

# BERNOULLI

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# Convergence rate of the powers of an operator. Applications to stochastic systems

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We extend the traditional operator theoretic approach for the study of dynamical systems in order to handle the problem of non-geometric convergence. We show that the probabilistic treatment developed and popularized under Richard Tweedie's impulsion, can be placed into an operator framework in the spirit of Yosida–Kakutani's approach. General theorems as well as specific results for Markov chains are given. Application examples to general classes of Markov chains and dynamical systems are presented.

**Keywords:** Markov chains

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# Spectral analysis of sample autocovariance matrices of a class of linear time series in moderately high dimensions

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*Dedicated to the memory of Peter Gavin Hall*

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This article is concerned with the spectral behavior of  $p$ -dimensional linear processes in the moderately high-dimensional case when both dimensionality  $p$  and sample size  $n$  tend to infinity so that  $p/n \rightarrow 0$ . It is shown that, under an appropriate set of assumptions, the empirical spectral distributions of the renormalized and symmetrized sample autocovariance matrices converge almost surely to a nonrandom limit distribution supported on the real line. The key assumption is that the linear process is driven by a sequence of  $p$ -dimensional real or complex random vectors with i.i.d. entries possessing zero mean, unit variance and finite fourth moments, and that the  $p \times p$  linear process coefficient matrices are Hermitian and simultaneously diagonalizable. Several relaxations of these assumptions are discussed. The results put forth in this paper can help facilitate inference on model parameters, model diagnostics and prediction of future values of the linear process.

**Keywords:** empirical spectral distribution; high-dimensional statistics; limiting spectral distribution; Stieltjes transform

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# Probability approximation of point processes with Papangelou conditional intensity

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We give general bounds in the Gaussian and Poisson approximations of innovations (or Skorohod integrals) defined on the space of point processes with Papangelou conditional intensity. We apply the general results to Gibbs point processes with pair potential and determinantal point processes. In particular, we provide explicit error bounds and quantitative limit theorems for stationary, inhibitory and finite range Gibbs point processes with pair potential and  $\beta$ -Ginibre point processes.

**Keywords:** Chen–Stein’s method; determinantal point process; Gaussian approximation; Gibbs point process; Ginibre point process; innovation; Papangelou intensity; Poisson approximation; Poisson process; Skorohod integral; Stein’s method

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# The geometric foundations of Hamiltonian Monte Carlo

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Although Hamiltonian Monte Carlo has proven an empirical success, the lack of a rigorous theoretical understanding of the algorithm has in many ways impeded both principled developments of the method and use of the algorithm in practice. In this paper, we develop the formal foundations of the algorithm through the construction of measures on smooth manifolds, and demonstrate how the theory naturally identifies efficient implementations and motivates promising generalizations.

**Keywords:** differential geometry; disintegration; fiber bundle; Hamiltonian Monte Carlo; Markov chain Monte Carlo; Riemannian geometry; symplectic geometry; smooth manifold

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# Large-sample approximations for variance-covariance matrices of high-dimensional time series

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Distributional approximations of (bi-) linear functions of sample variance-covariance matrices play a critical role to analyze vector time series, as they are needed for various purposes, especially to draw inference on the dependence structure in terms of second moments and to analyze projections onto lower dimensional spaces as those generated by principal components. This particularly applies to the high-dimensional case, where the dimension  $d$  is allowed to grow with the sample size  $n$  and may even be larger than  $n$ . We establish large-sample approximations for such bilinear forms related to the sample variance-covariance matrix of a high-dimensional vector time series in terms of strong approximations by Brownian motions and the uniform (in the dimension) consistent estimation of their covariances. The results cover weakly dependent as well as many long-range dependent linear processes and are valid for uniformly  $\ell_1$ -bounded projection vectors, which arise, either naturally or by construction, in many statistical problems extensively studied for high-dimensional series. Among those problems are sparse financial portfolio selection, sparse principal components, the LASSO, shrinkage estimation and change-point analysis for high-dimensional time series, which matter for the analysis of big data and are discussed in greater detail.

**Keywords:** big data; change-points; data science and analytics; long memory; multivariate analysis; portfolio analysis; principal component analysis; strong approximation; time series

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# Laws of the iterated logarithm for symmetric jump processes

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Based on two-sided heat kernel estimates for a class of symmetric jump processes on metric measure spaces, the laws of the iterated logarithm (LILs) for sample paths, local times and ranges are established. In particular, the LILs are obtained for  $\beta$ -stable-like processes on  $\alpha$ -sets with  $\beta > 0$ .

**Keywords:** law of the iterated logarithm; local time; range; sample path; stable-like process; symmetric jump processes

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# Cutting down $\mathbf{p}$ -trees and inhomogeneous continuum random trees

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We study a fragmentation of the  $\mathbf{p}$ -trees of Camarri and Pitman. We give exact correspondences between the  $\mathbf{p}$ -trees and trees which encode the fragmentation. We then use these results to study the fragmentation of the inhomogeneous continuum random trees (scaling limits of  $\mathbf{p}$ -trees) and give distributional correspondences between the initial tree and the tree encoding the fragmentation. The theorems for the inhomogeneous continuum random tree extend previous results by Bertoin and Miermont about the cut tree of the Brownian continuum random tree.

*Keywords:* cut tree; inhomogeneous continuum random trees;  $\mathbf{p}$ -tree; random cutting

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# A new rejection sampling method without using hat function

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This paper proposes a new exact simulation method, which simulates a realisation from a proposal density and then uses exact simulation of a Langevin diffusion to check whether the proposal should be accepted or rejected. Comparing to the existing *coupling from the past* method, the new method does not require constructing fast coalescence Markov chains. Comparing to the existing rejection sampling method, the new method does not require the proposal density function to bound the target density function. The new method is much more efficient than existing methods for certain problems. An application on exact simulation of the posterior of finite mixture models is presented.

**Keywords:** conditioned Brownian motion; coupling from the past; diffusion bridges; exact Monte Carlo simulation; Langevin diffusion; mixture models; rejection sampling

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# Spectral analysis of high-dimensional sample covariance matrices with missing observations

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We study high-dimensional sample covariance matrices based on independent random vectors with missing coordinates. The presence of missing observations is common in modern applications such as climate studies or gene expression micro-arrays. A weak approximation on the spectral distribution in the “large dimension  $d$  and large sample size  $n$ ” asymptotics is derived for possibly different observation probabilities in the coordinates. The spectral distribution turns out to be strongly influenced by the missingness mechanism. In the null case under the missing at random scenario where each component is observed with the same probability  $p$ , the limiting spectral distribution is a Marčenko–Pastur law shifted by  $(1 - p)/p$  to the left. As  $d/n \rightarrow y \in (0, 1)$ , the almost sure convergence of the extremal eigenvalues to the respective boundary points of the support of the limiting spectral distribution is proved, which are explicitly given in terms of  $y$  and  $p$ . Eventually, the sample covariance matrix is positive definite if  $p$  is larger than

$$1 - (1 - \sqrt{y})^2,$$

whereas this is not true any longer if  $p$  is smaller than this quantity.

**Keywords:** almost sure convergence of extremal eigenvalues; characterization of positive definiteness; limiting spectral distribution; sample covariance matrix with missing observations; Stieltjes transform

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# Convergence rates of Laplace-transform based estimators

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This paper considers the problem of estimating probabilities of the form  $\mathbb{P}(Y \leq w)$ , for a given value of  $w$ , in the situation that a sample of i.i.d. observations  $X_1, \dots, X_n$  of  $X$  is available, and where we explicitly know a functional relation between the Laplace transforms of the non-negative random variables  $X$  and  $Y$ . A plug-in estimator is constructed by calculating the Laplace transform of the empirical distribution of the sample  $X_1, \dots, X_n$ , applying the functional relation to it, and then (if possible) inverting the resulting Laplace transform and evaluating it in  $w$ . We show, under mild regularity conditions, that the resulting estimator is weakly consistent and has expected absolute estimation error  $O(n^{-1/2} \log(n+1))$ . We illustrate our results by two examples: in the first we estimate the distribution of the workload in an M/G/1 queue from observations of the input in fixed time intervals, and in the second we identify the distribution of the increments when observing a compound Poisson process at equidistant points in time (usually referred to as “decompounding”).

*Keywords:* decompounding; estimation in queues; Laplace transform

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# Randomized pivots for means of short and long memory linear processes

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In this paper, we introduce randomized pivots for the means of short and long memory linear processes. We show that, under the same conditions, these pivots converge in distribution to the same limit as that of their classical non-randomized counterparts. We also present numerical results that indicate that these randomized pivots significantly outperform their classical counterparts and as a result they lead to a more accurate inference about the population mean.

*Keywords:* central limit theorem; randomized pivots; short and long memory time-series

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# Strongly degenerate time inhomogeneous SDEs: Densities and support properties. Application to Hodgkin–Huxley type systems

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In this paper, we study the existence of densities for strongly degenerate stochastic differential equations (SDEs) whose coefficients depend on time and are not globally Lipschitz. In these models, neither local ellipticity nor the strong Hörmander condition is satisfied. In this general setting, we show that continuous transition densities indeed exist in all neighborhoods of points where the weak Hörmander condition is satisfied. We also exhibit regions where these densities remain positive. We then apply these results to stochastic Hodgkin–Huxley models with periodic input as a first step towards the study of ergodicity properties of such systems in the sense of Meyn and Tweedie (*Adv. in Appl. Probab.* **25** (1993) 487–517; *Adv. in Appl. Probab.* **25** (1993) 518–548).

**Keywords:** degenerate diffusion processes; Hodgkin–Huxley system; local Hörmander condition; Malliavin calculus; support theorem; time inhomogeneous diffusion processes

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# Extended generalised variances, with applications

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We consider a measure  $\psi_k$  of dispersion which extends the notion of Wilk's generalised variance for a  $d$ -dimensional distribution, and is based on the mean squared volume of simplices of dimension  $k \leq d$  formed by  $k + 1$  independent copies. We show how  $\psi_k$  can be expressed in terms of the eigenvalues of the covariance matrix of the distribution, also when a  $n$ -point sample is used for its estimation, and prove its concavity when raised at a suitable power. Some properties of dispersion-maximising distributions are derived, including a necessary and sufficient condition for optimality. Finally, we show how this measure of dispersion can be used for the design of optimal experiments, with equivalence to  $A$  and  $D$ -optimal design for  $k = 1$  and  $k = d$ , respectively. Simple illustrative examples are presented.

**Keywords:** design of experiments; dispersion; generalised variance; maximum-dispersion measure; optimal design; quadratic entropy

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# Multilevel Richardson–Romberg extrapolation

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We propose and analyze a Multilevel Richardson–Romberg (ML2R) estimator which combines the higher order bias cancellation of the Multistep Richardson–Romberg method introduced in [*Monte Carlo Methods Appl.* **13** (2007) 37–70] and the variance control resulting from Multilevel Monte Carlo (MLMC) paradigm (see [*Ann. Appl. Probab.* **24** (2014) 1585–1620, In *Large-Scale Scientific Computing* (2001) 58–67 Berlin]). Thus, in standard frameworks like discretization schemes of diffusion processes, the root mean squared error (RMSE)  $\varepsilon > 0$  can be achieved with our ML2R estimator with a global complexity of  $\varepsilon^{-2} \log(1/\varepsilon)$  instead of  $\varepsilon^{-2} (\log(1/\varepsilon))^2$  with the standard MLMC method, at least when the weak error  $\mathbf{E}[Y_h] - \mathbf{E}[Y_0]$  of the biased implemented estimator  $Y_h$  can be expanded at any order in  $h$  and  $\|Y_h - Y_0\|_2 = O(h^{\frac{1}{2}})$ . The ML2R estimator is then halfway between a regular MLMC and a virtual unbiased Monte Carlo. When the strong error  $\|Y_h - Y_0\|_2 = O(h^{\frac{\beta}{2}})$ ,  $\beta < 1$ , the gain of ML2R over MLMC becomes even more striking. We carry out numerical simulations to compare these estimators in two settings: vanilla and path-dependent option pricing by Monte Carlo simulation and the less classical Nested Monte Carlo simulation.

**Keywords:** Euler scheme; multilevel Monte Carlo estimator; multistep; nested Monte Carlo method; option pricing; Richardson–Romberg extrapolation

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# Efficiency transfer for regression models with responses missing at random

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We consider independent observations on a random pair  $(X, Y)$ , where the response  $Y$  is allowed to be missing at random but the covariate vector  $X$  is always observed. We demonstrate that characteristics of the conditional distribution of  $Y$  given  $X$  can be estimated efficiently using complete case analysis, that is, one can simply omit incomplete cases and work with an appropriate efficient estimator which remains efficient. This means in particular that we do not have to use imputation or work with inverse probability weights. Those approaches will never be better (asymptotically) than the above complete case method.

This *efficiency transfer* is a general result and holds true for all regression models for which the distribution of  $Y$  given  $X$  and the marginal distribution of  $X$  do not share common parameters. We apply it to the general homoscedastic semiparametric regression model. This includes models where the conditional expectation is modeled by a complex semiparametric regression function, as well as all basic models such as linear regression and nonparametric regression. We discuss estimation of various functionals of the conditional distribution, for example, of regression parameters and of the error distribution.

**Keywords:** complete case analysis; efficient estimation; efficient influence function; linear and nonlinear regression; nonparametric regression; partially linear regression; random coefficient model; tangent space; transfer principle

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# Inference under biased sampling and right censoring for a change point in the hazard function

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Length-biased survival data commonly arise in cross-sectional surveys and prevalent cohort studies on disease duration. Ignoring biased sampling leads to bias in estimating the hazard-of-failure and the survival-time in the population. We address estimating the location of a possible change-point of an otherwise smooth hazard function when the collected data form a biased sample from the target population and the data are subject to informative censoring. We provide two estimation methodologies, for the location and size of the change-point, adapted to two scenarios of the truncation distribution: known and unknown. While the estimators in the first case show gain in efficiency as compared to those in the second case, the latter is more robust to the form of the truncation distribution. In both cases, the change-point estimators can achieve the rate  $\mathcal{O}_p(1/n)$ . We study the asymptotic properties of the estimates and devise interval-estimators for the location and size of the change, paving the way towards making statistical inference about whether or not a change-point exists. Several simulated examples are discussed to assess the finite sample behavior of the estimators. The proposed methods are then applied to analyze a set of survival data collected on elderly Canadian citizen (aged 65+) suffering from dementia.

**Keywords:** biased sampling; change point; informative censoring; jump size; left truncation; prevalent cohort survival data; survival with dementia

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# A generalized divergence for statistical inference

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The power divergence (PD) and the density power divergence (DPD) families have proven to be useful tools in the area of robust inference. In this paper, we consider a superfamily of divergences which contains both of these families as special cases. The role of this superfamily is studied in several statistical applications, and desirable properties are identified and discussed. In many cases, it is observed that the most preferred minimum divergence estimator within the above collection lies outside the class of minimum PD or minimum DPD estimators, indicating that this superfamily has real utility, rather than just being a routine generalization. The limitation of the usual first order influence function as an effective descriptor of the robustness of the estimator is also demonstrated in this connection.

**Keywords:** breakdown point; divergence measure; influence function; robust estimation;  $S$ -divergence

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# Conditional convex orders and measurable martingale couplings

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Strassen's classical martingale coupling theorem states that two random vectors are ordered in the convex (resp. increasing convex) stochastic order if and only if they admit a martingale (resp. submartingale) coupling. By analysing topological properties of spaces of probability measures equipped with a Wasserstein metric and applying a measurable selection theorem, we prove a conditional version of this result for random vectors conditioned on a random element taking values in a general measurable space. We provide an analogue of the conditional martingale coupling theorem in the language of probability kernels, and discuss how it can be applied in the analysis of pseudo-marginal Markov chain Monte Carlo algorithms. We also illustrate how our results imply the existence of a measurable minimiser in the context of martingale optimal transport.

**Keywords:** conditional coupling; convex stochastic order; increasing convex stochastic order; martingale coupling; pointwise coupling; probability kernel

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# Sharp thresholds for Gibbs–non-Gibbs transitions in the fuzzy Potts model with a Kac-type interaction

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We investigate the Gibbs properties of the fuzzy Potts model on the  $d$ -dimensional torus with Kac interaction. We use a variational approach for profiles inspired by that of Fernández, den Hollander and Martínez [*J. Stat. Phys.* **156** (2014) 203–220] for their study of the Gibbs–non-Gibbs transitions of a dynamical Kac–Ising model on the torus. As our main result, we show that the mean-field thresholds dividing Gibbsian from non-Gibbsian behavior are sharp in the fuzzy Kac–Potts model with class size unequal two. On the way to this result, we prove a large deviation principle for color profiles with diluted total mass densities and use monotonicity arguments.

**Keywords:** diluted large deviation principles; fuzzy Kac–Potts model; Gibbs versus non-Gibbs; Kac model; large deviation principles; Potts model

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# On Stein operators for discrete approximations

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In this paper, a new method based on probability generating functions is used to obtain multiple Stein operators for various random variables closely related to Poisson, binomial and negative binomial distributions. Also, the Stein operators for certain compound distributions, where the random summand satisfies Panjer's recurrence relation, are derived. A well-known perturbation approach for Stein's method is used to obtain total variation bounds for the distributions mentioned above. The importance of such approximations is illustrated, for example, by the binomial convoluted with Poisson approximation to sums of independent and dependent indicator random variables.

**Keywords:** binomial distribution; compound Poisson distribution; Panjer's recursion; perturbation; Stein's method; total variation norm

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# Information criteria for multivariate CARMA processes

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Multivariate continuous-time ARMA( $p, q$ ) (MCARMA( $p, q$ )) processes are the continuous-time analog of the well-known vector ARMA( $p, q$ ) processes. They have attracted interest over the last years. Methods to estimate the parameters of an MCARMA process require an identifiable parametrization such as the Echelon form with a fixed Kronecker index, which is in the one-dimensional case the degree  $p$  of the autoregressive polynomial. Thus, the Kronecker index has to be known in advance before parameter estimation can be done. When this is not the case, information criteria can be used to estimate the Kronecker index and the degrees  $(p, q)$ , respectively. In this paper, we investigate information criteria for MCARMA processes based on quasi maximum likelihood estimation. Therefore, we first derive the asymptotic properties of quasi maximum likelihood estimators for MCARMA processes in a misspecified parameter space. Then, we present necessary and sufficient conditions for information criteria to be strongly and weakly consistent, respectively. In particular, we study the well-known Akaike Information Criterion (AIC) and the Bayesian Information Criterion (BIC) as special cases.

**Keywords:** AIC; BIC; CARMA process; consistency; information criteria; law of iterated logarithm; Kronecker index; quasi maximum likelihood estimation

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# From trees to seeds: On the inference of the seed from large trees in the uniform attachment model

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We study the influence of the seed in random trees grown according to the uniform attachment model, also known as uniform random recursive trees. We show that different seeds lead to different distributions of limiting trees from a total variation point of view. To do this, we construct statistics that measure, in a certain well-defined sense, global “balancedness” properties of such trees. Our paper follows recent results on the same question for the preferential attachment model.

**Keywords:** random trees; seed tree; statistical inference; uniform attachment

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# Guided proposals for simulating multi-dimensional diffusion bridges

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A Monte Carlo method for simulating a multi-dimensional diffusion process conditioned on hitting a fixed point at a fixed future time is developed. Proposals for such diffusion bridges are obtained by superimposing an additional guiding term to the drift of the process under consideration. The guiding term is derived via approximation of the target process by a simpler diffusion processes with known transition densities. Acceptance of a proposal can be determined by computing the likelihood ratio between the proposal and the target bridge, which is derived in closed form. We show under general conditions that the likelihood ratio is well defined and show that a class of proposals with guiding term obtained from linear approximations fall under these conditions.

**Keywords:** change of measure; data augmentation; linear processes; multidimensional diffusion bridge

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