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LIMIT THEOREMS FOR MARKOV WALKS CONDITIONED TO STAY POSITIVE UNDER A SPECTRAL GAP ASSUMPTION

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Abstract. Consider a Markov chain $(X_n)_{n\geqslant 0}$ with values in the state space $\mathbb{X}$. Let f be a real function on $\mathbb{X}$ and set $S_n = \sum_{i=1}^n f(X_i)$, $n \geqslant 1$. Let $\mathbb{P}_x$ be the probability measure generated by the Markov chain starting at $X_0 = x$. For a starting point $y \in \mathbb{R}$ denote by τ_y the first moment when the Markov walk $(y + S_n)_{n\geqslant 1}$ becomes non-positive. Under the condition that S_n has zero drift, we find the asymptotics of the probability $\mathbb{P}_x(\tau_y > n)$ and of the conditional law $\mathbb{P}_x\left(y + S_n \leqslant \cdot\sqrt{n} \,\middle|\, \tau_y > n\right)$ as $n \to +\infty$.

1. Introduction. Assume that on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$ we are given a sequence of random variables $(X_n)_{n\geqslant 1}$ with values in a measurable space $\mathbb{X}$. Let f be a real function on $\mathbb{X}$. Suppose that the random walk $S_n = \sum_{i=1}^n f(X_i)$, $n \geqslant 1$ has zero drift. For a starting point $y \in \mathbb{R}$ denote by τ_y the time at which $(y + S_n)_{n\geqslant 1}$ first passes into the interval $(-\infty, 0]$. We are interested in the asymptotic behaviour of the probability $\mathbb{P}(\tau_y > n)$ and of the conditional law of $\frac{y+S_n}{\sqrt{n}}$ given the event $\{\tau_y > n\} = \{S_1 > 0, \ldots, S_n > 0\}$ as $n \to +\infty$.

The case when f is the identity function and $(X_n)_{n\geqslant 1}$ are i.i.d. in $\mathbb{X} = \mathbb{R}$ has been extensively studied in the literature. We refer to Spitzer [31], Iglehart [23, 24], Bolthausen [2], Doney [12], Bertoin and Doney [1], Borovkov [3, 4], Caravenna [6], Vatutin and Wachtel [35] to cite only a few. Recent progress has been made for random walks with independent increments in $\mathbb{X} = \mathbb{R}^d$, see Eichelbacher and König [14], Denisov and Wachtel [11, 9] and Duraj [13]. However, to the best of our knowledge, the case of the Markov chains has been treated only in some special cases. Upper and lower bounds for $\mathbb{P}(\tau_y > n)$ have been obtained in Varapoulos [32], [33] for Markov chains with bounded jumps and in Dembo, Ding and Gao [7] for integrated random walks based on independent increments. An approximation of $\mathbb{P}(\tau_y > n)$ by the survival probability of the Brownian motion for Markov walk under moment conditions is given in Varopoulos [34]. Exact asymptotic behaviour was determined in Presman [29, 30] in the case of sums of random variables defined on a finite Markov chain under the additional assumption that the distributions have an absolute continuous component and in Denisov and Wachtel [10] for integrated random walks. The case of products of i.i.d. random matrices which reduces to the study of a particular Markov chain defined on a merely compact state space was considered in [20] and the case of affine walks in $\mathbb{R}$ has been treated in [18]. We also point out the work of Denisov, Korshunov and Wachtel [8] where a constructive analysis of harmonic functions for Markov chains with values in $\mathbb{N}$ is performed.

In this paper we determine the limit of the probability of the exit time τ_y and of the law of $y + S_n$ conditioned to stay positive for a Markov chain under the assumption that its

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transition operator has a spectral gap. In particular our results cover the case of Markov chains with compact state spaces and the affine random walks in $\mathbb{R}$ (see [18]) and $\mathbb{R}^d$ (see Gao, Guivarc'h and Le Page [16]). Our results apply also to the case of sums of i.i.d. random variables.

To present briefly the main results of the paper denote by $\mathbb{P}_x$ and $\mathbb{E}_x$ the probability and the corresponding expectation generated by the trajectories of a Markov chain $(X_n)_{n\geqslant 1}$ with the initial state $X_0 = x \in \mathbb{X}$. Let $\mathbf{Q}$ be the transition operator of the Markov chain $(X_n, y+S_n)_{n\geqslant 1}$ and let $\mathbf{Q}_+$ be the restriction of $\mathbf{Q}$ on $\mathbb{X} \times \mathbb{R}_+^*$. We show that under appropriate assumptions, there exists a $\mathbf{Q}_+$-harmonic function V with non-empty support $\operatorname{supp}(V)$ in $\mathbb{X} \times \mathbb{R}$ such that, for any $(x,y) \in \operatorname{supp}(V)$,

$$\mathbb{P}_x\left(\tau_y > n\right) \underset{n\to+\infty}{\sim} \frac{2V(x,y)}{\sqrt{2\pi n}\sigma} \tag{1.1}$$

and

$$\mathbb{P}_x\left(\frac{y+S_n}{\sigma\sqrt{n}} \leqslant t \,\middle|\, \tau_y > n \right) \underset{n\to+\infty}{\longrightarrow} \mathbf{\Phi}^+(t),$$

where $\mathbf{\Phi}^+(t) = 1 - \mathrm{e}^{-\frac{t^2}{2}}$ is the Rayleigh distribution function and σ is a positive real. Moreover, we complete this result by giving the behaviour of $\mathbb{P}_x\left(\tau_y > n\right)$ on the complement of $\operatorname{supp}(V)$: for any $(x,y) \notin \operatorname{supp}(V)$,

$$\mathbb{P}_x\left(\tau_y > n\right) \leqslant c_x\, \mathrm{e}^{-cn}, \tag{1.2}$$

where c_x depends on x and c is a constant. In the case of sums of i.i.d. real random variables, instead of (1.2), on $\operatorname{supp}(V)^c$ it holds $\mathbb{P}_x\left(\tau_y > n\right) = 0$. We give an example of a Markov chain for which the bound (1.2) is attained and also uniform versions of (1.1) and (1.2). A characterization of the $\operatorname{supp}(V)$ is given in point 4 of Theorem 2.2. For details we refer to Section 2.

To study the asymptotic behaviour of the probability $\mathbb{P}(\tau_y > n)$ for walks on the real line $\mathbb{R}$ one usually uses the Wiener-Hopf factorization (see Feller [15]). Unfortunately the Wiener-Hopf factorisation is not well suited for more general walks, as for example those with values in $\mathbb{R}^d$ or for walks with dependent increments. For random walks with dependent increments and for random walks with independent increments in $\mathbb{R}^d$, Varopoulos [34], Eichelbacher and König [14] and Denisov and Wachtel [11] have developed an alternative approach based on the existence of the harmonic function. Using the particular structure of the underlaying models such extensions where performed in Denisov and Wachtel [10] for integrated random walks, in [20] for products of random matrices and in [18] for affine random walks in $\mathbb{R}$. Despite these advances, there are still some major difficulties in transferring the harmonic function approach to the case of more general Markov chains. In this paper we extend it to Markov chains under spectral gap assumptions. Let us highlight below the key points of the proofs.

We start with the construction of a martingale approximation $(M_n)_{n\geqslant 1}$ for $(S_n)_{n\geqslant 1}$ following the approach of Gordin [17]. The control of the difference between S_n and M_n is one of the delicate points of the proof and depends on the properties of the Banach space $\mathscr{B}$ related to the spectral gap properties of the transition operator $\mathbf{P}$ of the Markov chain $(X_n)_{n\geqslant 1}$ (for details we refer to Section 2). Our martingale approximation is such that

$$(z+M_n) - (y+S_n) = r\left(X_n\right),$$

where $r(x) = \Theta(x) - f(x)$, $z = y + r(x)$ and Θ is the solution of the Poisson equation $\Theta - \mathbf{P}\Theta = f$. Under Hypothesis **M4** we can control $|r(x)|$ by $c(1 + N(x))$ where $N \in \mathscr{B}$ has bounded moments $\mathbb{E}_x^{1/\alpha}(N(X_n)^\alpha) \leqslant c(1 + N(x))$, for some $\alpha > 2$. Note that in the case of products of random matrices [20], $\sup_{n\geqslant 1} |S_n - M_n|$ is bounded by a constant $\mathbb{P}_x$-a.s. for any $x \in \mathbb{X}$, which simplifies greatly the proofs. The extension to the case when $\sup_{n\geqslant 1} |S_n - M_n|$ is not bounded turns out to be quite laborious even for particular examples. We refer to the case of affine Markov walks in [18], where we have benefited from the special structure of the model.

The next point is to prove the existence of a positive harmonic function. The starting idea is very simple. Let $V_n(x, y) := \mathbb{E}_x((y + S_n)\mathbb{1}_{\{\tau_y > n\}})$ be the expectation of the Markov walk $(y + S_n)_{n\geqslant 1}$ killed at τ_y. Since by the Markov property, $V_{n+1}(x, y) = \mathbf{Q}_+ V_n(x, y)$, taking the limit as $n \to +\infty$ under appropriate assumptions, yields that the function $V(x, y) = \lim_{n\to+\infty} V_n(x, y)$ is harmonic. Using the approximating martingale, the function V can be identified as $V(x, y) = -\mathbb{E}_x\left(M_{\tau_y}\right)$. To justify this approach, it is important to control uniformly in n the expectation $w_n := \mathbb{E}_x((z + M_n)\mathbb{1}_{\{\tau_y > n\}})$. Our key idea (in contrast to [20] and [18]) is the introduction of two extra stopping times T_z and $\hat{T}_z$: the first time when $(z + M_n)_{n\geqslant 1}$ leaves $\mathbb{R}_+^*$ and the first time larger that τ_y when $(z + M_n)_{n\geqslant 1}$ leaves $\mathbb{R}_+^*$, respectively, where as before $z = y + r(x)$. Clearly, $\hat{T}_z$ depends on τ_y and dominates both, τ_y and T_z. The relation of the time $\hat{T}_z$ to the exit times τ_y and T_z is explicitly given in Lemma 5.3 which is an application of the Markov property to $\hat{T}_z$. This property is useful to control uniformly in n the expectation $u_n := \mathbb{E}_x((z + M_n)\mathbb{1}_{\{\hat{T}_z > n\}})$, which is one of the crucial points of the proof. To establish this we note that the sequence $(u_n)_{n\geqslant 0}$ is increasing, since $((z + M_n)\mathbb{1}_{\{\hat{T}_z > n\}})_{n\geqslant 1}$ is a submartingale. In addition we show that it satisfies a recurrence equation, which implies its boundedness. Using the previous arguments we obtain a uniform control on the expectation w_n. All the details can be found in Sections 6 and 7. The proof of the (strict) positivity of V is also rather involved but uses similar arguments based on the subhamonicity of the function $\hat{W}(x, z) = -\mathbb{E}_x(M_{\hat{T}_z})$. (see Section 8).

Now we can turn to the tail behaviour of the exit time τ_y. It is inferred from that of the exit time τ_y^{bm} of the Brownian motion, using the Donsker invariance principle for sums defined on Markov chains with a the rate of convergence, recently proved in [19]. The result in [19] gives the explicit dependence of the constants on the norm $\|\boldsymbol{\delta}_x\|_{\mathscr{B}'}$ of the Dirac measure $\boldsymbol{\delta}_x$ and on the absolute moments $\mu_\alpha(x) = \sup_{n\geqslant 1} \mathbb{E}_x^{1/\alpha}(|f(X_n)|^\alpha)$ for some initial state $x \in \mathbb{X}$ and some $\alpha > 2$. To have a control on the constants we make use of Hypothesis **M4**. Note that for products of random matrices [20], $\|\boldsymbol{\delta}_x\|_{\mathscr{B}'}$ and $\mu_\alpha(x)$ are bounded uniformly in the initial state $x \in \mathbb{X}$, so that the rate of convergence invariance principle does not depend on the initial state. The case of when $\|\boldsymbol{\delta}_x\|_{\mathscr{B}'}$ and $\mu_\alpha(x)$ are not bounded was considered in [18] for affine Markov walks.

The paper is organized as follows. In Section 2 we introduce the necessary notations and state our main results. In Section 3 we give applications of the results of the paper to stochastic recursions in $\mathbb{R}^d$ and Markov chains with compact state space. In Section 4 we collect some preliminary results. In Section 5 we construct the appropriating martingale and state some of its properties and of the associated exit times. In Section 6 we prove that the expectations $\mathbb{E}_x((y + S_n)\mathbb{1}_{\{\tau_y > n\}})$ are bounded uniformly in n. Using the results of Sections 5 and 6, we establish in Section 7 the existence of a $\mathbf{Q}_+$-harmonic function and prove in Section 8 that this function is not identically zero. We determine the limit of the probability $\mathbb{P}_x(\tau_y > n)$ in Section 9 and that of the conditioned law of $(y+S_n)/(\sigma\sqrt{n})$ given the event $\{\tau_y > n\}$ in Section 10.

We end this section by agreeing on some basic notations. For the rest of the paper the symbol c denotes a positive constant depending on the all previously introduced constants. Sometimes, to stress the dependence of the constants on some parameters $\alpha, \beta, \dots$ we shall use the notations $c_\alpha, c_{\alpha,\beta}, \dots$. All these constants are likely to change their values every occurrence. For any real numbers u and v, denote by $u \wedge v = \min(u,v)$ the minimum between u and v. The indicator of an event A is denoted by $\mathbb{1}_A$. For any bounded measurable function f on $\mathbb{X}$, random variable X in $\mathbb{X}$ and event A, the integral $\int_{\mathbb{X}} f(x)\mathbb{P}(X \in \mathrm{d}x, A)$ means the expectation $\mathbb{E}\left(f(X);A\right) = \mathbb{E}\left(f(X)\mathbb{1}_A\right)$.

2. Main results. Let $(X_n)_{n\geqslant 0}$ be a Markov chain taking values in the measurable state space $(\mathbb{X}, \mathscr{X})$, defined on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$. For any given $x \in \mathbb{X}$, denote by $\mathbf{P}(x,\cdot)$ its transition probability, to which we associate the transition operator

$$\mathbf{P}g(x) = \int_{\mathbb{X}} g(x')\mathbf{P}(x, \mathrm{d}x'),$$

for any complex bounded measurable function g on $\mathbb{X}$. Denote by $\mathbb{P}_x$ and $\mathbb{E}_x$ the probability and the corresponding expectation generated by the finite dimensional distributions of the Markov chain $(X_n)_{n\geqslant 0}$ starting at $X_0 = x$. We remark that $\mathbf{P}g\left(x\right) = \mathbb{E}_x\left(g\left(X_1\right)\right)$ and $\mathbf{P}^n g\left(x\right) = \mathbb{E}_x\left(g\left(X_n\right)\right)$ for any g complex bounded measurable, $x \in \mathbb{X}$ and $n \geqslant 1$.

Let f be a real valued function defined on the state space $\mathbb{X}$ and let $\mathscr{B}$ be a Banach space of complex valued functions on $\mathbb{X}$ endowed with the norm $\left\|\cdot\right\|_{\mathscr{B}}$. Let $\left\|\cdot\right\|_{\mathscr{B}\to\mathscr{B}}$ be the operator norm on $\mathscr{B}$ and let $\mathscr{B}' = \mathscr{L}\left(\mathscr{B}, \mathbb{C}\right)$ be the topological dual of $\mathscr{B}$ endowed with the norm $\left\|\varphi\right\|_{\mathscr{B}'} = \sup_{h\in\mathscr{B}} \frac{|\varphi(h)|}{\|h\|_{\mathscr{B}}}$, for any $\varphi \in \mathscr{B}'$. Denote by e the unit function of $\mathbb{X}$: $e(x) = 1$, for any $x \in \mathbb{X}$ and by $\boldsymbol{\delta}_x$ the Dirac measure at $x \in \mathbb{X}$: $\boldsymbol{\delta}_x(g) = g(x)$, for any $g \in \mathscr{B}$.

Following [19], we assume the following hypotheses.

Hypothesis M1 (Banach space).

1. *The unit function e belongs to $\mathscr{B}$.*
2. *For any $x \in \mathbb{X}$, the Dirac measure $\boldsymbol{\delta}_x$ belongs to $\mathscr{B}'$.*
3. *The Banach space $\mathscr{B}$ is included in $L^1\left(\mathbf{P}(x,\cdot)\right)$, for any $x \in \mathbb{X}$.*
4. *There exists a constant $\kappa \in (0,1)$ such that for any $g \in \mathscr{B}$, the function $\mathrm{e}^{itf}\, g$ is in $\mathscr{B}$ for any t satisfying $|t| \leqslant \kappa$.*

Under the point 3 of **M1**, $\mathbf{P}g(x)$ exists for any $g \in \mathscr{B}$ and $x \in \mathbb{X}$.

Hypothesis M2 (Spectral gap).

1. *The map $g \mapsto \mathbf{P}g$ is a bounded operator on $\mathscr{B}$.*
2. *There exist constants $c_1 > 0$ and $c_2 > 0$ such that*

$$\mathbf{P} = \Pi + Q,$$

where Π *is a one-dimensional projector and* Q *is an operator on* $\mathscr{B}$ *satisfying* $\Pi Q = Q\Pi = 0$ *and for any* $n \geqslant 1$,

$$\|Q^n\|_{\mathscr{B}\to\mathscr{B}} \leqslant c_1\, \mathrm{e}^{-c_2 n}.$$

Since Π is a one-dimensional projector and e is an eigenvector of $\mathbf{P}$, there exists a linear form $\boldsymbol{\nu} \in \mathscr{B}'$, such that for any $g \in \mathscr{B}$,

$$\Pi g = \boldsymbol{\nu}(g) e. \tag{2.1}$$

When Hypotheses **M1** and **M2** hold, we set $\mathbf{P}_t g := \mathbf{P}\left(\mathrm{e}^{itf} g\right)$ for any $g \in \mathscr{B}$ and $t \in [-\kappa, \kappa]$. In particular $\mathbf{P}_0 = \mathbf{P}$.

Hypothesis M3 (Perturbed transition operator).

1. *For any* $|t| \leqslant \kappa$ *the map* $g \mapsto \mathbf{P}_t g$ *is a bounded operator on* $\mathscr{B}$.
2. *There exists a constant* $C_{\mathbf{P}} > 0$ *such that, for any* $n \geqslant 1$ *and* $|t| \leqslant \kappa$,

$$\|\mathbf{P}_t^n\|_{\mathscr{B}\to\mathscr{B}} \leqslant C_{\mathbf{P}}.$$

The following hypothesis will be important for establishing the main results.

Hypothesis M4 (Local integrability). *The Banach space* $\mathscr{B}$ *contains a sequence of real non-negative functions* $N, N_1, N_2, \dots$ *such that:*

1. *There exist* $\alpha > 2$ *and* $\gamma > 0$ *such that, for any* $x \in \mathbb{X}$,

$$\max\left\{|f(x)|^{1+\gamma}, \|\boldsymbol{\delta}_x\|_{\mathscr{B}'}, \mathbb{E}_x^{1/\alpha}\left(N\left(X_n\right)^{\alpha}\right)\right\} \leqslant c\left(1 + N(x)\right)$$

and

$$N(x)\mathbb{1}_{\{N(x)>l\}} \leqslant N_l(x), \quad \text{for any} \quad l \geqslant 1.$$

2. *There exists* $c > 0$ *such that, for any* $l \geqslant 1$,

$$\|N_l\|_{\mathscr{B}} \leqslant c.$$

3. *There exist* $\beta > 0$ *and* $c > 0$ *such that, for any* $l \geqslant 1$,

$$|\boldsymbol{\nu}\left(N_l\right)| \leqslant \frac{c}{l^{1+\beta}}.$$

A comment on Hypothesis **M4** seems to be appropriate. Although the function N belongs to the Banach space $\mathscr{B}$, the truncated function $x \mapsto N(x)\mathbb{1}_{\{N(x)>l\}}$ may not belong to $\mathscr{B}$. Fortunately, in many interesting cases, there exists an element N_l in $\mathscr{B}$ dominating it. We refer to Section 3, where we verify Hypothesis **M4** for stochastic recursions in $\mathbb{R}^d$ and for Markov chains with compact state space. Note also that the function f need not belong to the Banach space $\mathscr{B}$.

Under Hypotheses **M1**, **M2** and **M4**, we have, for any $x \in \mathbb{X}$ and $n \geqslant 1$,

$$\begin{aligned}
\mathbb{E}_x\left(N(X_n)\right) &= \boldsymbol{\nu}(N) + Q^n N(x) \\
&\leqslant |\boldsymbol{\nu}(N)| + \|Q^n\|_{\mathscr{B}\to\mathscr{B}} \|N\|_{\mathscr{B}} \|\boldsymbol{\delta}_x\|_{\mathscr{B}'} \\
&\leqslant c(1 + \mathrm{e}^{-cn} N(x))
\end{aligned} \tag{2.2}$$

and, in the same way, for any $x \in \mathbb{X}$, $l \geqslant 1$ and $n \geqslant 1$,

$$\mathbb{E}_x \left(N_l \left(X_n \right) \right) \leqslant \frac{c}{l^{1+\beta}} + c\,\mathrm{e}^{-cn} \left(1 + N(x) \right). \tag{2.3}$$

Moreover, from the point 1 of **M4**, one can easily verify that, for any $x \in \mathbb{X}$,

$$\mu_\alpha(x) := \sup_{n \geqslant 1} \mathbb{E}_x^{1/\alpha} \left(\left| f \left(X_n \right) \right|^\alpha \right) \leqslant c \left(1 + N(x)^{\frac{1}{1+\gamma}} \right). \tag{2.4}$$

The following proposition is proved in [19], where the bounds on the right follow from (2.4) and again **M4**.

Proposition 2.1. *Assume that the Markov chain $(X_n)_{n\geqslant 0}$ and the function f satisfy Hypotheses **M1-M4**.*

1. *There exists a constant μ such that, for any $x \in \mathbb{X}$ and $n \geqslant 1$,*

$$\left| \mathbb{E}_x \left(f(X_n) \right) - \mu \right| \leqslant c\,\mathrm{e}^{-cn} \left(1 + \mu_\alpha(x)^{1+\gamma} + \left\| \boldsymbol{\delta}_x \right\|_{\mathscr{B}'} \right) \leqslant c\,\mathrm{e}^{-cn} \left(1 + N(x) \right).$$

2. *There exists a constant $\sigma \geqslant 0$ such that, for any $x \in \mathbb{X}$ and $n \geqslant 1$,*

$$\sup_{m \geqslant 0} \left| \mathrm{Var}_x \left(\sum_{k=m+1}^{m+n} f(X_k) \right) - n\sigma^2 \right| \leqslant c \left(1 + \mu_\alpha(x)^{2+2\gamma} + \left\| \boldsymbol{\delta}_x \right\|_{\mathscr{B}'} \right) \leqslant c \left(1 + N(x)^2 \right),$$

where Var_x is the variance under $\mathbb{P}_x$.

We do not assume the existence of the stationary probability measure. If a stationary probability measure $\boldsymbol{\nu}'$ satisfying $\boldsymbol{\nu}' \left(N^2 \right) < +\infty$ exists then, under Hypotheses **M1-M4**, we have that $\boldsymbol{\nu}' = \boldsymbol{\nu}$ is necessarily unique and it holds (see [19])

$$\boldsymbol{\nu}(f) = \mu \quad \text{and} \quad \sigma^2 = \int_{\mathbb{X}} f^2(x) \boldsymbol{\nu}(\mathrm{d}x) + 2 \sum_{n=1}^{+\infty} \int_{\mathbb{X}} f(x) \mathbf{P}^n f(x) \boldsymbol{\nu}(\mathrm{d}x). \tag{2.5}$$

Hypothesis M5 (Centring and non-degeneracy). *We suppose that the constants μ and σ defined in Proposition 2.1 satisfy $\mu = 0$ and $\sigma > 0$.*

Under **M5** it follows from Proposition 2.1 that, for any $x \in \mathbb{X}$ and $n \geqslant 1$,

$$\left| \mathbb{E}_x \left(f(X_n) \right) \right| \leqslant c\,\mathrm{e}^{-cn} \left(1 + N(x) \right). \tag{2.6}$$

Let $y \in \mathbb{R}$ be a starting point and $(y + S_n)_{n \geqslant 0}$ be the Markov walk defined by $S_n := \sum_{k=1}^n f\left(X_k \right)$, $n \geqslant 1$ with $S_0 = 0$. Denote by τ_y the first moment when $y + S_n$ becomes non-positive:

$$\tau_y := \inf \left\{ k \geqslant 1 : y + S_k \leqslant 0 \right\}.$$

It is shown in Lemma 5.5 that for any $y \in \mathbb{R}$ and $x \in \mathbb{X}$, the stopping time τ_y is $\mathbb{P}_x$-a.s. finite. The asymptotic behaviour of the probability $\mathbb{P}_x \left(\tau_y > n \right)$ is determined by the harmonic function which we proceed to introduce. For any $(x, y) \in \mathbb{X} \times \mathbb{R}$, denote by

$\mathbf{Q}(x,y,\cdot)$ the transition probability of the Markov chain $(X_n, y+S_n)_{n\geqslant 0}$. The restriction of the measure $\mathbf{Q}(x,y,\cdot)$ on $\mathbb{X}\times\mathbb{R}_+^*$ is defined by

$$\mathbf{Q}_+(x,y,B) = \mathbf{Q}(x,y,B)$$

for any measurable set B on $\mathbb{X}\times\mathbb{R}_+^*$ and for any $(x,y)\in\mathbb{X}\times\mathbb{R}$. For any bounded measurable function $\varphi : \mathbb{X}\times\mathbb{R}\to\mathbb{R}$ set $\mathbf{Q}_+\varphi(x,y) = \int_{\mathbb{X}\times\mathbb{R}_+^*}\varphi(x',y')\mathbf{Q}_+(x,y,\mathrm{d}x'\times\mathrm{d}y')$, where $(x,y)\in\mathbb{X}\times\mathbb{R}$. A function $V:\mathbb{X}\times\mathbb{R}\to\mathbb{R}$ is said to be $\mathbf{Q}_+$-harmonic if

$$\mathbf{Q}_+V(x,y) = V(x,y), \qquad \text{for any } (x,y)\in\mathbb{X}\times\mathbb{R}.$$

We shall deal only with non-negative harmonic functions V. Denote by $\operatorname{supp}(V)$ the support of such a function V,

$$\operatorname{supp}(V) := \{(x,y)\in\mathbb{X}\times\mathbb{R} : V(x,y) > 0\}.$$

On complement of $\operatorname{supp}(V)$, the function V is 0. For any $\gamma>0$, consider the set

$$\mathscr{D}_\gamma := \left\{(x,y)\in\mathbb{X}\times\mathbb{R} : \exists n_0\geqslant 1,\ \mathbb{P}_x\left(y+S_{n_0} > \gamma\left(1+N\left(X_{n_0}\right)\right),\ \tau_y > n_0\right) > 0\right\}.$$

The following assertion proves the existence of a non-identically zero harmonic function.

THEOREM 2.2. *Assume Hypotheses **M1**-**M5**.*

1. *For any $x\in\mathbb{X}$, $y\in\mathbb{R}$, the sequence $\left(\mathbb{E}_x\left(y+S_n\,;\,\tau_y>n\right)\right)_{n\geqslant 0}$ converges to a real number $V(x,y)$:*
$$\mathbb{E}_x\left(y+S_n\,;\,\tau_y>n\right) \underset{n\to+\infty}{\longrightarrow} V(x,y).$$
2. *The function $V:\mathbb{X}\times\mathbb{R}\to\mathbb{R}$, defined in the previous point is $\mathbf{Q}_+$-harmonic, i.e. for any $x\in\mathbb{X}$, $y\in\mathbb{R}$,*
$$\mathbf{Q}_+V(x,y) = \mathbb{E}_x\left(V\left(X_1, y+S_1\right)\,;\,\tau_y>1\right) = V(x,y).$$
3. *For any $x\in\mathbb{X}$, the function $V(x,\cdot)$ is non-negative and non-decreasing on $\mathbb{R}$ and*
$$\lim_{y\to+\infty}\frac{V(x,y)}{y} = 1.$$
Moreover, for any $\delta>0$, $x\in\mathbb{X}$ and $y\in\mathbb{R}$,
$$(1-\delta)\max(y,0) - c_\delta\left(1+N(x)\right) \leqslant V(x,y) \leqslant (1+\delta)\max(y,0) + c_\delta\left(1+N(x)\right).$$
4. *There exists $\gamma_0>0$ such that, for any $\gamma\geqslant\gamma_0$,*
$$\operatorname{supp}(V) = \mathscr{D}_\gamma.$$

The following result gives the asymptotic of the exit probability for fixed $(x,y)\in\mathbb{X}\times\mathbb{R}$.

THEOREM 2.3. *Assume Hypotheses **M1**-**M5**.*

1. *For any* $(x,y) \in \operatorname{supp}(V)$,

$$\mathbb{P}_x\left(\tau_y > n\right) \underset{n\to+\infty}{\sim} \frac{2V(x,y)}{\sqrt{2\pi n}\sigma}.$$

2. *For any* $(x,y) \notin \operatorname{supp}(V)$ *and* $n \geqslant 1$,

$$\mathbb{P}_x\left(\tau_y > n\right) \leqslant c\,\mathrm{e}^{-cn}\left(1 + N(x)\right).$$

Now we complete the point 1 of the previous theorem by some estimations.

THEOREM 2.4. *Assume Hypotheses* ***M1-M5***.

1. *There exists* $\varepsilon_0 > 0$ *such that, for any* $\varepsilon \in (0,\varepsilon_0)$, $n \geqslant 1$ *and* $(x,y) \in \mathbb{X}\times\mathbb{R}$,

$$\left|\mathbb{P}_x\left(\tau_y > n\right) - \frac{2V(x,y)}{\sqrt{2\pi n}\sigma}\right| \leqslant c_\varepsilon \frac{\max(y,0) + \left(1 + y\mathbb{1}_{\{y>n^{1/2-\varepsilon}\}} + N(x)\right)^2}{n^{1/2+\varepsilon/16}}.$$

2. *Moreover, for any* $(x,y) \in \mathbb{X}\times\mathbb{R}$ *and* $n \geqslant 1$,

$$\mathbb{P}_x\left(\tau_y > n\right) \leqslant c\frac{1 + \max(y,0) + N(x)}{\sqrt{n}}.$$

Finally, we give the asymptotic of the conditional law of $y + S_n$.

THEOREM 2.5. *Assume Hypotheses* ***M1-M5***.

1. *For any* $(x,y) \in \operatorname{supp}(V)$ *and* $t \geqslant 0$,

$$\mathbb{P}_x\left(\frac{y+S_n}{\sigma\sqrt{n}} \leqslant t \,\middle|\, \tau_y > n\right) \underset{n\to+\infty}{\longrightarrow} \mathbf{\Phi}^+(t),$$

where $\mathbf{\Phi}^+(t) = 1 - \mathrm{e}^{-\frac{t^2}{2}}$ *is the Rayleigh distribution function.*

2. *Moreover there exists* $\varepsilon_0 > 0$ *such that, for any* $\varepsilon \in (0,\varepsilon_0)$, $n \geqslant 1$, $t_0 > 0$, $t \in [0,t_0]$ *and* $(x,y) \in \mathbb{X}\times\mathbb{R}$,

$$\left|\mathbb{P}_x\left(y + S_n \leqslant t\sqrt{n}\,,\, \tau_y > n\right) - \frac{2V(x,y)}{\sqrt{2\pi n}\sigma}\mathbf{\Phi}^+\left(\frac{t}{\sigma}\right)\right| \leqslant c_{\varepsilon,t_0}\frac{\max(y,0) + \left(1 + y\mathbb{1}_{\{y>n^{1/2-\varepsilon}\}} + N(x)\right)^2}{n^{1/2+\varepsilon/16}}.$$

We now comment on Theorems 2.2 and 2.3.

REMARK 2.6. The sets $(\mathscr{D}_\gamma)_{\gamma>0}$ are nested and become equal to $\operatorname{supp}(V)$ for large γ: we have $\mathscr{D}_{\gamma_1} \supseteq \mathscr{D}_{\gamma_2} \supseteq \mathscr{D}_\gamma = \operatorname{supp}(V)$, for $\gamma_1 \leqslant \gamma_2 \leqslant \gamma$, where γ is large enough (see Proposition 8.8).

Remark 2.7. The set $\operatorname{supp}(V)$ is not empty. More precisely there exists $\gamma_1 > 0$ such that

$$\{(x,y) \in \mathbb{X} \times \mathbb{R} : y > \gamma_1 (1 + N(x))\} \subseteq \operatorname{supp}(V),$$

see Proposition 8.8. Example 2.10 and Figure 1 illustrate this property.

Remark 2.8. When $(X_n)_{n\geqslant 1}$ are i.i.d., it is well known that $\mathbb{P}_x(\tau_y > n) = 0$ for any $(x,y) \notin \operatorname{supp}(V)$. When the sequence $(X_n)_{n\geqslant 1}$ is a Markov chain, instead of this, we have an exponential bound, see the point 2 of Theorem 2.3. We show that this bound is attained for some Markov walk. We refer for details to Example 2.11.

Example 2.9 (Random walks in $\mathbb{R}$). Suppose that $(X_n)_{n\geqslant 1}$ are i.i.d. real random variables of mean 0 and positive variance with finite absolute moments of order $p > 2$. In this case, one can take $N = N_l = 0$, $l \geqslant 0$. Therefore,

$$\mathscr{D}_\gamma := \{y \in \mathbb{R} : \exists n_0 \geqslant 1,\ \mathbb{P}(y + S_{n_0} > \gamma\,,\ \tau_y > n_0) > 0\}.$$

Since the walk $(y + S_n)_{n\geqslant 1}$ can increase at each step with positive probability, it follows that $\mathbb{P}(y + S_{n_0} > \gamma\,,\ \tau_y > n_0) > 0$ if and only if $\mathbb{P}(\tau_y > 1) = \mathbb{P}(y + X_1 > 0) > 0$. Thus, $[0, +\infty) \subseteq (-\max \operatorname{supp}(\boldsymbol{\mu}), +\infty) = \mathscr{D}_\gamma = \operatorname{supp}(V)$, for every $\gamma > 0$, where $\boldsymbol{\mu}$ is the common law of X_n and $\operatorname{supp}(\boldsymbol{\mu})$ is its support.

The following example is intended to illustrate Remark 2.7.

Example 2.10. Consider the following special case of the one dimensional stochastic recursion: $X_{n+1} = a_{n+1}X_n + b_{n+1}$ where $(a_i)_{i\geqslant 1}$ and $(b_i)_{i\geqslant 1}$ are two independent sequences of i.i.d. random variables. In this example we consider that the law of a_i is $\frac{1}{2}\boldsymbol{\delta}_{\{-1/2\}} + \frac{1}{2}\boldsymbol{\delta}_{\{1/2\}}$ and that of b_i is uniform on $[-1,1]$. The state space $\mathbb{X}$ is $\mathbb{R}$. The functions N and N_l are given by $N(x) = |x|^{1+\varepsilon}$ for some $\varepsilon > 0$, and $N_l(x) = N(x)\phi_l(|x|)$ with ϕ_l defined by (11.4). The Banach space satisfying **M1**-**M5** is constructed in Section 11 (see also [18]). One can verify that the domain of positivity of the function V is $\operatorname{supp}(V) = \{(x,y) \in \mathbb{R}^2 : y > -\frac{|x|}{2} - 1\} = \mathscr{D}_\gamma$, for all $\gamma > 0$. Obviously, $\{(x,y) \in \mathbb{X} \times \mathbb{R} : y > \frac{1}{2}\left(1 + |x|^{1+\varepsilon}\right)\} \subseteq \operatorname{supp}(V)$, see Figure 1.

The next example is intended to show that the inequality of the point 2 of Theorem 2.3 is attained.

Example 2.11. Consider the Markov walk $(X_n)_{n\geqslant 0}$ living on the finite state space $\mathbb{X} := \{-1\,;\,1\,;\,-3\,;\,7/6\}$ with the transition probabilities given in Figure 2. Suppose that f is the identity function on $\mathbb{X}$. It is easy to see that the assumptions stated in Remark 3.10 of Section 3.3 are satisfied and thereby so are Hypotheses **M1**-**M5**. In particular, **M4** holds with $N = N_l = 0$ for any $l \geqslant 1$. Now, when $x = 1$ and $y \in (1,3]$ or when $x = -1$ and $y \in (-1,2]$, one can check that the Markov walk $y + S_n$ stays positive if and only if the values of the variables X_i alternate between 1 and -1 and therefore, for such starting points (x,y), we have $\mathbb{P}_x(\tau_y > n) = \left(\frac{1}{2}\right)^n$. This shows that, when the random variables $(X_n)_{n\geqslant 1}$ form a Markov chain, the survival probability $\mathbb{P}_x(\tau_y > n)$ has an asymptotic

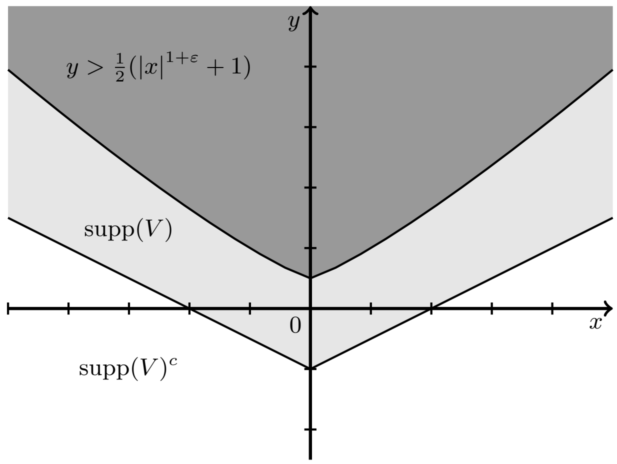

FIGURE 1.

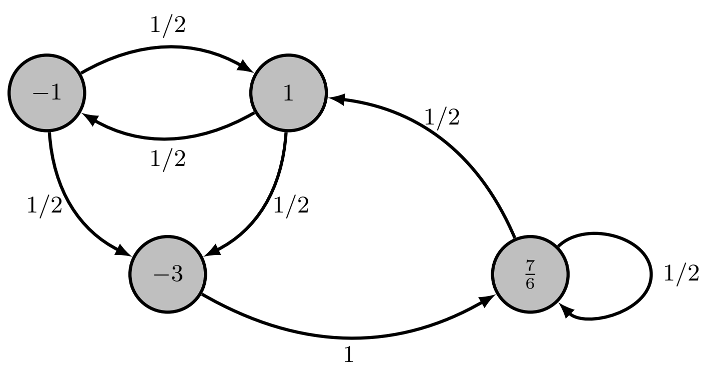

FIGURE 2.

behaviour different from that in the independent case where it can be either equivalent to $\frac{c_{x,y}}{\sqrt{n}}$ or 0.

In this example we can make explicit the support of the function V. Since $N = 0$, the function V is positive if and only if there exists an integer $n \geqslant 1$ such that $\mathbb{P}_x (y + S_n > \gamma\,,\, \tau_y > n) > 0$ for a γ large enough. This is possible only if the chain can reach the state $X_n = 7/6$ within a trajectory of $(y + S_k)_{n \geqslant k \geqslant 1}$ which stays positive, *i.e.* $\mathbb{P}_x (X_n = 7/6\,,\, \tau_y > n) > 0$. Consequently

$$\begin{aligned}\operatorname{supp}(V) &= \{-1\} \times (2, +\infty) \cup \{1\} \times (3, +\infty) \cup \{-3, 7/6\} \times (-7/6, +\infty) \\ &= \mathscr{D}_3 = \{(x, y) \in \mathbb{X} \times \mathbb{R} : \exists n \geqslant 1,\ \mathbb{P}_x (y + S_n > 3\,,\, \tau_y > n) > 0\}\,.\end{aligned}$$

To sum up, this model presents the three possible asymptotic behaviours of $\mathbb{P}_x (\tau_y > n)$: for any $(x, y) \in \operatorname{supp}(V) = \{-1\} \times (2, +\infty) \cup \{1\} \times (3, +\infty) \cup \{-3, 7/6\} \times (-7/6, +\infty)$,

$$\mathbb{P}_x (\tau_y > n) \underset{n \to +\infty}{\sim} \frac{2V(x, y)}{\sqrt{2\pi n}\sigma},$$

for any $(x, y) \in \{-1\} \times (-1, 2] \cup \{1\} \times (1, 3]$ and $n \geqslant 1$,

$$\mathbb{P}_x (\tau_y > n) = \left(\frac{1}{2}\right)^n,$$

for any $(x, y) \in \{-1\} \times (-\infty, -1] \cup \{1\} \times (-\infty, 1] \cup \{-3, 7/6\} \times (-\infty, -7/6]$ and $n \geqslant 1$,

$$\mathbb{P}_x (\tau_y > n) = 0.$$

3. Applications. We illustrate the results of Section 2 by considering three particular models.

3.1. *Affine random walk in $\mathbb{R}^d$ conditioned to stay in a half-space.* Let $d \geqslant 1$ be an integer and $(g_n)_{n\geqslant 1} = (A_n, B_n)_{n\geqslant 1}$ be a sequence of i.i.d. random elements in $\mathrm{GL}(d, \mathbb{R}) \times \mathbb{R}^d$ following the same distribution $\boldsymbol{\mu}$. Let $(X_n)_{n\geqslant 0}$ be the Markov chain on $\mathbb{R}^d$ defined by

$$X_0 = x \in \mathbb{R}^d, \qquad X_{n+1} = A_{n+1} X_n + B_{n+1}, \quad n \geqslant 1.$$

Set $S_n = \sum_{k=1}^n f(X_k)$, $n \geqslant 1$, where the function $f(x) = \langle u, x \rangle$ is the projection of the vector $x \in \mathbb{R}^d$ on the direction defined by the vector $u \in \mathbb{R}^d \setminus \{0\}$. For any $y \in \mathbb{R}$, consider the first time when the random walk $(y + S_n)_{n\geqslant 1}$ becomes non-positive:

$$\tau_y = \inf\{k \geqslant 1 : y + S_k \leqslant 0\}.$$

This stopping time coincides with the entry time of the affine walk $(\sum_{k=1}^n X_k)_{n\geqslant 0}$ in the closed half-subspace $\{s \in \mathbb{R}^d : \langle u, s \rangle \leqslant -y\}$.

Introduce the following hypothesis.

HYPOTHESIS 3.1.

1. *There exists a constant $\delta > 0$, such that*

$$\mathbb{E}\left(\|A_1\|^{2+2\delta}\right) < +\infty, \quad \mathbb{E}\left(|B_1|^{2+2\delta}\right) < +\infty$$

and

$$k(\delta) = \lim_{n\to+\infty} \mathbb{E}^{1/n}\left(\|A_n A_{n-1} \dots A_1\|^{2+2\delta}\right) < 1.$$

2. *There is no proper affine subspace of $\mathbb{R}^d$ which is invariant with respect to all the elements of the support of $\boldsymbol{\mu}$.*
3. *For any vector $v_0 \in \mathbb{R}^d \setminus \{0\}$,*

$$\mathbb{P}\left({}^t A_1^{-1} v_0 = {}^t A_2^{-1} v_0\right) < 1,$$

where ${}^t A$ is the transpose of A, for any $A \in \mathrm{GL}(d, \mathbb{R})$.

4. *The vector B_1 is centred: $\mathbb{E}(B_1) = 0$.*

PROPOSITION 3.2. *Under Hypothesis 3.1, Theorems 2.2–2.5 hold true.*

Proposition 3.2 is proved in Appendix 11 where we construct an appropriate Banach space $\mathscr{B}$ and show that Hypotheses **M1-M5** are satisfied with $N(x) = |x|^{1+\varepsilon}$, for some $\varepsilon > 0$ and with $N_l(x) = N(x)\phi_l(|x|)$, where ϕ_l is defined by (11.4).

REMARK 3.3. The set $\mathrm{supp}(V)$ depends on the law of (A_i, B_i). In the case when A_i are independent of B_i and the support of the law of $\langle u, B_i \rangle$ contains a sequence converging to $+\infty$, one can verify that $\mathrm{supp}(V) = \mathbb{R}^d \times \mathbb{R}$.

3.2. *Two components Markov chains in compact sets under the Doeblin-Fortet condition.* Let (X, d_X) be a compact metric space, $\mathscr{C}(X)$ and $\mathscr{L}(X)$ be the spaces of continuous and Lipschitz complex functions on X, respectively. Define

$$|h|_\infty = \sup_{x \in X} |h(x)|, \quad \forall h \in \mathscr{C}(X)$$

and

$$[h]_X = \sup_{\substack{(x,y) \in X \\ x \neq y}} \frac{|h(x) - h(y)|}{d_X(x,y)}, \quad \forall h \in \mathscr{L}(X).$$

We endow $\mathscr{C}(X)$ with the uniform norm $|\cdot|_\infty$ and $\mathscr{L}(X)$ with the norm $|\cdot|_{\mathscr{L}} = |\cdot|_\infty + [\cdot]_X$, respectively. Consider the space $\mathbb{X} := X \times X$ with the metric $d_{\mathbb{X}}$ on $\mathbb{X}$ defined by $d_{\mathbb{X}}((x_1,x_2),(y_1,y_2)) = d_X(x_1,y_1) + d_X(x_2,y_2)$, for any (x_1,x_2) and (y_1,y_2) in $\mathbb{X}$. Denote by $\mathscr{L}(\mathbb{X})$ the space of the Lipschitz complex function on $\mathbb{X}$ endowed with the norm $\|\cdot\|_{\mathscr{L}} = \|\cdot\|_\infty + [\cdot]_{\mathbb{X}}$, where

$$\|h\|_\infty = \sup_{x \in \mathbb{X}} |h(x)|, \quad \forall h \in \mathscr{C}(\mathbb{X})$$

and

$$[h]_{\mathbb{X}} = \sup_{\substack{(x,y) \in \mathbb{X} \\ x \neq y}} \frac{|h(x) - h(y)|}{d_{\mathbb{X}}(x,y)}, \quad \forall h \in \mathscr{L}(\mathbb{X}).$$

Following Guivarc'h and Hardy [21], consider a Markov chain $(\chi_n)_{n \geqslant 0}$ on X with transition probability P. Let $(X_n)_{n \geqslant 0}$ be the Markov chain on $\mathbb{X}$ defined by $X_n = (\chi_{n-1}, \chi_n)$, $n \geqslant 1$ and $X_0 = (0, \chi_0)$: its transition probability is given by

$$\mathbf{P}((x_1,x_2), \mathrm{d}y_1 \times \mathrm{d}y_2) = \boldsymbol{\delta}_{x_2}(\mathrm{d}y_1) P(x_2, \mathrm{d}y_2).$$

For a fixed real function f on $\mathbb{X}$, let $S_n := \sum_{k=1}^n f(X_n)$ be the associated Markov walk and, for any $y \in \mathbb{R}$, let $\tau_y := \inf\{n \geqslant 1 : y + S_n \leqslant 0\}$ be the associated exit time.

In order to apply the results stated in the previous section, we need some hypotheses on the function f and the operator P on $\mathscr{C}(X)$ defined by $Ph(x) = \int_X h(y) P(x, \mathrm{d}y)$ for any $x \in X$ and any $h \in \mathscr{C}(X)$.

HYPOTHESIS 3.4.

1. *For any h in $\mathscr{C}(X)$, respectively in $\mathscr{L}(X)$, the function Ph is an element of $\mathscr{C}(X)$, respectively of $\mathscr{L}(X)$.*
2. *There exist constants $n_0 \geqslant 1$, $0 < \rho < 1$ and $C > 0$ such that, for any function $h \in \mathscr{L}(X)$, we have*

$$|P^{n_0} h|_{\mathscr{L}} \leqslant \rho |h|_{\mathscr{L}} + C |h|_\infty$$

3. *The unique eigenvalue of P of modulus 1 is 1 and the associated eigenspace is generated by the function $\mathrm{e}\colon x \mapsto 1$, i.e. if there exist $\theta \in \mathbb{R}$ and $h \in \mathscr{L}(X)$ such that $Ph = \mathrm{e}^{i\theta} h$, then h is constant and $\mathrm{e}^{i\theta} = 1$.*

Under Hypothesis 3.4, one can check that conditions (a), (b), (c) and (d) of Chapter 3 in Norman [28] hold true and we can apply the theorem of Ionescu Tulcea and Marinescu [25] (see also [21]). Coupling this theorem with the point 3 of Hypothesis 3.4 we obtain the following proposition.

PROPOSITION 3.5.

1. *There exists a unique P-invariant probability ν on X.*
2. *For any $n \geqslant 1$ and $h \in \mathscr{L}(X)$,*

$$P^n h = \nu(h) + R^n h,$$

where R is an operator on $\mathscr{L}(X)$ with a spectral radius $r(R) < 1$.

Suppose that f and ν satisfy the following hypothesis.

HYPOTHESIS 3.6.

1. *The function f belongs to $\mathscr{L}(\mathbb{X})$.*
2. *The function f is centred, in the sense that*

$$\int_{\mathbb{X}} f(x,y) P(x,\mathrm{d}y) \nu(\mathrm{d}x) = 0.$$

3. *The function f is non-degenerated, that means that there is no function $h \in \mathscr{L}(X)$ such that*

$$f(x,y) = h(x) - h(y),$$

for P_ν-almost all (x,y), where $P_\nu(\mathrm{d}x \times \mathrm{d}y) = P(x,\mathrm{d}y)\nu(\mathrm{d}x)$.

Assuming Hypotheses 3.4 and 3.6, Guivarc'h and Hardy [21] have established that the sequence $(S_n/\sqrt{n})_{n\geqslant 1}$ converges weakly to a centred Gaussian random variable of variance $\sigma^2 > 0$, under the probability $\mathbb{P}_x$ generated by the finite dimensional distributions of the Markov chain $(X_n)_{n\geqslant 0}$ starting at $X_0 = x$, for any $x \in X$. Moreover, under the same hypotheses, we show in Appendix 12 that **M1**-**M5** are satisfied with $N = N_l = 0$, thereby proving the following assertion.

PROPOSITION 3.7. *Under Hypotheses 3.4 and 3.6, Theorems 2.2–2.5 hold true.*

3.3. *Markov chains in compact sets under spectral gap assumptions.* In this section we give sufficient conditions in order that a Markov chain with values in a compact set satisfy conditions **M1**-**M5**.

Let $(\mathbb{X}, d)$ be a compact metric space and $(X_n)_{n\geqslant 0}$ be a Markov chain living in $\mathbb{X}$. Denote by $\mathbf{P}$ the transition probability of $(X_n)_{n\geqslant 0}$ and by $\mathscr{C}(\mathbb{X})$ the Banach algebra of the continuous complex functions on $\mathbb{X}$ endowed with the uniform norm

$$|h|_\infty = \sup_{x\in\mathbb{X}} |h(x)|, \qquad h \in \mathscr{C}(\mathbb{X}).$$

Consider a real function f defined on $\mathbb{X}$, the transition operator $\mathbf{P}$ on $\mathscr{C}(\mathbb{X})$ associated to the transition probability of $(X_n)_{n\geqslant 0}$ and the unit function e defined on $\mathbb{X}$ by $e(x) = 1$, for any $x \in \mathbb{X}$.

HYPOTHESIS 3.8.

1. *For any $h \in \mathscr{C}(\mathbb{X})$, the function $\mathbf{P}h$ is an element of $\mathscr{C}(\mathbb{X})$.*
2. *The operator $\mathbf{P}$ has a unique invariant probability $\boldsymbol{\nu}$.*
3. *For any $n \geqslant 1$,*
$$\mathbf{P}^n = \Pi + Q^n,$$
where Π is the one-dimensional projector on $\mathscr{C}(\mathbb{X})$ defined by $\Pi(h) = \boldsymbol{\nu}(h)e$, for any $h \in \mathscr{C}(\mathbb{X})$, Q is an operator on $\mathscr{C}(\mathbb{X})$ of spectral radius $r(Q) < 1$ satisfying $\Pi Q = Q\Pi = 0$.
4. *The function f belongs to $\mathscr{C}(\mathbb{X})$ and is $\boldsymbol{\nu}$-centred, i.e. $\boldsymbol{\nu}(f) = 0$.*
5. *The function f is non-degenerated, that is there is no function $h \in \mathscr{C}(\mathbb{X})$ such that*
$$f(X_1) = h(X_0) - h(X_1), \qquad \mathbb{P}_{\boldsymbol{\nu}}\text{-a.s.},$$
where $\mathbb{P}_{\boldsymbol{\nu}}$ is the probability generated by the finite dimensional distributions of the Markov chain $(X_n)_{n\geqslant 0}$ when the initial law of X_0 is $\boldsymbol{\nu}$.

Consider the Markov walk $S_n = \sum_{k=1}^n f(X_k)$. It is well known, that under Hypothesis 3.8 the normalized sum $S_n/\sqrt{n}$ converges in law to a centred normal distribution of variance $\sigma^2 > 0$ with respect to the probability $\mathbb{P}_x$ generated by the finite dimensional distributions of the Markov chain $(X_n)_{n\geqslant 0}$ starting at $X_0 = x$, for any $x \in \mathbb{X}$.

PROPOSITION 3.9. *Under Hypothesis 3.8, Theorems 2.2–2.5 hold true.*

All the elements of the proof are contained in the proof of Proposition 3.7 (see Appendix 12), which therefore is left to the reader. In particular Hypothesis **M4** holds with $N = N_l = 0$.

REMARK 3.10. As a special example of the compact case, consider the Markov chain $(X_n)_{n\geqslant 1}$ taking values in a finite space $\mathbb{X}$. Assume that $(X_n)_{n\geqslant 1}$ is aperiodic and irreducible with transition matrix $\mathbf{P}$. Let f be a finite function on $\mathbb{X}$. We shall verify Hypotesis 3.8. The Banach space $\mathscr{B}$ consists of all finite real functions on $\mathbb{X}$, therefore condition 1 is obvious. Moreover, there is a unique invariant measure $\boldsymbol{\nu}$, which proves condition 2. According to Perron-Frobenius theorem, the transition matrix $\mathbf{P}$ admits 1 as the only simple eigenvalue of modulus 1, which implies condition 3. Assume in addition that $\boldsymbol{\nu}(f) = 0$ (which is condition 4) and that there exists a path $x_0, \dots, x_n$ in $\mathbb{X}$ such that $\mathbf{P}(x_0, x_1) > 0, \dots, \mathbf{P}(x_{n-1}, x_n) > 0, \mathbf{P}(x_n, x_0) > 0$ and $f(x_0) + \cdots + f(x_n) \neq 0$ (which implies condition 5). Thus all the conclusions of Theorems 2.2–2.5 hold true.

4. Preliminary statements.

4.1. *Results for the Brownian motion.* Let $(B_t)_{t\geqslant 0}$ be the standard Brownian motion with values in $\mathbb{R}$ living on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Define the exit time

$$\tau_y^{bm} = \inf\{t \geqslant 0 : y + \sigma B_t \leqslant 0\}, \tag{4.1}$$

where $\sigma > 0$. The following affirmations are due to Lévy [27].

Lemma 4.1. *For any $y > 0$, $0 \leqslant a \leqslant b$ and $n \geqslant 1$,*

$$\mathbb{P}\left(\tau_y^{bm} > n\,,\, y + \sigma B_n \in [a,b]\right) = \frac{1}{\sqrt{2\pi n}\sigma} \int_a^b \left(\mathrm{e}^{-\frac{(s-y)^2}{2n\sigma^2}} - \mathrm{e}^{-\frac{(s+y)^2}{2n\sigma^2}} \right) \mathrm{d}s.$$

Lemma 4.2.

1. *For any $y > 0$,*

$$\mathbb{P}\left(\tau_y^{bm} > n\right) \leqslant c\frac{y}{\sqrt{n}}.$$

2. *For any sequence of real numbers $(\theta_n)_{n\geqslant 0}$ such that $\theta_n \underset{n\to+\infty}{\longrightarrow} 0$,*

$$\sup_{y\in[0;\theta_n\sqrt{n}]} \left(\frac{\mathbb{P}\left(\tau_y^{bm} > n\right)}{\frac{2y}{\sqrt{2\pi n}\sigma}} - 1 \right) = O(\theta_n^2).$$

4.2. *Strong approximation.* Under hypotheses **M1-M5** it is proved in [19] that there is a version of the Markov walk $(S_n)_{n\geqslant 0}$ and of the standard Brownian motion $(B_t)_{t\geqslant 0}$ living on the same probability space which are close enough in the following sense:

Proposition 4.3. *There exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0]$, without loss of generality one can reconstruct the sequence $(S_n)_{n\geqslant 0}$ together with a continuous time Brownian motion $(B_t)_{t\in\mathbb{R}_+}$, such that for any $x \in \mathbb{X}$ and $n \geqslant 1$,*

$$\mathbb{P}_x\left(\sup_{0\leqslant t\leqslant 1} \left| S_{\lfloor tn \rfloor} - \sigma B_{tn} \right| > n^{1/2-\varepsilon} \right) \leqslant \frac{c_\varepsilon}{n^\varepsilon}(1 + N(x)), \tag{4.2}$$

where σ is defined in the point 2 of Proposition 2.1.

In the original result the right-hand side in (4.2) is $c_\varepsilon n^{-\varepsilon}\left(1 + \mu_\alpha(x) + \|\boldsymbol{\delta}_x\|_{\mathscr{B}'}\right)^\alpha \leqslant c_\varepsilon n^{-\varepsilon}(1 + N(x))^\alpha$ with $\alpha > 2$, by the point 1 of the Hypothesis **M5**. To obtain the result of Proposition 4.3 it suffices to take the power $1/\alpha$ on the both sides and to use the obvious inequality $p < p^{1/\alpha}$, for $p \in [0, 1]$.

Using Proposition 4.3 we easily deduce the following result.

Corollary 4.4. *There exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, $x \in \mathbb{R}$ and $n \geqslant 1$,*

$$\sup_{t\in\mathbb{R}} \left| \mathbb{P}_x\left(\frac{S_n}{\sqrt{n}} \leqslant t \right) - \int_{-\infty}^t \mathrm{e}^{-\frac{u^2}{2\sigma^2}} \frac{\mathrm{d}u}{\sqrt{2\pi}\sigma} \right| \leqslant \frac{c_\varepsilon}{n^\varepsilon}\left(1 + N(x)\right).$$

5. Martingale approximation and related assertions. In this section we construct an approximating martingale for the Markov walk $(S_n)_{n\geqslant 0}$, which will be used subsequently to define the harmonic function. We also state some useful properties.

Consider Θ the real valued function defined on $\mathbb{X}$ by:

$$\Theta(x) = f(x) + \sum_{k=1}^{+\infty} \mathbf{P}^k f(x), \qquad \forall x \in \mathbb{X}.$$

It is well known that Θ is the solution of the Poisson equation

$$\Theta - \mathbf{P}\Theta = f.$$

For any $x \in \mathbb{X}$, let

$$r(x) = \mathbf{P}\Theta(x) = \Theta(x) - f(x) = \sum_{k=1}^{+\infty} \mathbf{P}^k f(x).$$

Following Gordin [17], define the process $(M_n)_{n\geqslant 0}$ by setting $M_0 = 0$ and, for any $n \geqslant 1$,

$$M_n = \sum_{k=1}^{n} \left[\Theta\left(X_k\right) - \mathbf{P}\Theta\left(X_{k-1}\right)\right] = \sum_{k=1}^{n} \left[\Theta\left(X_k\right) - r\left(X_{k-1}\right)\right].$$

For any $x \in \mathbb{X}$, we have that $(M_n)_{n\geqslant 0}$ is a zero mean $\mathbb{P}_x$-martingale with respect to the natural filtration $(\mathscr{F}_n)_{n\geqslant 0}$. Denote by ξ_n the increments of the martingale $(M_n)_{n\geqslant 0}$: for any $n \geqslant 1$,

$$\xi_n := \Theta\left(X_n\right) - r\left(X_{n-1}\right).$$

In the sequel it will be convenient to consider the martingale $(z + M_n)_{n\geqslant 1}$ starting at

$$z = y + r(x).$$

The reason for this is the following approximation which is an easy consequence of the definition of the martingale $(z + M_n)_{n\geqslant 1}$: for any $x \in \mathbb{X}$ and $y \in \mathbb{R}$, we have

$$z + M_n = y + S_n + r\left(X_n\right). \tag{5.1}$$

From (2.6) we deduce the following assertion.

Lemma 5.1. *The functions Θ and r exist on $\mathbb{X}$ and for any $x \in \mathbb{X}$,*

$$|\Theta(x)| \leqslant c\left(1 + N(x)\right) \qquad \text{and} \qquad |r(x)| \leqslant c\left(1 + N(x)\right).$$

We show that the moments of order $p \in [1, \alpha]$ of the martingale $(M_n)_{n\geqslant 0}$ are bounded.

Lemma 5.2.

1. *For any $p \in [1, \alpha]$, $x \in \mathbb{X}$ and $n \geqslant 1$,*

$$\mathbb{E}_x^{1/p}\left(|M_n|^p\right) \leqslant c_p \sqrt{n}\left(1 + N(x)\right).$$

2. For any $x \in \mathbb{X}$ and $n \geqslant 1$,

$$\mathbb{E}_x\left(|M_n|\right) \leqslant c\left(\sqrt{n} + N(x)\right).$$

PROOF. First we control the increments ξ_n. By Lemma 5.1, for any $n \geqslant 1$,

$$|\xi_n| \leqslant c\left(1 + N\left(X_n\right) + N\left(X_{n-1}\right)\right). \tag{5.2}$$

So, using the point 1 of Hypothesis **M4** and (2.2), for any $n \geqslant 1$,

$$\mathbb{E}_x^{1/p}\left(|\xi_n|^p\right) \leqslant c_p\left(1 + N(x)\right) \qquad \forall p \in [1, \alpha], \tag{5.3}$$

$$\mathbb{E}_x\left(|\xi_n|\right) \leqslant c + c\,\mathrm{e}^{-cn} N(x). \tag{5.4}$$

Proof of the claim 1. By Burkholder's inequality, for $2 < p \leqslant \alpha$,

$$\mathbb{E}_x^{1/p}\left(|M_n|^p\right) \leqslant c_p \mathbb{E}_x^{1/p}\left(\left(\sum_{k=1}^n \xi_k^2\right)^{p/2}\right).$$

Using Hölder's inequality with the exponents $u = p/2 > 1$ and $v = \frac{p}{p-2}$, we obtain

$$\mathbb{E}_x^{1/p}\left(|M_n|^p\right) \leqslant c_p \mathbb{E}_x^{1/p}\left[\left(\sum_{k=1}^n \xi_k^{2u}\right)^{\frac{p}{2u}} n^{\frac{p}{2v}}\right] = c_p n^{\frac{p-2}{2p}}\left(\sum_{k=1}^n \mathbb{E}_x\left[|\xi_k|^p\right]\right)^{1/p}.$$

From (5.3), for any $p \in (2, \alpha]$,

$$\mathbb{E}_x^{1/p}\left(|M_n|^p\right) \leqslant c_p n^{\frac{p-2}{2p}}\left(\sum_{k=1}^n c_p\left(1 + N(x)\right)^p\right)^{1/p} \leqslant c_p\sqrt{n}\left(1 + N(x)\right). \tag{5.5}$$

Using the Jensen inequality for $p \in [1, 2]$, we obtain the claim 1.

Proof of the claim 2. Consider $\varepsilon \in (0, 1/2)$. By (5.4),

$$\begin{aligned}
\mathbb{E}_x\left(|M_n|\right) &\leqslant \sum_{k=1}^{\lfloor n^\varepsilon \rfloor} \mathbb{E}_x\left(|\xi_k|\right) + \mathbb{E}_x\left(\left|M_n - M_{\lfloor n^\varepsilon \rfloor}\right|\right) \\
&\leqslant cn^\varepsilon + cN(x) + \mathbb{E}_x\left(\left|M_n - M_{\lfloor n^\varepsilon \rfloor}\right|\right).
\end{aligned}$$

Since $(X_n, M_n)_{n \geqslant 0}$ is a Markov chain, by the Markov property, the claim 1 and (2.2),

$$\begin{aligned}
\mathbb{E}_x\left(|M_n|\right) &\leqslant cn^\varepsilon + cN(x) + \mathbb{E}_x\left(\mathbb{E}\left(\left|M_n - M_{\lfloor n^\varepsilon \rfloor}\right| \,\middle|\, \mathscr{F}_{\lfloor n^\varepsilon \rfloor}\right)\right) \\
&\leqslant cn^\varepsilon + cN(x) + \mathbb{E}_x\left[c\left(n - \lfloor n^\varepsilon \rfloor\right)^{1/2}\left(1 + N\left(X_{\lfloor n^\varepsilon \rfloor}\right)\right)\right] \\
&\leqslant c\sqrt{n} + c_\varepsilon N(x).
\end{aligned}$$

□

A key point in the proof of the existence and of the positivity of the harmonic function is the introduction of the following stopping times. Let T_z be the first time when the martingale $(z+M_n)_{n\geqslant 1}$ becomes non-positive, and let $\hat{T}_z$ be the first time, after the time τ_y, when the martingale $(z+M_n)_{n\geqslant 1}$ becomes non-positive. Precisely, for any $x\in\mathbb{X}$, $z\in\mathbb{R}$ and $y=z-r(x)$, set

$$T_z := \inf\{k\geqslant 1 : z+M_k\leqslant 0\} \quad \text{and} \quad \hat{T}_z := \inf\{k\geqslant \tau_y : z+M_k\leqslant 0\}. \tag{5.6}$$

The finiteness of the stopping times τ_y, T_z and $\hat{T}_z$ is proved in Lemmas 5.5, 5.6 and 5.7 below. Now we point out some elementary facts which will be helpful in the sequel. First, the stopping time $\hat{T}_z$ is such that $\tau_y\leqslant\hat{T}_z$ and $T_z\leqslant\hat{T}_z$. Since τ_y is the exit time of $(y+S_n)_{n\geqslant 0}$, by the Markov property,

$$\mathbb{P}_x(\tau_y>n) = \int_{\mathbb{X}\times\mathbb{R}} \mathbb{P}_{x'}(\tau_{y'}>n-k)\,\mathbb{P}_x\left(X_k\in \mathrm{d}x',\, z+M_k\in\mathrm{d}z',\, \tau_y>k\right). \tag{5.7}$$

A similar expression holds true for T_z. Unfortunately, (5.7) does not hold for $\hat{T}_z$. Instead we have a more sophisticated expression given by the following lemma. We shall use repeatedly the same trick for more complicated functionals, as for example $\mathbb{E}_x\left(z+M_n\,;\,\hat{T}_z>n\right)$.

LEMMA 5.3. *For any $x\in\mathbb{X}$, $z\in\mathbb{R}$, $n\geqslant 1$, $k\leqslant n$ and $y=z-r(x)$,*

$$\begin{aligned}\mathbb{P}_x\left(\hat{T}_z>n\right) &= \int_{\mathbb{X}\times\mathbb{R}} \mathbb{P}_{x'}\left(\hat{T}_{z'}>n-k\right)\mathbb{P}_x\left(X_k\in\mathrm{d}x',\, z+M_k\in\mathrm{d}z',\,\tau_y>k\right)\\ &\quad+\int_{\mathbb{X}\times\mathbb{R}}\mathbb{P}_{x'}(T_{z'}>n-k)\,\mathbb{P}_x\left(X_k\in\mathrm{d}x',\, z+M_k\in\mathrm{d}z',\,\tau_y\leqslant k,\,\hat{T}_z>k\right).\end{aligned}$$

PROOF. Since $\hat{T}_z\geqslant\tau_y$, for any $k\leqslant n$, we have

$$\mathbb{P}_x\left(\hat{T}_z>n\right) = \mathbb{P}_x(\tau_y>n) + \sum_{i=1}^{n-k}\mathbb{P}_x\left(\tau_y=i+k,\,\hat{T}_z>n\right) + \mathbb{P}_x\left(\tau_y\leqslant k,\,\hat{T}_z>n\right).$$

By the Markov property and (5.1), with $y'=z'-r(x')$,

$$\begin{aligned}\mathbb{P}_x\left(\hat{T}_z>n\right) = &\int_{\mathbb{X}\times\mathbb{R}}\mathbb{P}_{x'}(\tau_{y'}>n-k)\,\mathbb{P}_x\left(X_k\in\mathrm{d}x',\,z+M_k\in\mathrm{d}z',\,\tau_y>k\right)\\ &+\sum_{i=1}^{n-k}\int_{\mathbb{X}\times\mathbb{R}}\mathbb{P}_{x'}\left(\tau_{y'}=i,\,z'+M_i>0,\,\dots,\,z'+M_{n-k}>0\right)\\ &\qquad\times\mathbb{P}_x\left(X_k\in\mathrm{d}x',\,z+M_k\in\mathrm{d}z',\,\tau_y>k\right)\\ &+\int_{\mathbb{X}\times\mathbb{R}}\mathbb{P}_{x'}(T_{z'}>n-k)\,\mathbb{P}_x\left(X_k\in\mathrm{d}x',\,z+M_k\in\mathrm{d}z',\,\tau_y\leqslant k,\right.\\ &\qquad\qquad\left. z+M_{\tau_y}>0,\,\dots,\,z+M_k>0\right).\end{aligned}$$

Putting together the first two terms we get the result. □

The following lemma will be useful in the next sections.

LEMMA 5.4. *For any $x \in \mathbb{X}$, $z \in \mathbb{R}$, the sequence $\left((z+M_n) \mathbb{1}_{\left\{ \hat{T}_z > n \right\}} \right)_{n \geqslant 0}$ is a $\mathbb{P}_x$-submartingale.*

PROOF. Let $x \in \mathbb{X}$, $z \in \mathbb{R}$. For any $n \geqslant 0$,

$$\begin{aligned} &\mathbb{E}_x \left((z+M_{n+1}) \mathbb{1}_{\left\{ \hat{T}_z > n+1 \right\}} \,\middle|\, \mathscr{F}_n \right) \\ &\quad = \mathbb{E}_x \left((z+M_{n+1}) \mathbb{1}_{\left\{ \hat{T}_z > n \right\}} \,\middle|\, \mathscr{F}_n \right) - \mathbb{E}_x \left((z+M_{n+1}) \mathbb{1}_{\left\{ \hat{T}_z = n+1 \right\}} \,\middle|\, \mathscr{F}_n \right) \\ &\quad = (z+M_n) \mathbb{1}_{\left\{ \hat{T}_z > n \right\}} - \mathbb{E}_x \left(\left(z + M_{\hat{T}_z} \right) \mathbb{1}_{\left\{ \hat{T}_z = n+1 \right\}} \,\middle|\, \mathscr{F}_n \right). \end{aligned}$$

By the definition of $\hat{T}_z$ we have $z + M_{\hat{T}_z} \leqslant 0$ $\mathbb{P}_x$-a.s. and the result follows. □

We end this section by proving the finiteness of τ_y, T_z and $\hat{T}_z$.

LEMMA 5.5. *For any $x \in \mathbb{X}$ and $y \in \mathbb{R}$,*

$$\tau_y < +\infty \quad \mathbb{P}_x\text{-}a.s.$$

PROOF. Let $x \in \mathbb{X}$. Assume first that $y > 0$. Since $\{\tau_y > n\}$ is a non-increasing sequence of events,

$$\mathbb{P}_x(\tau_y = +\infty) = \lim_{n\to+\infty} \mathbb{P}_x(\tau_y > n) = \lim_{n\to+\infty} \mathbb{P}_x(y + S_k > 0, \forall k \leqslant n).$$

Using Proposition 4.3,

$$\mathbb{P}_x(y + S_k > 0, \forall k \leqslant n) \leqslant \frac{c_\varepsilon}{n^\varepsilon}(1 + N(x)) + \mathbb{P}\left(\tau^{bm}_{y+n^{1/2-\varepsilon}} > n \right).$$

Thus, by the point 1 of Lemma 4.2,

$$\mathbb{P}_x(\tau_y > n) \leqslant \frac{c_\varepsilon}{n^\varepsilon}(1 + N(x)) + c\frac{y + n^{1/2-\varepsilon}}{\sqrt{n}} \leqslant \frac{c_\varepsilon}{n^\varepsilon}(1 + y + N(x)). \tag{5.8}$$

When $y \leqslant 0$, we have, for any $y' > 0$, $\mathbb{P}_x(\tau_y > n) \leqslant \mathbb{P}_x\left(\tau_{y'} > n\right)$. Taking the limit when $y' \to 0$, we obtain that

$$\mathbb{P}_x(\tau_y > n) \leqslant \frac{c_\varepsilon}{n^\varepsilon}(1 + N(x)). \tag{5.9}$$

From (5.8) and (5.9) it follows that, for any $y \in \mathbb{R}$,

$$\mathbb{P}_x(\tau_y > n) \leqslant \frac{c_\varepsilon}{n^\varepsilon}(1 + \max(y,0) + N(x)). \tag{5.10}$$

Taking the limit as $n \to +\infty$, we conclude that $\tau_y < +\infty$ $\mathbb{P}_x$-a.s. □

The same result can be obtained for the exit time T_z of the martingale $(z + M_n)_{n\geqslant 0}$.

LEMMA 5.6. *For any $x \in \mathbb{X}$ and $z \in \mathbb{R}$,*

$$T_z < +\infty \quad \mathbb{P}_x\text{-}a.s.$$

PROOF. Let $x \in \mathbb{X}$, $z \in \mathbb{R}$ and $y = z - r(x)$. Assume first that $y = z - r(x) > 0$. Following the proof of Lemma 5.5,

$$\mathbb{P}_x\left(T_z = +\infty\right) = \lim_{n \to +\infty} \mathbb{P}_x\left(z + M_k > 0,\, \forall k \leqslant n\right).$$

By (5.1) the martingale $(z+M_n)_{n \geqslant 0}$ is relied to the Markov walk $(y+S_n)_{n\geqslant 0}$, which gives

$$\mathbb{P}_x\left(z + M_k > 0,\, \forall k \leqslant n\right) \leqslant \mathbb{P}_x\left(y + S_k > -n^{1/2-\varepsilon},\, \forall k \leqslant n\right) + \mathbb{P}_x\left(\max_{1 \leqslant k \leqslant n} \left|r\left(X_k\right)\right| > n^{1/2-\varepsilon}\right). \tag{5.11}$$

On the one hand, in the same way as in the proof of Lemma 5.5,

$$\mathbb{P}_x\left(y + S_k > -n^{1/2-\varepsilon},\, \forall k \leqslant n\right) \leqslant \frac{c_\varepsilon}{n^\varepsilon}\left(1 + N(x)\right) + \mathbb{P}_x\left(\tau^{bm}_{y+2n^{1/2-\varepsilon}} > n\right). \tag{5.12}$$

On the other hand, using Lemma 5.1, for n large enough,

$$\mathbb{P}_x\left(\max_{1 \leqslant k \leqslant n} \left|r\left(X_k\right)\right| > n^{1/2-\varepsilon}\right) \leqslant \sum_{k=1}^{\lfloor n^\varepsilon \rfloor} \mathbb{E}_x\left(\frac{cN\left(X_k\right)}{n^{1/2-\varepsilon}}\right) + \sum_{k=\lfloor n^\varepsilon \rfloor+1}^{n} \mathbb{E}_x\left(\frac{cN_l\left(X_k\right)}{n^{1/2-\varepsilon}}\right),$$

where $l = cn^{1/2-\varepsilon}$. So, using (2.3) and taking $\varepsilon \leqslant \min\left(\frac{1}{6}, \frac{\beta}{2(3+\beta)}\right)$, we obtain

$$\mathbb{P}_x\left(\max_{1 \leqslant k \leqslant n} \left|r\left(X_k\right)\right| > n^{1/2-\varepsilon}\right) \leqslant \frac{c_\varepsilon}{n^\varepsilon}\left(1 + N(x)\right). \tag{5.13}$$

Putting together (5.11), (5.12) and (5.13) and using the point 1 of Lemma 4.2, we have, for $z > r(x)$,

$$\mathbb{P}_x\left(T_z > n\right) \leqslant \frac{c_\varepsilon}{n^\varepsilon}\left(1 + N(x)\right) + c\frac{y + 2n^{1/2-\varepsilon}}{\sqrt{n}} \leqslant \frac{c_\varepsilon}{n^\varepsilon}\left(1 + \max(z,0) + N(x)\right).$$

Since $z \mapsto T_z$ is non-decreasing, we obtain the same bound for any $z \in \mathbb{R}$,

$$\mathbb{P}_x\left(T_z > n\right) \leqslant \frac{c_\varepsilon}{n^\varepsilon}\left(1 + \max(z,0) + N(x)\right). \tag{5.14}$$

Taking the limit as $n \to +\infty$ we conclude that $T_z < +\infty$ $\mathbb{P}_x$-a.s. □

LEMMA 5.7. *For any $x \in \mathbb{X}$ and $z \in \mathbb{R}$,*

$$\hat{T}_z < +\infty \quad \mathbb{P}_x\text{-}a.s.$$

PROOF. In order to apply Lemmas 5.5 and 5.6, we write, with $y = z - r(x)$,

$$\mathbb{P}_x\left(\hat{T}_z > n\right) \leqslant \mathbb{P}_x\left(\tau_y > \lfloor n/2 \rfloor\right) + \int_{\mathbb{X}\times\mathbb{R}} \mathbb{P}_{x'}\left(T_{z'} > n - \lfloor n/2 \rfloor\right) \mathbb{P}_x\left(X_{\lfloor n/2 \rfloor} \in \mathrm{d}x',\right.$$
$$\left. z + M_{\lfloor n/2 \rfloor} \in \mathrm{d}z',\, \tau_y \leqslant \lfloor n/2 \rfloor,\, \hat{T}_z > \lfloor n/2 \rfloor\right).$$

Using (5.10), (5.14) and the definition of y, we have

$$\mathbb{P}_x\left(\hat{T}_z > n\right) \leqslant \frac{c_\varepsilon}{n^\varepsilon}\left(1+\max(y,0)+N(x)\right) + \frac{c_\varepsilon}{n^\varepsilon}\mathbb{E}_x\left(1+z+M_{\lfloor n/2 \rfloor}+N\left(X_{\lfloor n/2 \rfloor}\right)\,;\,\tau_y \leqslant \lfloor n/2 \rfloor\,,\,\hat{T}_z > \lfloor n/2 \rfloor\right).$$

By the point 1 of Hypothesis **M4**,

$$\mathbb{P}_x\left(\hat{T}_z > n\right) \leqslant \frac{c_\varepsilon}{n^\varepsilon}\left(1+\max(y,0)+N(x)\right) + \frac{c_\varepsilon}{n^\varepsilon}\mathbb{E}_x\left(z+M_{\lfloor n/2 \rfloor}\,;\,\hat{T}_z > \lfloor n/2 \rfloor\right) - \frac{c_\varepsilon}{n^\varepsilon}\mathbb{E}_x\left(z+M_{\lfloor n/2 \rfloor}\,;\,\tau_y > \lfloor n/2 \rfloor\right).$$

Using (5.1), we see that on the event $\{\tau_y > \lfloor n/2 \rfloor\}$ we have $z+M_{\lfloor n/2 \rfloor} > r\left(X_{\lfloor n/2 \rfloor}\right)$. Then, by Lemma 5.1 and the point 1 of Hypothesis **M4**,

$$\mathbb{P}_x\left(\hat{T}_z > n\right) \leqslant \frac{c_\varepsilon}{n^\varepsilon}\left(1+\max(y,0)+N(x)\right) + \frac{c_\varepsilon}{n^\varepsilon}\mathbb{E}_x\left(z+M_{\lfloor n/2 \rfloor}\,;\,\hat{T}_z > \lfloor n/2 \rfloor\right).$$

Using Lemma 6.4, we have

$$\mathbb{P}_x\left(\hat{T}_z > n\right) \leqslant \frac{c_\varepsilon}{n^\varepsilon}\left(1+\max(y,0)+N(x)\right).$$

Finally, we conclude that

$$\mathbb{P}_x\left(\hat{T}_z = +\infty\right) = \lim_{n\to+\infty}\mathbb{P}_x\left(\hat{T}_z > n\right) = 0.$$

□

6. Integrability of the killed martingale and of the killed Markov walk.

The goal of this section is to show that the expectations of the martingale $(z+M_n)_{n\geqslant 0}$ killed at $\hat{T}_z$ and of the Markov walk $(y+S_n)_{n\geqslant 0}$ killed at τ_y are bounded uniformly in n.

We start by establishing two auxiliary bounds of order $n^{1/2-2\varepsilon}$ for the expectations of the martingale $(z+M_n)_{n\geqslant 0}$ killed at T_z or at $\hat{T}_z$.

LEMMA 6.1. *There exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0,\varepsilon_0)$, $x \in \mathbb{X}$, $z \in \mathbb{R}$ and $n \geqslant 1$, it holds*

$$\mathbb{E}_x\left(z+M_n\,;\,T_z > n\right) \leqslant \max(z,0) + c_\varepsilon\left(n^{1/2-2\varepsilon}+N(x)\right).$$

PROOF. Using the fact that $(M_n)_{n\geqslant 0}$ is a zero mean martingale and the optional stopping theorem,

$$\mathbb{E}_x\left(z+M_n\,;\,T_z > n\right) = z - \mathbb{E}_x\left(z+M_n\,;\,T_z \leqslant n\right) = z - \mathbb{E}_x\left(z+M_{T_z}\,;\,T_z \leqslant n\right).$$

By the definition of T_z, on the event $\{T_z > 1\}$, we have

$$\xi_{T_z} = z + M_{T_z} - \left(z+M_{T_z-1}\right) < z + M_{T_z} \leqslant 0.$$

Using this inequality and (5.2), we obtain

$$\begin{aligned} \mathbb{E}_x\left(z+M_n\,;\,T_z>n\right) &\leqslant z\mathbb{P}_x\left(T_z>1\right)+\mathbb{E}_x\left(|\xi_1|\,;\,T_z=1\right)+\mathbb{E}_x\left(|\xi_{T_z}|\,;\,1<T_z\leqslant n\right) \\ &\leqslant \max(z,0)+c\mathbb{E}_x\left(1+N\left(X_{T_z}\right)+N\left(X_{T_z-1}\right)\,;\,T_z\leqslant n\right). \end{aligned} \tag{6.1}$$

We bound $\mathbb{E}_x\left(N\left(X_{T_z}\right)\,;\,T_z\leqslant n\right)$ as follows. Let ε be a real number in $(0,1/6)$ and set $l=\left\lfloor n^{1/2-2\varepsilon}\right\rfloor$. Using the point 1 of Hypothesis **M4** we write

$$\begin{aligned} \mathbb{E}_x\left(N\left(X_{T_z}\right)\,;\,T_z\leqslant n\right) &\leqslant n^{1/2-2\varepsilon}+\mathbb{E}_x\left(N\left(X_{T_z}\right)\,;\,N\left(X_{T_z}\right)>n^{1/2-2\varepsilon}\,,\,T_z\leqslant n\right) \\ &\leqslant n^{1/2-2\varepsilon}+\sum_{k=1}^{\lfloor n^\varepsilon\rfloor}\mathbb{E}_x\left(N\left(X_k\right)\right)+\sum_{k=\lfloor n^\varepsilon\rfloor+1}^{n}\mathbb{E}_x\left(N_l\left(X_k\right)\right). \end{aligned}$$

By (2.2) and (2.3),

$$\mathbb{E}_x\left(N\left(X_{T_z}\right)\,;\,T_z\leqslant n\right)\leqslant cn^{1/2-2\varepsilon}+cN(x)+\frac{cn}{l^{1+\beta}}+c\,\mathrm{e}^{-cn^\varepsilon}\left(1+N(x)\right).$$

Choosing $\varepsilon<\min(\frac{\beta}{4(2+\beta)},\frac{1}{6})$, we find that

$$\mathbb{E}_x\left(N\left(X_{T_z}\right)\,;\,T_z\leqslant n\right)\leqslant c_\varepsilon n^{1/2-2\varepsilon}+c_\varepsilon N(x). \tag{6.2}$$

In the same manner, we obtain that $\mathbb{E}_x\left(N\left(X_{T_z-1}\right)\,;\,T_z\leqslant n\right)\leqslant c_\varepsilon n^{1/2-2\varepsilon}+c_\varepsilon N(x)$. Consequently, from (6.2) and (6.1), we conclude the assertion of the lemma. □

LEMMA 6.2. *There exists $\varepsilon_0>0$ such that, for any $\varepsilon\in(0,\varepsilon_0)$, $x\in\mathbb{X}$, $z\in\mathbb{R}$ and $n\geqslant 1$, we have*

$$\mathbb{E}_x\left(z+M_n\,;\,\hat{T}_z>n\right)\leqslant\max(z,0)+c_\varepsilon\left(n^{1/2-2\varepsilon}+n^{2\varepsilon}N(x)\right).$$

PROOF. Let ε be a real number in $(0,1/4)$ and $n\geqslant 1$. Denoting $z_+:=z+n^{1/2-2\varepsilon}$ we have,

$$\begin{aligned} \mathbb{E}_x\left(z+M_n\,;\,\hat{T}_z>n\right) &= \underbrace{\mathbb{E}_x\left(z+M_n\,;\,T_{z_+}\leqslant n\,,\,\hat{T}_z>n\right)}_{=:J_1} \\ &\quad+\underbrace{\mathbb{E}_x\left(z+M_n\,;\,T_{z_+}>n\,,\,\hat{T}_z>n\right)}_{=:J_2}. \end{aligned} \tag{6.3}$$

Bound of J_1. Recall that $y=z-r(x)$. Using the definition of $\hat{T}_z$, we can see that on the event $\{\tau_y\leqslant k\,,\,\hat{T}_z>k\}$ it holds $z_++M_k>z+M_k>0$. So $\mathbb{P}_x\left(\tau_y\leqslant k\,,\,\hat{T}_z>k\,,\,T_{z_+}=k\right)=0$. Using this fact and the Markov property, in the same way as in the proof of Lemma 5.3,

$$\begin{aligned} J_1=\sum_{k=1}^{n}\int_{\mathbb{X}\times\mathbb{R}}&\mathbb{E}_{x'}\left(z'+M_{n-k}\,;\,\hat{T}_{z'}>n-k\right) \\ &\times\mathbb{P}_x\left(X_k\in\mathrm{d}x'\,,\,z+M_k\in\mathrm{d}z'\,,\,\tau_y>k\,,\,T_{z_+}=k\right). \end{aligned}$$

Since $z + M_{T_{z_+}} < 0$, using the point 2 of Lemma 5.2, we have

$$J_1 \leqslant c\mathbb{E}_x \left(\sqrt{n} + N\left(X_{T_{z_+}}\right) ; \tau_y > T_{z_+} , T_{z_+} \leqslant n \right).$$

By the approximation (5.1), on the event $\{\tau_y > T_{z_+}\}$, it holds

$$r\left(X_{T_{z_+}}\right) = z + M_{T_{z_+}} - \left(y + S_{T_{z_+}}\right) < -n^{1/2-2\varepsilon}.$$

Therefore, by Lemma 5.1,

$$\begin{aligned} J_1 &\leqslant cn^{2\varepsilon}\mathbb{E}_x \left(\left| r\left(X_{T_{z_+}}\right)\right| + N\left(X_{T_{z_+}}\right) ; \left| r\left(X_{T_{z_+}}\right)\right| > n^{1/2-2\varepsilon} , T_{z_+} \leqslant n \right) \\ &\leqslant cn^{2\varepsilon} + cn^{2\varepsilon}\mathbb{E}_x \left(N\left(X_{T_{z_+}}\right) ; T_{z_+} \leqslant n \right). \end{aligned}$$

Choosing ε small enough, by (6.2),

$$J_1 \leqslant cn^{2\varepsilon} + c_\varepsilon n^{2\varepsilon} \left(n^{1/2-4\varepsilon} + N(x) \right) \leqslant c_\varepsilon n^{1/2-2\varepsilon} + c_\varepsilon n^{2\varepsilon} N(x). \tag{6.4}$$

Bound of J_2. By Lemma 6.1, there exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$,

$$J_2 \leqslant \mathbb{E}_x \left(z_+ + M_n ; T_{z_+} > n \right) \leqslant \max(z, 0) + c_\varepsilon n^{1/2-2\varepsilon} + c_\varepsilon N(x).$$

Inserting this bound and (6.4) into (6.3), for any $\varepsilon \in (0, \varepsilon_0)$, we deduce the assertion of the lemma. □

Let ν_n be the first time when the martingale $z + M_n$ exceeds $n^{1/2-\varepsilon}$: for any $n \geqslant 1$, $\varepsilon \in (0, 1/2)$ and $z \in \mathbb{R}$,

$$\nu_n = \nu_{n,\varepsilon,z} := \min \left\{ k \geqslant 1 : z + M_k > n^{1/2-\varepsilon} \right\}.$$

The control on the joint law of ν_n and $\hat{T}_z$ is given by the following lemma.

LEMMA 6.3. *There exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, $\delta > 0$, $x \in \mathbb{X}$, $z \in \mathbb{R}$ and $n \geqslant 1$,*

$$\mathbb{P}_x \left(\nu_n > \delta n^{1-\varepsilon} , \hat{T}_z > \delta n^{1-\varepsilon} \right) \leqslant c_{\varepsilon,\delta}\, \mathrm{e}^{-c_{\varepsilon,\delta} n^{\varepsilon}} \left(1 + N(x)\right).$$

PROOF. Let $\varepsilon \in (0, 1/4)$ and $\delta > 0$. Without loss of generally, we assume that $n \geqslant c_{\varepsilon,\delta}$, where $c_{\varepsilon,\delta}$ is large enough. Set $K := \lfloor n^\varepsilon/2 \rfloor$. We split the interval $[1, \delta n^{1-\varepsilon}]$ by subintervals of length $l := \lfloor \delta n^{1-2\varepsilon} \rfloor$. For any $k \in \{1, \dots K\}$, introduce the event $A_{k,z} := \{\max_{1 \leqslant k' \leqslant k} (z + M_{k'l}) \leqslant n^{1/2-\varepsilon}\}$. Then

$$\mathbb{P}_x \left(\nu_n > \delta n^{1-\varepsilon} , \hat{T}_z > \delta n^{1-\varepsilon} \right) \leqslant \mathbb{P}_x \left(A_{2K,z} , \hat{T}_z > 2Kl \right). \tag{6.5}$$

By the Markov property, as in the proof of Lemma 5.3, with $y = z - r(x)$, we have

$$\begin{aligned} &\mathbb{P}_x \left(A_{2K,z} , \hat{T}_z > 2Kl \right) \\ &= \int_{\mathbb{X}\times\mathbb{R}} \mathbb{P}_{x'} \left(A_{2,z'} , \hat{T}_{z'} > 2l \right) \mathbb{P}_x \Big(X_{2(K-1)l} \in \mathrm{d}x' , z + M_{2(K-1)l} \in \mathrm{d}z' , \\ &\qquad\qquad A_{2(K-1),z} , \tau_y > 2(K-1)l \Big) \\ &\quad + \int_{\mathbb{X}\times\mathbb{R}} \mathbb{P}_{x'} \left(A_{2,z'} , T_{z'} > 2l \right) \mathbb{P}_x \Big(X_{2(K-1)l} \in \mathrm{d}x' , z + M_{2(K-1)l} \in \mathrm{d}z' , \\ &\qquad\qquad A_{2(K-1),z} , \tau_y \leqslant 2(K-1)l , \hat{T}_z > 2(K-1)l \Big). \end{aligned} \tag{6.6}$$

Moreover, with $y' = z' - r(x')$, we write also that

$$
\begin{aligned}
&\mathbb{P}_{x'}\left(A_{2,z'}\,,\,\hat{T}_{z'} > 2l\right) \\
&\quad = \int_{\mathbb{X}\times\mathbb{R}} \mathbb{P}_{x''}\left(A_{1,z''}\,,\,\hat{T}_{z''} > l\right) \mathbb{P}_{x'}\left(X_l \in \mathrm{d}x''\,,\, z' + M_l \in \mathrm{d}z''\,,\, A_{1,z'}\,,\,\tau_{y'} > l\right) \\
&\qquad + \int_{\mathbb{X}\times\mathbb{R}} \mathbb{P}_{x''}\left(A_{1,z''}\,,\,T_{z''} > l\right) \\
&\qquad\qquad \times \mathbb{P}_{x'}\left(X_l \in \mathrm{d}x''\,,\, z' + M_l \in \mathrm{d}z''\,,\, A_{1,z'}\,,\,\tau_{y'} \leqslant l\,,\,\hat{T}_{z'} > l\right).
\end{aligned} \tag{6.7}
$$

Bound of $\mathbb{P}_{x''}\left(A_{1,z''}\,,\,\hat{T}_{z''} > l\right)$. Note that on the event $\{\tau_{y'} > l\}$ we have $z' + M_l - r(X_l) = y' + S_l > 0$. Consequently, in the first integral of the right-hand side of (6.7), the integration over $\mathbb{X}\times\mathbb{R}$ can be replaced by the integration over $\{(x'', z'') \in \mathbb{X}\times\mathbb{R} : z'' - r(x'') > 0\}$. Therefore it is enough to bound $\mathbb{P}_{x''}\left(A_{1,z''}\,,\,\hat{T}_{z''} > l\right)$ for x'' and z'' satisfying $y'' = z'' - r(x'') > 0$. Using (5.1) we have,

$$
\begin{aligned}
\mathbb{P}_{x''}\left(A_{1,z''}\,,\,\hat{T}_{z''} > l\right) \leqslant \mathbb{P}_{x''}&\left(y'' + S_l \leqslant 2n^{1/2-\varepsilon}\,,\, |r(X_l)| \leqslant n^{1/2-\varepsilon}\right) \\
&+ \mathbb{P}_{x''}\left(|r(X_l)| > n^{1/2-\varepsilon}\right).
\end{aligned}
$$

Therefore, there exists a constant $c_{\varepsilon,\delta}$ such that

$$
\mathbb{P}_{x''}\left(A_{1,z''}\,,\,\hat{T}_{z''} > l\right) \leqslant \mathbb{P}_{x''}\left(\frac{S_l}{\sqrt{l}} \leqslant c_{\varepsilon,\delta}\right) + \mathbb{E}_{x''}\left(\frac{|r(X_l)|}{n^{1/2-\varepsilon}}\right).
$$

Using Corollary 4.4 and Lemma 5.1, there exists $\varepsilon_0 \in (0, 1/4)$, such that, for any $\varepsilon \in (0, \varepsilon_0)$,

$$
\mathbb{P}_{x''}\left(A_{1,z''}\,,\,\hat{T}_{z''} > l\right) \leqslant \int_{-\infty}^{c_{\varepsilon,\delta}} \mathrm{e}^{-\frac{u^2}{2\sigma^2}} \frac{\mathrm{d}u}{\sqrt{2\pi}\sigma} + \frac{c_\varepsilon}{l^\varepsilon}\left(1 + N(x'')\right) + \frac{c}{n^{1/2-\varepsilon}} \mathbb{E}_{x''}\left(1 + N(X_l)\right).
$$

Using the point 1 of Hypothesis **M4** and the fact that $l^\varepsilon \geqslant n^{\varepsilon/2}/c_{\varepsilon,\delta}$ for $\varepsilon < 1/4$, we have,

$$
\mathbb{P}_{x''}\left(A_{1,z''}\,,\,\hat{T}_{z''} > l\right) \leqslant q_{\varepsilon,\delta} + \frac{c_{\varepsilon,\delta}}{n^{\varepsilon/2}}\left(1 + N(x'')\right), \tag{6.8}
$$

with $q_{\varepsilon,\delta} := \int_{-\infty}^{c_{\varepsilon,\delta}} \mathrm{e}^{-\frac{u^2}{2\sigma^2}} \frac{\mathrm{d}u}{\sqrt{2\pi}\sigma} < 1$.

Bound of $\mathbb{P}_{x''}\left(A_{1,z''}\,,\,T_{z''} > l\right)$. On the event $\{T_{z''} > l\}$ we have $z'' + M_l > 0$. Using (5.1) and Corollary 4.4, in the same way as in the proof of the bound (6.8), we obtain

$$
\begin{aligned}
\mathbb{P}_{x''}\left(A_{1,z''}\,,\,T_{z''} > l\right) &\leqslant \mathbb{P}_{x''}\left(0 < z'' + M_l \leqslant n^{1/2-\varepsilon}\right) \\
&\leqslant \int_{\frac{-y''}{\sqrt{l}} - c_{\varepsilon,\delta}}^{\frac{-y''}{\sqrt{l}} + c_{\varepsilon,\delta}} \mathrm{e}^{-\frac{u^2}{2\sigma^2}} \frac{\mathrm{d}u}{\sqrt{2\pi}\sigma} + \frac{c_{\varepsilon,\delta}}{n^{\varepsilon/2}}\left(1 + N(x'')\right) \\
&\leqslant q_{\varepsilon,\delta} + \frac{c_{\varepsilon,\delta}}{n^{\varepsilon/2}}\left(1 + N(x'')\right).
\end{aligned} \tag{6.9}
$$

Bound of $\mathbb{P}_{x'}\left(A_{2,z'}\,,\, \hat{T}_{z'} > 2l\right)$. Inserting (6.8) and (6.9) into (6.7) and using (2.2), we have

$$\begin{aligned}\mathbb{P}_{x'}\left(A_{2,z'}\,,\, \hat{T}_{z'} > 2l\right) &\leqslant q_{\varepsilon,\delta} + \frac{c_{\varepsilon,\delta}}{n^{\varepsilon/2}} + \frac{c_{\varepsilon,\delta}}{n^{\varepsilon/2}} \mathbb{E}_{x'}\left(N\left(X_l\right)\right) \\ &\leqslant q_{\varepsilon,\delta} + \frac{c_{\varepsilon,\delta}}{n^{\varepsilon/2}} + c_{\varepsilon,\delta}\, \mathrm{e}^{-c_{\varepsilon,\delta} n^{1-2\varepsilon}}\, N(x'). \end{aligned} \tag{6.10}$$

Bound of $\mathbb{P}_{x'}\left(A_{2,z'}\,,\, T_{z'} > 2l\right)$. By the Markov property,

$$\begin{aligned}\mathbb{P}_{x'}\left(A_{2,z'}\,,\, T_{z'} > 2l\right) = \int_{\mathbb{X}\times\mathbb{R}} & \mathbb{P}_{x''}\left(A_{1,z''}\,,\, T_{z''} > l\right) \\ & \times \mathbb{P}_{x'}\left(X_l \in \mathrm{d}x''\,,\, z' + M_l \in \mathrm{d}z''\,,\, A_{1,z'}\,,\, T_{z'} > l\right).\end{aligned}$$

Using (6.9) to bound the probability inside the integral, we get

$$\mathbb{P}_{x'}\left(A_{2,z'}\,,\, T_{z'} > 2l\right) \leqslant q_{\varepsilon,\delta} + \frac{c_{\varepsilon,\delta}}{n^{\varepsilon/2}} + c_{\varepsilon,\delta}\, \mathrm{e}^{-c_{\varepsilon,\delta} n^{1-2\varepsilon}}\, N(x'). \tag{6.11}$$

Inserting the bounds (6.10) and (6.11) into (6.6), we find that

$$\begin{aligned}&\mathbb{P}_{x}\left(A_{2K,z}\,,\, \hat{T}_{z} > 2Kl\right) \\ &\quad \leqslant \left(q_{\varepsilon,\delta} + \frac{c_{\varepsilon,\delta}}{n^{\varepsilon/2}}\right) \mathbb{P}_{x}\left(A_{2(K-1),z}\,,\, \hat{T}_{z} > 2(K-1)l\right) + c_{\varepsilon,\delta}\, \mathrm{e}^{-c_{\varepsilon,\delta} n^{1-2\varepsilon}}\left(1 + N(x)\right).\end{aligned}$$

Iterating this inequality, we get

$$\mathbb{P}_{x}\left(A_{2K,z}\,,\, \hat{T}_{z} > 2Kl\right) \leqslant \left(q_{\varepsilon,\delta} + \frac{c_{\varepsilon,\delta}}{n^{\varepsilon/2}}\right)^K + c_{\varepsilon,\delta}\, \mathrm{e}^{-c_{\varepsilon,\delta} n^{1-2\varepsilon}}\left(1 + N(x)\right) \sum_{k=0}^{K-1} \left(q_{\varepsilon,\delta} + \frac{c_{\varepsilon,\delta}}{n^{\varepsilon/2}}\right)^k.$$

As $K = \lfloor n^{\varepsilon}/2 \rfloor$ and $q_{\varepsilon,\delta} < 1$ it follows that, for n large enough, $\left(q_{\varepsilon,\delta} + \frac{c_{\varepsilon,\delta}}{n^{\varepsilon/2}}\right)^K \leqslant c_{\varepsilon,\delta}\, \mathrm{e}^{-c_{\varepsilon,\delta} n^{\varepsilon}}$, which, in turn, implies

$$\mathbb{P}_{x}\left(A_{2K,z}\,,\, \hat{T}_{z} > 2Kl\right) \leqslant c_{\varepsilon,\delta}\, \mathrm{e}^{-c_{\varepsilon,\delta} n^{\varepsilon}}\left(1 + N(x)\right).$$

□

LEMMA 6.4. *There exists* $\varepsilon_0 > 0$ *such that, for any* $\varepsilon \in (0, \varepsilon_0)$, $x \in \mathbb{X}$, $z \in \mathbb{R}$, $n \geqslant 2$ *and any integer* $k_0 \in \{2, \dots, n\}$,

$$\mathbb{E}_{x}\left(z + M_n\,;\, \hat{T}_{z} > n\right) \leqslant \left(1 + \frac{c_{\varepsilon}}{k_0^{\varepsilon}}\right)\left(\max(z,0) + cN(x)\right) + c_{\varepsilon} k_0^{1/2}.$$

PROOF. Set for brevity $u_n := \mathbb{E}_{x}\left(z + M_n\,;\, \hat{T}_{z} > n\right)$. By Lemma 5.4, the sequence $(u_n)_{n\geqslant 1}$ is non-decreasing. Let $\varepsilon \in (0, 1/2)$. We shall prove below that, for $n \geqslant 2$,

$$u_n \leqslant \left(1 + \frac{c_{\varepsilon}}{n^{\varepsilon}}\right) u_{\lfloor n^{1-\varepsilon} \rfloor} + c_{\varepsilon}\, \mathrm{e}^{-c_{\varepsilon} n^{\varepsilon}}\left(1 + N(x)\right). \tag{6.12}$$

Using Lemma 9.1 of [18], we obtain that for any $n \geqslant 2$ and $k_0 \in \{2, \dots, n\}$,

$$u_n \leqslant \left(1 + \frac{c_\varepsilon}{{k_0}^\varepsilon}\right) u_{k_0} + c_\varepsilon \, \mathrm{e}^{-c_\varepsilon k_0^\varepsilon} \left(1 + N(x)\right).$$

Next, by the point 2 of Lemma 5.2, $u_{k_0} \leqslant \mathbb{E}_x\left(|M_{k_0}|\right) \leqslant c\left(\sqrt{k_0} + N(x)\right)$, so that

$$u_n \leqslant \left(1 + \frac{c_\varepsilon}{k_0^\varepsilon}\right) \left(\max(z,0) + cN(x)\right) + c_\varepsilon k_0^{1/2},$$

which proves Lemma 6.4.

Establishing (6.12) is rather tedious. In the proof we make use of Lemmas 6.2 and 6.1. Consider the stopping time $\nu_n^\varepsilon := \nu_n + \lfloor n^\varepsilon \rfloor$. Then,

$$\begin{aligned} u_n \leqslant & \underbrace{\mathbb{E}_x\left(z + M_n \,;\, \hat{T}_z > n \,,\, \nu_n^\varepsilon > \left\lfloor n^{1-\varepsilon} \right\rfloor\right)}_{=:J_1} \\ & + \underbrace{\mathbb{E}_x\left(z + M_n \,;\, \hat{T}_z > n \,,\, \nu_n^\varepsilon \leqslant \left\lfloor n^{1-\varepsilon} \right\rfloor\right)}_{=:J_2}. \end{aligned} \tag{6.13}$$

Bound of J_1. Set $m_\varepsilon = \lfloor n^{1-\varepsilon} \rfloor - \lfloor n^\varepsilon \rfloor$ and recall that $y = z - r(x)$. Using the fact that $\{\nu_n^\varepsilon > \lfloor n^{1-\varepsilon} \rfloor\} = \{\nu_n > m_\varepsilon\}$ and the Markov property, as in the proof of Lemma 5.3,

$$\begin{aligned} J_1 = & \int_{\mathbb{X}\times\mathbb{R}} \mathbb{E}_{x'}\left(z' + M_{n-m_\varepsilon} \,;\, \hat{T}_{z'} > n - m_\varepsilon\right) \\ & \qquad \times \mathbb{P}_x\left(X_{m_\varepsilon} \in \mathrm{d}x' \,,\, z + M_{m_\varepsilon} \in \mathrm{d}z' \,,\, \tau_y > m_\varepsilon \,,\, \nu_n > m_\varepsilon\right) \\ & + \int_{\mathbb{X}\times\mathbb{R}} \mathbb{E}_{x'}\left(z' + M_{n-m_\varepsilon} \,;\, T_{z'} > n - m_\varepsilon\right) \\ & \qquad \times \mathbb{P}_x\left(X_{m_\varepsilon} \in \mathrm{d}x' \,,\, z + M_{m_\varepsilon} \in \mathrm{d}z' \,,\, \tau_y \leqslant m_\varepsilon \,,\, \hat{T}_z > m_\varepsilon \,,\, \nu_n > m_\varepsilon\right). \end{aligned}$$

On the event $\{\nu_n > m_\varepsilon\}$, we have $z' = z + M_{m_\varepsilon} \leqslant n^{1/2-\varepsilon} \leqslant n^{1/2}$. Moreover by the point 2 of Lemma 5.2, $\mathbb{E}_{x'}\left(|M_{n-m_\varepsilon}|\right) \leqslant cn^{1/2} + cN(x')$. Therefore,

$$J_1 \leqslant c\mathbb{E}_x\left(n^{1/2} + N\left(X_{m_\varepsilon}\right) \,;\, \hat{T}_z > m_\varepsilon \,,\, \nu_n > m_\varepsilon\right).$$

Set $m_\varepsilon' = m_\varepsilon - \lfloor n^\varepsilon \rfloor = \lfloor n^{1-\varepsilon} \rfloor - 2\lfloor n^\varepsilon \rfloor$. Using the Markov property and (2.2),

$$\begin{aligned} J_1 & \leqslant c \int_{\mathbb{X}} \left[n^{1/2} + \mathbb{E}_{x'}\left(N\left(X_{\lfloor n^\varepsilon \rfloor}\right)\right)\right] \mathbb{P}_x\left(X_{m_\varepsilon'} \in \mathrm{d}x' \,,\, \hat{T}_z > m_\varepsilon' \,,\, \nu_n > m_\varepsilon'\right) \\ & \leqslant cn^{1/2} \mathbb{P}_x\left(\hat{T}_z > m_\varepsilon' \,,\, \nu_n > m_\varepsilon'\right) + c\,\mathrm{e}^{-cn^\varepsilon}\, \mathbb{E}_x\left(N\left(X_{m_\varepsilon'}\right)\right). \end{aligned}$$

By Lemma 6.3 and the point 1 of Hypothesis **M4**,

$$J_1 \leqslant c_\varepsilon n^{1/2}\, \mathrm{e}^{-c_\varepsilon n^\varepsilon} \left(1 + N(x)\right) + c\,\mathrm{e}^{-cn^\varepsilon} \left(1 + N(x)\right) \leqslant c_\varepsilon\, \mathrm{e}^{-c_\varepsilon n^\varepsilon} \left(1 + N(x)\right). \tag{6.14}$$

Bound of J_2. By the Markov property, as in the proof of Lemma 5.3, we have

$$J_2 = \sum_{k=1}^{\lfloor n^{1-\varepsilon} \rfloor} \int_{\mathbb{X}\times\mathbb{R}} \mathbb{E}_{x'}\left(z' + M_{n-k} \,;\, \hat{T}_{z'} > n-k \right) \times \mathbb{P}_x\left(X_k \in \mathrm{d}x' \,,\, z + M_k \in \mathrm{d}z' \,,\, \tau_y > k \,,\, \nu_n^\varepsilon = k \right) + \int_{\mathbb{X}\times\mathbb{R}} \mathbb{E}_{x'}\left(z' + M_{n-k} \,;\, T_{z'} > n-k \right) \times \mathbb{P}_x\left(X_k \in \mathrm{d}x' \,,\, z + M_k \in \mathrm{d}z' \,,\, \tau_y \leqslant k \,,\, \hat{T}_z > k \,,\, \nu_n^\varepsilon = k \right).$$

By Lemmas 6.2 and 6.1,

$$J_2 \leqslant \underbrace{c_\varepsilon \mathbb{E}_x\left(n^{1/2-2\varepsilon} + n^{2\varepsilon} N\left(X_{\nu_n^\varepsilon} \right) \,;\, \hat{T}_z > \nu_n^\varepsilon \,,\, \nu_n^\varepsilon \leqslant \left\lfloor n^{1-\varepsilon} \right\rfloor \right)}_{=:J_{21}} + \underbrace{\mathbb{E}_x\left(\max\left(z + M_{\nu_n^\varepsilon}, 0 \right) \,;\, \hat{T}_z > \nu_n^\varepsilon \,,\, \nu_n^\varepsilon \leqslant \left\lfloor n^{1-\varepsilon} \right\rfloor \right)}_{=:J_{22}}. \tag{6.15}$$

Bound of J_{21}. Using the Markov property and (2.2),

$$J_{21} \leqslant c_\varepsilon \int_{\mathbb{X}} \mathbb{E}_{x'}\left(n^{1/2-2\varepsilon} + n^{2\varepsilon} N\left(X_{\lfloor n^\varepsilon \rfloor} \right) \right) \mathbb{P}_x\left(X_{\nu_n} \in \mathrm{d}x' \,,\, \hat{T}_z > \nu_n \,,\, \nu_n \leqslant \left\lfloor n^{1-\varepsilon} \right\rfloor \right)$$
$$\leqslant c_\varepsilon \mathbb{E}_x\left(n^{1/2-2\varepsilon} + \mathrm{e}^{-c_\varepsilon n^\varepsilon} N\left(X_{\nu_n} \right) \,;\, \hat{T}_z > \nu_n \,,\, \nu_n \leqslant \left\lfloor n^{1-\varepsilon} \right\rfloor \right).$$

Again by (2.2),

$$\mathbb{E}_x\left(\mathrm{e}^{-c_\varepsilon n^\varepsilon} N\left(X_{\nu_n} \right) \,;\, \hat{T}_z > \nu_n \,,\, \nu_n \leqslant \left\lfloor n^{1-\varepsilon} \right\rfloor \right) \leqslant \mathrm{e}^{-c_\varepsilon n^\varepsilon} \sum_{k=1}^{\lfloor n^{1-\varepsilon} \rfloor} \mathbb{E}_x\left(N\left(X_k \right) \,;\, \nu_n = k \right) \leqslant c_\varepsilon \, \mathrm{e}^{-c_\varepsilon n^\varepsilon} \, n^{1-\varepsilon} \left(1 + N(x) \right). \tag{6.16}$$

Therefore,

$$J_{21} \leqslant \underbrace{c_\varepsilon \mathbb{E}_x\left(n^{1/2-2\varepsilon} \,;\, \hat{T}_z > \nu_n \,,\, \nu_n \leqslant \left\lfloor n^{1-\varepsilon} \right\rfloor \right)}_{=:J_{21}'} + c_\varepsilon \, \mathrm{e}^{-c_\varepsilon n^\varepsilon} \left(1 + N(x) \right). \tag{6.17}$$

By the definition of ν_n, we have $n^{1/2-2\varepsilon} < \frac{z+M_{\nu_n}}{n^\varepsilon}$. So

$$J_{21}' \leqslant \frac{c_\varepsilon}{n^\varepsilon} \mathbb{E}_x\left(z + M_{\nu_n} \,;\, \hat{T}_z > \nu_n \,,\, \nu_n \leqslant \left\lfloor n^{1-\varepsilon} \right\rfloor \right).$$

Using Lemma 5.4,

$$J_{21}' \leqslant \frac{c_\varepsilon}{n^\varepsilon} \mathbb{E}_x\left(z + M_{\lfloor n^{1-\varepsilon} \rfloor} \,;\, \hat{T}_z > \left\lfloor n^{1-\varepsilon} \right\rfloor \right) - \frac{c_\varepsilon}{n^\varepsilon} \underbrace{\mathbb{E}_x\left(z + M_{\lfloor n^{1-\varepsilon} \rfloor} \,;\, \hat{T}_z > \left\lfloor n^{1-\varepsilon} \right\rfloor \,,\, \nu_n > \left\lfloor n^{1-\varepsilon} \right\rfloor \right)}_{=:J_{21}''}. \tag{6.18}$$

Note that on the event $\{\tau_y > \lfloor n^{1-\varepsilon} \rfloor\}$, by (5.1), we have $z + M_{\lfloor n^{1-\varepsilon} \rfloor} > r\left(X_{\lfloor n^{1-\varepsilon} \rfloor}\right)$ while on the event $\{\tau_y \leqslant \lfloor n^{1-\varepsilon} \rfloor, \hat{T}_z > \lfloor n^{1-\varepsilon} \rfloor\}$ we have $z + M_{\lfloor n^{1-\varepsilon} \rfloor} > 0$. Therefore, by the definition of $\hat{T}_z$,

$$\begin{aligned} -J_{21}'' &\leqslant -\mathbb{E}_x\left(r\left(X_{\lfloor n^{1-\varepsilon} \rfloor} \right) ; \tau_y > \left\lfloor n^{1-\varepsilon} \right\rfloor , \nu_n > \left\lfloor n^{1-\varepsilon} \right\rfloor \right) \\ &\leqslant c\mathbb{E}_x\left(1 + N\left(X_{\lfloor n^{1-\varepsilon} \rfloor} \right) ; \hat{T}_z > \left\lfloor n^{1-\varepsilon} \right\rfloor , \nu_n > \left\lfloor n^{1-\varepsilon} \right\rfloor \right). \end{aligned}$$

Using the Markov property and (2.2),

$$\begin{aligned} -J_{21}'' &\leqslant c\mathbb{E}_x\left(1 + \mathrm{e}^{-cn^{\varepsilon}} N\left(X_{m_\varepsilon} \right) ; \hat{T}_z > m_\varepsilon , \nu_n > m_\varepsilon \right) \\ &\leqslant c\mathbb{P}_x\left(\nu_n > m_\varepsilon , \hat{T}_z > m_\varepsilon \right) + c\,\mathrm{e}^{-cn^{\varepsilon}} \left(1 + N(x) \right). \end{aligned}$$

By Lemma 6.3,

$$-J_{21}'' \leqslant c_\varepsilon\, \mathrm{e}^{-c_\varepsilon n^{\varepsilon}} \left(1 + N(x) \right). \tag{6.19}$$

Putting together (6.19) and (6.18),

$$J_{21}' \leqslant \frac{c_\varepsilon}{n^\varepsilon} \mathbb{E}_x\left(z + M_{\lfloor n^{1-\varepsilon} \rfloor} ; \hat{T}_z > \left\lfloor n^{1-\varepsilon} \right\rfloor \right) + c_\varepsilon e^{-c_\varepsilon n^{\varepsilon}} \left(1 + N(x) \right). \tag{6.20}$$

From (6.20) and (6.17) it follows that

$$J_{21} \leqslant \frac{c_\varepsilon}{n^\varepsilon} \mathbb{E}_x\left(z + M_{\lfloor n^{1-\varepsilon} \rfloor} ; \hat{T}_z > \left\lfloor n^{1-\varepsilon} \right\rfloor \right) + c_\varepsilon\, \mathrm{e}^{-c_\varepsilon n^{\varepsilon}} \left(1 + N(x) \right). \tag{6.21}$$

Bound of J_{22}. On the event $\{\hat{T}_z > \nu_n^\varepsilon , \tau_y \leqslant \nu_n^\varepsilon\}$ we have $z + M_{\nu_n^\varepsilon} > 0$. Consequently

$$\begin{aligned} J_{22} &= \mathbb{E}_x\left(z + M_{\nu_n^\varepsilon} ; \hat{T}_z > \nu_n^\varepsilon , \nu_n^\varepsilon \leqslant \left\lfloor n^{1-\varepsilon} \right\rfloor \right) \\ &\quad + \mathbb{E}_x\left(\max\left(z + M_{\nu_n^\varepsilon}, 0 \right) - \left(z + M_{\nu_n^\varepsilon} \right) ; \tau_y > \nu_n^\varepsilon , \nu_n^\varepsilon \leqslant \left\lfloor n^{1-\varepsilon} \right\rfloor \right). \end{aligned}$$

By Lemma 5.4,

$$\begin{aligned} J_{22} \leqslant{}& \mathbb{E}_x\left(z + M_{\lfloor n^{1-\varepsilon} \rfloor} ; \hat{T}_z > \left\lfloor n^{1-\varepsilon} \right\rfloor \right) \\ & - \underbrace{\mathbb{E}_x\left(z + M_{\lfloor n^{1-\varepsilon} \rfloor} ; \hat{T}_z > \left\lfloor n^{1-\varepsilon} \right\rfloor , \nu_n^\varepsilon > \left\lfloor n^{1-\varepsilon} \right\rfloor \right)}_{=:J_{22}''} \\ & \underbrace{-\mathbb{E}_x\left(z + M_{\nu_n^\varepsilon} ; z + M_{\nu_n^\varepsilon} < 0 , \tau_y > \nu_n^\varepsilon , \nu_n^\varepsilon \leqslant \left\lfloor n^{1-\varepsilon} \right\rfloor \right)}_{=:J_{22}'}. \end{aligned} \tag{6.22}$$

In the same way as in the proof of the bound of J_{21}'', replacing ν_n by ν_n^ε, one can prove that

$$-J_{22}'' \leqslant c_\varepsilon\, \mathrm{e}^{-c_\varepsilon n^{\varepsilon}} \left(1 + N(x) \right). \tag{6.23}$$

Moreover, using (5.1), on the event $\{\tau_y > \nu_n^\varepsilon\}$, we have $-(z+M_{\nu_n^\varepsilon}) < -r\left(X_{\nu_n^\varepsilon}\right)$. So, by Lemma 5.1 and the Markov property

$$\begin{aligned} J'_{22} &\leqslant \mathbb{E}_x\left(\left|r\left(X_{\nu_n^\varepsilon}\right)\right| ; \hat{T}_z > \nu_n^\varepsilon , \nu_n^\varepsilon \leqslant \left\lfloor n^{1-\varepsilon}\right\rfloor\right) \\ &\leqslant \mathbb{E}_x\left(c\left(1+N\left(X_{\nu_n^\varepsilon}\right)\right) ; \hat{T}_z > \nu_n , \nu_n \leqslant \left\lfloor n^{1-\varepsilon}\right\rfloor\right) \\ &= c\int_{\mathbb{X}} \mathbb{E}_{x'}\left(1+N\left(X_{\lfloor n^\varepsilon\rfloor}\right)\right)\mathbb{P}_x\left(X_{\nu_n}\in \mathrm{d}x' , \hat{T}_z > \nu_n , \nu_n \leqslant \left\lfloor n^{1-\varepsilon}\right\rfloor\right). \end{aligned}$$

Using (2.2),

$$J'_{22} \leqslant c\mathbb{E}_x\left(1+\mathrm{e}^{-cn^\varepsilon} N\left(X_{\nu_n}\right) ; \hat{T}_z > \nu_n , \nu_n \leqslant \left\lfloor n^{1-\varepsilon}\right\rfloor\right).$$

Therefore, from (6.16) with the notation J'_{21} from (6.17),

$$J'_{22} \leqslant J'_{21} + c_\varepsilon\,\mathrm{e}^{-c_\varepsilon n^\varepsilon}\left(1+N(x)\right). \tag{6.24}$$

With (6.20), (6.22) and (6.23) we obtain,

$$J_{22} \leqslant \left(1+\frac{c_\varepsilon}{n^\varepsilon}\right)u_{\lfloor n^{1-\varepsilon}\rfloor} + c_\varepsilon\,\mathrm{e}^{-c_\varepsilon n^\varepsilon}\left(1+N(x)\right). \tag{6.25}$$

Inserting (6.25) and (6.21) into (6.15),

$$J_{2} \leqslant \left(1+\frac{c_\varepsilon}{n^\varepsilon}\right)u_{\lfloor n^{1-\varepsilon}\rfloor} + c_\varepsilon\,\mathrm{e}^{-c_\varepsilon n^\varepsilon}\left(1+N(x)\right). \tag{6.26}$$

Now, inserting (6.14) and (6.26) into (6.13), we find (6.12). □

COROLLARY 6.5. *There exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0,\varepsilon_0)$, $x \in \mathbb{X}$, $y \in \mathbb{R}$, $n \geqslant 2$ and any integer $k_0 \in \{2,\dots,n\}$,*

$$\mathbb{E}_x\left(y+S_n ; \tau_y > n\right) \leqslant \left(1+\frac{c_\varepsilon}{k_0^\varepsilon}\right)\left(\max(y,0)+cN(x)\right) + c_\varepsilon k_0^{1/2}.$$

PROOF. First, using the definition of $\hat{T}_z$ and Lemma 6.4, with $z = y + r(x)$,

$$\begin{aligned} \mathbb{E}_x\left(z+M_n ; \tau_y > n\right) &= \mathbb{E}_x\left(z+M_n ; \hat{T}_z > n\right) - \mathbb{E}_x\left(z+M_n ; \tau_y \leqslant n , \hat{T}_z > n\right) \\ &\leqslant \mathbb{E}_x\left(z+M_n ; \hat{T}_z > n\right) \qquad (6.27) \\ &\leqslant \left(1+\frac{c_\varepsilon}{k_0^\varepsilon}\right)\left(\max(z,0)+cN(x)\right) + c_\varepsilon k_0^{1/2}. \qquad (6.28) \end{aligned}$$

Now, using (5.1), Lemma 5.1 and (2.2),

$$\begin{aligned} \mathbb{E}_x\left(y+S_n ; \tau_y > n\right) &= \mathbb{E}_x\left(z+M_n ; \tau_y > n\right) - \mathbb{E}_x\left(r\left(X_n\right) ; \tau_y > n\right) \\ &\leqslant \mathbb{E}_x\left(z+M_n ; \tau_y > n\right) + c\left(1+\mathrm{e}^{-cn} N(x)\right) \\ &\leqslant \left(1+\frac{c_\varepsilon}{k_0^\varepsilon}\right)\left(\max(z,0)+cN(x)\right) + c_\varepsilon k_0^{1/2}. \end{aligned}$$

Using the definition of z concludes the proof. □

7. Existence and properties of the harmonic function. The idea is very simple. Set for brevity $V_n(x,y) := \mathbb{E}_x\left(y+S_n\,;\,\tau_y>n\right)$. By the Markov property $V_{n+1}(x,y) = \mathbf{Q}_+V_n(x,y)$. We show that $\lim_{n\to\infty} V_n(x,y)$ exists and is equal to $V(x,y) := -\mathbb{E}_x(M_{\tau_y})$. Then the harmonicity of V follows by the Lebesgue dominated convergence theorem. The key point of the proof is the integrability of the random variable M_{τ_y}. To justify the applicability of the Lebesgue dominated convergence theorem we use Lemma 6.4. We also shall establish some properties of V. They will be deduced from those of the following two functions: $W(x,z) := -\mathbb{E}_x(M_{T_z})$ and $\hat{W}(x,z) := -\mathbb{E}_x(M_{\hat{T}_z})$. The strict positivity of V is technically more delicate and therefore is deferred to the next section.

LEMMA 7.1. *Let $x\in\mathbb{X}$, $y\in\mathbb{R}$ and $z=y+r(x)$. The random variables $M_{\hat{T}_z}$, M_{T_z} and M_{τ_y} are integrable and*

$$\max\left\{\mathbb{E}_x\left(\left|M_{\hat{T}_z}\right|\right),\mathbb{E}_x\left(\left|M_{T_z}\right|\right),\mathbb{E}_x\left(\left|M_{\tau_y}\right|\right)\right\} \leqslant c\left(1+|z|+N(x)\right) < +\infty.$$

In particular, the following functions are well defined, for any $x\in\mathbb{X}$, $y\in\mathbb{R}$ and $z\in\mathbb{R}$,

$$V(x,y) := -\mathbb{E}_x\left(M_{\tau_y}\right),\quad W(x,z) := -\mathbb{E}_x\left(M_{T_z}\right)\quad \textit{and}\quad \hat{W}(x,z) := -\mathbb{E}_x\left(M_{\hat{T}_z}\right).$$

PROOF. Let $n\geqslant 1$. The stopping times $\tau_y\wedge n$, $T_z\wedge n$ and $\hat{T}_z\wedge n$ are bounded and satisfy $\tau_y\wedge n\leqslant \hat{T}_z\wedge n$ and $T_z\wedge n\leqslant\hat{T}_z\wedge n$. Since $(|M_n|)_{n\geqslant 0}$ is a submartingale, we have

$$\max\left\{\mathbb{E}_x\left(\left|M_{\tau_y\wedge n}\right|\right),\mathbb{E}_x\left(\left|M_{T_z\wedge n}\right|\right)\right\} \leqslant \mathbb{E}_x\left(\left|M_{\hat{T}_z\wedge n}\right|\right). \tag{7.1}$$

Using the optional stopping theorem,

$$\begin{aligned}
\mathbb{E}_x\left(\left|M_{\hat{T}_z\wedge n}\right|\right) \leqslant\; & -\mathbb{E}_x\left(z+M_{\hat{T}_z}\,;\,\hat{T}_z\leqslant n\right) + \mathbb{E}_x\left(|z+M_n|\,;\,\tau_y>n\right)\\
& + \mathbb{E}_x\left(z+M_n\,;\,\tau_y\leqslant n\,,\,\hat{T}_z>n\right)+|z|\\
=\; & -\mathbb{E}_x\left(z+M_n\,;\,\hat{T}_z\leqslant n\right) - 2\mathbb{E}_x\left(z+M_n\,;\,z+M_n\leqslant 0\,,\,\tau_y>n\right)\\
& + \mathbb{E}_x\left(z+M_n\,;\,\tau_y>n\right) + \mathbb{E}_x\left(z+M_n\,;\,\tau_y\leqslant n\,,\,\hat{T}_z>n\right)+|z|\\
=\; & -z+2\mathbb{E}_x\left(z+M_n\,;\,\hat{T}_z>n\right)\\
& -2\mathbb{E}_x\left(z+M_n\,;\,z+M_n\leqslant 0\,,\,\tau_y>n\right)+|z|.
\end{aligned}$$

On the event $\{z+M_n\leqslant 0\,,\,\tau_y>n\}$, by (5.1), it holds $|z+M_n|\leqslant|r\left(X_n\right)|$. Therefore, by Lemma 5.1 and the point 1 of Hypothesis **M4**, we have

$$-2\mathbb{E}_x\left(z+M_n\,;\,z+M_n\leqslant 0\,,\,\tau_y>n\right)\leqslant c\left(1+N(x)\right),$$

Using Lemma 6.4,

$$\mathbb{E}_x\left(\left|M_{\hat{T}_z}\right|\,;\,\hat{T}_z\leqslant n\right)\leqslant \mathbb{E}_x\left(\left|M_{\hat{T}_z\wedge n}\right|\right)\leqslant c\left(1+|z|+N(x)\right). \tag{7.2}$$

By the Lebesgue monotone convergence theorem and the fact that $\hat{T}_z < +\infty$, we deduce that $M_{\hat{T}_z}$ is $\mathbb{P}_x$-integrable and

$$\mathbb{E}_x\left(\left|M_{\hat{T}_z}\right|\right) \leqslant c\,(1+|z|+N(x))\,.$$

In the same manner, using (7.1), (7.2) and Lemmas 5.5 and 5.6, we conclude that M_{τ_y} and M_{T_z} are $\mathbb{P}_x$-integrable and

$$\max\left\{\mathbb{E}_x\left(\left|M_{\tau_y}\right|\right), \mathbb{E}_x\left(\left|M_{T_z}\right|\right)\right\} \leqslant c\,(1+|z|+N(x))\,.$$

The assertion of the lemma follows obviously from the last two inequalities. □

PROPOSITION 7.2.

1. *Let $x \in \mathbb{X}$, $y \in \mathbb{R}$ and $z = y + r(x)$. Then*

$$V(x,y) = \lim_{n\to+\infty} \mathbb{E}_x\left(z+M_n\,;\,\tau_y>n\right) = \lim_{n\to+\infty} \mathbb{E}_x\left(y+S_n\,;\,\tau_y>n\right)$$

and

$$W(x,z) = \lim_{n\to+\infty} \mathbb{E}_x\left(z+M_n\,;\,T_z>n\right),$$
$$\hat{W}(x,z) = \lim_{n\to+\infty} \mathbb{E}_x\left(z+M_n\,;\,\hat{T}_z>n\right).$$

2. *For any $x \in \mathbb{X}$, the functions $y \mapsto V(x,y)$, $z \mapsto W(x,z)$ and $z \mapsto \hat{W}(x,z)$ are non-decreasing on $\mathbb{R}$.*
3. *There exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0,\varepsilon_0)$, $x \in \mathbb{X}$, $z \in \mathbb{R}$ and any integer $k_0 \geqslant 2$,*

$$\hat{W}(x,z) \leqslant \left(1+\frac{c_\varepsilon}{k_0^\varepsilon}\right)(\max(z,0)+cN(x)) + c_\varepsilon k_0^{1/2} \tag{7.3}$$

and, for any $x \in \mathbb{X}$, $y \in \mathbb{R}$ and $z = y + r(x)$,

$$0 \leqslant \min\left\{V(x,y), W(x,z)\right\} \leqslant \max\left\{V(x,y), W(x,z)\right\} \leqslant \hat{W}(x,y). \tag{7.4}$$

In particular, for any $x \in \mathbb{X}$ and $y \in \mathbb{R}$,

$$0 \leqslant V(x,y) \leqslant c\,(1+\max(y,0)+N(x))\,. \tag{7.5}$$

4. *For any $x \in \mathbb{X}$ and $y \in \mathbb{R}$,*

$$V(x,y) = \mathbf{Q}_+ V(x,y) := \mathbb{E}_x\left(V(X_1, y+S_1)\,;\,\tau_y>1\right)$$

and $\left(V\left(X_n, y+S_n\right)\mathbb{1}_{\{\tau_y>n\}}\right)_{n\geqslant 0}$ is a $\mathbb{P}_x$-martingale.

PROOF. *Claim 1.* Let υ be any of the stopping times τ_y, T_z, or $\hat{T}_z$. By the martingale property, for $n \geqslant 1$,

$$\mathbb{E}_x\left(z+M_n\,;\,\upsilon>n\right) = z\mathbb{P}_x\left(\upsilon>n\right) - \mathbb{E}_x\left(M_\upsilon\,;\,\upsilon\leqslant n\right).$$

Using Lemmas 5.5, 5.6, 5.7, 7.1 and the Lebesgue dominated convergence theorem,

$$\mathbb{E}_x\left(z+M_n\,;\,\upsilon>n\right)=-\mathbb{E}_x\left(M_\upsilon\right).$$

Moreover, by (5.1),

$$\mathbb{E}_x\left(y+S_n\,;\,\tau_y>n\right)=\mathbb{E}_x\left(z+M_n\,;\,\tau_y>n\right)-\mathbb{E}_x\left(r\left(X_n\right)\,;\,\tau_y>n\right).$$

Since, by Lemma 5.1, the point 1 of Hypothesis **M4** and Lemma 5.5, we have

$$\begin{aligned}\left|\mathbb{E}_x\left(r\left(X_n\right)\,;\,\tau_y>n\right)\right| &\leqslant c\mathbb{E}_x^{1/2}\left(\left(1+N\left(X_n\right)\right)^2\right)\mathbb{P}_x^{1/2}\left(\tau_y>n\right)\\ &\leqslant c\left(1+N(x)\right)\mathbb{P}_x^{1/2}\left(\tau_y>n\right)\underset{n\to+\infty}{\longrightarrow}0,\end{aligned}\tag{7.6}$$

the claim 1 follows.

Proof of the claim 2. Let $x\in\mathbb{X}$. For any $y'\leqslant y$, we obviously have $\tau_{y'}\leqslant\tau_y$. Therefore, for $n\geqslant 1$,

$$\mathbb{E}_x\left(y'+S_n\,;\,\tau_{y'}>n\right)\leqslant\mathbb{E}_x\left(y+S_n\,;\,\tau_{y'}>n\right)\leqslant\mathbb{E}_x\left(y+S_n\,;\,\tau_y>n\right).$$

Taking the limit as $n\to+\infty$ and using the claim 1, it follows that $V(x,y')\leqslant V(x,y)$. In the same way $W(x,z')\leqslant W(x,z)$ for $z'\leqslant z$. To prove the monotonicity of $\hat{W}$, we note that, for any $z'\leqslant z$, $y'=z'-r(x)$ and $y=z-r(x)$, we have $\hat{T}_{z'}=\min\{k\geqslant\tau_{y'}:z'+M_k\leqslant 0\}\leqslant\min\{k\geqslant\tau_y:z'+M_k\leqslant 0\}\leqslant\hat{T}_z$. So

$$\begin{aligned}\mathbb{E}_x\left(z'+M_n\,;\,\hat{T}_{z'}>n\right) &\leqslant\mathbb{E}_x\left(z+M_n\,;\,\hat{T}_{z'}>n\,,\,\hat{T}_z>n\right)\\ &\leqslant\mathbb{E}_x\left(y+S_n\,;\,\tau_y>n\right)+\mathbb{E}_x\left(\left|r\left(X_n\right)\right|\,;\,\tau_y>n\right)\\ &\qquad+\mathbb{E}_x\left(z+M_n\,;\,\tau_y\leqslant n\,,\,\hat{T}_z>n\right)\\ &\leqslant\mathbb{E}_x\left(z+M_n\,;\,\hat{T}_z>n\right)+2\mathbb{E}_x\left(\left|r\left(X_n\right)\right|\,;\,\tau_y>n\right).\end{aligned}$$

As in (7.6), taking the limit as $n\to+\infty$, by the claim 1, we have $\hat{W}(x,z')\leqslant\hat{W}(x,z)$.

Proof of the claim 3. The inequality (7.3) is a direct consequence of the claim 1 and Lemma 6.4. Moreover, taking the limit as $n\to\infty$ in (6.27), we get $V(x,y)\leqslant\hat{W}(x,z)$.

To bound W, we write, for $n\geqslant 1$,

$$\begin{aligned}\mathbb{E}_x\left(z+M_n\,;\,T_z>n\right) &\leqslant\mathbb{E}_x\left(z+M_n\,;\,\tau_y\leqslant n\,,\,\hat{T}_z>n\,,\,T_z>n\right)\\ &\qquad+\mathbb{E}_x\left(z+M_n\,;\,z+M_n>0\,,\,\tau_y>n\,,\,T_z>n\right).\end{aligned}$$

Since $z+M_n>0$ on the event $\{\tau_y\leqslant n\,,\,\hat{T}_z>n\}$,

$$\begin{aligned}\mathbb{E}_x\left(z+M_n\,;\,T_z>n\right) &\leqslant\mathbb{E}_x\left(z+M_n\,;\,\tau_y\leqslant n\,,\,\hat{T}_z>n\right)\\ &\qquad+\mathbb{E}_x\left(z+M_n\,;\,z+M_n>0\,,\,\tau_y>n\right)\\ &=\mathbb{E}_x\left(z+M_n\,;\,\hat{T}_z>n\right)\\ &\qquad-\mathbb{E}_x\left(z+M_n\,;\,z+M_n\leqslant 0\,,\,\tau_y>n\right).\end{aligned}$$

Using the approximation (5.1),

$$\mathbb{E}_x\left(z+M_n\,;\,T_z>n\right)\leqslant\mathbb{E}_x\left(z+M_n\,;\,\hat{T}_z>n\right)+\mathbb{E}_x\left(\left|r\left(X_n\right)\right|\,;\,\tau_y>n\right). \tag{7.7}$$

As in (7.6), using the claim 1,

$$W(x,z)\leqslant\hat{W}(x,z).$$

Now, since $y+S_n$ is positive on the event $\{\tau_y>n\}$, by the claim 1, we see that $V(x,y)\geqslant 0$ and in the same way, $W(x,z)\geqslant 0$. This proves (7.4).

Inequality (7.5) follows from (7.3) and (7.4).

Proof of the claim 4. By the Markov property, for $n\geqslant 1$,

$$\begin{aligned}V_{n+1}(x,y)&:=\mathbb{E}_x\left(y+S_{n+1}\,;\,\tau_y>n+1\right)\\&=\int_{\mathbb{X}\times\mathbb{R}}V_n(x',y')\mathbb{P}_x\left(X_1\in\mathrm{d}x'\,,\,y+S_1\in\mathrm{d}y'\,,\,\tau_y>1\right),\end{aligned} \tag{7.8}$$

where, by Corollary 6.5, $V_n(x',y')\leqslant c\left(1+|y'|+N\left(x'\right)\right)$ and by the point 1 of Hypothesis **M4**,

$$\mathbb{E}_x\left(1+|y+S_1|+N\left(X_1\right)\right)\leqslant c\left(1+|y|+N(x)\right)<+\infty.$$

Taking the limit in (7.8), by the Lebesgue dominated convergence theorem, we have

$$V(x,y)=\mathbf{Q}_+V(x,y):=\mathbb{E}_x\left(V\left(X_1,y+S_1\right)\,;\,\tau_y>1\right).$$

□

8. Positivity of the harmonic function. The aim of this section is to prove that the harmonic function V is non-identically zero and to precise its support.

For any $x\in\mathbb{X}$, $z\in\mathbb{R}$ and $n\geqslant 0$, denote for brevity,

$$\hat{W}_n(x,z)=\hat{W}\left(X_n,z+M_n\right)\mathbb{1}_{\left\{\hat{T}_z>n\right\}}. \tag{8.1}$$

Although it is easy to verify that $\hat{W}(x,z)\geqslant z$ (see Lemma 8.1) which, in turn, ensures that $\hat{W}(x,z)>0$ for any $z>0$, it is not straightforward to give a lower bound for the function V. We show that $V(x,y)=\lim_{n\to+\infty}\mathbb{E}_x(\hat{W}_n(x,z)\,;\,\tau_y>n)$ (Lemma 8.2) and use the fact that $(\hat{W}_n(x,z)\mathbb{1}_{\{\tau_y>n\}})_{n\geqslant 0}$ is a $\mathbb{P}_x$-supermartingale (Lemma 8.1). By a recurrent procedure similar to that used in Lemma 6.4, we obtain a lower bound for V (Lemma 8.6) which subsequently is used to prove the positivity of V (Lemma 8.8).

LEMMA 8.1.

1. *For any $x\in\mathbb{X}$ and $z\in\mathbb{R}$,*

$$\hat{W}(x,z)\geqslant z.$$

2. *For any $x\in\mathbb{X}$,*

$$\lim_{z\to+\infty}\frac{\hat{W}(x,z)}{z}=1.$$

3. The function $\hat{W}$ is subharmonic, i.e. for any $x \in \mathbb{X}$, $z \in \mathbb{R}$ and $n \geqslant 0$,

$$\mathbb{E}_x \left(\hat{W}_n(x,z) \right) \geqslant \hat{W}(x,z).$$

4. For any $x \in \mathbb{X}$ and $z \in \mathbb{R}$, $\left(\hat{W}_n(x,z) \mathbb{1}_{\{\tau_y > n\}} \right)_{n \geqslant 0}$ is a $\mathbb{P}_x$-supermartingale.

PROOF. *Claim 1.* By the Doob optional theorem and the definition of $\hat{T}_z$, for any $n \geqslant 1$,

$$\mathbb{E}_x \left(z + M_n \,;\, \hat{T}_z > n \right) = z - \mathbb{E}_x \left(z + M_{\hat{T}_z} \,;\, \hat{T}_z \leqslant n \right) \geqslant z.$$

Taking the limit as $n \to +\infty$ and using the point 1 of Proposition 7.2 proves the claim 1.

Proof of the claim 2. By the claim 1, $\liminf_{z \to +\infty} \hat{W}(x,z)/z \geqslant 1$. Moreover, by (7.3), for any $k_0 \geqslant 2$,

$$\limsup_{z \to \infty} \frac{\hat{W}(x,z)}{z} \leqslant \left(1 + \frac{c_\varepsilon}{{k_0}^\varepsilon} \right).$$

Taking the limit as $k_0 \to +\infty$, the claim follows.

Proof of the claim 3. Recall the notation $y = z - r(x)$. Using the Markov property, as in the proof of Lemma 5.3, for any $k \geqslant 1$,

$$\begin{aligned} \mathbb{E}_x \left(z + M_{n+k} \,;\, \hat{T}_z > n+k \right) = \int_{\mathbb{X} \times \mathbb{R}} & \mathbb{E}_{x'} \left(z' + M_n \,;\, \hat{T}_{z'} > n \right) \\ & \times \mathbb{P}_x \left(X_k \in \mathrm{d}x' \,,\, z + M_k \in \mathrm{d}z' \,,\, \tau_y > k \right) \\ + \int_{\mathbb{X} \times \mathbb{R}} & \mathbb{E}_{x'} \left(z' + M_n \,;\, T_{z'} > n \right) \\ & \times \mathbb{P}_x \left(X_k \in \mathrm{d}x' \,,\, z + M_k \in \mathrm{d}z' \,,\, \tau_y \leqslant k \,,\, \hat{T}_z > k \right). \end{aligned} \tag{8.2}$$

We shall find the limits as $n \to +\infty$ of the two terms in the right hand side. By Lemmas 6.4 and 5.1, $\mathbb{E}_{x'} \left(z' + M_n \,;\, \hat{T}_{z'} > n \right) \leqslant c \left(1 + |y'| + N(x') \right)$, with $y' = z' - r(x')$. Moreover by the point 1 of Hypothesis **M4**, $\mathbb{E}_x \left(1 + |y + S_k| + N(X_k) \right) \leqslant ck \left(1 + |y| + N(x) \right) < +\infty$. So, by the Lebesgue dominated convergence theorem and the point 1 of Proposition 7.2,

$$\begin{aligned} \int_{\mathbb{X} \times \mathbb{R}} \mathbb{E}_{x'} \left(z' + M_n \,;\, \hat{T}_{z'} > n \right) & \mathbb{P}_x \left(X_k \in \mathrm{d}x' \,,\, z + M_k \in \mathrm{d}z' \,,\, \tau_y > k \right) \\ & \underset{n \to +\infty}{\longrightarrow} \mathbb{E}_x \left(\hat{W} \left(X_k, z + M_k \right) \,;\, \tau_y > k \right). \end{aligned} \tag{8.3}$$

Moreover, using (7.7), Lemmas 6.4 and 5.1 and the point 1 of Hypothesis **M4**,

$$\mathbb{E}_{x'} \left(z' + M_n \,;\, T_{z'} > n \right) \leqslant c \left(1 + |z'| + N(x') \right).$$

Again, by the Lebesgue dominated convergence theorem and the point 1 of Proposition 7.2, we have

$$\begin{aligned} \int_{\mathbb{X} \times \mathbb{R}} \mathbb{E}_{x'} \left(z' + M_n \,;\, T_{z'} > n \right) & \mathbb{P}_x \left(X_k \in \mathrm{d}x' \,,\, z + M_k \in \mathrm{d}z' \,,\, \tau_y \leqslant k \,,\, \hat{T}_z > k \right) \\ & \underset{n \to +\infty}{\longrightarrow} \mathbb{E}_x \left(W \left(X_k, z + M_k \right) \,;\, \tau_y \leqslant k \,,\, \hat{T}_z > k \right). \end{aligned} \tag{8.4}$$

Putting together (8.2), (8.3), (8.4) and using the point 1 of Proposition 7.2,

$$\begin{aligned}\hat{W}(x,z) &= \mathbb{E}_x\left(\hat{W}\left(X_k, z+M_k\right) \,;\, \tau_y > k\right) \\ &\quad + \mathbb{E}_x\left(W\left(X_k, z+M_k\right) \,;\, \tau_y \leqslant k\,,\, \hat{T}_z > k\right). \end{aligned} \tag{8.5}$$

Now, taking into account (7.4) and the identity $\{\tau_y > k\} = \{\tau_y > k,\ \hat{T}_z > k\}$, we obtain the claim 3.

Proof of the claim 4. By the point 3 of Proposition 7.2, W is a non-negative function. Therefore, using (8.5),

$$\hat{W}(x,z) \geqslant \mathbb{E}_x\left(\hat{W}\left(X_1, z+M_1\right) \,;\, \tau_y > 1\right),$$

which implies that $\left(\hat{W}_n(x,z)\mathbb{1}_{\{\tau_y>n\}}\right)_{n\geqslant 0}$ is a supermartingale. □

LEMMA 8.2. *For any $x \in \mathbb{X}$, $y \in \mathbb{R}$ and $z = y + r(x)$,*

$$V(x,y) = \lim_{n\to+\infty} \mathbb{E}_x\left(\hat{W}_n(x,z)\,;\, \tau_y > n\right).$$

PROOF. For any $n \geqslant 1$, $x \in \mathbb{X}$, $y \in \mathbb{R}$ and $z = y + r(x)$,

$$\mathbb{E}_x\left(z+M_n\,;\,\tau_y>n\right) = \mathbb{E}_x\left(z+M_n\,;\,\hat{T}_z>n\right) - \mathbb{E}_x\left(z+M_n\,;\,\tau_y\leqslant n\,,\,\hat{T}_z>n\right).$$

By the point 1 of Lemma 8.1, $z + M_n \leqslant \hat{W}_n(x,z)$ and therefore

$$\begin{aligned}\mathbb{E}_x\left(z+M_n\,;\,\tau_y>n\right) &\geqslant \mathbb{E}_x\left(z+M_n\,;\,\hat{T}_z>n\right) - \mathbb{E}_x\left(\hat{W}_n(x,z)\right) \\ &\quad + \mathbb{E}_x\left(\hat{W}_n(x,z)\,;\,\tau_y>n\right).\end{aligned} \tag{8.6}$$

Moreover, by (7.3), for any $\delta > 0$,

$$\begin{aligned}\mathbb{E}_x\left(\hat{W}_n(x,z)\right) \leqslant&\ (1+\delta)\,\mathbb{E}_x\left(z+M_n\,;\,\hat{T}_z>n\right) + c_\delta \mathbb{E}_x\left(1+N\left(X_n\right)\,;\,\hat{T}_z>n\right) \\ &- (1+\delta)\mathbb{E}_x\left(z+M_n\,;\,z+M_n<0\,,\,\tau_y>n\right).\end{aligned}$$

On the event $\{z+M_n<0\,,\,\tau_y>n\}$, by (5.1), it holds $r\left(X_n\right) < z + M_n < 0$. Therefore, using Lemma 5.1,

$$\mathbb{E}_x\left(\hat{W}_n(x,z)\right) \leqslant (1+\delta)\,\mathbb{E}_x\left(z+M_n\,;\,\hat{T}_z>n\right) + c_\delta \mathbb{E}_x\left(1+N\left(X_n\right)\,;\,\hat{T}_z>n\right).$$

By the Markov property and (2.2),

$$\begin{aligned}\mathbb{E}_x\left(1+N\left(X_n\right)\,;\,\hat{T}_z>n\right) &\leqslant c\mathbb{E}_x\left(1+\mathrm{e}^{-cn/2}\,N\left(X_{\lfloor n/2\rfloor}\right)\,;\,\hat{T}_z>\lfloor n/2\rfloor\right) \\ &\leqslant c\mathbb{P}_x\left(\hat{T}_z>\lfloor n/2\rfloor\right) + c\,\mathrm{e}^{-cn}\left(1+N(x)\right).\end{aligned}$$

By Lemma 5.7 and the point 1 of Proposition 7.2,

$$\lim_{n\to+\infty}\mathbb{E}_x\left(\hat{W}_n(x,z)\right) \leqslant (1+\delta)\,\hat{W}(x,z). \tag{8.7}$$

Taking the limit as $n \to +\infty$ in (8.6) and using the previous bound, we obtain that

$$V(x,y) \geqslant -\delta \hat{W}(x,z) + \lim_{n\to+\infty} \mathbb{E}_x \left(\hat{W}_n(x,z)\,;\, \tau_y > n \right).$$

Since this inequality holds true for any $\delta > 0$ small enough, we obtain the bound

$$\lim_{n\to+\infty} \mathbb{E}_x \left(\hat{W}_n(x,z)\,;\, \tau_y > n \right) \leqslant V(x,y). \tag{8.8}$$

Now, by the point 1 of Lemma 8.1,

$$\mathbb{E}_x \left(z + M_n\,;\, \tau_y > n \right) \leqslant \mathbb{E}_x \left(\hat{W}(X_n, z + M_n)\,;\, \tau_y > n \right).$$

Taking the limit as $n \to +\infty$ and using the point 1 of Proposition 7.2, we obtain that

$$V(x,y) \leqslant \lim_{n\to+\infty} \mathbb{E}_x \left(\hat{W}_n(x,z)\,;\, \tau_y > n \right).$$

Together with (8.8), this concludes the proof. □

REMARK 8.3. Taking the limit in the point 3 of Lemma 8.1, we can deduce that $\lim_{n\to+\infty} \mathbb{E}_x \left(\hat{W}_n(x,z) \right) \geqslant \hat{W}(x,z)$. Coupling this result with (8.7), it follows that

$$\lim_{n\to+\infty} \mathbb{E}_x \left(\hat{W}_n(x,z) \right) = \hat{W}(x,z).$$

LEMMA 8.4. *There exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, $n \geqslant 1$, $x \in \mathbb{X}$, $z \in \mathbb{R}$ and $y = z - r(x)$, we have*

$$\mathbb{E}_x \left(\hat{W}_n(x,z)\,;\, \tau_y > n \right) \geqslant \hat{W}(x,z) + c\min(z,0) - c_\varepsilon \left(n^{1/2-2\varepsilon} + n^{2\varepsilon} N(x) \right).$$

PROOF. Using the point 3 of Lemma 8.1, the bound (7.3) and the point 1 of Hypothesis **M4**, we have, for any $n \geqslant 1$,

$$\begin{aligned}
\mathbb{E}_x \left(\hat{W}_n(x,z)\,;\, \tau_y > n \right) &= \mathbb{E}_x \left(\hat{W}_n(x,z) \right) - \mathbb{E}_x \left(\hat{W}_n(x,z)\,;\, \tau_y \leqslant n \right) \\
&\geqslant \hat{W}(x,z) - c\mathbb{E}_x \left(z + M_n\,;\, \tau_y \leqslant n\,,\, \hat{T}_z > n \right) - c\left(1 + N(x)\right).
\end{aligned}$$

Again by the point 1 of **M4**, Lemma 6.2 and the Doob optional stopping theorem,

$$\begin{aligned}
\mathbb{E}_x \left(\hat{W}_n(x,z)\,;\, \tau_y > n \right) &\geqslant \hat{W}(x,z) - c\left[\mathbb{E}_x \left(z + M_n\,;\, \hat{T}_z > n \right) - \mathbb{E}_x \left(z + M_n\,;\, \tau_y > n \right) \right] \\
&\qquad - c\left(1 + N(x)\right) \\
&\geqslant \hat{W}(x,z) - c\left[\max(z,0) - z + \mathbb{E}_x \left(z + M_{\tau_y}\,;\, \tau_y \leqslant n \right) \right] \\
&\qquad - c_\varepsilon \left(n^{1/2-2\varepsilon} + n^{2\varepsilon} N(x) \right) - c\left(1 + N(x)\right).
\end{aligned}$$

By (5.1), $z + M_{\tau_y} \leqslant r\left(X_{\tau_y}\right)$. Therefore, in the same way as in the proof of (6.2),

$$\mathbb{E}_x \left(z + M_{\tau_y}\,;\, \tau_y \leqslant n \right) \leqslant c\mathbb{E}_x \left(1 + N\left(X_{\tau_y}\right)\,;\, \tau_y \leqslant n \right) \leqslant c_\varepsilon n^{1/2-2\varepsilon} + c_\varepsilon N(x).$$

Together with the previous bound, this implies that

$$\mathbb{E}_x \left(\hat{W}_n(x,z)\,;\, \tau_y > n \right) \geqslant \hat{W}(x,z) + c\min(z,0) - c_\varepsilon \left(n^{1/2-2\varepsilon} + n^{2\varepsilon} N(x) \right).$$

□

LEMMA 8.5. *There exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, $n \geqslant 2$, $k_0 \in \{2, \dots, n\}$, $x \in \mathbb{X}$ and $z \in \mathbb{R}$, with $y = z - r(x)$, we have*

$$\mathbb{E}_x \left(\hat{W}_n(x,z) \,;\, \tau_y > n \right) \geqslant \mathbb{E}_x \left(\hat{W}_{k_0}(x,z) \,;\, \tau_y > k_0 \right) - \frac{c_\varepsilon}{{k_0}^\varepsilon} \left(\max(z,0) + 1 + N(x) \right).$$

PROOF. Let $\varepsilon \in (0,1)$. Set for brevity $u_n := \mathbb{E}_x(\hat{W}_n(x,z) \,;\, \tau_y > n)$ for $n \geqslant 1$. By the point 4 of Lemma 8.1, the sequence $(u_n)_{n \geqslant 1}$ is non-increasing. We shall prove that

$$u_n \geqslant u_{\lfloor n^{1-\varepsilon} \rfloor} - \frac{c_\varepsilon}{n^\varepsilon} \left(\max(z,0) + 1 + N(x) \right). \tag{8.9}$$

By Lemma 9.2 of [18] on the convergence of recursively bounded non-increasing sequences, we conclude that, for any $n \geqslant 2$ and $k_0 \in \{2, \dots, n\}$,

$$u_n \geqslant u_{k_0} - \frac{c_\varepsilon}{{k_0}^\varepsilon} \left(\max(z,0) + 1 + N(x) \right),$$

which proves the assertion of the lemma.

It remains to establish (8.9). Consider the stopping time $\nu_n^\varepsilon = \nu_n + \lfloor n^\varepsilon \rfloor$. By the Markov property, with $y' = z' - r(x')$,

$$\begin{aligned} u_n &\geqslant \mathbb{E}_x \left(\hat{W}_n(x,z) \,;\, \tau_y > n \,,\, \nu_n^\varepsilon \leqslant \left\lfloor n^{1-\varepsilon} \right\rfloor \right) \\ &= \sum_{k=\lfloor n^\varepsilon \rfloor + 1}^{\lfloor n^{1-\varepsilon} \rfloor} \int_{\mathbb{X} \times \mathbb{R}} \mathbb{E}_{x'} \left(\hat{W}_{n-k}(x',z') \,;\, \tau_{y'} > n-k \right) \\ &\qquad\qquad \mathbb{P}_x \left(X_k \in \mathrm{d}x' \,,\, z + M_k \in \mathrm{d}z' \,,\, \tau_y > k \,,\, \nu_n^\varepsilon = k \right). \end{aligned}$$

Using Lemma 8.4, we obtain,

$$\begin{aligned} u_n &\geqslant \mathbb{E}_x \left(\hat{W}_{\nu_n^\varepsilon}(x,z) \,;\, \tau_y > \nu_n^\varepsilon \,,\, \nu_n^\varepsilon \leqslant \left\lfloor n^{1-\varepsilon} \right\rfloor \right) \\ &\quad + c\mathbb{E}_x \left(\min \left(z + M_{\nu_n^\varepsilon}, 0 \right) \,;\, \tau_y > \nu_n^\varepsilon \,,\, \nu_n^\varepsilon \leqslant \left\lfloor n^{1-\varepsilon} \right\rfloor \right) \\ &\quad - c_\varepsilon \mathbb{E}_x \left(n^{1/2 - 2\varepsilon} + n^{2\varepsilon} N \left(X_{\nu_n^\varepsilon} \right) \,;\, \tau_y > \nu_n^\varepsilon \,,\, \nu_n^\varepsilon \leqslant \left\lfloor n^{1-\varepsilon} \right\rfloor \right). \end{aligned}$$

On the event $\{z + M_{\nu_n^\varepsilon} \leqslant 0 \,,\, \tau_y > \nu_n^\varepsilon\}$, by (5.1), we have $0 \geqslant z + M_{\nu_n^\varepsilon} \geqslant r\left(X_{\nu_n^\varepsilon}\right)$. Therefore, by Lemma 5.1,

$$\begin{aligned} &\mathbb{E}_x \left(\min \left(z + M_{\nu_n^\varepsilon}, 0 \right) \,;\, \tau_y > \nu_n^\varepsilon \,,\, \nu_n^\varepsilon \leqslant \left\lfloor n^{1-\varepsilon} \right\rfloor \right) \\ &\qquad\qquad \geqslant -c\mathbb{E}_x \left(1 + N \left(X_{\nu_n^\varepsilon} \right) \,;\, \tau_y > \nu_n^\varepsilon \,,\, \nu_n^\varepsilon \leqslant \left\lfloor n^{1-\varepsilon} \right\rfloor \right). \end{aligned}$$

Consequently, using the point 4 of Lemma 8.1 and (2.2),

$$\begin{aligned} u_n &\geqslant \mathbb{E}_x \left(\hat{W}_{\lfloor n^{1-\varepsilon} \rfloor}(x,z) \,;\, \tau_y > \left\lfloor n^{1-\varepsilon} \right\rfloor \,,\, \nu_n^\varepsilon \leqslant \left\lfloor n^{1-\varepsilon} \right\rfloor \right) \\ &\quad - c_\varepsilon \mathbb{E}_x \left(n^{1/2 - 2\varepsilon} + \mathrm{e}^{-c_\varepsilon n^\varepsilon} N \left(X_{\nu_n} \right) \,;\, \tau_y > \nu_n \,,\, \nu_n \leqslant \left\lfloor n^{1-\varepsilon} \right\rfloor \right). \end{aligned}$$

By the definition of ν_n, we have $n^{1/2-2\varepsilon} \leqslant (z + M_{\nu_n})/n^{\varepsilon}$. Then as in (6.16),

$$\begin{aligned} u_n \geqslant & \, \mathbb{E}_x \left(\hat{W}_{\lfloor n^{1-\varepsilon} \rfloor}(x,z) \,;\, \tau_y > \left\lfloor n^{1-\varepsilon} \right\rfloor \,,\, \nu_n^{\varepsilon} \leqslant \left\lfloor n^{1-\varepsilon} \right\rfloor \right) \\ & - \frac{c_\varepsilon}{n^\varepsilon} \mathbb{E}_x \left(z + M_{\nu_n} \,;\, \tau_y > \nu_n \,,\, \nu_n \leqslant \left\lfloor n^{1-\varepsilon} \right\rfloor \right) \\ & - c_\varepsilon \, \mathrm{e}^{-c_\varepsilon n^\varepsilon} \left(1 + N(x) \right). \end{aligned}$$

Rearranging the terms, we have

$$\begin{aligned} u_n \geqslant & \, u_{\lfloor n^{1-\varepsilon} \rfloor} - c_\varepsilon \, \mathrm{e}^{-c_\varepsilon n^\varepsilon} \left(1 + N(x) \right) \\ & - \frac{c_\varepsilon}{n^\varepsilon} \underbrace{\mathbb{E}_x \left(z + M_{\nu_n} \,;\, \tau_y > \nu_n \,,\, \nu_n \leqslant \left\lfloor n^{1-\varepsilon} \right\rfloor \right)}_{=:I_1} \\ & - \underbrace{\mathbb{E}_x \left(\hat{W}_{\lfloor n^{1-\varepsilon} \rfloor}(x,z) \,;\, \tau_y > \left\lfloor n^{1-\varepsilon} \right\rfloor \,,\, \nu_n^{\varepsilon} > \left\lfloor n^{1-\varepsilon} \right\rfloor \right)}_{=:I_2}. \end{aligned} \tag{8.10}$$

Bound of I_1. To bound I_1 we use the facts that, by the definition of ν_n, $z + M_{\nu_n} > n^{1/2-\varepsilon} > 0$ and that $\hat{T}_z \geqslant \tau_y$. Taking into account Lemma 5.4, we have

$$\begin{aligned} I_1 & \leqslant \mathbb{E}_x \left(z + M_{\lfloor n^{1-\varepsilon} \rfloor} \,;\, \hat{T}_z > \left\lfloor n^{1-\varepsilon} \right\rfloor \,,\, \nu_n \leqslant \left\lfloor n^{1-\varepsilon} \right\rfloor \right) \\ & \leqslant \mathbb{E}_x \left(z + M_{\lfloor n^{1-\varepsilon} \rfloor} \,;\, \hat{T}_z > \left\lfloor n^{1-\varepsilon} \right\rfloor \right) - J_{21}'', \end{aligned}$$

where J_{21}'' is defined in (6.18). Now, it follows from Lemma 5.4 and the point 1 of Proposition 7.2, that $(\mathbb{E}_x(z + M_{\lfloor n^{1-\varepsilon} \rfloor} \,;\, \hat{T}_z > \lfloor n^{1-\varepsilon} \rfloor))_{n \geqslant 0}$ is a non-decreasing sequence which converges to $\hat{W}(x,z)$ and so $\mathbb{E}_x(z + M_{\lfloor n^{1-\varepsilon} \rfloor} \,;\, \hat{T}_z > \lfloor n^{1-\varepsilon} \rfloor) \leqslant \hat{W}(x,z)$. Using (6.19), we find that

$$I_1 \leqslant \hat{W}(x,z) + c_\varepsilon \, \mathrm{e}^{-c_\varepsilon n^\varepsilon} \left(1 + N(x) \right). \tag{8.11}$$

Bound of I_2. By (8.1) and (7.3),

$$\begin{aligned} I_2 \leqslant & \, c\mathbb{E}_x \left(z + M_{\lfloor n^{1-\varepsilon} \rfloor} \left(1 - \mathbb{1}_{\left\{ z + M_{\lfloor n^{1-\varepsilon} \rfloor} < 0 \right\}} \right) \,;\, \hat{T}_z > \left\lfloor n^{1-\varepsilon} \right\rfloor \,,\, \nu_n^{\varepsilon} > \left\lfloor n^{1-\varepsilon} \right\rfloor \right) \\ & + c\mathbb{E}_x \left(1 + N \left(X_{\lfloor n^{1-\varepsilon} \rfloor} \right) \,;\, \hat{T}_z > \left\lfloor n^{1-\varepsilon} \right\rfloor \,,\, \nu_n^{\varepsilon} > \left\lfloor n^{1-\varepsilon} \right\rfloor \right). \end{aligned}$$

On the event $\{ z + M_{\lfloor n^{1-\varepsilon} \rfloor} < 0 \,,\, \hat{T}_z > \lfloor n^{1-\varepsilon} \rfloor \} = \{ z + M_{\lfloor n^{1-\varepsilon} \rfloor} < 0 \,,\, \tau_y > \lfloor n^{1-\varepsilon} \rfloor \}$, it holds $z + M_{\lfloor n^{1-\varepsilon} \rfloor} > r \left(X_{\lfloor n^{1-\varepsilon} \rfloor} \right)$. Therefore, using Lemma 5.1,

$$I_2 \leqslant c\mathbb{E}_x \left(z + M_{\lfloor n^{1-\varepsilon} \rfloor} + 1 + N \left(X_{\lfloor n^{1-\varepsilon} \rfloor} \right) \,;\, \hat{T}_z > \left\lfloor n^{1-\varepsilon} \right\rfloor \,,\, \nu_n^{\varepsilon} > \left\lfloor n^{1-\varepsilon} \right\rfloor \right).$$

By Lemma 5.4,

$$\mathbb{E}_x \left(z + M_{\lfloor n^{1-\varepsilon} \rfloor} \,;\, \hat{T}_z > \left\lfloor n^{1-\varepsilon} \right\rfloor \,,\, \nu_n^{\varepsilon} > \left\lfloor n^{1-\varepsilon} \right\rfloor \right) \leqslant J_1,$$

where J_1 is defined in (6.13). Using inequalities (6.14), (2.2) and Lemma 6.3, with $m_\varepsilon = \lfloor n^{1-\varepsilon} \rfloor - \lfloor n^\varepsilon \rfloor$, we obtain

$$\begin{aligned} I_2 &\leqslant c_\varepsilon \, \mathrm{e}^{-c_\varepsilon n^\varepsilon} \left(1 + N(x)\right) + c\mathbb{E}_x \left(1 + \mathrm{e}^{-cn^\varepsilon} N\left(X_{m_\varepsilon}\right) ; \hat{T}_z > m_\varepsilon \, , \, \nu_n > m_\varepsilon \right) \\ &\leqslant c_\varepsilon \, \mathrm{e}^{-c_\varepsilon n^\varepsilon} \left(1 + N(x)\right). \end{aligned} \tag{8.12}$$

Putting together (8.12), (8.11) and (8.10) and using (7.3), we obtain (8.9), which completes the proof of the lemma. □

Proposition 8.6.

1. For any $\delta \in (0,1)$, $x \in \mathbb{X}$ and $y > 0$,

$$V(x,y) \geqslant (1-\delta)\, y - c_\delta \left(1 + N(x)\right).$$

2. For any $x \in \mathbb{X}$,

$$\lim_{y \to +\infty} \frac{V(x,y)}{y} = 1.$$

Proof. *Claim 1.* By Lemmas 8.5 and 8.2, we immediately have, with $z = y + r(x)$,

$$V(x,y) \geqslant \mathbb{E}_x \left(\hat{W}_{k_0}(x,z) \, ; \, \tau_y > k_0 \right) - \frac{c_\varepsilon}{{k_0}^\varepsilon} \left(\max(z,0) + 1 + N(x)\right).$$

Using the point 1 of Lemma 8.1 and the point 2 of Lemma 5.2,

$$\begin{aligned} V(x,y) &\geqslant \mathbb{E}_x \left(z + M_{k_0} \, ; \, \tau_y > k_0\right) - \frac{c_\varepsilon}{{k_0}^\varepsilon} \left(\max(z,0) + 1 + N(x)\right) \\ &\geqslant z\mathbb{P}_x \left(\tau_y > k_0\right) - c\left(\sqrt{k_0} + N(x)\right) - \frac{c_\varepsilon}{{k_0}^\varepsilon} \left(\max(z,0) + 1 + N(x)\right). \end{aligned}$$

Since, by the union bound and the Markov inequality,

$$\mathbb{P}_x \left(\tau_y > k_0\right) \geqslant \mathbb{P}_x \left(\max_{1 \leqslant k \leqslant k_0} \left| f\left(X_k\right)\right| < \frac{y}{k_0} \right) \geqslant 1 - \frac{c{k_0}^2 \left(1 + N(x)\right)}{y},$$

we obtain that, by the definition of z,

$$V(x,y) \geqslant \left(1 - \frac{c_\varepsilon}{{k_0}^\varepsilon}\right) y - c_\varepsilon {k_0}^2 \left(1 + N(x)\right). \tag{8.13}$$

Let $\delta \in (0,1)$. Taking k_0 large enough, we obtain the desired inequality.

Proof of the claim 2. By the claim 1, for any $\delta \in (0,1)$ and $x \in \mathbb{X}$, we have that $\liminf_{y \to +\infty} V(x,y)/y \geqslant 1 - \delta$. Taking the limit as $\delta \to 0$, we obtain the lower bound. Now by (7.4) and (7.3), for any integer $k_0 \geqslant 2$, $y \in \mathbb{R}$ and $z = y + r(x)$,

$$V(x,y) \leqslant \hat{W}(x,z) \leqslant \left(1 + \frac{c_\varepsilon}{k_0^\varepsilon}\right) \left(\max(z,0) + cN(x)\right) + c_\varepsilon k_0^{1/2}.$$

Using the definition of z, we conclude that

$$\limsup_{y \to +\infty} \frac{V(x,y)}{y} \leqslant \lim_{k_0 \to +\infty} \left(1 + \frac{c_\varepsilon}{{k_0}^\varepsilon}\right) = 1.$$

□

Now, for any $\gamma > 0$, consider the stopping time:

$$\zeta_\gamma := \inf \left\{ k \geqslant 1 : |y + S_k| > \gamma \left(1 + N\left(X_k\right)\right) \right\}. \tag{8.14}$$

The control on the tail of ζ_γ is given by the following Lemma.

LEMMA 8.7. *For any $\gamma > 0$, $x \in \mathbb{X}$, $y \in \mathbb{R}$ and $n \geqslant 1$,*

$$\mathbb{P}_x \left(\zeta_\gamma > n \right) \leqslant c \,\mathrm{e}^{-c_\gamma n} \left(1 + N\left(x\right)\right).$$

PROOF. The reasoning is very close to that of the proof of the Lemma 6.3. Let $\gamma > 0$. Consider the integer $l \geqslant 1$ which will be chosen later. Define $K := \left\lfloor \frac{n}{2l} \right\rfloor$ and introduce the event $A_{k,y}^\gamma := \bigcap_{k' \in \{1,\dots,k\}} \left\{ |y + S_{k'l}| \leqslant \gamma \left(1 + N\left(X_{k'l}\right)\right) \right\}$. We have

$$\mathbb{P}_x \left(\zeta_\gamma > n \right) \leqslant \mathbb{P}_x \left(A_{2K,y}^\gamma \right).$$

By the Markov property,

$$\begin{aligned} \mathbb{P}_x \left(A_{2K,y}^\gamma \right) = \int_{\mathbb{X}\times\mathbb{R}} \int_{\mathbb{X}\times\mathbb{R}} & \mathbb{P}_{x''} \left(A_{1,y''}^\gamma \right) \mathbb{P}_{x'} \left(X_l \in \mathrm{d}x'' \,,\, y' + S_l \in \mathrm{d}y'' \,,\, A_{1,y'}^\gamma \right) \\ & \times \mathbb{P}_x \left(X_{2(K-1)l} \in \mathrm{d}x' \,,\, y + S_{2(K-1)l} \in \mathrm{d}y' \,,\, A_{2(K-1),y}^\gamma \right). \end{aligned} \tag{8.15}$$

We write

$$\begin{aligned} \mathbb{P}_{x''} \left(A_{1,y''}^\gamma \right) &\leqslant \mathbb{P}_{x''} \left(\left| y'' + S_l \right| \leqslant 2\gamma\sqrt{l} \right) + \mathbb{P}_{x''} \left(N\left(X_l\right) > \sqrt{l} \right) \\ &\leqslant \mathbb{P}_{x''} \left(\frac{-y''}{\sqrt{l}} - 2\gamma \leqslant \frac{S_l}{\sqrt{l}} \leqslant \frac{-y''}{\sqrt{l}} + 2\gamma \right) + \mathbb{E}_{x''} \left(\frac{N\left(X_l\right)}{\sqrt{l}} \right). \end{aligned}$$

By Corollary 4.4 and the point 1 of Hypothesis **M4**, there exists $\varepsilon_0 \in (0, 1/4)$ such that, for any $\varepsilon \in (0, \varepsilon_0)$,

$$\mathbb{P}_{x''} \left(A_{1,y''}^\gamma \right) \leqslant \int_{\frac{-y''}{\sqrt{l}} - 2\gamma}^{\frac{-y''}{\sqrt{l}} + 2\gamma} \mathrm{e}^{-\frac{u^2}{2\sigma^2}} \frac{\mathrm{d}u}{\sqrt{2\pi}\sigma} + \frac{2c_\varepsilon}{l^\varepsilon} \left(1 + N(x'')\right) + \frac{c}{\sqrt{l}} \left(1 + N\left(x''\right)\right).$$

Set $q_\gamma := \int_{-2\gamma}^{2\gamma} \mathrm{e}^{-\frac{u^2}{2\sigma^2}} \frac{\mathrm{d}u}{\sqrt{2\pi}\sigma} < 1$. From (8.15), we obtain

$$\begin{aligned} \mathbb{P}_x \left(A_{2K,y}^\gamma \right) \leqslant \int_{\mathbb{X}\times\mathbb{R}} & \left(q_\gamma + \frac{c_\varepsilon}{l^\varepsilon} + \frac{c_\varepsilon}{l^\varepsilon} \mathbb{E}_{x'} \left(N\left(X_l\right) \right) \right) \\ & \times \mathbb{P}_x \left(X_{2(K-1)l} \in \mathrm{d}x' \,,\, y + S_{2(K-1)l} \in \mathrm{d}y' \,,\, A_{2(K-1),y}^\gamma \right) \\ \leqslant & \left(q_\gamma + \frac{c_\varepsilon}{l^\varepsilon} \right) \mathbb{P}_x \left(A_{2(K-1),y}^\gamma \right) + c_\varepsilon \,\mathrm{e}^{-c_\varepsilon l} \,\mathbb{E}_x \left(N\left(X_{2(K-1)l} \right) \,;\, A_{2(K-1),y}^\gamma \right). \end{aligned}$$

For brevity, set $p_K = \mathbb{P}_x \left(A_{2K,y}^\gamma \right)$ and $E_K = \mathbb{E}_x \left(N\left(X_{2Kl}\right) \,;\, A_{2K,y}^\gamma \right)$. Then, the previous inequality can be rewritten as

$$p_K \leqslant \left(q_\gamma + \frac{c_\varepsilon}{l^\varepsilon} \right) p_{K-1} + c_\varepsilon \,\mathrm{e}^{-c_\varepsilon l} \, E_{K-1}. \tag{8.16}$$

Moreover, from (2.2), we have

$$E_K \leqslant c p_{K-1} + c\,\mathrm{e}^{-c2l}\, E_{K-1}. \tag{8.17}$$

Using (8.16) and (8.17), we write that

$$\begin{pmatrix} p_K \\ E_K \end{pmatrix} \leqslant A_l \begin{pmatrix} p_{K-1} \\ E_{K-1} \end{pmatrix} \tag{8.18}$$

where

$$A_l := \begin{pmatrix} q_\gamma + \frac{c_\varepsilon}{l^\varepsilon} & c_\varepsilon\,\mathrm{e}^{-c_\varepsilon l} \\ c & c\,\mathrm{e}^{-cl} \end{pmatrix} \underset{l\to+\infty}{\longrightarrow} A = \begin{pmatrix} q_\gamma & 0 \\ c & 0 \end{pmatrix}.$$

Since the spectral radius q_γ of A is less than 1, we can choose $l = l(\varepsilon,\gamma)$ large enough such that the spectral radius $\rho_{\varepsilon,\gamma}$ of A_l is less than 1. Iterating (8.18), we get

$$p_K \leqslant c\rho_{\varepsilon,\gamma}^K \max\left(p_1, E_1\right) \leqslant c\rho_{\varepsilon,\gamma}^K \left(1 + N(x)\right).$$

Taking into account that $K \geqslant c_{\varepsilon,\gamma} n$, we obtain

$$\mathbb{P}_x\left(A_{2K,y}^\gamma\right) \leqslant c\,\mathrm{e}^{-c_\gamma n}\left(1 + N(x)\right).$$

□

Now we shall establish some properties of the set $\mathscr{D}_\gamma$ introduced in Section 2. It is easy to see that, for any $\gamma > 0$,

$$\mathscr{D}_\gamma = \left\{(x,y) \in \mathbb{X}\times\mathbb{R} : \exists n_0 \geqslant 1, \mathbb{P}_x\left(\zeta_\gamma \leqslant n_0\,,\ \tau_y > n_0\right) > 0\right\},$$

where ζ_γ is defined by (8.14).

Proposition 8.8.

1. *For any $\gamma_1 \leqslant \gamma_2$, it holds $\mathscr{D}_{\gamma_1} \supseteq \mathscr{D}_{\gamma_2}$.*
2. *For any $\gamma > 0$, there exists $c_\gamma > 0$ such that*

$$\mathscr{D}_\gamma^c \subseteq \left\{(x,y) \in \mathbb{X}\times\mathbb{R} : \mathbb{P}_x\left(\tau_y > n\right) \leqslant \mathrm{e}^{-c_\gamma n}\left(1 + N(x)\right),\ n \geqslant 1\right\}.$$

3. *For any $\gamma > 0$, the domain of positivity of the function V is included in $\mathscr{D}_\gamma$:*

$$\mathrm{supp}(V) = \left\{(x,y) \in \mathbb{X}\times\mathbb{R} : V(x,y) > 0\right\} \subseteq \mathscr{D}_\gamma.$$

4. *There exists $\gamma_0 > 0$ such that for any $\gamma \geqslant \gamma_0$,*

$$\mathrm{supp}(V) = \mathscr{D}_\gamma.$$

Moreover,

$$\left\{(x,y) \in \mathbb{X}\times\mathbb{R}_+^* : y > \frac{\gamma_0}{2}\left(1 + N(x)\right)\right\} \subseteq \mathrm{supp}(V).$$

Proof. *Claim 1.* For any $\gamma_1 \leqslant \gamma_2$, we have $\zeta_{\gamma_1} \leqslant \zeta_{\gamma_2}$ and the claim 1 follows.

Claim 2. Fix $\gamma > 0$. By the definition of $\mathscr{D}_\gamma$, for any $(x,y) \in \mathscr{D}_\gamma^c$ and $n \geqslant 1$,

$$0 = \mathbb{P}_x \left(\zeta_\gamma \leqslant n \,,\, \tau_y > n \right) = \mathbb{P}_x \left(\tau_y > n \right) - \mathbb{P}_x \left(\zeta_\gamma > n \,,\, \tau_y > n \right).$$

From this, using Lemma 8.7, we obtain

$$\mathbb{P}_x \left(\tau_y > n \right) = \mathbb{P}_x \left(\zeta_\gamma > n \,,\, \tau_y > n \right) \leqslant \mathbb{P}_x \left(\zeta_\gamma > n \right) \leqslant \mathrm{e}^{-c_\gamma n} \left(1 + N\left(x\right) \right).$$

Claim 3. Fix $\gamma > 0$. Using the claim 2 and Lemma 5.2, we have, for any $(x,y) \in \mathscr{D}_\gamma^c$, $z = y + r(x)$ and $n \geqslant 1$,

$$\begin{aligned}\mathbb{E}_x \left(z + M_n \,;\, \tau_y > n \right) &\leqslant |z|\, \mathbb{P}_x \left(\tau_y > n \right) + \mathbb{E}_x^{1/2} \left(|M_n|^2 \right) \mathbb{P}_x^{1/2} \left(\tau_y > n \right) \\ &\leqslant |z| \left(1 + N\left(x\right) \right) \mathrm{e}^{-c_\gamma n} + c\sqrt{n} \left(1 + N(x) \right)^{3/2} \mathrm{e}^{-c_\gamma n}.\end{aligned}$$

Taking the limit when $n \to +\infty$, by the point 1 of Proposition 7.2, we get

$$V(x,y) = 0,$$

and we conclude that $\mathscr{D}_\gamma^c \subseteq \operatorname{supp}(V)^c$.

Claim 4. By the point 1 of Proposition 8.6, taking $\delta = 1/2$, there exists $\gamma_0 > 0$ such that, for any $x \in \mathbb{X}$ and $y > 0$,

$$V(x,y) \geqslant \frac{y}{2} - \frac{\gamma_0}{4} \left(1 + N(x) \right). \tag{8.19}$$

Now, fix $(x,y) \in \mathscr{D}_{\gamma_0}$ and let $n_0 \geqslant 1$ be an integer such that $\mathbb{P}_x \left(\zeta_{\gamma_0} \leqslant n_0 \,,\, \tau_y > n_0 \right) > 0$. By the point 4 of Proposition 7.2,

$$\begin{aligned}V(x,y) &= \mathbb{E}_x \left(V\left(X_{n_0}, y + S_{n_0} \right) \,;\, \tau_y > n_0 \right) \\ &\geqslant \mathbb{E}_x \left(V\left(X_{n_0}, y + S_{n_0} \right) \,;\, \tau_y > n_0 \,,\, \zeta_{\gamma_0} \leqslant n_0 \right).\end{aligned}$$

By the Doob optional stopping theorem, (8.19) and the definition of ζ_{γ_0} (see (8.14)),

$$\begin{aligned}V(x,y) &\geqslant \mathbb{E}_x \left(V\left(X_{\zeta_{\gamma_0}}, y + S_{\zeta_{\gamma_0}} \right) \,;\, \tau_y > \zeta_{\gamma_0} \,,\, \zeta_{\gamma_0} \leqslant n_0 \right) \\ &\geqslant \frac{1}{2} \mathbb{E}_x \left(y + S_{\zeta_{\gamma_0}} - \frac{\gamma_0}{2} \left(1 + N\left(X_{\zeta_{\gamma_0}} \right) \right) \,;\, \tau_y > \zeta_{\gamma_0} \,,\, \zeta_{\gamma_0} \leqslant n_0 \right) \\ &\geqslant \frac{1}{2} \mathbb{E}_x \left(\frac{\gamma_0}{2} \left(1 + N\left(X_{\zeta_{\gamma_0}} \right) \right) \,;\, \tau_y > \zeta_{\gamma_0} \,,\, \zeta_{\gamma_0} \leqslant n_0 \right) \\ &\geqslant \frac{\gamma_0}{4} \mathbb{P}_x \left(\tau_y > n_0 \,,\, \zeta_{\gamma_0} \leqslant n_0 \right).\end{aligned}$$

Now, since n_0 has been chosen such that the last probability is strictly positive, we get that $V(x,y) > 0$. This proves that $\mathscr{D}_{\gamma_0} \subseteq \operatorname{supp}(V)$. Using the claims 1 and 3, for any $\gamma \geqslant \gamma_0$, we obtain that $\mathscr{D}_\gamma \subseteq \mathscr{D}_{\gamma_0} \subseteq \operatorname{supp}(V) \subseteq \mathscr{D}_\gamma$ and so $\mathscr{D}_\gamma = \mathscr{D}_{\gamma_0} = \operatorname{supp}(V)$. Using (8.19) proves the second assertion of the claim 4. □

Proof of Theorem 2.2. The claim 1 is proved by the point 1 of Proposition 7.2 ; the claim 2 is proved by the point 4 of Proposition 7.2 ; the claim 3 is proved by the points 2 and 3 of Proposition 7.2 and by Proposition 8.6 ; the claim 4 is proved by the point 4 of Proposition 8.8.

9. Asymptotic behaviour of the exit time.

9.1. *Preliminary results.*

LEMMA 9.1. *There exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, $x \in \mathbb{X}$, $y \in \mathbb{R}$ and $z = y + r(x)$,*

$$E_1 := \mathbb{E}_x\left(z + M_{\nu_n}\,;\, \tau_y > \nu_n\,,\, \nu_n \leqslant \left\lfloor n^{1-\varepsilon}\right\rfloor\right) \leqslant c_\varepsilon\left(1 + \max(y,0) + N(x)\right), \quad n \geqslant 1,$$
$$E_2 := \mathbb{E}_x\left(z + M_{\nu_n^{\varepsilon^2}}\,;\, \tau_y > \nu_n^{\varepsilon^2}\,,\, \nu_n^{\varepsilon^2} \leqslant \left\lfloor n^{1-\varepsilon}\right\rfloor\right) \underset{n\to\infty}{\longrightarrow} V(x,y).$$

Moreover, for any $n \geqslant 1$, $\varepsilon \in (0, \varepsilon_0)$, $x \in \mathbb{X}$ and $y \in \mathbb{R}$,

$$|E_2 - V(x,y)| \leqslant \frac{c_\varepsilon}{n^{\varepsilon/8}}\left(1 + \max(y,0) + N(x)\right).$$

PROOF. Using the fact $\{\tau_y > \nu_n\} \subseteq \{\hat{T}_z > \nu_n\}$ and Lemma 5.4, for $n \geqslant 1$,

$$E_1 \leqslant \mathbb{E}_x\left(z + M_{\lfloor n^{1-\varepsilon}\rfloor}\,;\, \hat{T}_z > \left\lfloor n^{1-\varepsilon}\right\rfloor\right) - J_{21}'',$$

where J_{21}'' is defined in (6.18) and by (6.19) the quantity $-J_{21}''$ does not exceed $c_\varepsilon\,\mathrm{e}^{-c_\varepsilon n^\varepsilon}(1+N(x))$. Again, by Lemma 5.4 and the point 1 of Proposition 7.2, we have that $(\mathbb{E}_x(z + M_n\,;\, \hat{T}_z > n))_{n\geqslant 0}$ is a non-decreasing sequence which converges to $\hat{W}(x,z)$. So, using the point 3 of Proposition 7.2 and the fact that $z = y + r(x)$,

$$E_1 \leqslant \hat{W}(x,z) + c_\varepsilon\,\mathrm{e}^{-c_\varepsilon n^\varepsilon}\left(1 + N(x)\right) \leqslant c_\varepsilon\left(1 + \max(y,0) + N(x)\right). \tag{9.1}$$

By the point 4 of Proposition 7.2, we have

$$\begin{aligned}V(x,y) = {} & \mathbb{E}_x\left(V\left(X_n, y + S_n\right)\,;\, \tau_y > n\,,\, \nu_n^{\varepsilon^2} \leqslant \left\lfloor n^{1-\varepsilon}\right\rfloor\right)\\ & + \mathbb{E}_x\left(V\left(X_n, y + S_n\right)\,;\, \tau_y > n\,,\, \nu_n^{\varepsilon^2} > \left\lfloor n^{1-\varepsilon}\right\rfloor\right).\end{aligned}$$

Using the point 3 of Proposition 7.2, for any $k_0 \geqslant 2$,

$$\begin{aligned}V(x,y) \leqslant {} & \mathbb{E}_x\left(V\left(X_{\nu_n^{\varepsilon^2}}, y + S_{\nu_n^{\varepsilon^2}}\right)\,;\, \tau_y > \nu_n^{\varepsilon^2}\,,\, \nu_n^{\varepsilon^2} \leqslant \left\lfloor n^{1-\varepsilon}\right\rfloor\right)\\ & + c\mathbb{E}_x\left(\max\left(z + M_n, 0\right) + 1 + N\left(X_n\right)\,;\, \tau_y > n\,,\, \nu_n^{\varepsilon^2} > \left\lfloor n^{1-\varepsilon}\right\rfloor\right)\\ \leqslant {} & \left(1 + \frac{c_\varepsilon}{k_0{}^\varepsilon}\right)E_2 + c_\varepsilon\mathbb{E}_x\left(\sqrt{k_0} + N\left(X_{\nu_n^{\varepsilon^2}}\right)\,;\, \tau_y > \nu_n^{\varepsilon^2}\,,\, \nu_n^{\varepsilon^2} \leqslant \left\lfloor n^{1-\varepsilon}\right\rfloor\right)\\ & \underbrace{-c_\varepsilon\mathbb{E}_x\left(z + M_{\nu_n^{\varepsilon^2}}\,;\, z + M_{\nu_n^{\varepsilon^2}} < 0\,,\, \tau_y > \nu_n^{\varepsilon^2}\,,\, \nu_n^{\varepsilon^2} \leqslant \left\lfloor n^{1-\varepsilon}\right\rfloor\right)}_{=J_{22}'(\varepsilon^2)}\\ & + c\mathbb{E}_x\left(z + M_n + \left|r\left(X_n\right)\right| + 1 + N\left(X_n\right)\,;\, \tau_y > n\,,\, \nu_n^{\varepsilon^2} > \left\lfloor n^{1-\varepsilon}\right\rfloor\right).\end{aligned}$$

From the previous bound, using the Markov property, the bound (2.2) and the approximation (5.1), we get

$$\begin{aligned} V(x,y) \leqslant & \left(1+\frac{c_\varepsilon}{{k_0}^\varepsilon}\right) E_2 + J_{22}'(\varepsilon^2) + c\,\underbrace{\mathbb{E}_x\left(z+M_n\,;\,\hat{T}_z > n\,,\,\nu_n^{\varepsilon^2} > \left\lfloor n^{1-\varepsilon}\right\rfloor\right)}_{=J_1(\varepsilon^2)} \\ & + c_\varepsilon \mathbb{E}_x\left(\sqrt{k_0} + \mathrm{e}^{-cn^{\varepsilon^2}} N\left(X_{\nu_n}\right)\,;\,\tau_y > \nu_n\,,\,\nu_n \leqslant \left\lfloor n^{1-\varepsilon}\right\rfloor\right) \\ & + c\mathbb{E}_x\left(1+\mathrm{e}^{-c_\varepsilon n} N\left(X_{\lfloor n^{1-\varepsilon}\rfloor}\right)\,;\,\tau_y > \left\lfloor n^{1-\varepsilon}\right\rfloor\,,\,\nu_n^{\varepsilon^2} > \left\lfloor n^{1-\varepsilon}\right\rfloor\right). \end{aligned}$$

Proceeding in the same way as for the bound (6.24),

$$\begin{aligned} J_{22}'(\varepsilon^2) &\leqslant c_\varepsilon \mathbb{E}_x\left(1+\mathrm{e}^{-cn^{\varepsilon^2}} N\left(X_{\nu_n}\right)\,;\,\tau_y > \nu_n\,,\,\nu_n \leqslant \left\lfloor n^{1-\varepsilon}\right\rfloor\right) \\ &\leqslant \frac{c_\varepsilon}{n^{1/2-\varepsilon}} E_1 + c_\varepsilon\,\mathrm{e}^{-c_\varepsilon n^{\varepsilon^2}}\left(1+N(x)\right). \end{aligned}$$

Moreover, similarly as for the bound (6.14), we have

$$J_1(\varepsilon^2) \leqslant c_\varepsilon\,\mathrm{e}^{-c_\varepsilon n^{\varepsilon^2}}\left(1+N(x)\right).$$

Taking into account these bounds and using Lemma 6.3,

$$V(x,y) \leqslant \left(1+\frac{c_\varepsilon}{{k_0}^\varepsilon}\right) E_2 + \frac{c_\varepsilon\sqrt{k_0}}{n^{1/2-\varepsilon}} E_1 + c_\varepsilon\,\mathrm{e}^{-c_\varepsilon n^{\varepsilon^2}}\left(1+N(x)\right). \tag{9.2}$$

Analagously, by (8.13) and (5.1), we have the lower bound

$$\begin{aligned} V(x,y) &\geqslant \mathbb{E}_x\left(V\left(X_{\nu_n^{\varepsilon^2}}, y+S_{\nu_n^{\varepsilon^2}}\right)\,;\,\tau_y > \nu_n^{\varepsilon^2}\,,\,\nu_n^{\varepsilon^2} \leqslant \left\lfloor n^{1-\varepsilon}\right\rfloor\right) \\ &\geqslant \left(1-\frac{c_\varepsilon}{{k_0}^\varepsilon}\right) E_2 - c_\varepsilon {k_0}^2 \mathbb{E}_x\left(1+N\left(X_{\nu_n^{\varepsilon^2}}\right)\,;\,\tau_y > \nu_n^{\varepsilon^2}\,,\,\nu_n^{\varepsilon^2} \leqslant \left\lfloor n^{1-\varepsilon}\right\rfloor\right) \\ &\geqslant \left(1-\frac{c_\varepsilon}{{k_0}^\varepsilon}\right) E_2 - \frac{c_\varepsilon {k_0}^2}{n^{1/2-\varepsilon}} E_1 - c_\varepsilon {k_0}^2\,\mathrm{e}^{-c_\varepsilon n^{\varepsilon^2}}\left(1+N(x)\right). \end{aligned} \tag{9.3}$$

Taking $k_0 = n^{1/4-\varepsilon}$ in (9.3) and (9.2), we conclude that, for any $\varepsilon \in (0,1/8)$,

$$\left|V(x,y) - E_2\right| \leqslant \frac{c_\varepsilon}{n^{\varepsilon/8}} E_2 + \frac{c_\varepsilon}{n^\varepsilon}\left(E_1 + 1 + N(x)\right).$$

Again, using (9.3),

$$\left|V(x,y) - E_2\right| \leqslant \frac{c_\varepsilon}{n^{\varepsilon/8}} V(x,y) + \frac{c_\varepsilon}{n^\varepsilon}\left(E_1 + 1 + N(x)\right).$$

Finally, employing (9.1) and (7.5),

$$\left|V(x,y) - E_2\right| \leqslant \frac{c_\varepsilon}{n^{\varepsilon/8}}\left(1 + \max(y,0) + N(x)\right).$$

□

LEMMA 9.2. *There exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, $x \in \mathbb{X}$, $y \in \mathbb{R}$ and $n \geqslant 1$,*

$$\mathbb{P}_x(\tau_y > n) \leqslant \frac{c_\varepsilon}{n^{1/2-\varepsilon}} (1 + \max(y,0) + N(x)).$$

Moreover, summing this bound, for any $\varepsilon \in (0, \varepsilon_0)$, $x \in \mathbb{X}$, $y \in \mathbb{R}$ and $n \geqslant 1$, we have

$$\sum_{k=1}^{\lfloor n^{1-\varepsilon} \rfloor} \mathbb{P}_x(\tau_y > k) \leqslant c_\varepsilon (1 + \max(y,0) + N(x)) n^{1/2+\varepsilon/2}.$$

PROOF. Using Lemma 6.3 and Lemma 9.1, with $z = y + r(x)$ and $n \geqslant 1$,

$$\begin{aligned}
\mathbb{P}_x(\tau_y > n) &\leqslant \mathbb{P}_x\left(\tau_y > n\,,\, \nu_n \leqslant \left\lfloor n^{1-\varepsilon} \right\rfloor\right) + \mathbb{P}_x\left(\hat{T}_z > n\,,\, \nu_n > \left\lfloor n^{1-\varepsilon} \right\rfloor\right) \\
&\leqslant \mathbb{E}_x\left(\frac{z + M_{\nu_n}}{n^{1/2-\varepsilon}}\,;\, \tau_y > n\,,\, \nu_n \leqslant \left\lfloor n^{1-\varepsilon} \right\rfloor\right) + c_\varepsilon\, \mathrm{e}^{-c_\varepsilon n^\varepsilon} (1 + N(x)) \\
&\leqslant \frac{c_\varepsilon}{n^{1/2-\varepsilon}} (1 + \max(y,0) + N(x)).
\end{aligned}$$

□

LEMMA 9.3. *There exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, $x \in \mathbb{X}$, $y \in \mathbb{R}$ and $z = y + r(x)$,*

$$E_3 := \mathbb{E}_x\left(z + M_{\nu_n}\,;\, z + M_{\nu_n} > n^{1/2-\varepsilon/2}\,,\, \tau_y > \nu_n\,,\, \nu_n \leqslant \left\lfloor n^{1-\varepsilon} \right\rfloor\right) \underset{n\to+\infty}{\longrightarrow} 0.$$

More precisely, for any $n \geqslant 1$, $\varepsilon \in (0, \varepsilon_0)$, $x \in \mathbb{X}$, $y \in \mathbb{R}$ and $z = y + r(x)$,

$$E_3 \leqslant c_\varepsilon \frac{\max(y,0) + \left(1 + y\mathbb{1}_{\{y > n^{1/2-2\varepsilon}\}} + N(x)\right)^2}{n^\varepsilon}.$$

PROOF. Notice that when $\nu_n \neq 1$ the following inclusion holds:

$$\{z + M_{\nu_n} > n^{1/2-\varepsilon/2}\} \subseteq \{\xi_{\nu_n} > n^{1/2-\varepsilon/2} - n^{1/2-\varepsilon} \geqslant c_\varepsilon n^{1/2-\varepsilon/2}\}.$$

Therefore,

$$\begin{aligned}
E_3 \leqslant\ & \underbrace{\mathbb{E}_x\left(z + M_{\nu_n}\,;\, \nu_n \leqslant 2\lfloor n^\varepsilon \rfloor\right)}_{=:E_{30}} \\
& + \underbrace{\sum_{k=2\lfloor n^\varepsilon \rfloor+1}^{\lfloor n^{1-\varepsilon} \rfloor} \mathbb{E}_x\left(z + M_k\,;\, \xi_k > c_\varepsilon n^{1/2-\varepsilon/2}\,,\, \tau_y > k\,,\, \nu_n = k\right)}_{=:E_{31}}. \qquad (9.4)
\end{aligned}$$

Bound of E_{30}. For $y \leqslant n^{1/2-2\varepsilon}$, by the Markov inequality and Lemma 5.2,

$$\mathbb{P}_x\left(\nu_n \leqslant 2\lfloor n^\varepsilon \rfloor\right) \leqslant \sum_{k=1}^{2\lfloor n^\varepsilon \rfloor} \mathbb{P}_x\left(r(x) + M_k > c_\varepsilon n^{1/2-\varepsilon}\right) \leqslant \frac{c_\varepsilon (1 + N(x))}{n^{1/2-3\varepsilon}}.$$

For $y > n^{1/2-2\varepsilon}$, in the same way, we have $\mathbb{P}_x\left(\nu_n \leqslant 2\left\lfloor n^\varepsilon \right\rfloor\right) \leqslant \frac{c_\varepsilon(1+y+N(x))}{n^{1/2-3\varepsilon}}$. Putting together these bounds, we get, for any $y \in \mathbb{R}$,

$$\mathbb{P}_x\left(\nu_n \leqslant 2\left\lfloor n^\varepsilon \right\rfloor\right) \leqslant \frac{c_\varepsilon\left(1 + y\mathbb{1}_{\{y>n^{1/2-2\varepsilon}\}} + N(x)\right)}{n^{1/2-3\varepsilon}}. \tag{9.5}$$

Using Lemma 5.2,

$$\begin{aligned} E_{30} &\leqslant z\mathbb{P}_x\left(\nu_n \leqslant 2\left\lfloor n^\varepsilon \right\rfloor\right) + \sum_{k=1}^{2\lfloor n^\varepsilon \rfloor} \mathbb{E}_x^{1/2}\left(\left|M_k\right|^2\right)\mathbb{P}_x^{1/2}\left(\nu_n \leqslant 2\left\lfloor n^\varepsilon \right\rfloor\right) \\ &\leqslant \frac{c_\varepsilon\left(1 + y\mathbb{1}_{\{y>n^{1/2-2\varepsilon}\}} + N(x)\right)^2}{n^\varepsilon}. \end{aligned} \tag{9.6}$$

Bound of E_{31}. Changing the index of summation ($j = k - \lfloor n^\varepsilon \rfloor$) and using the Markov property,

$$\begin{aligned} E_{31} \leqslant\ & \underbrace{\sum_{j=\lfloor n^\varepsilon \rfloor+1}^{\lfloor n^{1-\varepsilon} \rfloor} \int_{\mathbb{X}\times\mathbb{R}} \max(z',0)\mathbb{P}_{x'}\left(\xi_{\lfloor n^\varepsilon \rfloor} > c_\varepsilon n^{1/2-\varepsilon/2}\right) \times\mathbb{P}_x\left(X_j \in \mathrm{d}x'\,,\, z+M_j \in \mathrm{d}z'\,,\, \tau_y > j\right)}_{=:E_{32}} \\ & + \underbrace{\sum_{j=\lfloor n^\varepsilon \rfloor+1}^{\lfloor n^{1-\varepsilon} \rfloor} \int_{\mathbb{X}\times\mathbb{R}} \mathbb{E}_{x'}^{1/2}\left(\left|M_{\lfloor n^\varepsilon \rfloor}\right|^2\right)\mathbb{P}_{x'}^{1/2}\left(\xi_{\lfloor n^\varepsilon \rfloor} > c_\varepsilon n^{1/2-\varepsilon/2}\right) \times\mathbb{P}_x\left(X_j \in \mathrm{d}x'\,,\, z+M_j \in \mathrm{d}z'\,,\, \tau_y > j\right)}_{=:E_{33}}. \end{aligned} \tag{9.7}$$

Bound of E_{32}. Using (5.2), the Markov inequality and (2.3) with $l = \left\lfloor c_\varepsilon n^{1/2-\varepsilon/2} \right\rfloor$,

$$\begin{aligned} \mathbb{P}_{x'}\left(\xi_{\lfloor n^\varepsilon \rfloor} > c_\varepsilon n^{1/2-\varepsilon/2}\right) &\leqslant \mathbb{P}_{x'}\left(N\left(X_{\lfloor n^\varepsilon \rfloor}\right) > c_\varepsilon n^{1/2-\varepsilon/2}\right) \\ &\quad + \mathbb{P}_{x'}\left(N\left(X_{\lfloor n^\varepsilon \rfloor-1}\right) > c_\varepsilon n^{1/2-\varepsilon/2}\right) \\ &\leqslant \frac{1}{l}\mathbb{E}_{x'}\left(N_l\left(X_{\lfloor n^\varepsilon \rfloor}\right)\right) + \frac{1}{l}\mathbb{E}_{x'}\left(N_l\left(X_{\lfloor n^\varepsilon \rfloor-1}\right)\right) \\ &\leqslant \frac{c}{l^{2+\beta}} + \frac{c}{l}\,\mathrm{e}^{-cn^\varepsilon}\left(1+N(x')\right). \end{aligned}$$

Choosing $\varepsilon > 0$ small enough we find that

$$\mathbb{P}_{x'}\left(\xi_{\lfloor n^\varepsilon \rfloor} > c_\varepsilon n^{1/2-\varepsilon/2}\right) \leqslant \frac{c_\varepsilon}{n^{1+\beta/4}} + c_\varepsilon\,\mathrm{e}^{-c_\varepsilon n^\varepsilon} N(x'). \tag{9.8}$$

By the definition of E_{32} in (9.7),

$$\begin{aligned} E_{32} \leqslant\ & \frac{c_\varepsilon}{n^{1+\beta/4}} \sum_{j=\lfloor n^\varepsilon \rfloor+1}^{\lfloor n^{1-\varepsilon} \rfloor} \left[\mathbb{E}_x\left(z+M_j\,;\,\tau_y > j\right) + \mathbb{E}_x\left(\left|r\left(X_j\right)\right|\right)\right] \\ & + c_\varepsilon\,\mathrm{e}^{-c_\varepsilon n^\varepsilon} \sum_{j=\lfloor n^\varepsilon \rfloor+1}^{\lfloor n^{1-\varepsilon} \rfloor} \left[\max(z,0)\mathbb{E}_x\left(N\left(X_j\right)\right) + \mathbb{E}_x^{1/2}\left(\left|M_j\right|^2\right)\mathbb{E}_x^{1/2}\left(N\left(X_j\right)^2\right)\right]. \end{aligned}$$

Using (6.28), Lemma 5.2 and the point 1 of Hypothesis **M4**, we find that

$$E_{32} \leqslant c_\varepsilon \frac{\max(y,0) + \left(1 + y\mathbb{1}_{\{y>n^{1/2-2\varepsilon}\}} + N(x)\right)(1+N(x))}{n^{\beta/4}}. \tag{9.9}$$

Bound of E_{33}. Using (9.8) and Lemma 5.2, we have

$$E_{33} \leqslant \sum_{j=\lfloor n^\varepsilon \rfloor+1}^{\lfloor n^{1-\varepsilon} \rfloor} \mathbb{E}_x \left(n^{\varepsilon/2} \left(1 + N\left(X_j\right)\right) \left(\frac{c_\varepsilon}{n^{1/2+\beta/8}} + c_\varepsilon \,\mathrm{e}^{-c_\varepsilon n^\varepsilon} N\left(X_j\right)^{1/2} \right) ; \tau_y > j \right).$$

By the Markov property,

$$E_{33} \leqslant c_\varepsilon \,\mathrm{e}^{-c_\varepsilon n^\varepsilon} \left(1+N(x)\right)^{3/2} + \frac{c_\varepsilon}{n^{1/2+\beta/8-\varepsilon/2}} \sum_{j=1}^{\lfloor n^{1-\varepsilon} \rfloor} \mathbb{E}_x \left(1 + \mathrm{e}^{-cn^\varepsilon} N\left(X_j\right) ; \tau_y > j \right).$$

Using Lemma 9.2,

$$E_{33} \leqslant c_\varepsilon \frac{\max(y,0) + (1+N(x))^{3/2}}{n^{\beta/8-3\varepsilon/2}}. \tag{9.10}$$

With (9.10), (9.9) and (9.7), for $\varepsilon > 0$ small enough, we find that

$$E_{31} \leqslant c_\varepsilon \frac{\max(y,0) + \left(1 + y\mathbb{1}_{\{y>n^{1/2-2\varepsilon}\}} + N(x)\right)(1+N(x))}{n^{\varepsilon}}.$$

This bound, together with (9.6) and (9.4), proves the lemma. □

LEMMA 9.4. *There exists* $\varepsilon_0 > 0$ *such that, for any* $\varepsilon \in (0,\varepsilon_0)$, $x \in \mathbb{X}$, $y \in \mathbb{R}$ *and* $z = y + r(x)$,

$$E_4 := \mathbb{E}_x \left(z + M_{\nu_n^{\varepsilon^2}} ; z + M_{\nu_n^{\varepsilon^2}} > n^{1/2-\varepsilon/4}, \tau_y > \nu_n^{\varepsilon^2}, \nu_n^{\varepsilon^2} \leqslant \left\lfloor n^{1-\varepsilon} \right\rfloor \right) \underset{n\to+\infty}{\longrightarrow} 0.$$

More precisely, for any $n \geqslant 1$, $\varepsilon \in (0,\varepsilon_0)$, $x \in \mathbb{X}$, $y \in \mathbb{R}$ *and* $z = y + r(x)$,

$$E_4 \leqslant c_\varepsilon \frac{\max(y,0) + \left(1 + y\mathbb{1}_{\{y>n^{1/2-2\varepsilon}\}} + N(x)\right)^2}{n^{\varepsilon/2}}.$$

PROOF. We shall apply Lemma 9.3. For this we write, for any $n \geqslant 1$,

$$\begin{aligned} E_4 = {} & \underbrace{\mathbb{E}_x \left(z + M_{\nu_n^{\varepsilon^2}} ; z + M_{\nu_n^{\varepsilon^2}} > n^{1/2-\varepsilon/4}, z + M_{\nu_n} > n^{1/2-\varepsilon/2}, \tau_y > \nu_n^{\varepsilon^2}, \nu_n^{\varepsilon^2} \leqslant \left\lfloor n^{1-\varepsilon} \right\rfloor \right)}_{=:E_{41}} \\ & + \underbrace{\mathbb{E}_x \left(z + M_{\nu_n^{\varepsilon^2}} ; z + M_{\nu_n^{\varepsilon^2}} > n^{1/2-\varepsilon/4}, z + M_{\nu_n} \leqslant n^{1/2-\varepsilon/2}, \tau_y > \nu_n^{\varepsilon^2}, \nu_n^{\varepsilon^2} \leqslant \left\lfloor n^{1-\varepsilon} \right\rfloor \right)}_{=:E_{42}}. \end{aligned} \tag{9.11}$$

Bound of E_{41}. By the Markov property,

$$E_{41} = \sum_{k=1}^{\lfloor n^{1-\varepsilon}\rfloor - \lfloor n^{\varepsilon^2}\rfloor} \int_{\mathbb{X}\times\mathbb{R}} \mathbb{E}_{x'}\left(z' + M_{\lfloor n^{\varepsilon^2}\rfloor} \,;\, z' + M_{\lfloor n^{\varepsilon^2}\rfloor} > n^{1/2-\varepsilon/4} \,,\, \tau_{y'} > \left\lfloor n^{\varepsilon^2}\right\rfloor \right)$$
$$\times \mathbb{P}_x\left(X_k \in \mathrm{d}x' \,,\, z + M_k \in \mathrm{d}z' \,,\, z + M_k > n^{1/2-\varepsilon/2} \,,\, \tau_y > k \,,\, \nu_n = k \right),$$

where $y' = z' - r(x')$. Moreover, for any $x' \in \mathbb{X}$, $z' \in \mathbb{R}$, using (6.28), we have

$$\begin{aligned}
&\mathbb{E}_{x'}\left(z' + M_{\lfloor n^{\varepsilon^2}\rfloor} \,;\, z' + M_{\lfloor n^{\varepsilon^2}\rfloor} > n^{1/2-\varepsilon/4} \,,\, \tau_{y'} > \left\lfloor n^{\varepsilon^2}\right\rfloor \right) \\
&\qquad \leqslant \mathbb{E}_{x'}\left(z' + M_{\lfloor n^{\varepsilon^2}\rfloor} \,;\, z' + M_{\lfloor n^{\varepsilon^2}\rfloor} > 0 \,,\, \tau_{y'} > \left\lfloor n^{\varepsilon^2}\right\rfloor \right) \\
&\qquad \leqslant \mathbb{E}_{x'}\left(z' + M_{\lfloor n^{\varepsilon^2}\rfloor} \,;\, \tau_{y'} > \left\lfloor n^{\varepsilon^2}\right\rfloor \right) + \mathbb{E}_{x'}\left(\left| r\left(X_{n^{\varepsilon^2}} \right) \right| \right) \\
&\qquad \leqslant c_\varepsilon \max(z', 0) + c_\varepsilon \left(1 + N(x')\right).
\end{aligned}$$

Consequently,

$$\begin{aligned}
E_{41} &\leqslant c_\varepsilon E_3 + c_\varepsilon \mathbb{E}_x\left(1 + N\left(X_{\nu_n}\right) \,;\, z + M_{\nu_n} > n^{1/2-\varepsilon/2} \,,\, \tau_y > \nu_n \,,\, \nu_n \leqslant \left\lfloor n^{1-\varepsilon}\right\rfloor \right) \\
&\leqslant 2c_\varepsilon E_3 + c_\varepsilon \mathbb{E}_x\left(N\left(X_{\nu_n}\right) \,;\, N\left(X_{\nu_n}\right) > n^{1/2-\varepsilon} \,,\, \tau_y > \nu_n \,,\, \nu_n \leqslant \left\lfloor n^{1-\varepsilon}\right\rfloor \right) \\
&\quad + c_\varepsilon \mathbb{E}_x\Big(n^{1/2-\varepsilon} \,;\, N\left(X_{\nu_n}\right) \leqslant n^{1/2-\varepsilon} \,,\, z + M_{\nu_n} > n^{1/2-\varepsilon/2} \,, \\
&\qquad\qquad \tau_y > \nu_n \,,\, \nu_n \leqslant \left\lfloor n^{1-\varepsilon}\right\rfloor \Big)
\end{aligned}$$

$$\text{(9.12)} \qquad \leqslant 3c_\varepsilon E_3 + c_\varepsilon \underbrace{\mathbb{E}_x\left(N\left(X_{\nu_n}\right) \,;\, N\left(X_{\nu_n}\right) > n^{1/2-\varepsilon} \,,\, \tau_y > \nu_n \,,\, \nu_n \leqslant \left\lfloor n^{1-\varepsilon}\right\rfloor \right)}_{=:E'_{41}}.$$

Denoting $l = \left\lfloor n^{1/2-\varepsilon} \right\rfloor$ and using the point 1 of **M4** and (2.3), we have

$$\begin{aligned}
E'_{41} &\leqslant \mathbb{E}_x\left(\frac{N\left(X_{\nu_n}\right)^2}{n^{1/2-\varepsilon}} \,;\, \nu_n \leqslant \lfloor n^\varepsilon \rfloor \right) + \sum_{k=\lfloor n^\varepsilon\rfloor+1}^{\lfloor n^{1-\varepsilon}\rfloor} \mathbb{E}_x\left(N_l\left(X_k\right) \,;\, \tau_y > k \,,\, \nu_n = k \right) \\
&\leqslant \frac{cn^\varepsilon \left(1 + N(x)\right)^2}{n^{1/2-\varepsilon}} + \sum_{k=1}^{\lfloor n^{1-\varepsilon}\rfloor} \left[\frac{c}{l^{1+\beta}} \mathbb{P}_x\left(\tau_y > k\right) + c\,\mathrm{e}^{-cn^\varepsilon}\, \mathbb{E}_x\left(1 + N\left(X_k\right)\right) \right].
\end{aligned}$$

Using Lemma 9.2 and taking $\varepsilon > 0$ small enough,

$$E'_{41} \leqslant c_\varepsilon \frac{\max(y,0) + \left(1 + N(x)\right)^2}{n^{\min(1,\beta)/4}}. \tag{9.13}$$

In conjunction with Lemma 9.3, from (9.12) we obtain that, for some $\varepsilon > 0$,

$$E_{41} \leqslant c_\varepsilon \frac{\max(y,0) + \left(1 + y\mathbb{1}_{\{y > n^{1/2-2\varepsilon}\}} + N(x)\right)^2}{n^\varepsilon}. \tag{9.14}$$

Bound of E_{42}. For any $z' \in (0, n^{1/2-\varepsilon/2}]$, we have

$$\left(z' + M_{\lfloor n^{\varepsilon^2} \rfloor}\right) \mathbb{P}_{x'}(z' + M_{\lfloor n^{\varepsilon^2} \rfloor} > n^{1/2-\varepsilon/4}) \leqslant z'\mathbb{P}_{x'}(M_{\lfloor n^{\varepsilon^2} \rfloor} > c_\varepsilon n^{1/2-\varepsilon/4}) + \left|M_{\lfloor n^{\varepsilon^2} \rfloor}\right|.$$

Therefore, by the Markov property,

$$\begin{aligned} E_{42} \leqslant & \underbrace{\int_{\mathbb{X}\times\mathbb{R}} z'\mathbb{P}_{x'}\left(M_{\lfloor n^{\varepsilon^2} \rfloor} > c_\varepsilon n^{1/2-\varepsilon/4}\right) \mathbb{P}_x\left(X_{\nu_n} \in \mathrm{d}x' \,,\, z + M_{\nu_n} \in \mathrm{d}z' \,,\right. \\ & \qquad \left. z + M_{\nu_n} \leqslant n^{1/2-\varepsilon/2} \,,\, \tau_y > \nu_n \,,\, \nu_n \leqslant \left\lfloor n^{1-\varepsilon} \right\rfloor \right)}_{=:E_{43}} \\ & + \underbrace{\int_{\mathbb{X}\times\mathbb{R}} \mathbb{E}_{x'}\left(\left|M_{\lfloor n^{\varepsilon^2} \rfloor}\right|\right) \mathbb{P}_x\left(X_{\nu_n} \in \mathrm{d}x' \,,\, z + M_{\nu_n} \in \mathrm{d}z' \,,\right. \\ & \qquad \left. z + M_{\nu_n} \leqslant n^{1/2-\varepsilon/2} \,,\, \tau_y > \nu_n \,,\, \nu_n \leqslant \left\lfloor n^{1-\varepsilon} \right\rfloor \right)}_{=:E_{44}} . \end{aligned} \tag{9.15}$$

Bound of E_{43}. Using Lemma 5.2,

$$\mathbb{P}_{x'}\left(M_{\lfloor n^{\varepsilon^2} \rfloor} > c_\varepsilon n^{1/2-\varepsilon/4}\right) \leqslant \frac{c_\varepsilon n^{\varepsilon^2}\left(1 + N(x')\right)}{n^{1/2-\varepsilon/4}}.$$

Therefore, we have

$$\begin{aligned} E_{43} &\leqslant \mathbb{E}_x\left(\frac{c_\varepsilon}{n^{3\varepsilon/4-\varepsilon^2}}\left(z + M_{\nu_n}\right) \mathbb{1}_{\left\{N(X_{\nu_n}) \leqslant n^{1/2-\varepsilon}\right\}} + \frac{c_\varepsilon}{n^{\varepsilon/4-\varepsilon^2}} N\left(X_{\nu_n}\right) \mathbb{1}_{\left\{N(X_{\nu_n}) > n^{1/2-\varepsilon}\right\}} \,;\right. \\ &\qquad \left. z + M_{\nu_n} \leqslant n^{1/2-\varepsilon/2} \,,\, \tau_y > \nu_n \,,\, \nu_n \leqslant \left\lfloor n^{1-\varepsilon} \right\rfloor \right) \\ &\leqslant \frac{c_\varepsilon}{n^{3\varepsilon/4-\varepsilon^2}} E_1 + \frac{c_\varepsilon}{n^{\varepsilon/4-\varepsilon^2}} E_{41}'. \end{aligned}$$

By Lemma 9.1 and (9.13), we obtain for some small $\varepsilon > 0$,

$$E_{43} \leqslant c_\varepsilon \frac{\max(y,0) + \left(1 + N(x)\right)^2}{n^{\varepsilon/2}}. \tag{9.16}$$

Bound of E_{44}. Again by Lemma 5.2, $\mathbb{E}_{x'}\left(\left|M_{\lfloor n^{\varepsilon^2} \rfloor}\right|\right) \leqslant n^{\varepsilon^2}\left(1 + N(x')\right)$. Consequently,

$$\begin{aligned} E_{44} \leqslant \; & \frac{c_\varepsilon}{n^{\varepsilon-\varepsilon^2}} \mathbb{E}_x\left(z + M_{\nu_n} \,;\, N\left(X_{\nu_n}\right) \leqslant n^{1/2-2\varepsilon} \,,\, \tau_y > \nu_n \,,\, \nu_n \leqslant \left\lfloor n^{1-\varepsilon} \right\rfloor \right) \\ & + c_\varepsilon n^{\varepsilon^2} \mathbb{E}_x\left(N\left(X_{\nu_n}\right) \,;\, N\left(X_{\nu_n}\right) > n^{1/2-2\varepsilon} \,,\, \tau_y > \nu_n \,,\, \nu_n \leqslant \left\lfloor n^{1-\varepsilon} \right\rfloor \right). \end{aligned}$$

Proceeding exactly as in the proof of the bound of E_{41}' but with $l = \left\lfloor n^{1/2-2\varepsilon} \right\rfloor$, we obtain, by Lemma 9.1,

$$E_{44} \leqslant c_\varepsilon \frac{\max(y,0) + \left(1 + N(x)\right)^2}{n^{\varepsilon/2}}.$$

Putting together this bound with (9.16) and (9.15), we find that

$$E_{42} \leqslant c_\varepsilon \frac{\max(y,0) + \left(1 + N(x)\right)^2}{n^{\varepsilon/2}}.$$

So, using (9.11) and (9.14), we obtain the second assertion. The first one is an easy consequence of the second one. □

The following results are similar to that provided by Lemmas 9.1 and 9.4 (see E_2 and E_4 respectively).

LEMMA 9.5. *There exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, $x \in \mathbb{X}$ and $y \in \mathbb{R}$,*

$$F_2 := \mathbb{E}_x \left(y + S_{\nu_n^{\varepsilon^2}} \,;\, \tau_y > \nu_n^{\varepsilon^2} \,,\, \nu_n^{\varepsilon^2} \leqslant \left\lfloor n^{1-\varepsilon} \right\rfloor \right) \underset{n\to\infty}{\longrightarrow} V(x,y),$$
$$F_4 := \mathbb{E}_x \left(y + S_{\nu_n^{\varepsilon^2}} \,;\, y + S_{\nu_n^{\varepsilon^2}} > n^{1/2-\varepsilon/8} \,,\, \tau_y > \nu_n^{\varepsilon^2} \,,\, \nu_n^{\varepsilon^2} \leqslant \left\lfloor n^{1-\varepsilon} \right\rfloor \right) \underset{n\to+\infty}{\longrightarrow} 0.$$

More precisely, for any $n \geqslant 1$, $\varepsilon \in (0, \varepsilon_0)$, $x \in \mathbb{X}$ and $y \in \mathbb{R}$,

$$|F_2 - V(x,y)| \leqslant \frac{c_\varepsilon}{n^{\varepsilon/8}} \left(1 + \max(y,0) + N(x) \right)$$

and

$$F_4 \leqslant c_\varepsilon \frac{\max(y,0) + \left(1 + y \mathbb{1}_{\{y > n^{1/2-2\varepsilon}\}} + N(x) \right)^2}{n^{\varepsilon/2}}.$$

PROOF. By (5.1), for any $n \geqslant 1$,

$$|F_2 - E_2| \leqslant \underbrace{\mathbb{E}_x \left(\left| r\left(X_{\nu_n^{\varepsilon^2}} \right) \right| \,;\, \tau_y > \nu_n^{\varepsilon^2} \,,\, \nu_n^{\varepsilon^2} \leqslant \left\lfloor n^{1-\varepsilon} \right\rfloor \right)}_{=:F_2'}.$$

Using the Markov property, the definition of ν_n and Lemma 9.1,

$$\begin{aligned} F_2' &\leqslant c \mathbb{E}_x \left(1 + \mathrm{e}^{-cn^{\varepsilon^2}} N\left(X_{\nu_n} \right) \,;\, \tau_y > \nu_n \,,\, \nu_n \leqslant \left\lfloor n^{1-\varepsilon} \right\rfloor \right) \\ &\leqslant \frac{c}{n^{1/2-\varepsilon}} E_1 + c\, \mathrm{e}^{-cn^{\varepsilon^2}} \left(1 + N(x) \right) \\ &\leqslant \frac{c_\varepsilon}{n^{1/2-\varepsilon}} \left(1 + \max(y,0) + N(x) \right). \end{aligned} \tag{9.17}$$

Therefore, by Lemma 9.1,

$$|F_2 - V(x,y)| \leqslant |E_2 - V(x,y)| + F_2' \leqslant \frac{c_\varepsilon}{n^{\varepsilon/8}} \left(1 + \max(y,0) + N(x) \right).$$

Now we shall control F_4. Recall the notation $z = y + r(x)$. By equation (5.1), we note that on the event

$$\left\{ z + M_{\nu_n^{\varepsilon^2}} \leqslant n^{1/2-\varepsilon/4} \right\} \cap \left\{ y + S_{\nu_n^{\varepsilon^2}} > n^{1/2-\varepsilon/8} \right\}$$

we have $\left| r\left(X_{\nu_n^{\varepsilon^2}} \right) \right| > c_\varepsilon n^{1/2-\varepsilon/8}$. Therefore,

$$y + S_{\nu_n^{\varepsilon^2}} \leqslant n^{1/2-\varepsilon/4} - r\left(X_{\nu_n^{\varepsilon^2}} \right) \leqslant \left(\frac{c_\varepsilon}{n^{\varepsilon/8}} + 1 \right) \left| r\left(X_{\nu_n^{\varepsilon^2}} \right) \right|,$$

which implies that

$$F_4 \leqslant \mathbb{E}_x \left(y + S_{\nu_n^{\varepsilon^2}} \,;\, z + M_{\nu_n^{\varepsilon^2}} > n^{1/2-\varepsilon/4} \,,\, \tau_y > \nu_n^{\varepsilon^2} \,,\, \nu_n^{\varepsilon^2} \leqslant \left\lfloor n^{1-\varepsilon} \right\rfloor \right) + c_\varepsilon F_2'.$$

By (5.1), Lemma 9.4 and (9.17), we conclude that

$$F_4 \leqslant E_4 + F_2' + c_\varepsilon F_2' \leqslant c_\varepsilon \frac{\max(y,0) + \left(1 + y \mathbb{1}_{\{y > n^{1/2-2\varepsilon}\}} + N(x) \right)^2}{n^{\varepsilon/2}}.$$

□

9.2. *Proof of Theorem 2.3.* Assume that $(x,y) \in \mathbb{X} \times \mathbb{R}$. Let $(B_t)_{t\geqslant 0}$ be the Brownian motion defined by Proposition 4.3. For any $k \geqslant 1$, consider the event

$$A_k = \left\{ \sup_{0\leqslant t\leqslant 1} \left| S_{\lfloor tk \rfloor} - \sigma B_{tk} \right| \leqslant k^{1/2-2\varepsilon} \right\} \tag{9.18}$$

and denote by $\overline{A}_k$ its complement. Let $n \geqslant 1$ and remind that $\nu_n^{\varepsilon^2} = \nu_n + \left\lfloor n^{\varepsilon^2} \right\rfloor > \left\lfloor n^{\varepsilon^2} \right\rfloor$. With the previous notation, we write

$$\begin{aligned} \mathbb{P}_x\left(\tau_y > n\right) = \mathbb{P}_x\left(\tau_y > n\,,\, \nu_n^{\varepsilon^2} > \left\lfloor n^{1-\varepsilon} \right\rfloor\right) \\ + \underbrace{\sum_{k=\left\lfloor n^{\varepsilon^2}\right\rfloor+1}^{\left\lfloor n^{1-\varepsilon}\right\rfloor} \int_{\mathbb{X}\times\mathbb{R}} \mathbb{P}_{x'}\left(\tau_{y'} > n-k\,,\, \overline{A}_{n-k}\right) \mathbb{P}_x\left(X_k \in \mathrm{d}x'\,,\, y+S_k \in \mathrm{d}y'\,,\, \tau_y > k\,,\, \nu_n^{\varepsilon^2} = k\right)}_{=:J_1} \\ + \underbrace{\sum_{k=\left\lfloor n^{\varepsilon^2}\right\rfloor+1}^{\left\lfloor n^{1-\varepsilon}\right\rfloor} \int_{\mathbb{X}\times\mathbb{R}} \mathbb{P}_{x'}\left(\tau_{y'} > n-k\,,\, A_{n-k}\right) \mathbb{P}_x\left(X_k \in \mathrm{d}x'\,,\, y+S_k \in \mathrm{d}y'\,,\, \tau_y > k\,,\, \nu_n^{\varepsilon^2} = k\right)}_{=:J_2}. \end{aligned} \tag{9.19}$$

Bound of J_1. Since $n-k \geqslant c_\varepsilon n$, for any $k \leqslant \left\lfloor n^{1-\varepsilon} \right\rfloor$, by Proposition 4.3, we have

$$\mathbb{P}_{x'}\left(\tau_{y'} > n-k\,,\, \overline{A}_{n-k}\right) \leqslant \mathbb{P}_{x'}\left(\overline{A}_{n-k}\right) \leqslant \frac{c_\varepsilon\left(1+N(x')\right)}{n^{2\varepsilon}}.$$

So, using the fact that $n^{1/2-\varepsilon} \leqslant z + M_{\nu_n}$ and Lemma 9.1,

$$\begin{aligned} J_1 &\leqslant \frac{c_\varepsilon}{n^{2\varepsilon}} \mathbb{E}_x\left(1 + \mathrm{e}^{-cn^{\varepsilon^2}} N\left(X_{\nu_n}\right) \,;\, \tau_y > \nu_n\,,\, \nu_n \leqslant \left\lfloor n^{1-\varepsilon} \right\rfloor\right) \\ &\leqslant \frac{c_\varepsilon}{n^{1/2+\varepsilon}} E_1 + c_\varepsilon\, \mathrm{e}^{-c_\varepsilon n^{\varepsilon^2}} \left(1+N(x)\right) \\ &\leqslant \frac{c_\varepsilon\left(1+\max(y,0)+N(x)\right)}{n^{1/2+\varepsilon}}. \end{aligned} \tag{9.20}$$

Bound of J_2. We split J_2 into two terms:

$$
\begin{aligned}
J_2 = & \underbrace{\sum_{k=\lfloor n^{\varepsilon^2} \rfloor+1}^{\lfloor n^{1-\varepsilon} \rfloor} \int_{\mathbb{X}\times\mathbb{R}} \mathbb{P}_{x'}\left(\tau_{y'} > n-k\,,\, A_{n-k}\right) \times\mathbb{P}_x\left(X_k \in \mathrm{d}x'\,,\, y+S_k \in \mathrm{d}y'\,,\, y+S_k > n^{1/2-\varepsilon/8}\,,\, \tau_y > k\,,\, \nu_n^{\varepsilon^2} = k\right)}_{=:J_3} \\
& + \underbrace{\sum_{k=\lfloor n^{\varepsilon^2} \rfloor+1}^{\lfloor n^{1-\varepsilon} \rfloor} \int_{\mathbb{X}\times\mathbb{R}} \mathbb{P}_{x'}\left(\tau_{y'} > n-k\,,\, A_{n-k}\right) \times\mathbb{P}_x\left(X_k \in \mathrm{d}x'\,,\, y+S_k \in \mathrm{d}y'\,,\, y+S_k \leqslant n^{1/2-\varepsilon/8}\,,\, \tau_y > k\,,\, \nu_n^{\varepsilon^2} = k\right)}_{=:J_4}. \qquad (9.21)
\end{aligned}
$$

Bound of J_3. With $y'_+ = y' + (n-k)^{1/2-2\varepsilon}$, we have

$$
\mathbb{P}_{x'}\left(\tau_{y'} > n-k\,,\, A_{n-k}\right) \leqslant \mathbb{P}_{x'}\left(\tau_{y'_+}^{bm} > n-k\right), \qquad (9.22)
$$

where τ_y^{bm} is defined in (4.1). By the point 1 of Lemma 4.2 and Lemma 9.5,

$$
\begin{aligned}
J_3 &\leqslant \frac{c_\varepsilon}{\sqrt{n}} \mathbb{E}_x\left(y + S_{\nu_n^{\varepsilon^2}} + n^{1/2-2\varepsilon}\,;\, y + S_{\nu_n^{\varepsilon^2}} > n^{1/2-\varepsilon/8}\,,\, \tau_y > \nu_n^{\varepsilon^2}\,,\, \nu_n^{\varepsilon^2} \leqslant \left\lfloor n^{1-\varepsilon} \right\rfloor\right) \\
&\leqslant \frac{2c_\varepsilon}{\sqrt{n}} F_4 \\
&\leqslant c_\varepsilon \frac{\max(y,0) + \left(1 + y\mathbb{1}_{\{y>n^{1/2-2\varepsilon}\}} + N(x)\right)^2}{n^{1/2+\varepsilon/2}}. \qquad (9.23)
\end{aligned}
$$

Upper bound of J_4. For $y' \leqslant n^{1/2-\varepsilon/8}$ and any $k \leqslant \lfloor n^{1-\varepsilon} \rfloor$, it holds $y'_+ \leqslant 2n^{1/2-\varepsilon/8} \leqslant c_\varepsilon (n-k)^{1/2-\varepsilon/8}$. Therefore, by (9.22) and the point 2 of Lemma 4.2 with $\theta_m = c_\varepsilon m^{-\varepsilon/8}$ and $m = n-k$, we have

$$
J_4 \leqslant \sum_{k=\lfloor n^{\varepsilon^2} \rfloor+1}^{\lfloor n^{1-\varepsilon} \rfloor} \int_{\mathbb{X}\times\mathbb{R}} \frac{2\left(1+\theta_{n-k}^2\right)}{\sqrt{2\pi(n-k)}\sigma} \mathbb{E}_x\left(y + S_k + (n-k)^{1/2-2\varepsilon}\,; \right. \left. y + S_k \leqslant n^{1/2-\varepsilon/8}\,,\, \tau_y > k\,,\, \nu_n^{\varepsilon^2} = k\right).
$$

Since $\frac{2\left(1+\theta_{n-k}^2\right)}{\sqrt{2\pi(n-k)}\sigma} \leqslant \frac{2}{\sqrt{2\pi n}\sigma}\left(1 + \frac{c_\varepsilon}{n^{\varepsilon/4}}\right)$ and $n^{1/2-\varepsilon} \leqslant z + M_{\nu_n}$, we get

$$
\begin{aligned}
J_4 &\leqslant \frac{2}{\sqrt{2\pi n}\sigma}\left(1 + \frac{c_\varepsilon}{n^{\varepsilon/4}}\right) \mathbb{E}_x\left(y + S_{\nu_n^{\varepsilon^2}} + n^{1/2-2\varepsilon}\,;\, y + S_{\nu_n^{\varepsilon^2}} \leqslant n^{1/2-\varepsilon/8}\,, \right. \\
&\qquad\qquad \left. \tau_y > \nu_n^{\varepsilon^2}\,,\, \nu_n^{\varepsilon^2} \leqslant \left\lfloor n^{1-\varepsilon} \right\rfloor\right) \\
&\leqslant \frac{2}{\sqrt{2\pi n}\sigma}\left(1 + \frac{c_\varepsilon}{n^{\varepsilon/4}}\right) F_2 + \frac{c_\varepsilon}{n^{1/2+\varepsilon}} E_1.
\end{aligned}
$$

By Lemmas 9.1, 9.5 and (7.5),

$$J_4 \leqslant \frac{2V(x,y)}{\sqrt{2\pi n}\sigma} + \frac{c_\varepsilon \left(1+\max(y,0)+N(x)\right)}{n^{1/2+\varepsilon/8}}. \tag{9.24}$$

Lower bound of J_4. With $y'_- = y' - (n-k)^{1/2-2\varepsilon}$, we have $\mathbb{P}_{x'}\left(\tau_{y'} > n-k\,,\, A_{n-k}\right) \geqslant \mathbb{P}_{x'}\left(\tau_{y'_-}^{bm} > n-k\right) - \mathbb{P}_{x'}\left(\overline{A}_{n-k}\right)$. Considering the event $\{y+S_k > (n-k)^{1/2-2\varepsilon}\}$ and repeating the arguments used to bound J_1 (see (9.20)), we obtain

$$\begin{aligned} J_4 \geqslant \sum_{k=\lfloor n^{\varepsilon^2}\rfloor+1}^{\lfloor n^{1-\varepsilon}\rfloor} \int_{\mathbb{X}\times\mathbb{R}} \mathbb{P}_{x'}\left(\tau_{y'_-}^{bm} > n-k\right) \mathbb{P}_x\big(X_k \in \mathrm{d}x'\,,\, y+S_k \in \mathrm{d}y'\,,& \\ y+S_k \leqslant n^{1/2-\varepsilon/8}\,,\, y+S_k > (n-k)^{1/2-2\varepsilon}\,,\, \tau_y > k\,,\, \nu_n^{\varepsilon^2} = k\big)& \\ - \frac{c_\varepsilon \left(1+\max(y,0)+N(x)\right)}{n^{1/2+\varepsilon}}.& \end{aligned}$$

Using the point 2 of Lemma 4.2 and Proposition 4.3,

$$\begin{aligned} J_4 &\geqslant \frac{2}{\sqrt{2\pi n}\sigma}\left(1-\frac{c_\varepsilon}{n^{\varepsilon/4}}\right)\mathbb{E}_x\Big(y+S_{\nu_n^{\varepsilon^2}} - (n-\nu_n^{\varepsilon^2})^{1/2-2\varepsilon}\,; \\ &\qquad y+S_{\nu_n^{\varepsilon^2}} > (n-\nu_n^{\varepsilon^2})^{1/2-2\varepsilon}\,,\, y+S_{\nu_n^{\varepsilon^2}} \leqslant n^{1/2-\varepsilon/8}\,,\, \tau_y > \nu_n^{\varepsilon^2}\,,\, \nu_n^{\varepsilon^2} \leqslant \left\lfloor n^{1-\varepsilon}\right\rfloor\Big) \\ &\quad - \frac{c_\varepsilon \left(1+\max(y,0)+N(x)\right)}{n^{1/2+\varepsilon}} \\ &\geqslant \frac{2}{\sqrt{2\pi n}\sigma}\left(1-\frac{c_\varepsilon}{n^{\varepsilon/4}}\right)F_2 - \frac{c_\varepsilon}{\sqrt{n}}F_4 - \frac{c_\varepsilon}{n^{1/2+\varepsilon}}E_1 - \frac{c_\varepsilon \left(1+\max(y,0)+N(x)\right)}{n^{1/2+\varepsilon}}. \end{aligned}$$

By Lemmas 9.1, 9.5 and (7.5),

$$J_4 \geqslant \frac{2V(x,y)}{\sqrt{2\pi n}\sigma} - c_\varepsilon \frac{\max(y,0) + \left(1+y\mathbb{1}_{\{y>n^{1/2-2\varepsilon}\}}+N(x)\right)^2}{n^{1/2+\varepsilon/8}}. \tag{9.25}$$

Putting together (9.25), (9.24), (9.23) and (9.21),

$$\left| J_2 - \frac{2V(x,y)}{\sqrt{2\pi n}\sigma}\right| \leqslant c_\varepsilon \frac{\max(y,0) + \left(1+y\mathbb{1}_{\{y>n^{1/2-2\varepsilon}\}}+N(x)\right)^2}{n^{1/2+\varepsilon/8}}.$$

Taking into account (9.20), (9.19) and Lemma 6.3, we conclude that, for any $(x,y) \in \mathbb{X}\times\mathbb{R}$,

$$\left| \mathbb{P}_x\left(\tau_y > n\right) - \frac{2V(x,y)}{\sqrt{2\pi n}\sigma}\right| \leqslant c_\varepsilon \frac{\max(y,0) + \left(1+y\mathbb{1}_{\{y>n^{1/2-2\varepsilon}\}}+N(x)\right)^2}{n^{1/2+\varepsilon/8}}. \tag{9.26}$$

Taking the limit as $n \to +\infty$ in (9.26), we obtain the point 1 of Theorem 2.3. The point 2 of Theorem 2.3 is an immediate consequence of the points 2 and 4 of Proposition 8.8.

9.3. *Proof of Theorem 2.4.* The point 1 of Theorem 2.4 is exactly (9.26). In order to prove the point 2 of Theorem 2.4, we shall first establish a bound for $\mathbb{P}_x(\tau_y > n)$ when $z = y + r(x) \geqslant n^{1/2-\varepsilon}$, $n \geqslant 1$. Set $m_\varepsilon = n - \lfloor n^\varepsilon \rfloor$. By the Markov property,

$$\begin{aligned} \mathbb{P}_x(\tau_y > n) = \int_{\mathbb{X}\times\mathbb{R}} & \mathbb{P}_{x'}(\tau_{y'} > m_\varepsilon) \\ & \times \mathbb{P}_x\left(X_{\lfloor n^\varepsilon \rfloor} \in \mathrm{d}x' \,,\, y + S_{\lfloor n^\varepsilon \rfloor} \in \mathrm{d}y' \,,\, \tau_y > \lfloor n^\varepsilon \rfloor \right). \end{aligned} \tag{9.27}$$

For any $x' \in \mathbb{X}$ and $y' > 0$, using A_{m_ε} defined by (9.18), we have

$$\mathbb{P}_{x'}(\tau_{y'} > m_\varepsilon) \leqslant \mathbb{P}_{x'}\left(\tau_{y'_+}^{bm} > m_\varepsilon \right) + \mathbb{P}_{x'}\left(\overline{A}_{m_\varepsilon} \right),$$

where $\tau_{y'_+}^{bm}$ is defined by (4.1) and $y'_+ = y' + m_\varepsilon^{1/2-2\varepsilon}$. By the point 1 of Lemma 4.2 and Proposition 4.3,

$$\mathbb{P}_{x'}(\tau_{y'} > m_\varepsilon) \leqslant \frac{c y'_+}{\sqrt{m_\varepsilon}} + \frac{c_\varepsilon}{m_\varepsilon^{2\varepsilon}}\left(1 + N(x')\right) \leqslant \frac{c_\varepsilon y'}{\sqrt{n}} + \frac{c_\varepsilon}{n^{2\varepsilon}} + \frac{c_\varepsilon}{n^{2\varepsilon}} N(x').$$

Introducing this bound in (9.27), we get

$$\mathbb{P}_x(\tau_y > n) \leqslant \frac{c_\varepsilon}{\sqrt{n}} \mathbb{E}_x\left(y + S_{\lfloor n^\varepsilon \rfloor} \,,\, \tau_y > \lfloor n^\varepsilon \rfloor \right) + \frac{c_\varepsilon}{n^{2\varepsilon}} + \frac{c_\varepsilon}{n^{2\varepsilon}} \mathbb{E}_x\left(N\left(X_{\lfloor n^\varepsilon \rfloor} \right) \right).$$

Using Corollary 6.5, the inequality (2.2) and the fact that $n^{1/2-\varepsilon} \leqslant z$, we find

$$\mathbb{P}_x(\tau_y > n) \leqslant \frac{c_\varepsilon \left(z + N(x) \right)}{\sqrt{n}}. \tag{9.28}$$

Now, for any $x \in \mathbb{X}$, $z \in \mathbb{R}$ and $y = z - r(x)$, using the Markov property, (9.28) and the fact that $\sqrt{n - \nu_n} \geqslant c_\varepsilon \sqrt{n}$ on the event $\{\nu_n \leqslant \lfloor n^{1-\varepsilon} \rfloor\}$, we have

$$\begin{aligned} \mathbb{P}_x(\tau_y > n) \leqslant & \frac{c_\varepsilon}{\sqrt{n}} \mathbb{E}_x\left(z + M_{\nu_n} + N\left(X_{\nu_n} \right) ;\, \tau_y > \nu_n \,,\, \nu_n \leqslant \left\lfloor n^{1-\varepsilon} \right\rfloor \right) \\ & + \mathbb{P}_x\left(\tau_y > n \,,\, \nu_n > \left\lfloor n^{1-\varepsilon} \right\rfloor \right). \end{aligned}$$

Using Lemma 6.3 and the fact that $N(X_{\nu_n}) \leqslant z + M_{\nu_n}$ on the event $\{N(X_{\nu_n}) \leqslant n^{1/2-\varepsilon}\}$, with $l = \left\lfloor n^{1/2-\varepsilon} \right\rfloor$, it holds

$$\begin{aligned} \mathbb{P}_x(\tau_y > n) \leqslant & \frac{c_\varepsilon}{\sqrt{n}} \mathbb{E}_x\left(\left(z + M_{\nu_n} \right) \left(1 + \mathbb{1}_{\left\{ N(X_{\nu_n}) \leqslant n^{1/2-\varepsilon} \right\}} \right) ;\, \tau_y > \nu_n \,,\, \nu_n \leqslant \left\lfloor n^{1-\varepsilon} \right\rfloor \right) \\ & + \frac{c_\varepsilon}{\sqrt{n}} \mathbb{E}_x\left(N_l\left(X_{\nu_n} \right) ;\, \tau_y > \nu_n \,,\, \nu_n \leqslant \left\lfloor n^{1-\varepsilon} \right\rfloor \right) + c_\varepsilon \,\mathrm{e}^{-c_\varepsilon n^\varepsilon} \left(1 + N(x) \right) \\ \leqslant & \frac{2c_\varepsilon}{\sqrt{n}} E_1 + \frac{c_\varepsilon}{\sqrt{n}} \sum_{k=1}^{\lfloor n^\varepsilon \rfloor} \mathbb{E}_x\left(N_l\left(X_k \right) \right) \\ & + \frac{c_\varepsilon}{\sqrt{n}} \sum_{k=\lfloor n^\varepsilon \rfloor + 1}^{\left\lfloor n^{1-\varepsilon} \right\rfloor} \mathbb{E}_x\left(N_l\left(X_k \right) ;\, \tau_y > k \right) + c_\varepsilon \,\mathrm{e}^{-c_\varepsilon n^\varepsilon} \left(1 + N(x) \right). \end{aligned}$$

By (2.3) and the Markov property,

$$\begin{aligned}\mathbb{P}_x\left(\tau_y > n\right) &\leqslant \frac{c_\varepsilon}{\sqrt{n}} E_1 + \frac{c_\varepsilon}{\sqrt{n}} \left(\frac{cn^\varepsilon}{l^{1+\beta}} + (1+N(x)) \right) + c_\varepsilon \,\mathrm{e}^{-c_\varepsilon n^\varepsilon} \left(1+N(x)\right) \\ &\quad + \frac{c_\varepsilon}{\sqrt{n}} \sum_{j=1}^{\lfloor n^{1-\varepsilon} \rfloor - \lfloor n^\varepsilon \rfloor} \left[\frac{c}{l^{1+\beta}} \mathbb{P}_x\left(\tau_y > j\right) + c\,\mathrm{e}^{-cn^\varepsilon}\, \mathbb{E}_x\left(\left(1+N\left(X_j\right)\right)\right) \right] \\ &\leqslant \frac{c_\varepsilon}{\sqrt{n}} E_1 + \frac{c_\varepsilon \left(1+N(x)\right)}{\sqrt{n}} + \frac{c_\varepsilon}{\sqrt{n}} \frac{c}{l^{1+\beta}} \sum_{j=1}^{\lfloor n^{1-\varepsilon} \rfloor} \mathbb{P}_x\left(\tau_y > j\right).\end{aligned}$$

Using Lemmas 9.1 and 9.2, we deduce the point 2 of Theorem 2.4.

10. Asymptotic behaviour of the conditioned Markov walk. In this section, we prove Theorem 2.5. The arguments are similar to those given in Section 9. We also keep the same notations. Assume that $(x,y) \in \mathbb{X} \times \mathbb{R}$ and let $t_0 > 0$ be a positive real. For any $t \in [0,t_0]$ and $n \geqslant 1$, we write

$$\begin{aligned}&\mathbb{P}_x\left(y+S_n \leqslant t\sqrt{n}\,,\, \tau_y > n\right) \\ &= \mathbb{P}_x\left(y+S_n \leqslant t\sqrt{n}\,,\, \tau_y > n\,,\, \nu_n^{\varepsilon^2} > \left\lfloor n^{1-\varepsilon} \right\rfloor\right) \\ &\quad + \underbrace{\sum_{k=\lfloor n^{\varepsilon^2} \rfloor +1}^{\lfloor n^{1-\varepsilon} \rfloor} \int_{\mathbb{X}\times\mathbb{R}} \mathbb{P}_{x'}\left(y'+S_{n-k} \leqslant t\sqrt{n}\,,\, \tau_{y'} > n-k\,,\, \overline{A}_{n-k}\right) \times \mathbb{P}_x\left(X_k \in \mathrm{d}x'\,,\, y+S_k \in \mathrm{d}y'\,,\, \tau_y > k\,,\, \nu_n^{\varepsilon^2} = k\right)}_{=:L_1} \\ &\quad + \underbrace{\sum_{k=\lfloor n^{\varepsilon^2} \rfloor +1}^{\lfloor n^{1-\varepsilon} \rfloor} \int_{\mathbb{X}\times\mathbb{R}} \mathbb{P}_{x'}\left(y'+S_{n-k} \leqslant t\sqrt{n}\,,\, \tau_{y'} > n-k\,,\, A_{n-k}\right). \times \mathbb{P}_x\left(X_k \in \mathrm{d}x'\,,\, y+S_k \in \mathrm{d}y'\,,\, \tau_y > k\,,\, \nu_n^{\varepsilon^2} = k\right)}_{=:L_2}. \end{aligned} \tag{10.1}$$

Bound of L_1. With J_1 defined in (9.19) and with the bound (9.20), we have,

$$L_1 \leqslant J_1 \leqslant \frac{c_\varepsilon\left(1+\max(y,0)+N(x)\right)}{n^{1/2+\varepsilon}}. \tag{10.2}$$

Bound of L_2. According to whether $y+S_k \leqslant n^{1/2-\varepsilon/8}$ or not, we write

$$L_2 = \underbrace{\sum_{k=\lfloor n^{\varepsilon^2}\rfloor+1}^{\lfloor n^{1-\varepsilon}\rfloor} \int_{\mathbb{X}\times\mathbb{R}} \mathbb{P}_{x'}\left(y'+S_{n-k} \leqslant t\sqrt{n}\,,\, \tau_{y'} > n-k\,,\, A_{n-k}\right) \times \mathbb{P}_x\left(X_k \in \mathrm{d}x'\,,\, y+S_k \in \mathrm{d}y'\,,\, y+S_k > n^{1/2-\varepsilon/8}\,,\, \tau_y > k\,,\, \nu_n^{\varepsilon^2} = k\right)}_{=:L_3}$$

$$+ \underbrace{\sum_{k=\lfloor n^{\varepsilon^2}\rfloor+1}^{\lfloor n^{1-\varepsilon}\rfloor} \int_{\mathbb{X}\times\mathbb{R}} \mathbb{P}_{x'}\left(y'+S_{n-k} \leqslant t\sqrt{n}\,,\, \tau_{y'} > n-k\,,\, A_{n-k}\right) \times \mathbb{P}_x\left(X_k \in \mathrm{d}x'\,,\, y+S_k \in \mathrm{d}y'\,,\, y+S_k \leqslant n^{1/2-\varepsilon/8}\,,\, \tau_y > k\,,\, \nu_n^{\varepsilon^2} = k\right)}_{=:L_4}. \tag{10.3}$$

Bound of L_3. With J_3 defined in (9.21) and with the bound (9.23), we have

$$L_3 \leqslant J_3 \leqslant c_\varepsilon \frac{\max(y,0) + \left(1 + y\mathbb{1}_{\{y>n^{1/2-2\varepsilon}\}} + N(x)\right)^2}{n^{1/2+\varepsilon/2}}. \tag{10.4}$$

Bound of L_4. We start with the upper bound. Set $y'_+ = y'+(n-k)^{1/2-2\varepsilon}$ and $t_+ = t+\frac{2}{n^{2\varepsilon}}$. Note that on the event $\{y'+S_{n-k} \leqslant t\sqrt{n}\,,\, \tau_{y'} > n-k\,,\, A_{n-k}\}$ we have $y'_+ + \sigma B_{n-k} \leqslant t_+\sqrt{n}$ and $\tau_{y'_+}^{bm} > n-k$. Therefore, by Lemma 4.1,

$$\mathbb{P}_{x'}\left(y'+S_{n-k} \leqslant t\sqrt{n}\,,\, \tau_{y'} > n-k\,,\, A_{n-k}\right) \leqslant \frac{2}{\sqrt{2\pi}} \int_0^{\frac{t_+\sqrt{n}}{\sigma\sqrt{n-k}}} \mathrm{e}^{-s^2/2}\,\mathrm{sh}\left(s\frac{y'_+}{\sqrt{n-k}\sigma}\right)\mathrm{d}s.$$

We shall use the following bounds:

$$\mathrm{sh}(u) \leqslant u\left(1+\frac{u^2}{6}\,\mathrm{ch}(u)\right), \qquad \text{for } u \geqslant 0,$$

$$\frac{y'_+}{\sigma\sqrt{n-k}} \leqslant \frac{y'_+}{\sigma\sqrt{n}}\left(1+\frac{c_\varepsilon}{n^\varepsilon}\right) \leqslant \frac{c_\varepsilon}{n^{\varepsilon/8}}, \qquad \text{for } y' \leqslant n^{1/2-\varepsilon/8} \text{ and } k \leqslant \left\lfloor n^{1-\varepsilon}\right\rfloor,$$

$$\frac{t_+\sqrt{n}}{\sigma\sqrt{n-k}} \leqslant \frac{t}{\sigma} + \frac{c_{\varepsilon,t_0}}{n^\varepsilon} \leqslant c_{\varepsilon,t_0}, \qquad \text{for } k \leqslant \left\lfloor n^{1-\varepsilon}\right\rfloor.$$

Consequently,

$$\begin{aligned}
&\mathbb{P}_{x'}\left(y'+S_{n-k} \leqslant t\sqrt{n}\,,\, \tau_{y'} > n-k\,,\, A_{n-k}\right) \\
&\quad\leqslant \frac{2y'_+}{\sqrt{2\pi n}\sigma}\left(1+\frac{c_\varepsilon}{n^\varepsilon}\right) \int_0^{\frac{t_+\sqrt{n}}{\sigma\sqrt{n-k}}} s\,\mathrm{e}^{-s^2/2}\left(1+\frac{c_\varepsilon s^2}{n^{\varepsilon/4}}\,\mathrm{ch}\left(c_\varepsilon s\right)\right)\mathrm{d}s \\
&\quad\leqslant \frac{2y'_+}{\sqrt{2\pi n}\sigma}\left(1+\frac{c_\varepsilon}{n^\varepsilon}\right)\left(1+\frac{c_{\varepsilon,t_0}}{n^{\varepsilon/4}}\right)\left(\int_0^{\frac{t}{\sigma}} s\,\mathrm{e}^{-s^2/2}\,\mathrm{d}s + \int_{\frac{t}{\sigma}}^{\frac{t_+\sqrt{n}}{\sigma\sqrt{n-k}}} s\,\mathrm{e}^{-s^2/2}\,\mathrm{d}s\right) \\
&\quad\leqslant \frac{2y'_+}{\sqrt{2\pi n}\sigma}\left(1+\frac{c_{\varepsilon,t_0}}{n^{\varepsilon/4}}\right)\left(1-\mathrm{e}^{-\frac{t^2}{2\sigma^2}}+\frac{c_{\varepsilon,t_0}}{n^\varepsilon}\right).
\end{aligned}$$

This implies the upper bound (with F_2 and E_1 from Lemmas 9.5 and 9.1, respectively)

$$\begin{aligned} L_4 &\leqslant \frac{2}{\sqrt{2\pi n}\sigma}\left(1+\frac{c_{\varepsilon,t_0}}{n^{\varepsilon/4}}\right)\left(1-\mathrm{e}^{-\frac{t^2}{2\sigma^2}}+\frac{c_{\varepsilon,t_0}}{n^{\varepsilon}}\right)F_2+\frac{c_{\varepsilon,t_0}}{n^{1/2+\varepsilon}}E_1 \\ &\leqslant \frac{2V(x,y)}{\sqrt{2\pi n}\sigma}\left(1-\mathrm{e}^{-\frac{t^2}{2\sigma^2}}\right)+\frac{c_{\varepsilon,t_0}\left(1+\max(y,0)+N(x)\right)}{n^{1/2+\varepsilon/8}}. \end{aligned}$$

The proof of the lower bound of L_4, being similar, is left to the reader:

$$L_4 \geqslant \frac{2V(x,y)}{\sqrt{2\pi n}\sigma}\left(1-\mathrm{e}^{-\frac{t^2}{2\sigma^2}}\right)-c_{\varepsilon,t_0}\frac{\max(y,0)+\left(1+y\mathbb{1}_{\{y>n^{1/2-2\varepsilon}\}}+N(x)\right)^2}{n^{1/2+\varepsilon/8}}.$$

Combining the upper and the lower bounds of L_4 and (10.4) with (10.3) we obtain an asymptotic developpement of L_2. Implementing this developpement and the bound (10.2) into (10.1) and using Lemma 6.3, we conclude that

$$\begin{aligned} &\left|\mathbb{P}_x\left(y+S_n\leqslant t\sqrt{n}\,,\,\tau_y>n\right)-\frac{2V(x,y)}{\sqrt{2\pi n}\sigma}\left(1-\mathrm{e}^{-\frac{t^2}{2\sigma^2}}\right)\right| \\ &\qquad\qquad\leqslant c_{\varepsilon,t_0}\frac{\max(y,0)+\left(1+y\mathbb{1}_{\{y>n^{1/2-2\varepsilon}\}}+N(x)\right)^2}{n^{1/2+\varepsilon/8}}. \end{aligned}$$

Using the asymptotic of $\mathbb{P}_x(\tau_y>n)$ provided by Theorem 2.3 finishes the proof of Theorem 2.5.

11. Appendix: proofs for affine random walks in $\mathbb{R}^d$.

In this section we prove Proposition 3.2. For this we verify that Hypotheses **M1-M5** hold true on an appropriate Banach space which we proceed to introduce. Let $\delta>0$ be the constant from Hypothesis 3.1. Denote by $\mathscr{C}(\mathbb{R}^d)$ the space of continuous complex valued functions on $\mathbb{R}^d$. Let ε and θ be two positive numbers satisfying

$$1+\varepsilon<\theta<2<2+2\varepsilon<2+2\delta.$$

For any function $h\in\mathscr{C}(\mathbb{R}^d)$ introduce the norm $\left\|h\right\|_{\theta,\varepsilon}=\left|h\right|_\theta+\left[h\right]_\varepsilon$, where

$$\left|h\right|_\theta=\sup_{x\in\mathbb{R}^d}\frac{|h(x)|}{(1+|x|)^\theta},\qquad \left[h\right]_\varepsilon=\sup_{x\neq y}\frac{|h(x)-h(y)|}{|x-y|^\varepsilon(1+|x|)(1+|y|)}$$

and consider the Banach space

$$\mathscr{B}:=\mathscr{L}_{\theta,\varepsilon}=\left\{h\in\mathscr{C}\left(\mathbb{R}^d\right):\left\|h\right\|_{\theta,\varepsilon}<+\infty\right\}.$$

*Proof of **M1**.* Conditions 1, 2 and 3 of **M1** can be easily verified under the point 1 of Hypothesis 3.1 and the fact that $\theta<2+2\delta$ and $\left\|\boldsymbol{\delta}_x\right\|_{\mathscr{B}'}\leqslant(1+|x|)^\theta$, for any $x\in\mathbb{R}^d$.

We verify the point 4 of Hypothesis **M1**. For any $(x,y)\in\mathbb{R}^d\times\mathbb{R}^d$ and $t\in\mathbb{R}$, we have $\left|\mathrm{e}^{itf(x)}-\mathrm{e}^{itf(y)}\right|\leqslant|t|\left|f(x)-f(y)\right|\leqslant|t|\left|u\right|\left|x-y\right|$ and $\left|\mathrm{e}^{itf(x)}-\mathrm{e}^{itf(y)}\right|\leqslant 2$. Therefore, we write

$$\left|\mathrm{e}^{itf(x)}-\mathrm{e}^{itf(y)}\right|\leqslant 2^{1-\varepsilon}\left|t\right|^\varepsilon\left|u\right|^\varepsilon\left|x-y\right|^\varepsilon.$$

Supposing that $|x| \leqslant |y|$, we obtain, for any $h \in \mathscr{L}_{\theta,\varepsilon}$,

$$\left|\mathrm{e}^{itf(x)}\, h(x) - \mathrm{e}^{itf(y)}\, h(y)\right| \leqslant \left|\mathrm{e}^{itf(x)} - \mathrm{e}^{itf(y)}\right| |h|_\theta \,(1+|x|)^\theta + |h(x)-h(y)|\,.$$

Since $\theta < 2$, we have $\left[\mathrm{e}^{itf}\, h - \mathrm{e}^{itf}\, h\right]_\varepsilon \leqslant 2^{1-\varepsilon}\,|t|^\varepsilon\,|u|^\varepsilon\,|h|_\theta + [h]_\varepsilon$. Consequently, $\left\|\mathrm{e}^{itf}\, h\right\|_{\theta,\varepsilon} \leqslant (1+2^{1-\varepsilon}\,|t|^\varepsilon\,|u|^\varepsilon)\,\|h\|_{\theta,\varepsilon}$ and the point 4 is verified.

*Proof of **M2** and **M3**.* We shall verify that the conditions of the theorem of Ionescu-Tulcea and Marinescu are satisfied (see [28] and [25]). We start by establishing two lemmas.

LEMMA 11.1. *Assume Hypothesis 3.1.*

1. *There exists a constant $c > 0$ such that, for any $t \in \mathbb{R}$, $n \geqslant 1$, and $h \in \mathscr{L}_{\theta,\varepsilon}$,*

$$|\mathbf{P}_t^n h|_\theta \leqslant c\,|h|_\theta\,.$$

2. *There exist constants c_1, c_2 and $\rho < 1$ such that, for any $n \geqslant 1$, $h \in \mathscr{L}_{\theta,\varepsilon}$ and $t \in \mathbb{R}$,*

$$[\mathbf{P}_t^n h]_\varepsilon \leqslant c_1 \rho^n\,[h]_\varepsilon + c_2\,|t|^\varepsilon\,|h|_\theta\,.$$

3. *For any $t \in \mathbb{R}$, the operator $\mathbf{P}_t$ is compact from $(\mathscr{B}, \|\cdot\|_{\theta,\varepsilon})$ to $(\mathscr{C}\left(\mathbb{R}^d\right), |\cdot|_\theta)$.*

PROOF. *Claim 1.* For any $x \in \mathbb{R}^d$,

$$|\mathbf{P}_t^n h(x)| = \left|\mathbb{E}_x\left(\mathrm{e}^{itS_n}\, h\left(X_n\right)\right)\right| \leqslant 3^\theta\,|h|_\theta \left(1 + \mathbb{E}\left(\|\Pi_n\|^\theta\right)|x|^\theta + \mathbb{E}\left(\left|X_n^0\right|^\theta\right)\right),$$

with $\Pi_n = A_n A_{n-1}\dots A_1$ and $X_n^0 = g_n \dots g_1 \cdot 0 = \sum_{k=1}^n A_n \dots A_{k+1} B_k$. By the point 1 of Hypothesis 3.1, there exist $c(\delta) > 0$ and $0 < \rho(\delta) < 1$ such that, for any $n \geqslant 1$,

$$\mathbb{E}^{\frac{2+2\delta}{\theta}}\left(\|\Pi_n\|^\theta\right) \leqslant \mathbb{E}\left(\|\Pi_n\|^{2+2\delta}\right) \leqslant c(\delta)\rho(\delta)^n \underset{n\to+\infty}{\longrightarrow} 0,$$

from which it follows that

$$\mathbb{E}\left(\left|X_n^0\right|^\theta\right) \leqslant \left(\sum_{k=1}^n \mathbb{E}^{1/\theta}\left(\|\Pi_n\|^\theta\right)\mathbb{E}^{1/\theta}\left(|B_1|^\theta\right)\right)^\theta < +\infty.$$

This proves the claim 1.

Proof of the claim 2. For any $x \neq y \in \mathbb{R}^d$, with $|x| \leqslant |y|$, we have

$$\begin{aligned}
&|\mathbf{P}_t^n h(x) - \mathbf{P}_t^n h(y)| \\
&\quad\leqslant \mathbb{E}\left(2^{1-\varepsilon}\,|t|^\varepsilon\,|u|^\varepsilon\left(\sum_{k=1}^n \|\Pi_k\|\right)^\varepsilon |x-y|^\varepsilon\,|h|_\theta\left(1+\|\Pi_n\|\,|x| + \left|X_n^0\right|\right)^\theta\right) \\
&\qquad + \mathbb{E}\left([h]_\varepsilon\,\|\Pi_n\|^\varepsilon\,|x-y|^\varepsilon\left(1+\|\Pi_n\|\,|x|+\left|X_n^0\right|\right)\left(1+\|\Pi_n\|\,|y|+\left|X_n^0\right|\right)\right).
\end{aligned}$$

Since $\theta < 2$, we obtain that

$$[\mathbf{P}_t^n h]_\varepsilon \leqslant 2^{1-\varepsilon}\,|t|^\varepsilon\,|u|^\varepsilon\,C_2(n)\,|h|_\theta + C_1(n)\,[h]_\varepsilon\,,$$

where

$$C_1(n) = \mathbb{E}\left(\|\Pi_n\|^{\varepsilon} \left(1 + \|\Pi_n\| + \left|X_n^0\right|\right)^2 \right)$$

and

$$C_2(n) = \mathbb{E}\left(\left(\sum_{k=1}^n \|\Pi_k\| \right)^{\varepsilon} \left(1 + \|\Pi_n\| + \left|X_n^0\right|\right)^{\theta} \right).$$

Since $2+2\varepsilon < 2+2\delta = p$, by the Hölder inequality,

$$\begin{aligned} C_1(n) &\leqslant \mathbb{E}^{\frac{\varepsilon}{1+\varepsilon}} \left(\|\Pi_n\|^{1+\varepsilon} \right) \mathbb{E}^{\frac{1}{1+\varepsilon}} \left(\left(1 + \|\Pi_n\| + \left|X_n^0\right|\right)^{2+2\varepsilon} \right) \\ &\leqslant c(\delta)^{\frac{\varepsilon}{p}} \rho(\delta)^{\frac{n\varepsilon}{p}} 3^2 \left(1 + c(\delta)^{\frac{2}{p}} + \left(\frac{c(\delta)^{\frac{1}{p}} \mathbb{E}^{\frac{1}{p}} \left(|B_1|^p\right)}{1-\rho(\delta)^{\frac{1}{p}}} \right)^2 \right), \end{aligned}$$

which shows that $C_1(n)$ converges exponentially fast to 0. In the same way, taking into account that $\theta < 2$ we show that $C_2(n)$ is bounded:

$$\begin{aligned} C_2(n) &\leqslant \left(\sum_{k=1}^n \mathbb{E}^{\frac{1}{1+\varepsilon}} \left(\|\Pi_k\|^{1+\varepsilon} \right) \right)^{\varepsilon} \mathbb{E}^{\frac{1}{1+\varepsilon}} \left(\left(1 + \|\Pi_n\| + \left|X_n^0\right|\right)^{2+2\varepsilon} \right) \\ &\leqslant \left(\frac{c(\delta)^{\frac{1}{p}}}{1-\rho(\delta)^{\frac{1}{p}}} \right)^{\varepsilon} 3^2 \left(1 + c(\delta)^{\frac{2}{p}} + \left(\frac{c(\delta)^{\frac{1}{p}} \mathbb{E}^{\frac{1}{p}} \left(|B_1|^p\right)}{1-\rho(\delta)^{\frac{1}{p}}} \right)^2 \right). \end{aligned}$$

Proof of the claim 3. Let B be a bounded subset of $\mathscr{B}$, $(h_n)_{n\geqslant 0}$ be a sequence in B and K be a compact of $\mathbb{R}^d$. Using the claim 1, it follows that, for any $x \in K$ and $n \geqslant 0$,

$$|\mathbf{P}_t h_n(x)| \leqslant c \left|h_n\right|_{\theta} (1+|x|)^{\theta} \leqslant c_K,$$

which implies that the set $\mathscr{A} = \{\mathbf{P}_t h_n : n \geqslant 0\}$ is uniformly bounded in $(\mathscr{C}(K), |\cdot|_{\infty})$, where $|\cdot|_{\infty}$ is the supremum norm. By the claims 1 and 2, we have that, for any $x, y \in K$ and $n \geqslant 0$,

$$|\mathbf{P}_t h_n(x) - \mathbf{P}_t h_n(y)| \leqslant [\mathbf{P}_t h_n]_{\varepsilon} |x-y|^{\varepsilon} (1+|x|)^{\theta} (1+|y|)^{\theta} \leqslant c_K \|h_n\|_{\mathscr{B}} |x-y|^{\varepsilon}$$

and, thereby, the set $\mathscr{A}$ is uniformly equicontinuous. By the theorem of Arzelà-Ascoli, we conclude that $\mathscr{A}$ is relatively compact in $(\mathscr{C}(K), |\cdot|_{\infty})$. Using a diagonal extraction, we deduce that there exist a subsequence $(n_k)_{k\geqslant 1}$ and a function $\varphi \in \mathscr{C}(\mathbb{R}^d)$ such that, for any compact $K \subset \mathbb{R}^d$,

$$\sup_{x\in K} |P_t h_{n_k}(x) - \varphi(x)| \underset{n\to+\infty}{\longrightarrow} 0.$$

Moreover, by the claims 1 and 2, for any $n \geqslant 1$ and $x \in \mathbb{R}^d$,

$$|P_t h_n(x)| \leqslant |P_t h_n(0)| + [P_t h_n]_{\varepsilon} |x|^{\varepsilon} (1+|x|) \leqslant c \left|h_n\right|_{\theta} + c \|h_n\|_{\mathscr{B}} |x|^{\varepsilon} (1+|x|).$$

Since B is bounded, we have $|P_t h_n(x)| \leqslant c(1+|x|)^{1+\varepsilon}$, for any $x \in \mathbb{R}^d$, as well as $\varphi(x) \leqslant c(1+|x|)^{1+\varepsilon}$, for any $x \in \mathbb{R}^d$. Consequently, for any $k \geqslant 1$ and $A > 0$,

$$\sup_{x\in\mathbb{R}^d} \frac{|P_t h_{n_k}(x) - \varphi(x)|}{(1+|x|)^{\theta}} \leqslant \sup_{|x|\leqslant A} |P_t h_{n_k}(x) - \varphi(x)| + 2c \sup_{|x|>A} \frac{(1+|x|)^{1+\varepsilon}}{(1+|x|)^{\theta}}.$$

Taking the limit as $k \to +\infty$ and then the limit as $A \to +\infty$, we can conclude that $\lim_{k\to+\infty} |P_t h_{n_k} - \varphi|_\theta = 0$.

□

LEMMA 11.2. *Assume Hypothesis 3.1.*

1. *The operator* $\mathbf{P}$ *has a unique invariant probability* $\boldsymbol{\nu}$ *which coincides with the distribution of the* $\mathbb{P}$*-a.s. convergent series* $Z := \sum_{k=1}^{+\infty} A_1 \dots A_{k-1} B_k$. *Moreover, the unique eigenvalue of modulus* 1 *of the operator* $\mathbf{P}$ *on* $\mathscr{B}$ *is* 1 *and the associated eigenspace is generated by the function* $e\colon x \mapsto 1$.
2. *Let* $t \in \mathbb{R}^*$. *If* $h \in \mathscr{B}$ *and* $z \in \mathbb{C}$ *of modulus* 1 *are such that*

$$\mathbf{P}_t h(x) = z h(x), \qquad x \in \operatorname{supp}(\boldsymbol{\nu}),$$

then $h = 0$ *on* $\operatorname{supp}(\boldsymbol{\nu})$.

PROOF. We proceed as in Guivarc'h and Le Page [22] and Buraczewski, Damek and Guivarc'h [5]. For any $g = (A, B) \in \mathrm{GL}(d, \mathbb{R}) \times \mathbb{R}^d$ and $x \in \mathbb{R}^d$, we set $g \cdot x = Ax + B$.

Proof of claim 1. Since $k(\delta) < 1$, the series $\sum_k \mathbb{E}^{\frac{1}{2+2\delta}}(|A_1 \dots A_{k-1} B_k|^{2+2\delta})$ converges and so the sequence $g_1 \dots g_n \cdot x = A_1 \dots A_n x + \sum_{k=1}^n A_1 \dots A_{k-1} B_k$ converges almost surely to $Z = \sum_{k=1}^{+\infty} A_1 \dots A_{k-1} B_k$ as $n \to +\infty$. Therefore, for any $\varphi \in \mathscr{B}$, the sequence $\varphi(g_1 \dots g_n \cdot x)$ converges to $\varphi(Z)$ almost surely as $n \to +\infty$. Moreover, since $|\varphi(x)| \leqslant |\varphi|_\theta (1 + |x|)^\theta$ and $\theta < 2 + 2\delta$, the sequence $(\varphi(g_1 \dots g_n \cdot x))_{n\geqslant 1}$ is uniformly integrable. So $\mathbf{P}^n \varphi(x)$ converges to $\mathbb{E}(\varphi(Z))$ as $n \to +\infty$. This proves that the distribution $\boldsymbol{\nu}$ of Z is the only invariant probability of $\mathbf{P}$.

Fix $z \in \mathbb{C}$ such that $|z| = 1$ and let $h \neq 0$ belonging to $\mathscr{B}$ be an eigenfunction of $\mathbf{P}$, so that $\mathbf{P}h = zh$. From the previous argument, it follows that, for any $x \in \mathbb{R}^d$,

$$z^n h(x) = \mathbf{P}^n h(x) \underset{n\to+\infty}{\longrightarrow} \boldsymbol{\nu}(h).$$

Since there exists $x \in \mathbb{R}^d$ such that $h(x) \neq 0$, the sequence $(z^n)_{n\geqslant 1}$ should be convergent which is possible only if $z = 1$. From this, we deduce that for any $x \in \mathbb{R}^d$, $h(x) = \mathbb{E}(h(Z))$ which implies that h is constant.

Proof of the claim 2. Our argument is by contradiction. Let $t \in \mathbb{R}^*$, $h \in \mathscr{B}$ and $z \in \mathbb{C}$ of modulus 1 be such that $\mathbf{P}_t h(x) = z h(x)$, for any $x \in \operatorname{supp}(\boldsymbol{\nu})$ and suppose that there exists $x_0 \in \operatorname{supp}(\boldsymbol{\nu})$ such that $h(x_0) \neq 0$.

First we establish that $|h|$ is constant on the support of the distribution $\boldsymbol{\nu}$. Since $\boldsymbol{\nu}$ is $\boldsymbol{\mu}$-invariant, for any $(g, x) \in \operatorname{supp}(\boldsymbol{\mu}) \times \operatorname{supp}(\boldsymbol{\nu})$ we have $g \cdot x \in \operatorname{supp}(\boldsymbol{\nu})$. From this fact it follows that $\mathbf{P}_t^n h(x) = z^n h(x)$, for any $n \geqslant 1$ and $x \in \operatorname{supp}(\boldsymbol{\nu})$. This implies that $|h|(x) \leqslant \mathbf{P}^n |h|(x)$, for any $x \in \operatorname{supp}(\boldsymbol{\nu})$. Note also that $|h|$ belongs to $\mathscr{B}$. Therefore, as we have seen in the proof of the first claim, we have, $\lim_{n\to+\infty} \mathbf{P}^n |h|(x) = \boldsymbol{\nu}(|h|) = \mathbb{E}(|h|(Z)) < +\infty$, for any $x \in \operatorname{supp}(\boldsymbol{\nu})$. So $|h|(x) \leqslant \int_{x'\in\mathbb{R}^d} |h|(x')\boldsymbol{\nu}(\mathrm{d}x')$, for any $x \in \operatorname{supp}(\boldsymbol{\nu})$. Since $|h|$ is continuous, this implies that $|h|$ is constant on the support of $\boldsymbol{\nu}$. In particular, this means that $h(x) \neq 0$ for any $x \in \operatorname{supp}(\boldsymbol{\nu})$.

Since the support of $\boldsymbol{\nu}$ is stable by all the elements of the support of $\boldsymbol{\mu}$, we deduce that the random variable $\xi_n(x) = \exp(it \langle u, \sum_{k=1}^n g_k \dots g_1 \cdot x\rangle) h(g_n \dots g_1 \cdot x)$ takes values on the sphere $\mathbb{S}_{\nu(|h|)} = \{a \in \mathbb{C} : |a| = \boldsymbol{\nu}(|h|)\}$, for all x in the support of $\boldsymbol{\nu}$. Moreover, the mean $z^n h(x)$ of $\xi_n(x)$ is also on $\mathbb{S}_{\nu(|h|)}$, which is possible only if $\xi_n(x)$ is a constant, for any $x \in \operatorname{supp}(\boldsymbol{\nu})$. Consequently, for any pair $x, y \in \operatorname{supp}(\boldsymbol{\nu})$, there exists an event $\Omega_{x,y}$ of $\mathbb{P}$-probability one such that on $\Omega_{x,y}$ it holds, for any $n \geqslant 1$,

$$\exp\left(it \left\langle u, \sum_{k=1}^{n} g_k \dots g_1 \cdot v \right\rangle \right) h\left(g_n \dots g_1 \cdot v\right) = z^n h(v),$$

with $v \in \{x, y\}$, from which we get

$$\frac{h\left(g_n \dots g_1 \cdot y\right)}{h\left(g_n \dots g_1 \cdot x\right)} = \frac{h(y)}{h(x)} \exp\left(it \left\langle \sum_{k=1}^{n} {}^t A_1 \dots {}^t A_k u, x - y \right\rangle \right). \tag{11.1}$$

In addition, for any $n \geqslant 1$,

$$\mathbb{E}\left(\left| \frac{h\left(g_n \dots g_1 \cdot y\right)}{h\left(g_n \dots g_1 \cdot x\right)} - 1 \right| \right) = \mathbb{E}\left(\left| \frac{h\left(g_1 \dots g_n \cdot y\right)}{h\left(g_1 \dots g_n \cdot x\right)} - 1 \right| \right).$$

Since, for $v \in \{x, y\}$, the sequence $h(g_1 \dots g_n \cdot v)$ converges a.s. to $h(Z)$ and since h is bounded with a constant modulus, we have by (11.1),

$$\begin{aligned} 0 &= \lim_{n \to +\infty} \mathbb{E}\left(\left| \frac{h\left(g_n \dots g_1 \cdot y\right)}{h\left(g_n \dots g_1 \cdot x\right)} - 1 \right| \right) \\ &= \lim_{n \to +\infty} \mathbb{E}\left(\left| \frac{h(y)}{h(x)} \exp\left(it \left\langle \sum_{k=1}^{n} {}^t A_1 \dots {}^t A_k u, x - y \right\rangle \right) - 1 \right| \right). \end{aligned}$$

Taking into account that the series $\sum_{k=1}^{n} {}^t A_1 \dots {}^t A_k$ converges a.s. to a random variable Z', we have for any $x, y \in \operatorname{supp}(\boldsymbol{\nu})$,

$$\mathbb{E}\left(\left| \frac{h(y)}{h(x)} \, \mathrm{e}^{it\langle Z'u, x-y\rangle} - 1 \right| \right) = 0. \tag{11.2}$$

Since the support of $\boldsymbol{\nu}$ is invariant by all the elements of the support of $\boldsymbol{\mu}$, by the point 2 of Hypothesis 3.1, we deduce that the support of $\boldsymbol{\nu}$ is not contained in an affine subspace of $\mathbb{R}^d$, *i.e.* for any $1 \leqslant j \leqslant d$, there exist $x_j, y_j \in \operatorname{supp}(\boldsymbol{\nu})$, such that the family $(v_j)_{1 \leqslant j \leqslant d} = (x_j - y_j)_{1 \leqslant j \leqslant d}$ generates $\mathbb{R}^d$. From (11.2), we conclude that for any $1 \leqslant j \leqslant d$,

$$\frac{h(y_j)}{h(x_j)} \, \mathrm{e}^{it\langle Z'u, v_j\rangle} = 1, \qquad \mathbb{P}\text{-a.s.}$$

Let θ_j be such that $\frac{h(x_j)}{h(y_j)} = \mathrm{e}^{i\theta_j}$. Denoting by $\boldsymbol{\eta}_u$ the distribution of $Z'u$, we obtain that $\langle Z'u, v_j \rangle \in \frac{\theta_j + 2\pi\mathbb{Z}}{t}$ $\mathbb{P}$-a.s. and so the support of $\boldsymbol{\eta}_u$ is discrete. Moreover, the measure $\boldsymbol{\eta}_u$ is invariant for the Markov chain $X'_{n+1} = {}^t A_{n+1}(X'_n + u)$ and so, for any Borel set B of $\mathbb{R}^d$,

$$\boldsymbol{\eta}_u\left(B\right) = \mathbb{E}\left(\int_{v \in \mathbb{R}^d} \mathbb{1}_B\left({}^t A_1 \left(v + u\right) \right) \boldsymbol{\eta}_u(\mathrm{d}v) \right). \tag{11.3}$$

Since $\boldsymbol{\eta}_u$ is discrete, the set $E_{max} = \{x \in \mathbb{R}^d : \boldsymbol{\eta}_u(\{x\}) = \max_{y\in\mathbb{R}^d} \boldsymbol{\eta}_u(\{y\})\}$ is non-empty and finite. Moreover, using (11.3) with $B = \{x\}$ and $x \in E_{max}$, we can see that the image ${}^tA_1^{-1}x - u$ belongs to E_{max} $\mathbb{P}$-a.s. Denoting by v_0 the barycentre of E_{max}, we find that

$$\mathbb{P}\left({}^tA_1^{-1}v_0 - u = v_0\right) = 1.$$

The fact that $u \neq 0$ implies that $v_0 \neq 0$. The latter implies that ${}^tA_1^{-1}v_0 = v_0 + u = {}^tA_2^{-1}v_0$ almost surely, which contradicts the point 3 of Hypothesis 3.1. □

The conditions (b), (c) and (d) of the theorem of Ionescu-Tulcea and Marinescu as stated in Chapter 3 of Norman [28] follow from points 1-3 of Lemma 11.1 repectively. It remains to show the condition (a). Let $(h_n)_{n\geqslant 0}$ be a sequence in $\mathscr{L}_{\theta,\varepsilon}$ satisfying $\|h_n\|_{\theta,\varepsilon} \leqslant K$, for any $n \geqslant 0$ and some constant K and suppose that there exists $h \in \mathscr{C}(\mathbb{R}^d)$ such that $\lim_{n\to+\infty} |h_n - h|_\theta = 0$. For any $x, y, z \in \mathbb{R}^d$ and $n \geqslant 0$,

$$\begin{aligned}
&\frac{|h(x) - h(y)|}{|x-y|^\varepsilon (1+|x|)(1+|y|)} + \frac{|h(z)|}{(1+|z|)^\theta} \\
&\leqslant |h_n - h|_\theta \left(\frac{(1+|x|)^\theta + (1+|y|)^\theta}{|x-y|^\varepsilon (1+|x|)(1+|y|)} + 1\right) + [h_n]_\varepsilon + |h_n|_\theta .
\end{aligned}$$

Taking the limit as $n \to +\infty$, shows that $h \in \mathscr{L}_{\theta,\varepsilon}$ and $\|h\|_{\theta,\varepsilon} \leqslant K$.

The theorem of Ionescu-Tulcea and Marinescu and the unicity of the one-dimensional projector proved in the point 1 of Lemma 11.2 imply Hypothesis **M2**. Hypothesis **M3** is obtained easily from Lemma 11.1.

The point 2 of Lemma 11.2 will be used latter to prove that $\sigma^2 > 0$.

*Proof of **M4**.* By the hypothesis $\alpha = \frac{2+2\delta}{1+\varepsilon} > 2$. Consider the function $N\colon \mathbb{R}^d \to \mathbb{R}_+$ defined by $N(x) = |x|^{1+\varepsilon}$. For any $x, y \in \mathbb{R}^d$ satisfying $|x| \leqslant |y|$,

$$|N(x) - N(y)| \leqslant (1+\varepsilon)\,|y|^\varepsilon\,|x-y|\,.$$

Using the fact that $|N(x) - N(y)| \leqslant 2\,|y|^{1+\varepsilon}$, we have

$$|N(x) - N(y)| \leqslant (1+\varepsilon)^\varepsilon 2^{1-\varepsilon}\,|y|^{\varepsilon^2 + (1+\varepsilon)(1-\varepsilon)}\,|x-y|^\varepsilon = c_\varepsilon\,|y|\,|x-y|^\varepsilon\,.$$

Together with $|N|_\theta < +\infty$, this proves that the function N is in $\mathscr{B} = \mathscr{L}_{\theta,\varepsilon}$.

Obviously $|f(x)|^{1+\varepsilon} = |\langle u, x\rangle|^{1+\varepsilon} \leqslant |u|^{1+\varepsilon}(1 + N(x))$. Moreover, for any $h \in \mathscr{L}_{\theta,\varepsilon}$,

$$|h(x)| \leqslant [h]_\varepsilon\,|x|^\varepsilon\,(1+|x|) + |h(0)| \leqslant 2\,\|h\|_{\theta,\varepsilon}\,(1 + N(x))$$

and so $\|\boldsymbol{\delta}_x\|_{\mathscr{B}'} \leqslant 2\,(1 + N(x))$. Note that for any $p \in [1, \alpha]$,

$$\mathbb{E}^{1/p}\left(N\left(g_n \dots g_1 \cdot x\right)^p\right) \leqslant 2^{1+\varepsilon}\left(\mathbb{E}^{1/p}\left(\|\Pi_n\|^{p(1+\varepsilon)}\right) N(x) + \mathbb{E}^{1/p}\left(|g_n \dots g_1 \cdot 0|^{p(1+\varepsilon)}\right)\right).$$

Since $p(1+\varepsilon) \leqslant 2 + 2\delta$, the previous inequality proves that $\mathbb{E}_x^{1/p}\left(N\left(X_n\right)^p\right) \leqslant c\,(1 + N(x))$. Thus, we proved the first inequality of the point 1 of **M4**.

For any $l \geqslant 1$, we consider the function ϕ_l on $\mathbb{R}_+$ defined by:

$$\phi_l(t) = \begin{cases} 0 & \text{if } t \leqslant l^{\frac{1}{1+\varepsilon}} - 1, \\ t - \left(l^{\frac{1}{1+\varepsilon}} - 1\right) & \text{if } t \in \left[l^{\frac{1}{1+\varepsilon}} - 1, l^{\frac{1}{1+\varepsilon}}\right], \\ 1 & \text{if } t \geqslant l^{\frac{1}{1+\varepsilon}}. \end{cases} \tag{11.4}$$

Define N_l on $\mathbb{R}^d$ by $N_l(x) = \phi_l(|x|)N(x)$. For any $x \in \mathbb{R}^d$, we have $N(x)\mathbb{1}_{\{N(x)>l\}} \leqslant N_l(x) \leqslant N(x)$ which implies that $|N_l|_\theta \leqslant |N|_\theta < +\infty$. Moreover, for any $x, y \in \mathbb{R}^d$ satisfying $|x| \leqslant |y|$, we have

$$|\phi_l(|y|) - \phi_l(|x|)| \leqslant \min\left(|y| - |x|, 1\right).$$

So

$$|N_l(y) - N_l(x)| \leqslant [N]_\varepsilon |x-y|^\varepsilon (1+|x|)(1+|y|) + |x|^{1+\varepsilon} |y-x|^\varepsilon.$$

Since $|x| \leqslant |y|$, we obtain that $[N_l]_\varepsilon \leqslant [N]_\varepsilon + 1 < +\infty$. Therefore, the function N_l belongs to $\mathscr{B} = \mathscr{L}_{\theta,\varepsilon}$, which finishes the proof of the point 1 of **M4**.

Moreover, $\|N_l\|_{\theta,\varepsilon} \leqslant \|N\|_{\theta,\varepsilon} + 1$ and, so the point 2 of **M4** is also established.

Since $\int_{\mathbb{X}} |x|^p \boldsymbol{\nu}(\mathrm{d}x) < +\infty$, for any $p \leqslant 2+2\delta$, we find that

$$\boldsymbol{\nu}(N_l) \leqslant \int_{\mathbb{X}} |x|^{1+\varepsilon} \mathbb{1}_{\left\{|x| \geqslant l^{\frac{1}{1+\varepsilon}} - 1\right\}} \boldsymbol{\nu}(\mathrm{d}x) \leqslant \frac{\int_{\mathbb{X}} |x|^{2+2\delta} \boldsymbol{\nu}(\mathrm{d}x)}{\left(l^{\frac{1}{1+\varepsilon}} - 1\right)^{2+2\delta-(1+\varepsilon)}}.$$

Choosing $\beta = \alpha - 2 > 0$, we obtain the point 3 of **M4**.

*Proof of **M5***. Using (2.5) and the point 4 of Hypothesis 3.1,

$$\mu = \int_{\mathbb{R}^d} \langle u, x \rangle \boldsymbol{\nu}(\mathrm{d}x) = \left\langle u, \mathbb{E}\left(\sum_{k=1}^{+\infty} A_1 \dots A_{k-1} B_k\right)\right\rangle = 0. \tag{11.5}$$

Now we prove that $\sigma^2 > 0$. For this, suppose the contrary: $\sigma^2 = 0$. One can easily check that the function f belongs to $\mathscr{B}$. Using **M2** and the fact that $\nu(f) = \mu = 0$, we deduce that $\sum_{n \geqslant 0} \|\mathbf{P}^n f\|_{\theta,\varepsilon} = \sum_{n\geqslant 0} \|Q^n f\|_{\theta,\varepsilon} < +\infty$ and therefore the series $\sum_{n \geqslant 0} \mathbf{P}^n f$ converges in $\left(\mathscr{B}, \|\cdot\|_{\theta,\varepsilon}\right)$. We denote by $\Theta \in \mathscr{B}$ its limit and notice that the function Θ satisfies the Poisson equation: $\Theta - \mathbf{P}\Theta = f$.

Using the bound (2.6), we have $\left|\sum_{n=1}^N f(x)\mathbf{P}^n f(x)\right| \leqslant c\left(1+N(x)\right)^2$. By the Lebesgue dominated convergence theorem, from (2.5), we obtain

$$\begin{aligned} \sigma^2 &= \int_{\mathbb{R}^d} f(x) \left(2\Theta(x) - f(x)\right) \boldsymbol{\nu}(\mathrm{d}x) \\ &= \int_{\mathbb{R}^d} \left(\Theta^2(x) - (\mathbf{P}\Theta)^2(x)\right) \boldsymbol{\nu}(\mathrm{d}x) \\ &= \int_{\mathrm{GL}(d,\mathbb{R})\times\mathbb{R}^d\times\mathbb{R}^d} \left(\Theta(g_1 \cdot x) - \mathbf{P}\Theta(x)\right)^2 \boldsymbol{\mu}(\mathrm{d}g_1)\boldsymbol{\nu}(\mathrm{d}x). \end{aligned}$$

As $\sigma^2 = 0$, we have $\Theta(g_1 \cdot x) = \mathbf{P}\Theta(x)$, *i.e.* $f(g_1 \cdot x) = \mathbf{P}\Theta(x) - \mathbf{P}\Theta(g_1 \cdot x)$, $\boldsymbol{\mu} \times \boldsymbol{\nu}$-a.s. Consequently, there exists a Borel subset B_0 of $\mathbb{R}^d$ such that $\boldsymbol{\nu}(B_0) = 1$ and for any $t \in \mathbb{R}$ and $x \in B_0$,

$$\int_{\mathrm{GL}(d,\mathbb{R})\times\mathbb{R}^d} \mathrm{e}^{it\langle u, g_1\cdot x\rangle} \mathrm{e}^{it\mathbf{P}\Theta(g_1\cdot x)} \boldsymbol{\mu}(\mathrm{d}g_1) = \mathrm{e}^{it\mathbf{P}\Theta(x)}.$$

Since the functions in the both sides are continuous, this equality holds for every $x \in \operatorname{supp}(\boldsymbol{\nu})$. Since $\Theta \in \mathscr{L}_{\theta,\varepsilon}$, the function $x \mapsto \mathrm{e}^{it\mathbf{P}\Theta(x)}$ belongs to $\mathscr{L}_{\theta,\varepsilon} \setminus \{0\}$. This contradicts the point 2 of Lemma 11.2 and we conclude that $\sigma^2 > 0$ and so **M5** holds true.

12. Appendix: proofs for compact Markov chains. In this section we prove Proposition 3.7. For this we show that **M1-M5** hold true with $N = N_l = 0$, for the Markov chain $(X_n)_{n\geqslant 1}$, the function f and the Banach space $\mathscr{L}(\mathbb{X})$ given in Section 3.2.

*Proof of **M1**.* Obviously the Dirac measure belongs to $\mathscr{L}(\mathbb{X})'$ and $\|\boldsymbol{\delta}_x\|_{\mathscr{L}(\mathbb{X})'} \leqslant 1$ for any $x \in \mathbb{X}$. For any $h \in \mathscr{L}(\mathbb{X})$ and $t \in \mathbb{R}$ the function $\mathrm{e}^{itf}\, h$ belongs to $\mathscr{L}(\mathbb{X})$ and

$$\left\| \mathrm{e}^{itf}\, h \right\|_{\mathscr{L}} \leqslant |t|\,[f]_{\mathbb{X}}\, \|h\|_\infty + \|h\|_{\mathscr{L}} \leqslant (|t|\,[f]_{\mathbb{X}} + 1)\, \|h\|_{\mathscr{L}}\,. \tag{12.1}$$

*Proof of **M2**.* Let (x_1,x_2) and (y_1,y_2) be two elements of $\mathbb{X}$ and $h \in \mathscr{L}(\mathbb{X})$. Since

$$\mathbf{P}h(x_1,x_2) = \int_X h(x_2,x')P(x_2,\mathrm{d}x'),$$

we have $\|\mathbf{P}h\|_\infty \leqslant \|h\|_\infty$. Denote by h_{x_2} the function $z \mapsto h(x_2,z)$, which is an element of $\mathscr{L}(X)$. Since $[h_{x_2}]_X \leqslant [h]_{\mathbb{X}}$ and $|h_{x_2}|_\infty \leqslant \|h\|_\infty$, we obtain also that

$$\begin{aligned}
|\mathbf{P}h(x_1,x_2) - \mathbf{P}h(y_1,y_2)| &= |Ph_{x_2}(x_2) - Ph_{y_2}(y_2)| \\
&\leqslant [Ph_{x_2}]_X\, d_X(x_2,y_2) + [h]_{\mathbb{X}}\, d_X(x_2,y_2) \\
&\leqslant (|P|_{\mathscr{L}\to\mathscr{L}}\, \|h\|_{\mathbb{X}} + [h]_{\mathbb{X}})\, d_X(x_2,y_2),
\end{aligned}$$

where $|P|_{\mathscr{L}\to\mathscr{L}}$ is the norm of the operator $P\colon \mathscr{L}(X) \to \mathscr{L}(X)$. Therefore $\mathbf{P}$ is a bounded operator on $\mathscr{L}(\mathbb{X})$ and $\|\mathbf{P}\|_{\mathscr{L}\to\mathscr{L}} \leqslant (1 + |P|_{\mathscr{L}\to\mathscr{L}})$. Now, for any $h \in \mathscr{L}(\mathbb{X})$, we define the function F_h by

$$F_h(x_2) := \int_X h(x_2,x')P(x_2,\mathrm{d}x') = \mathbf{P}h(x_1,x_2).$$

Notice that F_h belongs to $\mathscr{L}(X)$ and $|F_h|_{\mathscr{L}} \leqslant \|\mathbf{P}h\|_{\mathscr{L}}$. So by Proposition 3.5, for any $n \geqslant 2$, $(x_1,x_2) \in \mathbb{X}$ and $h \in \mathscr{L}(\mathbb{X})$,

$$\mathbf{P}^n h(x_1,x_2) = P^{n-1}F_h(x_2) = \nu(F_h) + R^{n-1}F_h(x_2) = \boldsymbol{\nu}(h)e(x_1,x_2) + Q^n h(x_1,x_2),$$

where the probability $\boldsymbol{\nu}$ is defined on $\mathbb{X}$ by

$$\boldsymbol{\nu}(h) = \nu(F_h) = \int_{X\times X} h(x',x'')P(x',\mathrm{d}x'')\nu(\mathrm{d}x'),$$

the function e is the unit function on $\mathbb{X}$, $e(x_1,x_2) = 1$, $\forall (x_1,x_2) \in \mathbb{X}$ and Q is the linear operator on $\mathscr{L}(\mathbb{X})$ defined by $Qh = R(F_h) = \mathbf{P}h - \boldsymbol{\nu}(h)$. By Proposition 3.5, the operator Q is bounded and for any $n \geqslant 1$, $\|Q^n\|_{\mathscr{L}\to\mathscr{L}} \leqslant \left|R^{n-1}\right|_{\mathscr{L}\to\mathscr{L}} \|\mathbf{P}\|_{\mathscr{L}\to\mathscr{L}} \leqslant c\,\mathrm{e}^{-cn}$. Since ν is invariant by P, one can easily verify that $\Pi Q = Q\Pi = 0$, where Π is the one-dimensional projector defined on $\mathscr{L}(\mathbb{X})$ by $\Pi h = \nu(h)e$.

*Proof of **M3**.* For any $t \in \mathbb{R}$, $h \in \mathscr{L}(\mathbb{X})$ and $(x_1,x_2) \in \mathbb{X}$,

$$\mathbf{P}_t h(x_1,x_2) = \int_X \mathrm{e}^{itf(x_2,x')}\, h(x_2,x')P(x_2,\mathrm{d}x') = \sum_{n=0}^{+\infty} \frac{i^n t^n}{n!} L_n(h)(x_1,x_2),$$

where $L_n(h) = \mathbf{P}(f^n h)$. Since $(\mathscr{L}(\mathbb{X}), \|\cdot\|_{\mathscr{L}})$ is a Banach algebra, it follows that L_n is a bounded operator on $\mathscr{L}(\mathbb{X})$ and $\|L_n\|_{\mathscr{L}\to\mathscr{L}} \leqslant \|\mathbf{P}\|_{\mathscr{L}\to\mathscr{L}} \|f\|_{\mathscr{L}}^n$. Consequently, the application $t \mapsto \mathbf{P}_t$ is analytic on $\mathbb{R}$ and so, by the analytic perturbation theory of linear operators (see [26]), there exists $\varepsilon_0 > 0$ such that, for any $|t| \leqslant \kappa$,

$$\mathbf{P}_t^n = \lambda_t^n \Pi_t + Q_t^n,$$

where λ_t is an eigenvalue of $\mathbf{P}_t$, Π_t is the projector on the one-dimensional eigenspace of λ_t and Q_t is an operator of spectral radius $r(Q_t) < |\lambda_t|$ such that $\Pi_t Q_t = Q_t \Pi_t = 0$. The functions $t \mapsto \lambda_t$, $t \mapsto \Pi_t$ and $t \mapsto Q_t$ are analytic on $[-\kappa, \kappa]$. Furthermore, for any $h \in \mathscr{L}(\mathbb{X})$ and $(x_1, x_2) \in \mathbb{X}$,

$$|\mathbf{P}_t h|(x_1, x_2) = \left| \int_X \mathrm{e}^{itf(x_2,x')} h(x_2, x') P(x_2, \mathrm{d}x') \right| \leqslant \|h\|_\infty$$

and necessarily $|\lambda_t| \leqslant 1$, for any $|t| \leqslant \kappa$. Consequently

$$\sup_{|t|\leqslant\kappa, n\geqslant 1} \|\mathbf{P}_t^n\|_{\mathscr{L}\to\mathscr{L}} \leqslant c.$$

*Proof of **M4** and **M5**.* Since for any $x \in \mathbb{X}$, $|f(x)| \leqslant |f|_\infty$ and $\|\boldsymbol{\delta}_x\|_{\mathscr{L}(\mathbb{X})'} \leqslant 1$, we can choose $N = 0$ and $N_l = 0$ for any $l \geqslant 1$ and Hypothesis **M4** is obviously satisfied.

Finally, Hypothesis 3.6 ensures that **M5** holds true.

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