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ON THE BOUNDARY OF THE SUPPORT OF SUPER-BROWNIAN MOTION

BY CARL MUELLER[1,*] LEONID MYTNIK[2,†] AND EDWIN PERKINS[3,‡]

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We study the density $X(t,x)$ of one-dimensional super-Brownian motion and find the asymptotic behaviour of $P(0 < X(t,x) \le a)$ as $a \downarrow 0$ as well as the Hausdorff dimension of the boundary of the support of $X(t,\cdot)$. The answers are in terms of the leading eigenvalue of the Ornstein–Uhlenbeck generator with a particular killing term. This work is motivated in part by questions of pathwise uniqueness for associated stochastic partial differential equations.

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Key words and phrases. Super-Brownian motion, Hausdorff dimension, stochastic partial differential equations.

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HOW TO INITIALIZE A SECOND CLASS PARTICLE?

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We identify the ballistically and diffusively rescaled limit distribution of the second class particle position in a wide range of asymmetric and symmetric interacting particle systems with established hydrodynamic behavior, respectively (including zero-range, misanthrope and many other models). The initial condition is a step profile, which in some classical cases of asymmetric models, gives rise to a rarefaction fan scenario. We also point out a model with nonconcave, nonconvex hydrodynamics, where the rescaled second class particle distribution has both continuous and discrete counterparts. The results follow from a substantial generalization of Ferrari and Kipnis' arguments (*Ann. Inst. H. Poincaré* **31** (1995) 143–154) for the totally asymmetric simple exclusion process. The main novelty is the introduction of a *signed* coupling measure as initial data, which nevertheless results in a proper probability initial distribution for the site of the second class particle and makes the extension possible. We also reveal in full generality a very interesting invariance property of the one-site marginal distribution of the process underneath the second class particle which in particular proves the intrinsicality of our choice for the initial distribution. Finally, we give a lower estimate on the probability of survival of a second class particle–antiparticle pair.

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MSC2010 subject classifications. Primary 60K35; secondary 82C22.

Key words and phrases. Second class particle, limit distribution, rarefaction fan, shock, hydrodynamic limit, collision probability.

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OUTLIERS IN THE SPECTRUM OF LARGE DEFORMED UNITARILY INVARIANT MODELS

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We characterize the possible outliers in the spectrum of large deformed unitarily invariant additive and multiplicative models, as well as the eigenvectors corresponding to them. We allow both the nondeformed unitarily invariant model and the perturbation matrix to have nontrivial limiting eigenvalue distributions and spiked outliers in their spectrum. The free subordination functions play a key role in this analysis.

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Key words and phrases. Random matrices, deformed unitarily invariant models, spikes and outliers, free probability and free subordination.

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BULK EIGENVALUE STATISTICS FOR RANDOM REGULAR GRAPHS

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We consider the uniform random d-regular graph on N vertices, with $d \in [N^{\alpha}, N^{2/3-\alpha}]$ for arbitrary $\alpha > 0$. We prove that in the bulk of the spectrum the local eigenvalue correlation functions and the distribution of the gaps between consecutive eigenvalues coincide with those of the Gaussian orthogonal ensemble.

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UNIVERSALITY OF CUTOFF FOR THE ISING MODEL

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On any locally-finite geometry, the stochastic Ising model is known to be contractive when the inverse-temperature β is small enough, via classical results of Dobrushin and of Holley in the 1970s. By a general principle proposed by Peres, the dynamics is then expected to exhibit cutoff. However, so far cutoff for the Ising model has been confirmed mainly for lattices, heavily relying on amenability and log Sobolev inequalities. Without these, cutoff was unknown at any fixed $\beta > 0$, no matter how small, even in basic examples such as the Ising model on a binary tree or a random regular graph.

We use the new framework of information percolation to show that, in any geometry, there is cutoff for the Ising model at high enough temperatures. Precisely, on any sequence of graphs with maximum degree d, the Ising model has cutoff provided that $\beta < \kappa/d$ for some absolute constant κ (a result which, up to the value of κ, is best possible). Moreover, the cutoff location is established as the time at which the sum of squared magnetizations drops to 1, and the cutoff window is $O(1)$, just as when $\beta = 0$.

Finally, the mixing time from almost every initial state is not more than a factor of $1 + \varepsilon_\beta$ faster then the worst one (with $\varepsilon_\beta \to 0$ as $\beta \to 0$), whereas the uniform starting state is at least $2 - \varepsilon_\beta$ times faster.

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INTERMITTENCY AND MULTIFRACTALITY: A CASE STUDY VIA PARABOLIC STOCHASTIC PDES[1]

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Let ξ denote space–time white noise, and consider the following stochastic partial differential equations on $\mathbb{R}_+ \times \mathbb{R}$: (i) $\dot{u} = \frac{1}{2}u'' + u\xi$, started identically at one; and (ii) $\dot{Z} = \frac{1}{2}Z'' + \xi$, started identically at zero. It is well known that the solution to (i) is intermittent, whereas the solution to (ii) is not. And the two equations are known to be in different universality classes.

We prove that the tall peaks of both systems are multifractals in a natural large-scale sense. Some of this work is extended to also establish the multifractal behavior of the peaks of stochastic PDEs on $\mathbb{R}_+ \times \mathbb{R}^d$ with $d \geq 2$. Gregory Lawler has asked us if intermittency is the same as multifractality. The present work gives a negative answer to this question.

As a byproduct of our methods, we prove also that the peaks of the Brownian motion form a large-scale monofractal, whereas the peaks of the Ornstein–Uhlenbeck process on $\mathbb{R}$ are multifractal.

Throughout, we make extensive use of the macroscopic fractal theory of Barlow and Taylor [*J. Phys. A* **22** (1989) 2621–2628; *Proc. Lond. Math. Soc.* (3) **64** (1992) 125–152]. We expand on aspects of the Barlow–Taylor theory, as well.

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THE FRONT LOCATION IN BRANCHING BROWNIAN MOTION WITH DECAY OF MASS

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We augment standard branching Brownian motion by adding a competitive interaction between nearby particles. Informally, when particles are in competition, the local resources are insufficient to cover the energetic cost of motion, so the particles' masses decay. In standard BBM, we may define the *front displacement* at time t as the greatest distance of a particle from the origin. For the model with masses, it makes sense to instead define the front displacement as the distance at which the local mass density drops from $\Theta(1)$ to $o(1)$. We show that one can find arbitrarily large times t for which this occurs at a distance $\Theta(t^{1/3})$ behind the front displacement for standard BBM.

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Key words and phrases. Branching Brownian motion, branching interacting particle systems, front propagation, consistent maximal displacement, Brownian motion.

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BSE'S, BSDE'S AND FIXED-POINT PROBLEMS

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In this paper, we introduce a class of backward stochastic equations (BSEs) that extend classical BSDEs and include many interesting examples of generalized BSDEs as well as semimartingale backward equations. We show that a BSE can be translated into a fixed-point problem in a space of random vectors. This makes it possible to employ general fixed-point arguments to establish the existence of a solution. For instance, Banach's contraction mapping theorem can be used to derive general existence and uniqueness results for equations with Lipschitz coefficients, whereas Schauder-type fixed-point arguments can be applied to non-Lipschitz equations. The approach works equally well for multidimensional as for one-dimensional equations and leads to results in several interesting cases such as equations with path-dependent coefficients, anticipating equations, McKean–Vlasov-type equations and equations with coefficients of superlinear growth.

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MSC2010 subject classifications. 60H10, 47H10.

Key words and phrases. Backward stochastic equation, backward stochastic differential equation, path-dependent coefficients, anticipating equations, McKean–Vlasov-type equations, coefficients of superlinear growth.

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A PHASE TRANSITION IN EXCURSIONS FROM INFINITY OF THE "FAST" FRAGMENTATION-COALESCENCE PROCESS

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An important property of Kingman's coalescent is that, starting from a state with an infinite number of blocks, over any positive time horizon, it transitions into an almost surely finite number of blocks. This is known as "coming down from infinity". Moreover, of the many different (exchangeable) stochastic coalescent models, Kingman's coalescent is the "fastest" to come down from infinity. In this article, we study what happens when we counteract this "fastest" coalescent with the action of an extreme form of fragmentation. We augment Kingman's coalescent, where any two blocks merge at rate $c > 0$, with a fragmentation mechanism where each block fragments at constant rate, $\lambda > 0$, into its constituent elements. We prove that there exists a phase transition at $\lambda = c/2$, between regimes where the resulting "fast" fragmentation-coalescence process is able to come down from infinity or not. In the case that $\lambda < c/2$, we develop an excursion theory for the fast fragmentation-coalescence process out of which a number of interesting quantities can be computed explicitly.

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MSC2010 subject classifications. 60J25, 60G09.
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LOCAL SINGLE RING THEOREM

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The single ring theorem, by Guionnet, Krishnapur and Zeitouni in *Ann. of Math. (2)* **174** (2011) 1189–1217, describes the empirical eigenvalue distribution of a large generic matrix with prescribed singular values, that is, an $N \times N$ matrix of the form $A = UTV$, with U, V some independent Haar-distributed unitary matrices and T a deterministic matrix whose singular values are the ones prescribed. In this text, we give a *local* version of this result, proving that it remains true at the microscopic scale $(\log N)^{-1/4}$. On our way to prove it, we prove a *matrix subordination result* for singular values of sums of non-Hermitian matrices, as Kargin did in *Ann. Probab.* **43** (2015) 2119–2150 for Hermitian matrices. This allows to prove a local law for the singular values of the sum of two non-Hermitian matrices and a delocalization result for singular vectors.

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CONVERGENCE OF THE CENTERED MAXIMUM OF LOG-CORRELATED GAUSSIAN FIELDS

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We show that the centered maximum of a sequence of logarithmically correlated Gaussian fields in any dimension converges in distribution, under the assumption that the covariances of the fields converge in a suitable sense. We identify the limit as a randomly shifted Gumbel distribution, and characterize the random shift as the limit in distribution of a sequence of random variables, reminiscent of the derivative martingale in the theory of branching random walk and Gaussian chaos. We also discuss applications of the main convergence theorem and discuss examples that show that for logarithmically correlated fields; some additional structural assumptions of the type we make are needed for convergence of the centered maximum.

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Key words and phrases. Gaussian processes, extremes values, log-correlated fields.

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VARIATIONAL REPRESENTATIONS FOR THE PARISI FUNCTIONAL AND THE TWO-DIMENSIONAL GUERRA–TALAGRAND BOUND[1]

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The validity of the Parisi formula in the Sherrington–Kirkpatrick model (SK) was initially proved by Talagrand [*Ann. of Math. (2)* **163** (2006) 221–263]. The central argument relied on a dedicated study of the coupled free energy via the two-dimensional Guerra–Talagrand (GT) replica symmetry breaking bound. It is believed that this bound and its higher dimensional generalization are highly related to the conjectures of temperature chaos and ultrametricity in the SK model, but a complete investigation remains elusive. Motivated by Bovier–Klimovsky [*Electron. J. Probab.* **14** (2009) 161–241] and Auffinger–Chen [*Comm. Math. Phys*. **335** (2015) 1429–1444] the aim of this paper is to present a novel approach to analyzing the Parisi functional and the two-dimensional GT bound in the mixed p-spin models in terms of optimal stochastic control problems. We compute the directional derivative of the Parisi functional and derive equivalent criteria for the Parisi measure. We demonstrate how our approach provides a simple and efficient control for the GT bound that yields several new results on Talagrand's positivity of the overlap and disorder chaos in Chatterjee [Disorder chaos and multiple valleys in spin glasses. Preprint] and Chen [*Ann. Probab.* **41** (2013) 3345–3391]. In particular, we provide some examples of the models containing odd p-spin interactions.

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THE VERTEX REINFORCED JUMP PROCESS AND A RANDOM SCHRÖDINGER OPERATOR ON FINITE GRAPHS

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We introduce a new exponential family of probability distributions, which can be viewed as a multivariate generalization of the inverse Gaussian distribution. Considered as the potential of a random Schrödinger operator, this exponential family is related to the random field that gives the mixing measure of the Vertex Reinforced Jump Process (VRJP), and hence to the mixing measure of the Edge Reinforced Random Walk (ERRW), the so-called magic formula. In particular, it yields by direct computation the value of the normalizing constants of these mixing measures, which solves a question raised by Diaconis. The results of this paper are instrumental in [Sabot and Zeng (2015)], where several properties of the VRJP and the ERRW are proved, in particular a functional central limit theorem in transient regimes, and recurrence of the 2-dimensional ERRW.

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MSC2010 subject classifications. Primary 60K37, 60K35, 82B44; secondary 81T25, 81T60.

Key words and phrases. Vertex-reinforced jump process, self-interacting random walks, random Schrödinger operator, supersymmetric hyperbolic nonlinear sigma model.

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NONEQUILIBRIUM ISOTHERMAL TRANSFORMATIONS IN A TEMPERATURE GRADIENT FROM A MICROSCOPIC DYNAMICS[1]

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We consider a chain of anharmonic oscillators immersed in a heat bath with a temperature gradient and a time-varying tension applied to one end of the chain while the other side is fixed to a point. We prove that under diffusive space–time rescaling the volume strain distribution of the chain evolves following a nonlinear diffusive equation. The stationary states of the dynamics are of nonequilibrium and have a positive entropy production, so the classical relative entropy methods cannot be used. We develop new estimates based on entropic hypocoercivity, that allow to control the distribution of the position configurations of the chain. The macroscopic limit can be used to model isothermal thermodynamic transformations between nonequilibrium stationary states.

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Key words and phrases. Hydrodynamic limits, relative entropy, hypocoercivity, non-equilibrium stationary states, isothermal transformations, Langevin heat bath.

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OIL AND WATER: A TWO-TYPE INTERNAL AGGREGATION MODEL

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We introduce a two-type internal DLA model which is an example of a nonunary abelian network. Starting with n "oil" and n "water" particles at the origin, the particles diffuse in $\mathbb{Z}$ according to the following rule: whenever some site $x \in \mathbb{Z}$ has at least 1 oil and at least 1 water particle present, it *fires* by sending 1 oil particle and 1 water particle each to an independent random neighbor $x \pm 1$. Firing continues until every site has at most one type of particles. We establish the correct order for several statistics of this model and identify the scaling limit under assumption of existence.

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MSC2010 subject classifications. 60J10, 60K35, 82B41, 82C22, 82C24.

Key words and phrases. Internal DLA, nonunary abelian network, odometer function, random walk, scaling limit.

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THE NUMBER OF OPEN PATHS IN ORIENTED PERCOLATION

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We study the number N_n of open paths of length n in supercritical oriented percolation on $\mathbb{Z}^d \times \mathbb{N}$, with $d \geq 1$, and we prove the existence of the connective constant for the supercritical oriented percolation cluster: on the percolation event $\{\inf N_n > 0\}$, $N_n^{1/n}$ almost surely converges to a positive deterministic constant.

The proof relies on the introduction of adapted sequences of regenerating times, on subadditive arguments and on the properties of the coupled zone in supercritical oriented percolation. This global convergence result can be deepened to give directional limits and can be extended to more general random linear recursion equations known as linear stochastic evolutions.

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Key words and phrases. Subadditive ergodic theorem, oriented percolation.

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THE HOFFMANN–JØRGENSEN INEQUALITY IN METRIC SEMIGROUPS[1]

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We prove a refinement of the inequality by Hoffmann–Jørgensen that is significant for three reasons. First, our result improves on the state-of-the-art even for real-valued random variables. Second, the result unifies several versions in the Banach space literature, including those by Johnson and Schechtman [*Ann. Probab.* **17** (1989) 789–808], Klass and Nowicki [*Ann. Probab.* **28** (2000) 851–862], and Hitczenko and Montgomery-Smith [*Ann. Probab.* **29** (2001) 447–466]. Finally, we show that the Hoffmann–Jørgensen inequality (including our generalized version) holds not only in Banach spaces but more generally, in a very primitive mathematical framework required to state the inequality: a metric semigroup $\mathscr{G}$. This includes normed linear spaces as well as all compact, discrete or (connected) abelian Lie groups.

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EXTREMAL EIGENVALUE CORRELATIONS IN THE GUE MINOR PROCESS AND A LAW OF FRACTIONAL LOGARITHM[1]

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Let $\lambda^{(N)}$ be the largest eigenvalue of the $N \times N$ GUE matrix which is the Nth element of the GUE minor process, rescaled to converge to the standard Tracy–Widom distribution. We consider the sequence $\{\lambda^{(N)}\}_{N\geq 1}$ and prove a law of fractional logarithm for the lim sup:

$$\limsup_{N\to\infty} \frac{\lambda^{(N)}}{(\log N)^{2/3}} = \left(\frac{1}{4}\right)^{2/3} \qquad \text{almost surely.}$$

For the lim inf, we prove the weaker result that there are constants $c_1, c_2 > 0$ so that

$$-c_1 \leq \liminf_{N\to\infty} \frac{\lambda^{(N)}}{(\log N)^{1/3}} \leq -c_2 \qquad \text{almost surely.}$$

We conjecture that in fact, $c_1 = c_2 = 4^{1/3}$.

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Key words and phrases. GUE, law of fractional logarithm, minor process.

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