED DOMAIN

BY TERRY LYONS[1] AND HAO NI[2]

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The signature of a path provides a top down description of the path in terms of its effects as a control [*Differential Equations Driven by Rough Paths* (2007) Springer]. The signature transforms a path into a group-like element in the tensor algebra and is an essential object in rough path theory. The expected signature of a stochastic process plays a similar role to that played by the characteristic function of a random variable. In [Chevyrev (2013)], it is proved that under certain boundedness conditions, the expected value of a random signature already determines the law of this random signature. It becomes of great interest to be able to compute examples of expected signatures and obtain the upper bounds for the decay rates of expected signatures. For instance, the computation for Brownian motion on [0, 1] leads to the "cubature on Wiener space" methodology [Lyons and Victoir, *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **460** (2004) 169–198]. In this paper we fix a bounded domain Γ in a Euclidean space E and study the expected signature of a Brownian path starting at $z \in \Gamma$ and stopped at the first exit time from Γ. We denote this tensor series valued function by $\Phi_\Gamma(z)$ and focus on the case $E = \mathbb{R}^d$. We show that $\Phi_\Gamma(z)$ satisfies an elliptic PDE system and a boundary condition. The equations determining Φ_Γ can be recursively solved; by an iterative application of Sobolev estimates we are able, under certain smoothness and boundedness condition of the domain Γ, to prove geometric bounds for the terms in $\Phi_\Gamma(z)$. However, there is still a gap and we have not shown that $\Phi_\Gamma(z)$ determines the law of the signature of this stopped Brownian motion even if Γ is a unit ball.

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MSC2010 subject classifications. 60J60, 60J65, 60J10, 60J35, 47D03, 47D07, 35K05, 35K08, 35K10, 35K51.

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The Annals of Probability
2015, Vol. 43, No. 5, 2763–2809
DOI: 10.1214/14-AOP951

MULTIFRACTAL ANALYSIS OF SUPERPROCESSES WITH STABLE BRANCHING IN DIMENSION ONE

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We show that density functions of a $(\alpha, 1, \beta)$-superprocesses are almost sure multifractal for $\alpha > \beta + 1$, $\beta \in (0, 1)$ and calculate the corresponding spectrum of singularities.

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MSC2010 subject classifications. Primary 60J80, 28A80; secondary 60G57.
Key words and phrases. Multifractal spectrum, superprocess, Hölder continuity, Hausdorff dimension.

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The Annals of Probability
2015, Vol. 43, No. 5, 2810–2840
DOI: 10.1214/14-AOP952

INDEPENDENCE RATIO AND RANDOM EIGENVECTORS IN TRANSITIVE GRAPHS

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A theorem of Hoffman gives an upper bound on the independence ratio of regular graphs in terms of the minimum $\lambda_{\min}$ of the spectrum of the adjacency matrix. To complement this result we use random eigenvectors to gain lower bounds in the vertex-transitive case. For example, we prove that the independence ratio of a 3-regular transitive graph is at least

$$q = \frac{1}{2} - \frac{3}{4\pi} \arccos\left(\frac{1 - \lambda_{\min}}{4}\right).$$

The same bound holds for infinite transitive graphs: we construct factor of i.i.d. independent sets for which the probability that any given vertex is in the set is at least $q - o(1)$.

We also show that the set of the distributions of factor of i.i.d. processes is not closed w.r.t. the weak topology provided that the spectrum of the graph is uncountable.

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MSC2010 subject classifications. 05C69, 05C50, 60G15.

Key words and phrases. Independent set, independence ratio, minimum eigenvalue, adjacency matrix, regular graph, transitive graph, factor of i.i.d., invariant Gaussian process.

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