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LIMIT THEOREMS FOR MARKOV WALKS CONDITIONED TO STAY POSITIVE UNDER A SPECTRAL GAP ASSUMPTION

BY ION GRAMA, RONAN LAUVERGNAT AND ÉMILE LE PAGE

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Consider a Markov chain $(X_n)_{n\geq 0}$ with values in the state space $\mathbb{X}$. Let f be a real function on $\mathbb{X}$ and set $S_n = \sum_{i=1}^n f(X_i)$, $n \geq 1$. Let $\mathbb{P}_x$ be the probability measure generated by the Markov chain starting at $X_0 = x$. For a starting point $y \in \mathbb{R}$, denote by τ_y the first moment when the Markov walk $(y + S_n)_{n\geq 1}$ becomes nonpositive. Under the condition that S_n has zero drift, we find the asymptotics of the probability $\mathbb{P}_x(\tau_y > n)$ and of the conditional law $\mathbb{P}_x(y + S_n \leq \cdot\sqrt{n} \mid \tau_y > n)$ as $n \to +\infty$.

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THE FOURTH MOMENT THEOREM ON THE POISSON SPACE[1]

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We prove a fourth moment bound without remainder for the normal approximation of random variables belonging to the Wiener chaos of a general Poisson random measure. Such a result—that has been elusive for several years—shows that the so-called ‘fourth moment phenomenon’, first discovered by Nualart and Peccati [*Ann. Probab.* **33** (2005) 177–193] in the context of Gaussian fields, also systematically emerges in a Poisson framework. Our main findings are based on Stein's method, Malliavin calculus and Mecke-type formulae, as well as on a methodological breakthrough, consisting in the use of carré-du-champ operators on the Poisson space for controlling residual terms associated with add-one cost operators. Our approach can be regarded as a successful application of Markov generator techniques to probabilistic approximations in a nondiffusive framework: as such, it represents a significant extension of the seminal contributions by Ledoux [*Ann. Probab.* **40** (2012) 2439–2459] and Azmoodeh, Campese and Poly [*J. Funct. Anal.* **266** (2014) 2341–2359]. To demonstrate the flexibility of our results, we also provide some novel bounds for the Gamma approximation of nonlinear functionals of a Poisson measure.

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Key words and phrases. Poisson functionals, multiple Wiener–Itô integrals, fourth moment theorem, carré-du-champ operator, Berry–Esseen bounds, Gaussian approximation, Gamma approximation, Malliavin calculus, Stein's method.

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ANCHORED EXPANSION, SPEED AND THE POISSON–VORONOI TESSELLATION IN SYMMETRIC SPACES

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We show that a random walk on a stationary random graph with positive anchored expansion and exponential volume growth has positive speed. We also show that two families of random triangulations of the hyperbolic plane, the hyperbolic Poisson–Voronoi tessellation and the hyperbolic Poisson–Delaunay triangulation, have 1-skeletons with positive anchored expansion. As a consequence, we show that the simple random walks on these graphs have positive hyperbolic speed. Finally, we include a section of open problems and conjectures on the topics of stationary geometric random graphs and the hyperbolic Poisson–Voronoi tessellation.

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Key words and phrases. Hyperbolic space, hyperbolic geometry, Poisson process, Voronoi tiling, tessellation, isoperimetric constant, anchored isoperimetric constant, expansion, unimodular random graph.

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OPTIMAL BILINEAR CONTROL OF NONLINEAR STOCHASTIC SCHRÖDINGER EQUATIONS DRIVEN BY LINEAR MULTIPLICATIVE NOISE[1]

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We analyze the bilinear optimal control problem of quantum mechanical systems with final observation governed by a stochastic nonlinear Schrödinger equation perturbed by a linear multiplicative Wiener process. The existence of an open-loop optimal control and first-order Lagrange optimality conditions are derived, via Skorohod's representation theorem, Ekeland's variational principle and the existence for the linearized dual backward stochastic equation. Moreover, our approach in particular applies to the deterministic case.

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RANDOM PARTITIONS OF THE PLANE VIA POISSONIAN COLORING AND A SELF-SIMILAR PROCESS OF COALESCING PLANAR PARTITIONS

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Plant differently colored points in the plane; then let random points ("Poisson rain") fall, and give each new point the color of the nearest existing point. Previous investigation and simulations strongly suggest that the colored regions converge (in some sense) to a random partition of the plane. We prove a weak version of this, showing that normalized empirical measures converge to Lebesgue measures on a random partition into measurable sets. Topological properties remain an open problem. In the course of the proof, which heavily exploits time-reversals, we encounter a novel self-similar process of coalescing planar partitions. In this process, sets $A(z)$ in the partition are associated with Poisson random points z, and the dynamics are as follows. Points are deleted randomly at rate 1; when z is deleted, its set $A(z)$ is adjoined to the set $A(z')$ of the nearest other point z'.

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SCALING LIMIT OF TWO-COMPONENT INTERACTING BROWNIAN MOTIONS

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This paper presents our study of the asymptotic behavior of a two-component system of Brownian motions undergoing certain form of singular interactions. In particular, the system is a combination of two different types of particles and the mechanical properties and the interaction parameters depend on the corresponding type of particles. We prove that the hydrodynamic limit of the empirical densities of two types is the solution of a partial differential equation known as the Maxwell–Stefan equation.

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LARGE EXCURSIONS AND CONDITIONED LAWS FOR RECURSIVE SEQUENCES GENERATED BY RANDOM MATRICES

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We study the large exceedance probabilities and large exceedance paths of the recursive sequence $V_n = M_n V_{n-1} + Q_n$, where $\{(M_n, Q_n)\}$ is an i.i.d. sequence, and M_1 is a $d \times d$ random matrix and Q_1 is a random vector, both with nonnegative entries. We impose conditions which guarantee the existence of a unique stationary distribution for $\{V_n\}$ and a Cramér-type condition for $\{M_n\}$. Under these assumptions, we characterize the distribution of the first passage time $T_u^A := \inf\{n : V_n \in uA\}$, where A is a general subset of $\mathbb{R}^d$, exhibiting that T_u^A/u^α converges to an exponential law for a certain $\alpha > 0$. In the process, we revisit and refine classical estimates for $\mathbb{P}(V \in uA)$, where V possesses the stationary law of $\{V_n\}$. Namely, for $A \subset \mathbb{R}^d$, we show that $\mathbb{P}(V \in uA) \sim C_A u^{-\alpha}$ as $u \to \infty$, providing, most importantly, a new characterization of the constant C_A. As a simple consequence of these estimates, we also obtain an expression for the extremal index of $\{|V_n|\}$. Finally, we describe the large exceedance paths via two conditioned limit theorems showing, roughly, that $\{V_n\}$ follows an exponentially-shifted Markov random walk, which we identify. We thereby generalize results from the theory of classical random walk to multivariate recursive sequences.

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PHASE TRANSITION FOR THE ONCE-REINFORCED RANDOM WALK ON $\mathbb{Z}^d$-LIKE TREES

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In this short paper, we consider the Once-reinforced random walk with reinforcement parameter a on trees with bounded degree which are transient for the simple random walk. On each of these trees, we prove that there exists an explicit critical parameter a_0 such that the Once-reinforced random walk is almost surely recurrent if $a > a_0$ and almost surely transient if $a < a_0$. This provides the first examples of phase transition for the Once-reinforced random walk.

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THE BROWNIAN LIMIT OF SEPARABLE PERMUTATIONS[1]

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We study uniform random permutations in an important class of pattern-avoiding permutations: the separable permutations. We describe the asymptotics of the number of occurrences of any fixed given pattern in such a random permutation in terms of the Brownian excursion. In the recent terminology of permutons, our work can be interpreted as the convergence of uniform random separable permutations towards a "Brownian separable permuton".

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Key words and phrases. Permutation patterns, Brownian excursion, permutons.

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CRITICAL DENSITY OF ACTIVATED RANDOM WALKS ON TRANSITIVE GRAPHS

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We consider the activated random walk model on general vertex-transitive graphs. A central question in this model is whether the critical density μ_c for sustained activity is strictly between 0 and 1. It was known that $\mu_c > 0$ on $\mathbb{Z}^d$, $d \geq 1$, and that $\mu_c < 1$ on $\mathbb{Z}$ for small enough sleeping rate. We show that $\mu_c \to 0$ as $\lambda \to 0$ in all vertex-transitive transient graphs, implying that $\mu_c < 1$ for small enough sleeping rate. We also show that $\mu_c < 1$ for any sleeping rate in any vertex-transitive graph in which simple random walk has positive speed. Furthermore, we prove that $\mu_c > 0$ in any vertex-transitive amenable graph, and that $\mu_c \in (0, 1)$ for any sleeping rate on regular trees.

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Key words and phrases. Interacting particle systems, random walks, absorbing states phase transitions.

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INDISTINGUISHABILITY OF THE COMPONENTS OF RANDOM SPANNING FORESTS[1]

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We prove that the infinite components of the Free Uniform Spanning Forest (FUSF) of a Cayley graph are indistinguishable by any invariant property, given that the forest is different from its wired counterpart. Similar result is obtained for the Free Minimal Spanning Forest (FMSF). We also show that with the above assumptions there can only be 0, 1 or infinitely many components, which solves the problem for the FUSF of Caylay graphs completely. These answer questions by Benjamini, Lyons, Peres and Schramm for Cayley graphs, which have been open up to now. Our methods apply to a more general class of percolations, those satisfying "weak insertion tolerance", and work beyond Cayley graphs, in the more general setting of unimodular random graphs.

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MSC2010 subject classifications. Primary 60D05; secondary 82B43.

Key words and phrases. Spanning forests, uniform spanning forest, minimal spanning forest, insertion tolerance, indistinguishability.

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WEAK SYMMETRIC INTEGRALS WITH RESPECT TO THE FRACTIONAL BROWNIAN MOTION

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The aim of this paper is to establish the weak convergence, in the topology of the Skorohod space, of the ν-symmetric Riemann sums for functionals of the fractional Brownian motion when the Hurst parameter takes the critical value $H = (4\ell + 2)^{-1}$, where $\ell = \ell(\nu) \geq 1$ is the largest natural number satisfying $\int_0^1 \alpha^{2j} \nu(d\alpha) = \frac{1}{2j+1}$ for all $j = 0, \dots, \ell - 1$. As a consequence, we derive a change-of-variable formula in distribution, where the correction term is a stochastic integral with respect to a Brownian motion that is independent of the fractional Brownian motion.

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MSC2010 subject classifications. 60G05, 60H07, 60G15, 60F17.

Key words and phrases. Fractional Brownian motion, Stratonovich integrals, Malliavin calculus, Itô formula in law.

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ON THE SPECTRAL RADIUS OF A RANDOM MATRIX: AN UPPER BOUND WITHOUT FOURTH MOMENT[1]

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Consider a square matrix with independent and identically distributed entries of zero mean and unit variance. It is well known that if the entries have a finite fourth moment, then, in high dimension, with high probability, the spectral radius is close to the square root of the dimension. We conjecture that this holds true under the sole assumption of zero mean and unit variance. In other words, that there are no outliers in the circular law. In this work, we establish the conjecture in the case of symmetrically distributed entries with a finite moment of order larger than two. The proof uses the method of moments combined with a novel truncation technique for cycle weights that might be of independent interest.

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MSC2010 subject classifications. 05C20, 15B52, 47A10, 05C80.
Key words and phrases. Combinatorics, digraph, spectral radius, random matrix, heavy tail.

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STOCHASTIC AIRY SEMIGROUP THROUGH TRIDIAGONAL MATRICES

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We determine the operator limit for large powers of random symmetric tridiagonal matrices as the size of the matrix grows. The result provides a novel expression in terms of functionals of Brownian motions for the Laplace transform of the Airy$_\beta$ process, which describes the largest eigenvalues in the β ensembles of random matrix theory. Another consequence is a Feynman–Kac formula for the stochastic Airy operator of Edelman–Sutton and Ramirez–Rider–Virag.

As a side result, we find that the difference between the area underneath a standard Brownian excursion and one half of the integral of its squared local times is a Gaussian random variable.

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MSC2010 subject classifications. Primary 60B20, 60H25; secondary 47D08, 60G55, 60J55.

Key words and phrases. Airy point process, Brownian bridge, Brownian excursion, Dumitriu–Edelman model, Feynman–Kac formula, Gaussian beta ensemble, intersection local time, moment method, path transformation, quantile transform, random matrix soft edge, random walk bridge, stochastic Airy operator, strong invariance principle, trace formula, Vervaat transform.

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ON THE MIXING TIME OF KAC'S WALK AND OTHER HIGH-DIMENSIONAL GIBBS SAMPLERS WITH CONSTRAINTS

BY NATESH S. PILLAI[*,1] AND AARON SMITH[†,2]

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Determining the total variation mixing time of Kac's random walk on the special orthogonal group SO(n) has been a long-standing open problem. In this paper, we construct a novel non-Markovian coupling for bounding this mixing time. The analysis of our coupling entails controlling the smallest singular value of a certain random matrix with highly dependent entries. The dependence of the entries in our matrix makes it not amenable to existing techniques in random matrix theory. To circumvent this difficulty, we extend some recent bounds on the smallest singular values of matrices with independent entries to our setting. These bounds imply that the mixing time of Kac's walk on the group SO(n) is between $C_1 n^2$ and $C_2 n^4 \log(n)$ for some explicit constants $0 < C_1, C_2 < \infty$, substantially improving on the bound of $O(n^5 \log(n)^2)$ in the preprint of Jiang [Jiang (2012)]. Our methods may also be applied to other high dimensional Gibbs samplers with constraints, and thus are of independent interest. In addition to giving analytical bounds on the mixing time, our approach allows us to compute rigorous estimates of the mixing time by simulating the eigenvalues of a random matrix.

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ERRATA TO "DISTANCE COVARIANCE IN METRIC SPACES"

BY RUSSELL LYONS[1]

Indiana University

We correct several statements and proofs in our paper, *Ann. Probab.* **41**, no. 5 (2013), 3284–3305.

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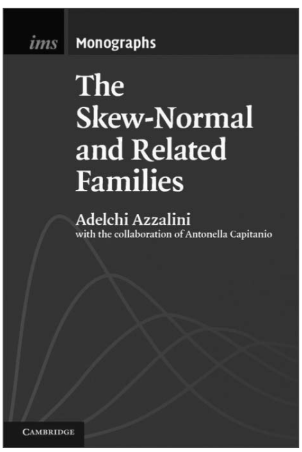

The Skew-Normal and Related Families

Adelchi Azzalini
in collaboration with Antonella Capitanio

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