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A Review of Nonparametric Hypothesis Tests of Isotropy Properties in Spatial Data

Zachary D. Weller and Jennifer A. Hoeting

Abstract. An important aspect of modeling spatially referenced data is appropriately specifying the covariance function of the random field. A practitioner working with spatial data is presented a number of choices regarding the structure of the dependence between observations. One of these choices is to determine whether or not an isotropic covariance function is appropriate. Isotropy implies that spatial dependence does not depend on the direction of the spatial separation between sampling locations. Misspecification of isotropy properties (directional dependence) can lead to misleading inferences, for example, inaccurate predictions and parameter estimates. A researcher may use graphical diagnostics, such as directional sample variograms, to decide whether the assumption of isotropy is reasonable. These graphical techniques can be difficult to assess, open to subjective interpretations, and misleading. Hypothesis tests of the assumption of isotropy may be more desirable. To this end, a number of tests of directional dependence have been developed using both the spatial and spectral representations of random fields. We provide an overview of nonparametric methods available to test the hypotheses of isotropy and symmetry in spatial data. We discuss important considerations in choosing a test, provide recommendations for implementing a test, compare several of the methods via a simulation study, and propose a number of open research questions. Several of the reviewed methods can be implemented in R using our package `spTest`, available on CRAN.

Key words and phrases: Isotropy, symmetry, nonparametric spatial covariance.

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Rank Tests from Partially Ordered Data Using Importance and MCMC Sampling Methods

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Abstract. We discuss distribution-free exact rank tests from partially ordered data that arise in various biological and other applications where the primary objective is to conduct testing of significance to assess the linear dependence or to compare different groups. The tests here are obtained by treating the usual rank statistics, based on the completely ordered data as "latent" or missing, and conceptualizing the "latent" p-value as the random probability under the null hypothesis of a test statistic that is as extreme, or more extreme, than the latent test statistics based on the completely ordered data. The latent p-value is then predicted by sampling linear extensions or the complete orderings that are consistent with the observed partially ordered data. The sampling methods explored here include importance sampling methods based on randomized topological sorting algorithms, Gibbs sampling methods, random-walk based Metropolis–Hasting sampling methods and random-walk based modern perfect Markov chain Monte Carlo sampling methods. We discuss running times of these sampling methods and their strength and weaknesses. A simulation experiment and three data examples are given. The simulation experiment illustrates how the exact rank tests from partially ordered data work when the desired result is known. The first data example concerns the light preference behavior of fruit flies and tests whether heterogeneity observed in average light-preference behavior can be explained by manipulations in serotonin signaling. The second one is a reanalysis of the lead absorption data in children of employees who worked in a lead battery factory and consolidates the results reported in Rosenbaum [*Ann. Statist.* **19** (1991) 1091–1097]. The third one reexamines the breast cosmesis data from Finkelstein [*Biometrics* **42** (1986) 845–854].

Key words and phrases: Exact tests, fuzzy p-values, Gibbs sampling, interval censoring, linear extensions, linear rank statistics, perfect MCMC, proportional hazard model, topological sorting.

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On Negative Outcome Control of Unobserved Confounding as a Generalization of Difference-in-Differences

Tamar Sofer, David B. Richardson, Elena Colicino, Joel Schwartz and Eric J. Tchetgen Tchetgen

Abstract. The *difference-in-differences* (DID) approach is a well-known strategy for estimating the effect of an exposure in the presence of unobserved confounding. The approach is most commonly used when pre- and post-exposure outcome measurements are available, and one can assume that the association of the unobserved confounder with the outcome is equal in the two exposure groups, and constant over time. Then one recovers the treatment effect by regressing the change in outcome over time on the exposure. In this paper, we interpret the difference-in-differences as a negative outcome control (NOC) approach. We show that the pre-exposure outcome is a negative control outcome, as it cannot be influenced by the subsequent exposure, and it is affected by both observed and unobserved confounders of the exposure-outcome association of interest. The relation between DID and NOC provides simple conditions under which negative control outcomes can be used to detect and correct for confounding bias. However, for general negative control outcomes, the DID-like assumption may be overly restrictive and rarely credible, because it requires that both the outcome of interest and the control outcome are measured on the same scale. Thus, we present a scale-invariant generalization of the DID that may be used in broader NOC contexts. The proposed approach is demonstrated in simulations and on a Normative Aging Study data set, in which Body Mass Index is used for NOC of the relationship between air pollution and inflammatory outcomes.

Key words and phrases: Location-scale models, quantile–quantile transformation, air pollution, inflammation.

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Quantum Annealing with Markov Chain Monte Carlo Simulations and D-Wave Quantum Computers

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Abstract. Quantum computation performs calculations by using quantum devices instead of electronic devices following classical physics and used by classical computers. Although general purpose quantum computers of practical scale may be many years away, special purpose quantum computers are being built with capabilities exceeding classical computers. One prominent case is the so-called D-Wave quantum computer, which is a computing hardware device built to implement quantum annealing for solving combinatorial optimization problems. Whether D-Wave computing hardware devices display a quantum behavior or can be described by a classical model has attracted tremendous attention, and it remains controversial to determine whether quantum or classical effects play a crucial role in exhibiting the computational input–output behaviors of the D-Wave devices. This paper consists of two parts where the first part provides a review of quantum annealing and its implementations, and the second part proposes statistical methodologies to analyze data generated from annealing experiments. Specifically, we introduce quantum annealing to solve optimization problems and describe D-Wave computing devices to implement quantum annealing. We illustrate implementations of quantum annealing using Markov chain Monte Carlo (MCMC) simulations carried out by classical computers. Computing experiments have been conducted to generate data and compare quantum annealing with classical annealing. We propose statistical methodologies to analyze computing experimental data from a D-Wave device and simulated data from the MCMC based annealing methods, and establish asymptotic theory and check finite sample performances for the proposed statistical methodologies. Our findings confirm bimodal histogram patterns displayed in input–output data from the D-Wave device and both U-shape and unimodal histogram patterns exhibited in input–output data from the MCMC based annealing methods. Further statistical explorations reveal possible sources for the U-shape patterns. On the other hand, our statistical analysis produces statistical evidence to indicate that input–output data from the D-Wave device are not consistent with the stochastic behaviors of any MCMC based annealing models under the study. We present a list of statistical research topics for the future study on quantum annealing and MCMC simulations.

Key words and phrases: Quantum annealing, quantum computing, Markov chain Monte Carlo, Ising model, ground state success probability, Hamiltonian, quantum bit (qubit).

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Markov Chains as Models in Statistical Mechanics

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Abstract. The Bernoulli [*Novi Commentarii Academiae Scientiarum Imperialis Petropolitanae* **14** (1769) 3–25]/Laplace [*Théorie Analytique des Probabilités* (1812) V. Courcier] urn model and the Ehrenfest and Ehrenfest [*Physikalische Zeitschrift* **8** (1907) 311–314] urn model for mixing are instances of simple Markov chain models called random walks. Both can be used to suggest a probabilistic resolution to the coexistence of irreversibility and recurrence in Boltzmann's H-Theorem. Marian von Smoluchowski [In *Sitzungsberichte der Akademie der Wissenschaften. Mathematisch-Naturwissenschaftliche Klasse* (1914) 2381–2405 Hölder] also modelled by a simple Markov chain, with analogous properties, have fluctuations over time in the number of particles contained in a small element of volume in a solution.This paper explores the themes of entropy, recurrence and reversibility within the framework of such Markov chains.

A branching process with immigration, in this respect like Smoluchowski's model, is introduced to accentuate common features of the spectral theory of all models. This is related to their reversibility, a key issue.

Key words and phrases: Ehrenfest, Smoluchowski, entropy and recurrence, reversible Markov chain, stochastic matrix, Krawtchouk, Hahn, Charlier, Meixner polynomials, branching process with immigration.

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Fractional Imputation in Survey Sampling: A Comparative Review

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Abstract. Fractional imputation (FI) is a relatively new method of imputation for handling item nonresponse in survey sampling. In FI, several imputed values with their fractional weights are created for each record with missing items. Each fractional weight represents the conditional probability of the imputed value given the observed data, and the parameters in the conditional probabilities are often computed by an iterative method such as the EM algorithm. The underlying model for FI can be fully parametric, semiparametric or nonparametric, depending on the plausibility of assumptions and the data structure.

In this paper, we give an overview of FI, introduce key ideas and methods to readers who are new to the FI literature, and highlight some new developments. We also provide guidance on practical implementation of FI and valid inferential tools after imputation. We demonstrate the empirical performance of FI with respect to multiple imputation using a pseudo finite population generated from a sample from the Monthly Retail Trade Survey conducted by the US Census Bureau.

Key words and phrases: Item nonresponse, missing at random, Monte Carlo EM, multiple imputation, synthetic imputation.

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A Conversation with Michael Woodroofe

Moulinath Banerjee and Bodhisattva Sen

Abstract. Michael Woodroofe was born in Corvallis on March 17, 1940, and grew up in a small town called Athena in Oregon. Michael graduated from McEwen High School in 1958 and entered Stanford University, from which he graduated four years later with a major in Mathematics. He earned his masters degree and Ph.D. from the mathematics department at the University of Oregon in 1964 and 1965, respectively.

Michael Woodroofe has had a distinguished career and is widely recognized as a preeminent statistician and probabilist. He has broad interests and has made deep and significant contributions in many areas in statistical inference and probability, including biased sampling, shape-restricted inference, sequential analysis, nonlinear renewal theory, modern nonparametric inference, statistics in astronomy and central limit theory for stationary processes. He has published more than 100 research articles, written a SIAM monograph and authored a book. He is a former Editor of *the Annals of Statistics*, a member of Phi Beta Kappa and a fellow of the Institute of Mathematical Statistics.

Michael's professional positions have included being on the faculty of the Department of Statistics at Carnegie Mellon University and the University of Michigan at Ann Arbor, where he has been on faculty for more than 40 years. He was a founding member of the Department of Statistics at the University Michigan in 1969, retaining a joint appointment with Mathematics, and served as the Chair of the Department of Statistics during 1977–1983. In addition, he has held visiting positions at Columbia University, Massachusetts Institute of Technology and Rutgers University.

Michael and his wife, Fran Woodroofe, reside in Ann Arbor. He is the father of one daughter, Caroline, and two sons, Russell and Blake.

Key words and phrases: Biased sampling, nonlinear renewal theory, sequential analysis, shape-restricted inference, statistics in astronomy.

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A Conversation with Arthur Cohen

Joseph Naus

Abstract. Arthur Cohen was born in 1933. He received his B.A. in mathematics from Brooklyn College in 1955, and then went to graduate studies in statistics at Columbia University. In 1957, he took leave from Columbia to serve for two years at the Communicable Disease Center, Public Health Services. He returned to Columbia, completed his studies and received his Ph.D. in mathematical statistics in 1963. Art joined the statistics department at Rutgers as an Assistant Professor, and two years later became Associate Professor. From 1968 through 1977, he served as chairman of the department during a critical period in its development. For 52 years, his wisdom has helped guide the department in its rise to excellence.

Art served as Editor of the *Annals of Statistics* for three years, Co-editor of the *Journal of Multivariate Analysis* for eleven years, as Associate Editor of the *Journal of the American Statistical Association* and the *Journal of Statistical Planning and Inference*, each for five years. Art has over 140 publications. In an influential series of fifty-two *Annals of Statistics* and *JASA* papers, Art and co-authors developed wide ranging and fundamental results in decision theory, admissibility, Bayes' procedures, sequential tests, complete class theorems, directional tests, order restricted inference and multiple testing. Art is a Fellow of the Institute of Mathematical Statistics, the American Statistical Association and the International Statistical Institute.

Key words and phrases: Admissibility, *Annals of Statistics* Editor, change points, Columbia Statistics, Communicable Disease Center, Epidemic Intelligence Service, ordered restricted inference, Public Health Service, Rutgers Statistics, step-up and down procedures, testimators, variable selection.

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A Conversation with Estate V. Khmaladze

Hira L. Koul and Roger Koenker

Abstract. Estate V. Khmaladze was born in Tbilisi, Georgia, on October 20, 1944. He earned his B.Sc. degree from the Javakhishvili Tbilisi State University in 1964, majoring in physics. and his Ph.D. in mathematics in 1971 and Doctor of Physical and Mathematical Sciences in 1988, both from the Moscow State University. From 1972 to 1990, he held appointments at the Razmadze Mathematical Institute in Tbilisi and interim appointments at the V. A. Steklov Mathematical Institute in Moscow. From 1990 to 1999, he was head of the Department of Probability and Mathematical Statistics of the Razmadze Institute. From 1996 to 2001, he was on the faculty of the Department of Statistics of the University of New South Wales. Since 2002, he holds the Chair in Statistics in the School of Mathematics and Statistics of Victoria University of Wellington, New Zealand. He is a Fellow of the Royal Society of New Zealand and of the Institute of Mathematical Statistics. In 2013, he was awarded the Javakhishvili Medal from Tbilisi I. Javakhishvili State University and was elected to be a Foreign Member of the Georgian Academy of Sciences in 2016. As the conversation reveals, Khmaladze's research ranges widely over statistical topics and beyond.

The conversation began in the old building of I. Javakhishvili Tbilisi State University during a conference on probability theory and mathematical statistics, September 6–12, 2015, and continued in the Research Center of Ilia University, Stephantsminda, during the subsequent workshop, 12–16, September, Georgia. Mount Kazbegi, 5047 m, with its white summit was occasionally visible not too far away. In what follows, the questions are put in italics while the Estate's answers appear in the standard font.

Key words and phrases: Khmaladze transform, asymptotically distribution-free GOF tests.

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