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Introduction to the Special Section on Missing Data

Julie Josse and Jerome P. Reiter

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Multiple Imputation: A Review of Practical and Theoretical Findings

Jared S. Murray

Abstract. Multiple imputation is a straightforward method for handling missing data in a principled fashion. This paper presents an overview of multiple imputation, including important theoretical results and their practical implications for generating and using multiple imputations. A review of strategies for generating imputations follows, including recent developments in flexible joint modeling and sequential regression/chained equations/fully conditional specification approaches. Finally, we compare and contrast different methods for generating imputations on a range of criteria before identifying promising avenues for future research.

Key words and phrases: Missing data, proper imputation, congeniality, chained equations, fully conditional specification, sequential regression multivariate imputation.

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Multiple Imputation for Multilevel Data with Continuous and Binary Variables

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Abstract. We present and compare multiple imputation methods for multilevel continuous and binary data where variables are systematically and sporadically missing. The methods are compared from a theoretical point of view and through an extensive simulation study motivated by a real dataset comprising multiple studies. The comparisons show that these multiple imputation methods are the most appropriate to handle missing values in a multilevel setting and why their relative performances can vary according to the missing data pattern, the multilevel structure and the type of missing variables. This study shows that valid inferences can only be obtained if the dataset includes a large number of clusters. In addition, it highlights that heteroscedastic multiple imputation methods provide more accurate inferences than homoscedastic methods, which should be reserved for data with few individuals per cluster. Finally, guidelines are given to choose the most suitable multiple imputation method according to the structure of the data.

Key words and phrases: Missing data, systematically missing values, multilevel data, mixed data, multiple imputation, joint modelling, fully conditional specification.

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Introduction to Double Robust Methods for Incomplete Data

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Abstract. Most methods for handling incomplete data can be broadly classified as inverse probability weighting (IPW) strategies or imputation strategies. The former model the occurrence of incomplete data; the latter, the distribution of the missing variables given observed variables in each missingness pattern. Imputation strategies are typically more efficient, but they can involve extrapolation, which is difficult to diagnose and can lead to large bias. Double robust (DR) methods combine the two approaches. They are typically more efficient than IPW and more robust to model misspecification than imputation. We give a formal introduction to DR estimation of the mean of a partially observed variable, before moving to more general incomplete-data scenarios. We review strategies to improve the performance of DR estimators under model misspecification, reveal connections between DR estimators for incomplete data and "design-consistent" estimators used in sample surveys, and explain the value of double robustness when using flexible data-adaptive methods for IPW or imputation.

Key words and phrases: Augmented inverse probability weighting, calibration estimators, data-adaptive methods, doubly robust, empirical likelihood, imputation, inverse probability weighting, missing data, semiparametric methods.

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Bayesian Approaches for Missing Not at Random Outcome Data: The Role of Identifying Restrictions

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Abstract. Missing data is almost always present in real datasets, and introduces several statistical issues. One fundamental issue is that, in the absence of strong uncheckable assumptions, effects of interest are typically not nonparametrically identified. In this article, we review the generic approach of the use of identifying restrictions from a likelihood-based perspective, and provide points of contact for several recently proposed methods. An emphasis of this review is on restrictions for nonmonotone missingness, a subject that has been treated sparingly in the literature. We also present a general, fully Bayesian, approach which is widely applicable and capable of handling a variety of identifying restrictions in a uniform manner.

Key words and phrases: Missing data, MNAR, mixture models, multiple imputation, nonignorable missingness, nonparametric Bayes.

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Causal Inference: A Missing Data Perspective

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Abstract. Inferring causal effects of treatments is a central goal in many disciplines. The potential outcomes framework is a main statistical approach to causal inference, in which a causal effect is defined as a comparison of the potential outcomes of the same units under different treatment conditions. Because for each unit at most one of the potential outcomes is observed and the rest are missing, causal inference is inherently a missing data problem. Indeed, there is a close analogy in the terminology and the inferential framework between causal inference and missing data. Despite the intrinsic connection between the two subjects, statistical analyses of causal inference and missing data also have marked differences in aims, settings and methods. This article provides a systematic review of causal inference from the missing data perspective. Focusing on ignorable treatment assignment mechanisms, we discuss a wide range of causal inference methods that have analogues in missing data analysis, such as imputation, inverse probability weighting and doubly robust methods. Under each of the three modes of inference—Frequentist, Bayesian and Fisherian randomization—we present the general structure of inference for both finite-sample and super-population estimands, and illustrate via specific examples. We identify open questions to motivate more research to bridge the two fields.

Key words and phrases: Assignment mechanism, ignorability, imputation, missing data mechanism, observational studies, potential outcome, propensity score, randomization, weighting.

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Flexible Low-Rank Statistical Modeling with Missing Data and Side Information

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Abstract. We explore a general statistical framework for low-rank modeling of matrix-valued data, based on convex optimization with a generalized nuclear norm penalty. We study several related problems: the usual low-rank matrix completion problem with flexible loss functions arising from generalized linear models; reduced-rank regression and multi-task learning; and generalizations of both problems where side information about rows and columns is available, in the form of features or smoothing kernels. We show that our approach encompasses maximum a posteriori estimation arising from Bayesian hierarchical modeling with latent factors, and discuss ramifications of the missing-data mechanism in the context of matrix completion. While the above problems can be naturally posed as rank-constrained optimization problems, which are nonconvex and computationally difficult, we show how to relax them via generalized nuclear norm regularization to obtain convex optimization problems. We discuss algorithms drawing inspiration from modern convex optimization methods to address these large scale convex optimization computational tasks. Finally, we illustrate our flexible approach in problems arising in functional data reconstruction and ecological species distribution modeling.

Key words and phrases: Matrix completion, nuclear norm regularization, matrix factorization, convex optimization, missing data.

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Missing Information Principle: A Unified Approach for General Truncated and Censored Survival Data Problems

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Abstract. It is well known that truncated survival data are subject to sampling bias, where the sampling weight depends on the underlying truncation time distribution. Recently, there has been a rising interest in developing methods to better exploit the information about the truncation time, thus the sampling weight function, to obtain more efficient estimation. In this paper, we propose to treat truncation and censoring as "missing data mechanism" and apply the missing information principle to develop a unified framework for analyzing left-truncated and right-censored data with unspecified or known truncation time distributions. Our framework is structured in a way that is easy to understand and enjoys a great flexibility for handling different types of models. Moreover, a new test for checking the independence between the underlying truncation time and survival time is derived along the same line. The proposed hypothesis testing procedure utilizes all observed data and hence can yield a much higher power than the conditional Kendall's tau test that only involves comparable pairs of observations under truncation. Simulation studies with practical sample sizes are conducted to compare the performance of the proposed method with its competitors. The proposed methodologies are applied to a dementia study and a nursing house study for illustration.

Key words and phrases: Kendall's tau, inverse probability weighted estimator, outcome-dependent sampling, prevalent sampling, self-consistency algorithm.

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Marie-France Bru and Bernard Bru on Dice Games and Contracts

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Abstract. This note introduces Marie-France and Bernard Bru's forthcoming book on the history of probability, especially its chapter on dice games, translated in this issue of *Statistical Science*, and its commentary on the history of fair price in the settlement of contracts.

As the Brus remind us, the traditions of counting chances in dice games and estimating fair price came together in the correspondence between Pascal and Fermat in 1654. To solve the problem of dividing the stakes in a prematurely halted game, Fermat used combinatorial principles that had been used for centuries to analyze dice games, while Pascal used principles that had been proposed in previous centuries by students of commercial arithmetic.

Key words and phrases: Dice games, emergence of probability, *De vetula*, expectation.

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Dice Games

Marie-France Bru and Bernard Bru

Abstract. Translated from the French by Glenn Shafer, the French text will appear as Chapter 1 of Volume 2 of *Les jeux de l'infini et du hasard*, by Marie-France and Bernard Bru, to be published by the Presses universitaires de Franche-Comté. The translation is published here with the permission of the publisher and the surviving author. The text has been edited to omit most references to other parts of the book. The authors extensive notes, which provide many additional references and historical details, have also been omitted.

Key words and phrases: *De vetula*, dice games, history of probability, Huygens, Jacob Bernoulli, Laplace, Montmort, normal approximation.

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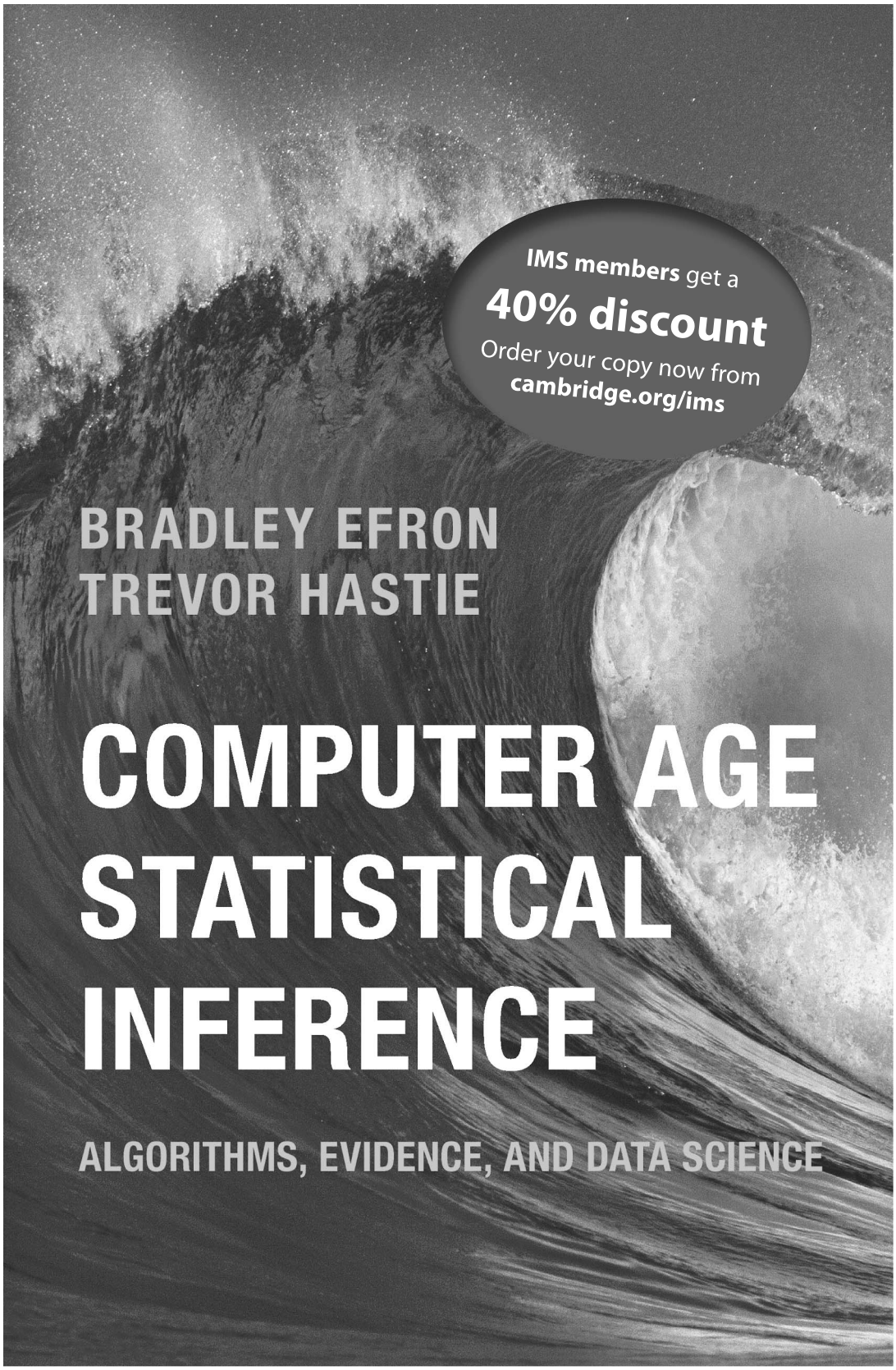
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