

A temporal perspective on the rate of convergence in first-passage percolation under a moment condition*

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Abstract

We study the rate of convergence in the celebrated Shape Theorem in first-passage percolation, obtaining the precise asymptotic rate of decay for the probability of linear order deviations under a moment condition. Our results are presented from a temporal perspective and complement previous work by the same author, in which the rate of convergence was studied from the standard spatial perspective.

1 Introduction

Consider first-passage percolation on the \mathbb{Z}^d nearest-neighbour lattice for $d \geq 2$. Large deviations were first studied in the context of first-passage percolation in the 1980s, in a pioneering work of Grimmett and Kesten [3]. In this work, together with the subsequent work of Kesten [4], the authors investigate the rate of convergence of travel times to the so-called time constant, and provide necessary and sufficient conditions for exponential decay for the probability of linear order deviations. For the exponential decay to hold, one requires finite moment of exponential order on the passage times.

It was only recently, in [1], that large deviations in the regime of polynomial decay of the probability tails were studied. This regime is highly interesting since it is in this regime that strong laws such as Kingman's Subadditive Ergodic Theorem [5], and Richardson's Shape Theorem [6] (whose precise conditions were determined by Cox and Durrett [2]) cease to hold. In this paper we complement the results from [1] by offering a temporal perspective on the rate of convergence in the Shape Theorem. These results are sharp in the regime of polynomial decay on the probability tails.

We will assume throughout that the edges of the \mathbb{Z}^d lattice are assigned independent non-negative random weights from some probability distribution satisfying $F(0) < p_c(d)$, where $p_c(d)$ denotes the critical probability for bond percolation on \mathbb{Z}^d . The resulting weighted graph induces a random (pseudo-)metric structure to \mathbb{Z}^d , where the distance $T(x, y)$ between two sites $x, y \in \mathbb{Z}^d$ is given by the minimal weight accumulated along paths connecting the two points. The existence of a *time constant* follows from a simple application of the Subadditive Ergodic Theorem [5]. In fact, without further assumptions

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on the edge weights, the limit

$$\mu(z) := \lim_{n \rightarrow \infty} \frac{T(0, nz)}{n}, \quad (1)$$

exists in probability, and is finite and non-zero for all $z \neq 0$, see [4].

Richardson [6] realize that the above convergence holds in all directions simultaneously, in that a large ball $\mathcal{B}_t := \{z \in \mathbb{Z}^d : T(0, z) \leq t\}$ in the metric T is asymptotically comparable to the deterministic ball $\mathcal{B}_t^\mu := \{z \in \mathbb{Z}^d : \mu(z) \leq t\}$. Cox and Durrett [2] provided the precise condition for Richardson's result to hold. Let Y denote the minimum of the $2d$ weights assigned to the edges adjacent to the origin. The result of Cox and Durrett says that if $\mathbb{E}[Y^d] < \infty$, then, almost surely, for every $\varepsilon \in (0, 1)$

$$\mathcal{B}_{(1-\varepsilon)t}^\mu \subset \mathcal{B}_t \subset \mathcal{B}_{(1+\varepsilon)t}^\mu \quad (2)$$

for large enough t . The given moment condition is also necessary for this conclusion.

In this paper we investigate in further detail the probability that (2) fails, and that \mathcal{B}_t deviates significantly from its asymptotic rate of growth. Let

$$\mathcal{T}_\varepsilon := \{t \geq 0 : \text{either inclusion in (2) fails}\}.$$

We will study the behaviour of $\mathbb{P}(t \in \mathcal{T}_\varepsilon)$ for large t and fixed $\varepsilon \in (0, 1)$.

The result of Cox and Durrett implies that \mathcal{T}_ε is almost surely bounded if and only if $\mathbb{E}[Y^d] < \infty$. As it turns out, not only is \mathcal{T}_ε unbounded unless $\mathbb{E}[Y^d] < \infty$, but $\mathbb{P}(t \in \mathcal{T}_\varepsilon)$ may in fact be bounded away from zero even in the case when $\mathbb{E}[Y^\alpha] < \infty$ for all $\alpha < d$. (See the remark at the end of the paper.) When $\mathbb{E}[Y^d] < \infty$, however, we show that in the regime of polynomial tails of the weight distribution, the decay of $\mathbb{P}(t \in \mathcal{T}_\varepsilon)$ is governed by the tails of Y .

Theorem. *Assume that $d \geq 2$, $F(0) < p_c(d)$ and $\mathbb{E}[Y^d] < \infty$. Then, for every $\varepsilon > 0$ and $q \geq 1$, there is $c = c(\varepsilon, d, q)$ such that for all $t \geq 1$*

$$ct^d \mathbb{P}(Y > t) \leq \mathbb{P}(t \in \mathcal{T}_\varepsilon) \leq \frac{1}{c} t^d \mathbb{P}(Y > ct) + \frac{1}{c} t^{-q}$$

Using Markov's inequality we find that for each $\alpha > 0$, the condition $\mathbb{E}[Y^{d+\alpha}] < \infty$ implies that $\mathbb{P}(t \in \mathcal{T}_\varepsilon) = O(t^{-\alpha})$. Another consequence of the theorem is the following characterization.

Corollary. *Assume that $F(0) < p_c(d)$. For every $\alpha \geq 0$, $\varepsilon > 0$ and $d \geq 2$,*

$$\mathbb{E}[Y^{d+\alpha}] < \infty \quad \Leftrightarrow \quad \int_0^\infty t^{\alpha-1} \mathbb{P}(t \in \mathcal{T}_\varepsilon) dt < \infty.$$

Together with Fubini's theorem we see that

$$\mathbb{E}|\mathcal{T}_\varepsilon| = \mathbb{E} \int_0^\infty 1_{\{t \in \mathcal{T}_\varepsilon\}} dt = \int_0^\infty \mathbb{P}(t \in \mathcal{T}_\varepsilon) dt,$$

where $|\cdot|$ denotes Lebesgue measure. Hence the conclusion of the corollary was for $\alpha = 1$ known already in [1, Theorem 2], but the statement for general $\alpha > 0$ was previously unknown. The proofs we present below will be heavily based on ideas and results from [1].

2 Proof

We first recall some results that will be required for the analysis: For every $\varepsilon > 0$ and $d \geq 2$ there exists constants $M = M(\varepsilon, d)$ and $\gamma = \gamma(\varepsilon, d)$ such that

$$\mathbb{P}(T(0, z) - \mu(z) < -\varepsilon x) \leq M e^{-\gamma x} \quad \text{for all } z \in \mathbb{Z}^d \text{ and } x \geq |z|. \quad (3)$$

If, in addition, $\mathbb{E}[Y^\alpha] < \infty$ for some $\alpha > 0$ and $q \geq 1$, then we may choose the constant $M = M(\alpha, \varepsilon, d, q)$ such that

$$\mathbb{P}(T(0, z) - \mu(z) > \varepsilon x) \leq M \mathbb{P}(Y > x/M) + M x^{-q} \quad \text{for all } z \in \mathbb{Z}^d \text{ and } x \geq |z|. \quad (4)$$

The former statement was first proved by Grimmett and Kesten [3, 4] for coordinate directions, and later extended in [1]. The latter statement is original in [1].

Fix $\varepsilon > 0$ and $q \geq 1$. We want to estimate the decay of $\mathbb{P}(t \in \mathcal{T}_\varepsilon)$. Define thus

$$\begin{aligned} A(z) &:= \{t \geq 0 : T(0, z) > t \text{ and } \mu(z) \leq (1 - \varepsilon)t\}, \\ B(z) &:= \{t \geq 0 : T(0, z) \leq t \text{ and } \mu(z) > (1 + \varepsilon)t\}. \end{aligned}$$

Note that $t \in \mathcal{T}_\varepsilon$ if and only if $t \in A(z) \cup B(z)$ for some $z \in \mathbb{Z}^d$. Moreover, $t \in A(z)$ indicates that the time to reach z is unusually long, whereas $t \in B(z)$ means that the time to reach z is exceptionally short.

We proceed with the proof of the theorem, and begin with the lower bound. This part of the argument is elementary, and does not require the bounds in (3) and (4).

2.1 The lower bound

Let D denote the set of all $z \in \mathbb{Z}^d$ such that $\mu(z) \leq (1 - \varepsilon)t$ and whose ℓ^1 -distance from the origin is even. Let $Y(z)$ denote the minimum weight among the $2d$ edges adjacent to z , and note that the $Y(z)$'s are independent for $z \in D$, as points in D are at ℓ^1 -distance at least 2. Since $A(z) \subset \mathcal{T}_\varepsilon$ and $T(0, z) \geq Y(z)$ for every $z \in \mathbb{Z}^d$, we obtain

$$\begin{aligned} \mathbb{P}(t \in \mathcal{T}_\varepsilon) &\geq \mathbb{P}(T(0, z) > t \text{ for some } z \in D) \\ &\geq \mathbb{P}(Y(z) > t \text{ for some } z \in D). \end{aligned} \quad (5)$$

An application of Cuchy-Schwarz's inequality shows that any non-negative random variable X satisfies $\mathbb{P}(X > 0) \geq \mathbb{E}[X]^2 / \mathbb{E}[X^2]$, and if X is binomially distributed with parameters n and p , then a further lower bound is given by $np / (1 + np)$. Applying this to (5) leaves us with

$$\mathbb{P}(t \in \mathcal{T}_\varepsilon) \geq \frac{|D| \mathbb{P}(Y > t)}{1 + |D| \mathbb{P}(Y > t)}. \quad (6)$$

By assumption $\mathbb{E}[Y^d] < \infty$, which implies that $t^d \mathbb{P}(Y > t) \leq \mathbb{E}[Y^d]$ via Markov's inequality. Since the set D grows as t^d , we obtain the required lower bound on $\mathbb{P}(t \in \mathcal{T}_\varepsilon)$.

2.2 The upper bound

We now continue with the upper bound. The union bound leaves us with

$$\mathbb{P}(t \in \mathcal{T}_\varepsilon) \leq \sum_{\mu(z) \leq (1-\varepsilon)t} \mathbb{P}(T(0, z) > t) + \sum_{\mu(z) > (1+\varepsilon)t} \mathbb{P}(T(0, z) \leq t),$$

and we will treat the two sums separately.

Let us first note that it is immediate from subadditivity and (1) that $\mu(kz) = k\mu(z)$ and $\mu(z + y) \leq \mu(z) + \mu(y)$ for all $k \geq 1$ and $y, z \in \mathbb{Z}^d$. In particular, μ is comparable to Euclidean distance. We also notice that the condition $x \geq |z|$ in (3) and (4) is not essential. The bounds can be extended to $x \geq \delta|z|$ by adjusting the constants γ and M . Based on these observations, we use (3) to find $M_1 = M_1(\varepsilon, d)$ and $\gamma_1 = \gamma_1(\varepsilon, d)$ so that

$$\begin{aligned} \sum_{\mu(z) > (1+\varepsilon)t} \mathbb{P}(T(0, z) \leq t) &\leq \sum_{\mu(z) \geq (1+\varepsilon)t} \mathbb{P}\left(T(0, z) - \mu(z) \leq -\frac{\varepsilon}{1+\varepsilon}\mu(z)\right) \\ &\leq \sum_{\mu(z) \geq t} M_1 e^{-\gamma_1 \mu(z)}, \end{aligned}$$

which is at most $M_2 e^{-\gamma_2 t}$ for $t \geq 1$, and some constants M_2 and γ_2 . Using (4) we may find $M_3 = M_3(\varepsilon, d, q)$ such that

$$\begin{aligned} \sum_{\mu(z) \leq (1-\varepsilon)t} \mathbb{P}(T(0, z) > t) &\leq \sum_{\mu(z) \leq (1-\varepsilon)t} \mathbb{P}(T(0, z) - \mu(z) > \varepsilon t) \\ &\leq \sum_{\mu(z) \leq t} \left(M_3 \mathbb{P}(Y \geq t/M_3) + \frac{M_3}{t^{d+q}} \right). \end{aligned}$$

Since the cardinality of the set \mathcal{B}_t^μ grows at the order of t^d we obtain, for some M_4 ,

$$\mathbb{P}(t \in \mathcal{T}_\varepsilon) \leq M_4 t^d \mathbb{P}(Y \geq t/M_3) + M_4 t^{-q} + M_2 e^{-\gamma_2 t},$$

for all $t \geq 1$, which gives the required upper bound.

Remark. Finally, we mention that the condition $\mathbb{E}[Y^d] < \infty$ cannot be relaxed in general. Consider for instance the case when $\mathbb{P}(Y > t) = (d-1)t^{-d}$ for $t \geq 1$. Then $\mathbb{E}[Y^\alpha]$ is finite for all $\alpha < d$ and infinite for $\alpha = d$, but the bound in (6) shows that $\mathbb{P}(t \in \mathcal{T}_\varepsilon)$ is bounded away from zero.

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