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GAUSSIAN PHASE TRANSITIONS AND CONIC INTRINSIC VOLUMES: STEINING THE STEINER FORMULA

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Intrinsic volumes of convex sets are natural geometric quantities that also play important roles in applications, such as linear inverse problems with convex constraints, and constrained statistical inference. It is a well-known fact that, given a closed convex cone $C \subset \mathbb{R}^d$, its conic intrinsic volumes determine a probability measure on the finite set $\{0, 1, \dots, d\}$, customarily denoted by $\mathcal{L}(V_C)$. The aim of the present paper is to provide a Berry–Esseen bound for the normal approximation of $\mathcal{L}(V_C)$, implying a general quantitative central limit theorem (CLT) for sequences of (correctly normalised) discrete probability measures of the type $\mathcal{L}(V_{C_n})$, $n \geq 1$. This bound shows that, in the high-dimensional limit, most conic intrinsic volumes encountered in applications can be approximated by a suitable Gaussian distribution. Our approach is based on a variety of techniques, namely: (1) Steiner formulae for closed convex cones, (2) Stein's method and second-order Poincaré inequality, (3) concentration estimates and (4) Fourier analysis. Our results explicitly connect the sharp phase transitions, observed in many regularised linear inverse problems with convex constraints, with the asymptotic Gaussian fluctuations of the intrinsic volumes of the associated descent cones. In particular, our findings complete and further illuminate the recent breakthrough discoveries by Amelunxen, Lotz, McCoy and Tropp [*Inf. Inference* **3** (2014) 224–294] and McCoy and Tropp [*Discrete Comput. Geom.* **51** (2014) 926–963] about the concentration of conic intrinsic volumes and its connection with threshold phenomena. As an additional outgrowth of our work we develop total variation bounds for normal approximations of the lengths of projections of Gaussian vectors on closed convex sets.

REFERENCES

[1] ADLER, R. J. and TAYLOR, J. E. (2007). *Random Fields and Geometry*. Springer, New York. MR2319516

[2] ALLENDOERFER, C. B. (1948). Steiner's formulae on a general S^{n+1}. *Bull. Amer. Math. Soc.* **54** 128–135. MR0025172

[3] AMELUNXEN, D., LOTZ, M., MCCOY, M. B. and TROPP, J. A. (2014). Living on the edge: Phase transitions in convex programs with random data. *Inf. Inference* **3** 224–294. MR3311453

[4] BAI, J. and SILVERSTEIN, J. W. (2006). *Spectral Analysis of Large Dimensional Random Matrices*. Springer, Berlin.

[5] BAYATI, M., LELARGE, M. and MONTANARI, A. (2015). Universality in polytope phase transitions and message passing algorithms. *Ann. Appl. Probab.* **25** 753–822. MR3313755

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[6] BIRGÉ, L. and MASSART, P. (1993). Rates of convergence for minimum contrast estimators. *Probab. Theory Related Fields* **97** 113–150. MR1240719

[7] BLASS, A. and SAGAN, B. E. (1998). Characteristic and Ehrhart polynomials. *J. Algebraic Combin.* **7** 115–126. MR1609889

[8] BOGACHEV, V. I. (1998). *Gaussian Measures*. *Mathematical Surveys and Monographs* **62**. Amer. Math. Soc., Providence, RI. MR1642391

[9] BOUCHERON, S., LUGOSI, G. and MASSART, P. (2013). *Concentration Inequalities*: *A Nonasymptotic Theory of Independence*, *with a Foreword by Michel Ledoux*. Oxford Univ. Press, Oxford. MR3185193

[10] BÜHLMANN, P. and VAN DE GEER, S. (2011). *Statistics for High-Dimensional Data*: *Methods*, *Theory and Applications*. Springer, Heidelberg. MR2807761

[11] CANDÈS, E. J. (2014). Mathematics of sparsity (and a few other things). In *Proceedings of the International Congress of Mathematicians*. Seoul, South Korea.

[12] CANDÈS, E. J., ROMBERG, J. and TAO, T. (2006). Robust uncertainty principles: Exact signal reconstruction from highly incomplete frequency information. *IEEE Trans. Inform. Theory* **52** 489–509. MR2236170

[13] CANDÈS, E. J., ROMBERG, J. K. and TAO, T. (2006). Stable signal recovery from incomplete and inaccurate measurements. *Comm. Pure Appl. Math.* **59** 1207–1223. MR2230846

[14] CHANDRASEKARAN, V., RECHT, B., PARRILO, P. A. and WILLSKY, A. S. (2012). The convex geometry of linear inverse problems. *Found. Comput. Math.* **12** 805–849. MR2989474

[15] CHATTERJEE, S. (2009). Fluctuations of eigenvalues and second order Poincaré inequalities. *Probab. Theory Related Fields* **143** 1–40. MR2449121

[16] CHATTERJEE, S. (2014). A new perspective on least squares under convex constraint. *Ann. Statist.* **42** 2340–2381. MR3269982

[17] CHATTERJEE, S., GUNTUBOYINA, A. and SEN, B. (2013). Improved risk bounds in isotonic regression. Preprint. Available at arXiv:1311.3765.

[18] CHEN, L. H. Y., GOLDSTEIN, L. and SHAO, Q. M. (2010). *Normal Approximation by Stein's Method*. Springer, Berlin.

[19] COXETER, H. S. M. and MOSER, W. O. J. (1972). *Generators and Relations for Discrete Groups*, 3rd ed. Springer, New York. MR0349820

[20] DAVIS, K. A. (2012). Constrained statistical inference: A hybrid of statistical theory, projective geometry and applied optimization techniques. *Prog. Appl. Math.* **4** 167–181.

[21] DONOHO, D. L. (2006). Compressed sensing. *IEEE Trans. Inform. Theory* **52** 1289–1306. MR2241189

[22] DONOHO, D. L. (2006). High-dimensional centrally symmetric polytopes with neighborliness proportional to dimension. *Discrete Comput. Geom.* **35** 617–652. MR2225676

[23] DONOHO, D. L. and TANNER, J. (2009). Counting faces of randomly projected polytopes when the projection radically lowers dimension. *J. Amer. Math. Soc.* **22** 1–53. MR2449053

[24] DRTON, M. and KLIVANS, C. J. (2010). A geometric interpretation of the characteristic polynomial of reflection arrangements. *Proc. Amer. Math. Soc.* **138** 2873–2887. MR2644900

[25] DURRETT, R. (2010). *Probability*: *Theory and Examples*, 4th ed. Cambridge Univ. Press, Cambridge. MR2722836

[26] DYKSTRA, R. (1991). Asymptotic normality for chi-bar-square distributions. *Canad. J. Statist.* **19** 297–306. MR1144147

[27] GLASAUER, S. (1995). Integralgeometrie konvexer Körper im sphärischen Raum. Thesis, Univ. Freiburg i. Br.

[28] HERGLOTZ, G. (1943). Über die Steinersche Formel für Parallelflächen. *Abh. Math. Sem. Hansischen Univ.* **15** 165–177. MR0015832

[29] HJORT, N. L. and POLLARD, D. (1993). Asymptotic for minimisers of convex processes. Preprint, Dept. of Statistics, Yale Univ.

[30] HORN, R. A. and JOHNSON, C. R. (1990). *Matrix Analysis*. Cambridge Univ. Press, Cambridge. MR1084815

[31] KLAIN, D. A. and ROTA, G.-C. (1997). *Introduction to Geometric Probability*. Cambridge Univ. Press, Cambridge. MR1608265

[32] LEDOUX, M. (2001). *The Concentration of Measure Phenomenon*. *Mathematical Surveys and Monographs* **89**. Amer. Math. Soc., Providence, RI. MR1849347

[33] LUK, M. (1994). Stein's method for the Gamma distribution and related statistical applications Ph.D. dissertation, Univ. Southern, California, Los Angeles, CA.

[34] MCCOY, M. B. and TROPP, J. A. (2014). From Steiner formulas for cones to concentration of intrinsic volumes. *Discrete Comput. Geom.* **51** 926–963. MR3216671

[35] NOURDIN, I. and PECCATI, G. (2009). Stein's method and exact Berry–Esseen asymptotics for functionals of Gaussian fields. *Ann. Probab.* **37** 2231–2261. MR2573557

[36] NOURDIN, I. and PECCATI, G. (2012). *Normal Approximations with Malliavin Calculus*: *From Stein's Method to Universality*. Cambridge Univ. Press, Cambridge. MR2962301

[37] NOURDIN, I., PECCATI, G. and REINERT, G. (2009). Second order Poincaré inequalities and CLTs on Wiener space. *J. Funct. Anal.* **257** 593–609. MR2527030

[38] ROCKAFELLAR, R. T. (1970). *Convex Analysis*. Princeton Univ. Press, Princeton, NJ. MR0274683

[39] ROCKAFELLAR, R. T. and WETS, R. J.-B. (1998). *Variational Analysis*. Springer, Berlin. MR1491362

[40] RUDELSON, M. and VERSHYNIN, R. (2008). On sparse reconstruction from Fourier and Gaussian measurements. *Comm. Pure Appl. Math.* **61** 1025–1045. MR2417886

[41] SANTALÓ, L. A. (1950). On parallel hypersurfaces in the elliptic and hyperbolic n-dimensional space. *Proc. Amer. Math. Soc.* **1** 325–330. MR0036532

[42] SCHNEIDER, R. and WEIL, W. (2008). *Stochastic and Integral Geometry*. Springer, Berlin. MR2455326

[43] SHAPIRO, A. (1985). Asymptotic distribution of test statistics in the analysis of moment structures under inequality constraints. *Biometrika* **72** 133–144. MR0790208

[44] SHAPIRO, A. (1988). Towards a unified theory of inequality constrained testing in multivariate analysis. *Int. Stat. Rev.* **56** 49–62. MR0963140

[45] SILVAPULLE, M. J. and SEN, P. K. (2005). *Constrained Statistical Inference*. Wiley, Hoboken, NJ. MR2099529

[46] TAYLOR, J. (2013). The geometry of least squares in the 21st century. *Bernoulli* **19** 1449–1464. MR3102558

[47] VAN DE GEER, S. (1990). Estimating a regression function. *Ann. Statist.* **18** 907–924. MR1056343

[48] WANG, Y. (1996). The L_2 risk of an isotonic estimate. *Comm. Statist. Theory Methods* **25** 281–294. MR1379445

[49] ZHANG, C.-H. (2002). Risk bounds in isotonic regression. *Ann. Statist.* **30** 528–555. MR1902898

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ACHIEVING NONZERO INFORMATION VELOCITY IN WIRELESS NETWORKS

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In wireless networks, where each node transmits independently of other nodes in the network (the ALOHA protocol), the expected delay experienced by a packet until it is successfully received at any other node is known to be infinite for the signal-to-interference-plus-noise-ratio (SINR) model with node locations distributed according to a Poisson point process. Consequently, the information velocity, defined as the limit of the ratio of the distance to the destination and the time taken for a packet to successfully reach the destination over multiple hops, is zero, as the distance tends to infinity. A nearest neighbor distance based power control policy is proposed to show that the expected delay required for a packet to be successfully received at the nearest neighbor can be made finite. Moreover, the information velocity is also shown to be nonzero with the proposed power control policy. The condition under which these results hold does not depend on the intensity of the underlying Poisson point process.

REFERENCES

[1] ANDREWS, J., WEBER, S., KOUNTOURIS, M. and HAENGGI, M. (2010). Random access transport capacity. *IEEE Trans. Wireless Commun.* **9** 2101–2111.

[2] ANDREWS, J. G., BACCELLI, F. and GANTI, R. K. (2011). A tractable approach to coverage and rate in cellular networks. *IEEE Trans. Commun.* **59** 3122–3134.

[3] BACCELLI, F., BŁASZCZYSZYN, B. and HAJI-MIRSADEGHI, M.-O. (2011). Optimal paths on the space–time SINR random graph. *Adv. in Appl. Probab.* **43** 131–150. MR2761151

[4] BACCELLI, F., BŁASZCZYSZYN, B. and MÜHLETHALER, P. (2006). An aloha protocol for multihop mobile wireless networks. *IEEE Trans. Inform. Theory* **52** 421–436. MR2236166

[5] BORBASH, S. A., EPHREMIDES, A. and MCGLYNN, M. J. (2007). An asynchronous neighbor discovery algorithm for wireless sensor networks. *Ad Hoc Networks* **5** 998–1016.

[6] DURRETT, R. (1999). Stochastic spatial models. *SIAM Rev.* **41** 677–718. MR1722998

[7] GANTI, R. K. and HAENGGI, M. (2009). Bounds on the information propagation delay in interference-limited aloha networks. In *International Symposium on Modeling and Optimization in Mobile, Ad Hoc, and Wireless Networks. WiOPT* 2009 1–7. IEEE, Seoul.

[8] GUPTA, P. and KUMAR, P. R. (2000). The capacity of wireless networks. *IEEE Trans. Inform. Theory* **46** 388–404. MR1748976

[9] HAENGGI, M. (2013). *Stochastic Geometry for Wireless Networks*. Cambridge Univ. Press, Cambridge. MR3184048

[10] HAMMERSLEY, J. M. and WELSH, D. J. A. (1965). First-passage percolation, subadditive processes, stochastic networks, and generalized renewal theory. In *Proc. Internat.*

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Res. Semin., Statist. Lab., Univ. California, Berkeley, CA 61–110. Springer, New York. MR0198576

[11] JACQUET, P., MANS, B., MUHLETHALER, P. and RODOLAKIS, G. (2009). Opportunistic routing in wireless ad hoc networks: Upper bounds for the packet propagation speed. *IEEE J. Sel. Areas Commun.* **27** 1192–1202.

[12] KESTEN, H. (1986). Aspects of first passage percolation. In *École D'été de Probabilités de Saint-Flour, XIV*—1984. *Lecture Notes in Math.* **1180** 125–264. Springer, Berlin. MR0876084

[13] KRENGEL, U. (1985). *Ergodic Theorems. De Gruyter Studies in Mathematics* **6**. de Gruyter, Berlin. MR0797411

[14] MOLLISON, D. (1977). Spatial contact models for ecological and epidemic spread. *J. R. Stat. Soc. Ser. B. Stat. Methodol.* **39** 283–326. MR0496851

[15] MOLLISON, D. (1978). Markovian contact processes. *Adv. in Appl. Probab.* **10** 85–108. MR0480208

[16] TSE, D. and VISWANATH, P. (2005). *Fundamentals of Wireless Communication*. Cambridge Univ. Press, New York, NY.

[17] VAZE, R. (2011). Throughput-delay-reliability tradeoff with ARQ in wireless ad hoc networks. *IEEE Trans. Wireless Commun.* **10** 2142–2149.

[18] WEBER, S. P., YANG, X., ANDREWS, J. G. and DE VECIANA, G. (2005). Transmission capacity of wireless ad hoc networks with outage constraints. *IEEE Trans. Inform. Theory* **51** 4091–4102. MR2243145

[19] YU, Y., GOVINDAN, R. and ESTRIN, D. (2001). Geographical and energy aware routing: A recursive data dissemination protocol for wireless sensor networks. Technical report ucla/csd-tr-01-0023, UCLA Computer Science Department.

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POLYNOMIAL CONVERGENCE TO EQUILIBRIUM FOR A SYSTEM OF INTERACTING PARTICLES

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We consider a stochastic particle system in which a finite number of particles interact with one another via a common energy tank. Interaction rate for each particle is proportional to the square root of its kinetic energy, as is consistent with analogous mechanical models. Our main result is that the rate of convergence to equilibrium for such a system is $\sim t^{-2}$, more precisely it is faster than a constant times $t^{-2+\varepsilon}$ for any $\varepsilon > 0$. A discussion of exponential vs. polynomial convergence for similar particle systems is included.

REFERENCES

[1] BALAJI, S. and MEYN, S. P. (2000). Multiplicative ergodicity and large deviations for an irreducible Markov chain. *Stochastic Process. Appl.* **90** 123–144. MR1787128

[2] BOWEN, R. (2008). *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms*, revised ed. *Lecture Notes in Math.* **470**. Springer, Berlin. MR2423393

[3] BUNIMOVICH, L., LIVERANI, C., PELLEGRINOTTI, A. and SUHOV, Y. (1992). Ergodic systems of *n* balls in a billiard table. *Comm. Math. Phys.* **146** 357–396. MR1165188

[4] CARLEN, E. A., CARVALHO, M. C. and LOSS, M. (2003). Determination of the spectral gap for Kac's master equation and related stochastic evolution. *Acta Math.* **191** 1–54. MR2020418

[5] CARLEN, E. A., CARVALHO, M. C. and LOSS, M. (2014). Spectral gap for the Kac model with hard sphere collisions. *J. Funct. Anal.* **266** 1787–1832. MR3146836

[6] CHERNOV, N. and MARKARIAN, R. (2006). *Chaotic Billiards. Mathematical Surveys and Monographs* **127**. Amer. Math. Soc., Providence, RI. MR2229799

[7] CHERNOV, N. and YOUNG, L. S. (2000). Decay of correlations for Lorentz gases and hard balls. In *Hard Ball Systems and the Lorentz Gas. Encyclopaedia Math. Sci.* **101** 89–120. Springer, Berlin. MR1805327

[8] DOUC, R., FORT, G. and GUILLIN, A. (2009). Subgeometric rates of convergence of *f*-ergodic strong Markov processes. *Stochastic Process. Appl.* **119** 897–923. MR2499863

[9] DOUC, R., FORT, G., MOULINES, E. and SOULIER, P. (2004). Practical drift conditions for subgeometric rates of convergence. *Ann. Appl. Probab.* **14** 1353–1377. MR2071426

[10] ECKMANN, J.-P. and JACQUET, P. (2007). Controllability for chains of dynamical scatterers. *Nonlinearity* **20** 1601–1617. MR2335075

[11] ECKMANN, J.-P. and MEJÍA-MONASTERIO, C. (2006). Thermal rectification in billiardlike systems. *Phys. Rev. Lett.* **97** 094301.

[12] ECKMANN, J.-P., MEJÍA-MONASTERIO, C. and ZABEY, E. (2006). Memory effects in nonequilibrium transport for deterministic Hamiltonian systems. *J. Stat. Phys.* **123** 1339–1360. MR2253882

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[13] ECKMANN, J.-P. and YOUNG, L.-S. (2006). Nonequilibrium energy profiles for a class of 1-D models. *Comm. Math. Phys.* **262** 237–267. MR2200889

[14] FORT, G. and ROBERTS, G. O. (2005). Subgeometric ergodicity of strong Markov processes. *Ann. Appl. Probab.* **15** 1565–1589. MR2134115

[15] GASPARD, P. and GILBERT, T. (2008). Heat conduction and Fourier's law in a class of many particle dispersing billiards. *New J. Phys.* **10** 103004.

[16] GASPARD, P. and GILBERT, T. (2008). Heat conduction and Fourier's law by consecutive local mixing and thermalization. *Phys. Rev. Lett.* **101** 020601.

[17] GASPARD, P. and GILBERT, T. (2008). On the derivation of Fourier's law in stochastic energy exchange systems. *J. Stat. Mech. Theory Exp.* **2008** P11021.

[18] GAVEAU, B. and KAC, M. (1986). A probabilistic formula for the quantum N-body problem and the nonlinear Schrödinger equation in operator algebra. *J. Funct. Anal.* **66** 308–322. MR0839104

[19] GRIGO, A., KHANIN, K. and SZÁSZ, D. (2012). Mixing rates of particle systems with energy exchange. *Nonlinearity* **25** 2349–2376. MR2956578

[20] HAIRER, M. (2010). Convergence of Markov processes. *Lecture Notes*. Available at http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.728.1839.

[21] JANVRESSE, E. (2001). Spectral gap for Kac's model of Boltzmann equation. *Ann. Probab.* **29** 288–304. MR1825150

[22] JARNER, S. F. and ROBERTS, G. O. (2002). Polynomial convergence rates of Markov chains. *Ann. Appl. Probab.* **12** 224–247. MR1890063

[23] JONES, G. L. (2004). On the Markov chain central limit theorem. *Probab. Surv.* **1** 299–320. MR2068475

[24] KAC, M. (1956). Foundations of kinetic theory. In *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability*, 1954–1955, *Vol. III* 171–197. Univ. California Press, Berkeley. MR0084985

[25] KHANIN, K. and YARMOLA, T. (2013). Ergodic properties of random billiards driven by thermostats. *Comm. Math. Phys.* **320** 121–147. MR3046992

[26] KONTOYIANNIS, I. and MEYN, S. P. (2005). Large deviations asymptotics and the spectral theory of multiplicatively regular Markov processes. *Electron. J. Probab.* **10** 61–123 (electronic). MR2120240

[27] LEFEVERE, R., MARIANI, M. and ZAMBOTTI, L. (2011). Large deviations of the current in stochastic collisional dynamics. *J. Math. Phys.* **52** 033302, 22. MR2814865

[28] LI, Y. (2015). On the stochastic behaviors of locally confined particle systems. *Chaos* **25** 073121, 14. MR3405854

[29] LI, Y. and YOUNG, L.-S. (2013). Existence of nonequilibrium steady state for a simple model of heat conduction. *J. Stat. Phys.* **152** 1170–1193. MR3101481

[30] LI, Y. and YOUNG, L.-S. (2014). Nonequilibrium steady states for a class of particle systems. *Nonlinearity* **27** 607–636. MR3177574

[31] LIN, K. K. and YOUNG, L.-S. (2010). Nonequilibrium steady states for certain Hamiltonian models. *J. Stat. Phys.* **139** 630–657. MR2638931

[32] LINDVALL, T. (2002). *Lectures on the Coupling Method*. Dover Publications, Mineola, NY. MR1924231

[33] MASLEN, D. K. (2003). The eigenvalues of Kac's master equation. *Math. Z.* **243** 291–331. MR1961868

[34] MEJIA-MONASTERIO, C., LARRALDE, H. and LEYVRAZ, F. (2001). Coupled normal heat and matter transport in a simple model system. *Phys. Rev. Lett.* **86** 5417.

[35] NUMMELIN, E. (1978). A splitting technique for Harris recurrent Markov chains. *Z. Wahrsch. Verw. Gebiete* **43** 309–318. MR0501353

[36] NUMMELIN, E. and TUOMINEN, P. (1983). The rate of convergence in Orey's theorem for Harris recurrent Markov chains with applications to renewal theory. *Stochastic Process. Appl.* **15** 295–311. MR0711187

[37] RATEITSCHAK, K., KLAGES, R. and NICOLIS, G. (2000). Thermostating by deterministic scattering: The periodic Lorentz gas. *J. Stat. Phys.* **99** 1339–1364. MR1773148

[38] REY-BELLET, L. and YOUNG, L.-S. (2008). Large deviations in non-uniformly hyperbolic dynamical systems. *Ergodic Theory Dynam. Systems* **28** 587–612. MR2408394

[39] SASADA, M. (2015). Spectral gap for stochastic energy exchange model with nonuniformly positive rate function. *Ann. Probab.* **43** 1663–1711. MR3353812

[40] SINAĬ, JA. G. (1970). Dynamical systems with elastic reflections. Ergodic properties of dispersing billiards. *Uspekhi Mat. Nauk* **25** 141–192. MR0274721

[41] TUOMINEN, P. and TWEEDIE, R. L. (1994). Subgeometric rates of convergence of f-ergodic Markov chains. *Adv. in Appl. Probab.* **26** 775–798. MR1285459

[42] WU, L. (2000). Uniformly integrable operators and large deviations for Markov processes. *J. Funct. Anal.* **172** 301–376. MR1753178

[43] WU, L. (2001). Large and moderate deviations and exponential convergence for stochastic damping Hamiltonian systems. *Stochastic Process. Appl.* **91** 205–238. MR1807683

[44] YARMOLA, T. (2011). Ergodicity of some open systems with particle–disk interactions. *Comm. Math. Phys.* **304** 665–688. MR2794543

[45] YARMOLA, T. (2013). Sub-exponential mixing of random billiards driven by thermostats. *Nonlinearity* **26** 1825–1837. MR3071443

[46] YARMOLA, T. (2014). Sub-exponential mixing of open systems with particle–disk interactions. *J. Stat. Phys.* **156** 473–492. MR3217533

[47] YOUNG, L.-S. (1998). Statistical properties of dynamical systems with some hyperbolicity. *Ann. of Math.* (2) **147** 585–650. MR1637655

[48] YOUNG, L.-S. (1999). Recurrence times and rates of mixing. *Israel J. Math.* **110** 153–188. MR1750438

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TWO-DIMENSIONAL VOLUME-FROZEN PERCOLATION: EXCEPTIONAL SCALES

BY JACOB VAN DEN BERG[*,†] AND PIERRE NOLIN[‡]

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We study a percolation model on the square lattice, where clusters "freeze" (stop growing) as soon as their volume (i.e., the number of sites they contain) gets larger than N, the parameter of the model. A model where clusters freeze when they reach *diameter* at least N was studied in van den Berg, de Lima and Nolin [*Random Structures Algorithms* **40** (2012) 220–226] and Kiss [*Probab. Theory Related Fields* **163** (2015) 713–768]. Using volume as a way to measure the size of a cluster—instead of diameter—leads, for large N, to a quite different behavior (contrary to what happens on the binary tree van den Berg, Kiss and Nolin [*Electron. Commun. Probab.* **17** (2012) 1–11], where the volume model and the diameter model are "asymptotically the same"). In particular, we show the existence of a sequence of "exceptional" length scales.

REFERENCES

[1] ALDOUS, D. J. (2000). The percolation process on a tree where infinite clusters are frozen. *Math. Proc. Cambridge Philos. Soc.* **128** 465–477. MR1744108

[2] VAN DEN BERG, J., KISS, D. and NOLIN, P. (2012). A percolation process on the binary tree where large finite clusters are frozen. *Electron. Commun. Probab.* **17** 1–11. MR2872571

[3] VAN DEN BERG, J., KISS, D. and NOLIN, P. (2015). Two-dimensional volume-frozen percolation: Deconcentration and prevalence of mesoscopic clusters. Preprint. Available at arXiv:1512.05335.

[4] VAN DEN BERG, J., DE LIMA, B. N. B. and NOLIN, P. (2012). A percolation process on the square lattice where large finite clusters are frozen. *Random Structures Algorithms* **40** 220–226. MR2877564

[5] VAN DEN BERG, J. and TÓTH, B. (2001). A signal-recovery system: Asymptotic properties, and construction of an infinite-volume process. *Stochastic Process. Appl.* **96** 177–190. MR1865354

[6] BORGS, C., CHAYES, J. T., KESTEN, H. and SPENCER, J. (2001). The birth of the infinite cluster: Finite-size scaling in percolation. *Comm. Math. Phys.* **224** 153–204. MR1868996

[7] GRIMMETT, G. (1999). *Percolation*, 2nd ed. *Grundlehren der Mathematischen Wissenschaften* **321**. Springer, Berlin. MR1707339

[8] KESTEN, H. (1980). The critical probability of bond percolation on the square lattice equals $\frac{1}{2}$. *Comm. Math. Phys.* **74** 41–59. MR0575895

[9] KESTEN, H. (1987). Scaling relations for 2D-percolation. *Comm. Math. Phys.* **109** 109–156. MR0879034

[10] KISS, D. (2015). Frozen percolation in two dimensions. *Probab. Theory Related Fields* **163** 713–768. MR3418754

MSC2010 subject classifications. 60K35, 82B43.

Key words and phrases. Frozen percolation, near-critical percolation, sol-gel transitions.

[11] LAWLER, G. F., SCHRAMM, O. and WERNER, W. (2001). Values of Brownian intersection exponents. I. Half-plane exponents. *Acta Math*. **187** 237–273. MR1879850

[12] LAWLER, G. F., SCHRAMM, O. and WERNER, W. (2001). Values of Brownian intersection exponents. II. Plane exponents. *Acta Math*. **187** 275–308. MR1879851

[13] LAWLER, G. F., SCHRAMM, O. and WERNER, W. (2002). One-arm exponent for critical 2D percolation. *Electron*. *J*. *Probab*. **7** 13 pp. (electronic). MR1887622

[14] LIGGETT, T. M. (2005). *Interacting Particle Systems*. Springer, Berlin. MR2108619

[15] NOLIN, P. (2008). Near-critical percolation in two dimensions. *Electron*. *J*. *Probab*. **13** 1562–1623. MR2438816

[16] SMIRNOV, S. (2001). Critical percolation in the plane: Conformal invariance, Cardy's formula, scaling limits. *C*. *R*. *Acad*. *Sci*. *Paris Sér*. *I Math*. **333** 239–244. MR1851632

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ONE-DIMENSIONAL RANDOM WALKS WITH SELF-BLOCKING IMMIGRATION

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We consider a system of independent one-dimensional random walkers where new particles are added at the origin at fixed rate whenever there is no older particle present at the origin. A Poisson ansatz leads to a semi-linear lattice heat equation and predicts that starting from the empty configuration the total number of particles grows as $c\sqrt{t}\log t$. We confirm this prediction and also describe the asymptotic macroscopic profile of the particle configuration.

REFERENCES

[1] ABRAMOWITZ, M. and STEGUN, I. A. (1964). *Handbook of Mathematical Functions with Formulas*, *Graphs*, *and Mathematical Tables*. *National Bureau of Standards Applied Mathematics Series* **55**. U.S. Government Printing Office, Washington, DC. MR0167642

[2] BIRKNER, M. (2003). Particle systems with locally dependent branching: Long-time behaviour, genealogy and critical parameters. Dissertation, Johann Wolfgang Goethe-Universität Frankfurt am Main.

[3] BIRKNER, M. and SUN, R. (2016). Low-dimensional lonely branching random walks die out. Manuscript in progress.

[4] CHEN, Z.-Q. and FAN, W.-T. (2016). Hydrodynamic limits and propagation of chaos for interacting random walks in domains. *Ann. Appl. Probab.* To appear. Available at arXiv:1311.2325.

[5] DURRETT, R. (1996). *Probability*: *Theory and Examples*, 2nd ed. Duxbury Press, Belmont, CA. MR1609153

[6] FELLER, W. (1971). *An Introduction to Probability Theory and Its Applications*. *Vol*. *II*, 2nd ed. Wiley, New York. MR0270403

[7] GARCIA, N. L. and KURTZ, T. G. (2006). Spatial birth and death processes as solutions of stochastic equations. *ALEA Lat*. *Am*. *J*. *Probab*. *Math*. *Stat*. **1** 281–303. MR2249658

[8] KIPNIS, C. and LANDIM, C. (1999). *Scaling Limits of Interacting Particle Systems*. *Grundlehren der Mathematischen Wissenschaften* **320**. Springer, Berlin. MR1707314

[9] NEWMAN, C. M., RAVISHANKAR, K. and SUN, R. (2005). Convergence of coalescing nonsimple random walks to the Brownian web. *Electron*. *J*. *Probab*. **10** 21–60. MR2120239

[10] YAU, H.-T. (1991). Relative entropy and hydrodynamics of Ginzburg–Landau models. *Lett*. *Math*. *Phys*. **22** 63–80. MR1121850

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Key words and phrases. Interacting random walks, density-dependent immigration, Poisson comparison, vacant time.

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NONEQUILIBRIUM FLUCTUATIONS OF ONE-DIMENSIONAL BOUNDARY DRIVEN WEAKLY ASYMMETRIC EXCLUSION PROCESSES

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We consider one-dimensional, boundary driven, weakly asymmetric exclusion processes in contact with reservoirs at fixed density. For a general set of initial measures and by using a microscopic Cole–Hopf transformation, we derive the nonequilibrium fluctuations which are given by a generalized Ornstein–Uhlenbeck process.

REFERENCES

[1] BERTINI, L., DE SOLE, A., GABRIELLI, D., JONA-LASINIO, G. and LANDIM, C. (2002). Macroscopic fluctuation theory for stationary non-equilibrium states. *J. Stat. Phys.* **107** 635–675. MR1898852

[2] BERTINI, L., LANDIM, C. and MOURRAGUI, M. (2009). Dynamical large deviations for the boundary driven weakly asymmetric exclusion process. *Ann. Probab.* **37** 2357–2403. MR2573561

[3] BROX, TH. and ROST, H. (1984). Equilibrium fluctuations of stochastic particle systems: The role of conserved quantities. *Ann. Probab.* **12** 742–759. MR0744231

[4] DAVIES, E. B. (1989). *Heat Kernels and Spectral Theory*. Cambridge Univ. Press, Cambridge. MR0990239

[5] DERRIDA, B., ENAUD, C., LANDIM, C. and OLLA, S. (2005). Fluctuations in the weakly asymmetric exclusion process with open boundary conditions. *J. Stat. Phys.* **118** 795–811. MR2130640

[6] DERRIDA, B., LEBOWITZ, J. L. and SPEER, E. R. (2002). Large deviation of the density profile in the steady state of the open symmetric simple exclusion process. *J. Stat. Phys.* **107** 599–634. MR1898851

[7] DE MASI, A., PRESUTTI, E. and SCACCIATELLI, E. (1989). The weakly asymmetric simple exclusion process. *Ann. Inst. Henri Poincaré Probab. Stat.* **25** 1–38. MR0995290

[8] DIACONIS, P. and SALOFF-COSTE, L. (1996). Logarithmic Sobolev inequalities for finite Markov chains. *Ann. Appl. Probab.* **6** 695–750. MR1410112

[9] DITTRICH, P. and GÄRTNER, J. (1991). A central limit theorem for the weakly asymmetric simple exclusion process. *Math. Nachr.* **151** 75–93. MR1121199

[10] FARFAN, J., LANDIM, C. and MOURRAGUI, M. (2011). Hydrostatics and dynamical large deviations of boundary driven gradient symmetric exclusion processes. *Stochastic Process. Appl.* **121** 725–758. MR2770905

[11] GÄRTNER, J. (1988). Convergence towards Burgers' equation and propagation of chaos for weakly asymmetric exclusion processes. *Stochastic Process. Appl.* **27** 233–260. MR0931030

MSC2010 subject classifications. 60k35.

Key words and phrases. Nonequilibrium fluctuations, weakly asymmetric exclusion, Dirichlet boundary conditions, Cole–Hopf transformation, Ornstein–Uhlenbeck process.

[12] JACOD, J. and SHIRYAEV, A. N. (2003). *Limit Theorems for Stochastic Processes*, 2nd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **288**. Springer, Berlin. MR1943877

[13] KIPNIS, C. and LANDIM, C. (1999). *Scaling Limits of Interacting Particle Systems*. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **320**. Springer, Berlin. MR1707314

[14] LANDIM, C. (1998). Decay to equilibrium in L^∞ of finite interacting particle systems in infinite volume. *Markov Process*. *Related Fields* **4** 517–534. MR1677057

[15] LANDIM, C. (2005). Gaussian estimates for symmetric simple exclusion processes. *Ann*. *Fac*. *Sci*. *Toulouse Math*. (6) **14** 683–703. MR2188588

[16] LANDIM, C., MILANÉS, A. and OLLA, S. (2008). Stationary and nonequilibrium fluctuations in boundary driven exclusion processes. *Markov Process*. *Related Fields* **14** 165–184. MR2437527

[17] REED, M. and SIMON, B. (1978). *Methods of Modern Mathematical Physics*. *Iv. Analysis of Operators*. Academic Press, New York. MR0493421

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MAXIMA OF A RANDOMIZED RIEMANN ZETA FUNCTION, AND BRANCHING RANDOM WALKS

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A recent conjecture of Fyodorov–Hiary–Keating states that the maximum of the absolute value of the Riemann zeta function on a typical bounded interval of the critical line is $\exp\{\log\log T - \frac{3}{4}\log\log\log T + O(1)\}$, for an interval at (large) height T. In this paper, we verify the first two terms in the exponential for a model of the zeta function, which is essentially a randomized Euler product. The critical element of the proof is the identification of an approximate tree structure, present also in the actual zeta function, which allows us to relate the maximum to that of a branching random walk.

REFERENCES

[1] ADDARIO-BERRY, L. and REED, B. (2009). Minima in branching random walks. *Ann. Probab.* **37** 1044–1079. MR2537549

[2] AÏDÉKON, E. (2013). Convergence in law of the minimum of a branching random walk. *Ann. Probab.* **41** 1362–1426. MR3098680

[3] AÏDÉKON, E., BERESTYCKI, J., BRUNET, É. and SHI, Z. (2013). Branching Brownian motion seen from its tip. *Probab. Theory Related Fields* **157** 405–451. MR3101852

[4] ARGUIN, L.-P., BOVIER, A. and KISTLER, N. (2013). The extremal process of branching Brownian motion. *Probab. Theory Related Fields* **157** 535–574. MR3129797

[5] BELIUS, D. and KISTLER, N. (2016). The subleading order of two dimensional cover times. *Probab. Theory Related Fields*. To appear. DOI:10.1007/s00440-015-0689-6.

[6] BHATTACHARYA, R. N. and RANGA RAO, R. (1976). *Normal Approximation and Asymptotic Expansions*. *Wiley Series in Probability and Mathematical Statistics*. Wiley, New York. MR0436272

[7] BISKUP, M. and LOUIDOR, O. (2016). Extreme local extrema of two-dimensional discrete Gaussian free field. *Comm. Math. Phys.* **345** 271–304. MR3509015

[8] BONDARENKO, A. and SEIP, K. (2015). Large GCD sums and extreme values of the Riemann zeta function. Preprint. Available at arXiv:1507.05840.

[9] BOURGADE, P. (2010). Mesoscopic fluctuations of the zeta zeros. *Probab. Theory Related Fields* **148** 479–500. MR2678896

[10] BRAMSON, M. D. (1978). Maximal displacement of branching Brownian motion. *Comm. Pure Appl. Math.* **31** 531–581. MR0494541

[11] BRAMSON, M., DING, J. and ZEITOUNI, O. (2014). Convergence in law of the maximum of nonlattice branching random walk. Preprint. Available at arXiv:1404.3423.

[12] BRAMSON, M., DING, J. and ZEITOUNI, O. (2016). Convergence in law of the maximum of the two-dimensional discrete Gaussian free field. *Comm. Pure Appl. Math.* **69** 62–123. MR3433630

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Key words and phrases. Extreme value theory, Riemann zeta function, branching random walk.

[13] FARMER, D. W., GONEK, S. M. and HUGHES, C. P. (2007). The maximum size of L-functions. *J. Reine Angew. Math.* **609** 215–236. MR2350784

[14] FYODOROV, Y. V. and BOUCHAUD, J.-P. (2008). Freezing and extreme-value statistics in a random energy model with logarithmically correlated potential. *J. Phys. A* **41** 372001. MR2430565

[15] FYODOROV, Y. V., HIARY, G. A. and KEATING, J. P. (2012). Freezing transition, characteristic polynomials of random matrices, and the Riemann zeta function. *Phys. Rev. Lett.* **108** 170601.

[16] FYODOROV, Y. V. and KEATING, J. P. (2014). Freezing transitions and extreme values: Random matrix theory, and disordered landscapes. *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **372** 20120503. MR3151088

[17] FYODOROV, Y. V. and SIMM, N. J. (2015). On the distribution of maximum value of the characteristic polynomial of gue random matrices. Preprint. Available at arXiv:1503.07110.

[18] GÖTZE, F. (1991). On the rate of convergence in the multivariate CLT. *Ann. Probab.* **19** 724–739. MR1106283

[19] HALÁSZ, G. (1983). On random multiplicative functions. In *Hubert Delange Colloquium (Orsay*, 1982). *Publ. Math. Orsay* **83** 74–96. Univ. Paris XI, Orsay. MR0728404

[20] HARPER, A. J. (2013). A note on the maximum of the Riemann zeta function, and log-correlated random variables. Preprint. Available at arXiv:1304.0677.

[21] KEATING, J. P. and SNAITH, N. C. (2000). Random matrix theory and $\zeta(1/2+it)$. *Comm. Math. Phys.* **214** 57–89. MR1794265

[22] KISTLER, N. (2015). Derrida's random energy models. From spin glasses to the extremes of correlated random fields. In *Correlated Random Systems: Five Different Methods. Lecture Notes in Math.* **2143** 71–120. Springer, Cham. MR3380419

[23] MADAULE, T. (2015). Maximum of a log-correlated Gaussian field. *Ann. Inst. Henri Poincaré Probab. Stat.* **51** 1369–1431. MR3414451

[24] MONTGOMERY, H. L. and VAUGHAN, R. C. (2007). *Multiplicative Number Theory. I. Classical Theory. Cambridge Studies in Advanced Mathematics* **97**. Cambridge Univ. Press, Cambridge. MR2378655

[25] SELBERG, A. (1946). Contributions to the theory of the Riemann zeta-function. *Arch. Math. Naturvidensk.* **48** 89–155. MR0020594

[26] SOUNDARARAJAN, K. (2008). Extreme values of zeta and L-functions. *Math. Ann.* **342** 467–486. MR2425151

[27] SOUNDARARAJAN, K. (2009). Moments of the Riemann zeta function. *Ann. of Math.* (2) **170** 981–993. MR2552116

[28] WEBB, C. (2011). Exact asymptotics of the freezing transition of a logarithmically correlated random energy model. *J. Stat. Phys.* **145** 1595–1619. MR2863721

[29] WEBB, C. (2015). The characteristic polynomial of a random unitary matrix and Gaussian multiplicative chaos—The L^2-phase. *Electron. J. Probab.* **20** 104. MR3407221

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DISTANCES BETWEEN NESTED DENSITIES AND A MEASURE OF THE IMPACT OF THE PRIOR IN BAYESIAN STATISTICS

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In this paper, we propose tight upper and lower bounds for the Wasserstein distance between any two univariate continuous distributions with probability densities p_1 and p_2 having nested supports. These explicit bounds are expressed in terms of the derivative of the likelihood ratio p_1/p_2 as well as the Stein kernel τ_1 of p_1. The method of proof relies on a new variant of Stein's method which manipulates Stein operators.

We give several applications of these bounds. Our main application is in Bayesian statistics: we derive explicit data-driven bounds on the Wasserstein distance between the posterior distribution based on a given prior and the no-prior posterior based uniquely on the sampling distribution. This is the first finite sample result confirming the well-known fact that with well-identified parameters and large sample sizes, reasonable choices of prior distributions will have only minor effects on posterior inferences if the data are benign.

REFERENCES

[1] AZZALINI, A. (1985). A class of distributions which includes the normal ones. *Scand. J. Statist.* **12** 171–178. MR0808153

[2] CHEN, L. H. Y., GOLDSTEIN, L. and SHAO, Q.-M. (2011). *Normal Approximation by Stein's Method. Probability and Its Applications* (*New York*). Springer, Heidelberg. MR2732624

[3] CHWIALKOWSKI, K., STRATHMANN, H. and GRETTON, A. (2016). A kernel test of goodness of fit. Preprint. Available at arXiv:1602.02964v3.

[4] DIACONIS, P. and FREEDMAN, D. (1986). On inconsistent Bayes estimates of location. *Ann. Statist.* **14** 68–87. MR0829556

[5] DIACONIS, P. and FREEDMAN, D. (1986). On the consistency of Bayes estimates. *Ann. Statist.* **14** 1–67. MR0829555

[6] DÖBLER, C. (2015). Stein's method of exchangeable pairs for the beta distribution and generalizations. *Electron. J. Probab.* **20** 109. MR3418541

[7] DÖBLER, C. (2015). Stein's method for the half-normal distribution with applications to limit theorems related to the simple symmetric random walk. *ALEA Lat. Am. J. Probab. Math. Stat.* **12** 171–191. MR3343481

[8] EDEN, R. and VÍQUEZ, J. (2015). Nourdin–Peccati analysis on Wiener and Wiener–Poisson space for general distributions. *Stochastic Process. Appl.* **125** 182–216. MR3274696

[9] EFRON, B. (1981). Nonparametric standard errors and confidence intervals. *Canad. J. Statist.* **9** 139–172. MR0640014

[10] EICHELSBACHER, P. and THÄLE, C. (2015). Malliavin–Stein method for variance-gamma approximation on Wiener space. *Electron. J. Probab.* **20** 123. MR3425543

MSC2010 subject classifications. Primary 60E15; secondary 62F15.

Key words and phrases. Stein's method, Bayesian analysis, prior distribution, posterior distribution.

[11] GAUNT, R. E. (2014). Variance-gamma approximation via Stein's method. *Electron*. *J*. *Probab*. **19** 38. MR3194737

[12] GOLDSTEIN, L. and REINERT, G. (2005). Distributional transformations, orthogonal polynomials, and Stein characterizations. *J*. *Theoret*. *Probab*. **18** 237–260. MR2132278

[13] GORHAM, J. and MACKEY, L. (2015). Measuring sample quality with Stein's method. *Adv*. *Neural Inf*. *Process*. *Syst*. 226–234.

[14] GORHAM, J. and MACKEY, L. (2016). Multivariate Stein factors for strongly log-concave distributions. *Electron*. *Commun*. *Probab*. **21**.

[15] HALLIN, M. and LEY, C. (2014). Skew-symmetric distributions and Fisher information: The double sin of the skew-normal. *Bernoulli* **20** 1432–1453. MR3217449

[16] KARLIN, S. and RUBIN, H. (1956). Distributions possessing a monotone likelihood ratio. *J*. *Amer*. *Statist*. *Assoc*. **51** 637–643. MR0104303

[17] LEY, C., REINERT, G. and SWAN, Y. (2016). Stein's method for comparison of univariate distributions. Preprint. Available at arXiv:1408.2998.

[18] LEY, C. and SWAN, Y. (2013). Local Pinsker inequalities via Stein's discrete density approach. *IEEE Trans*. *Inform*. *Theory* **59** 5584–5591. MR3096942

[19] LEY, C. and SWAN, Y. (2013). Stein's density approach and information inequalities. *Electron*. *Commun*. *Probab*. **18** 7. MR3019670

[20] LEY, C. and SWAN, Y. (2016). Parametric Stein operators and variance bounds. *Braz*. *J*. *Probab*. *Stat*. **30** 171–195. MR3481100

[21] NOURDIN, I. and PECCATI, G. (2012). *Normal Approximations with Malliavin Calculus*: *From Stein's Method to Universality*. *Cambridge Tracts in Mathematics* **192**. Cambridge Univ. Press, Cambridge. MR2962301

[22] NOURDIN, I., PECCATI, G. and SWAN, Y. (2014). Entropy and the fourth moment phenomenon. *J*. *Funct*. *Anal*. **266** 3170–3207. MR3158721

[23] NOURDIN, I., PECCATI, G. and SWAN, Y. (2014). Integration by parts and representation of information functionals. 2014 *IEEE International Symposium on Information Theory* (*ISIT*) 2217–2221.

[24] OATES, C. J., GIROLAMI, M. and CHOPIN, N. (2016). Control funtionals for Monte Carlo integration. *J*. *R*. *Stat*. *Soc*. *Ser*. *B*. *Stat*. *Methodol*. To appear. DOI:10.1111/rssb.12185.

[25] PIKE, J. and REN, H. (2014). Stein's method and the Laplace distribution. *ALEA Lat*. *Am*. *J*. *Probab*. *Math*. *Stat*. **11** 571–587. MR3283586

[26] ROSS, N. (2011). Fundamentals of Stein's method. *Probab*. *Surv*. **8** 210–293. MR2861132

[27] ROSS, S. M. (1996). *Stochastic Processes*, 2nd ed. *Wiley Series in Probability and Statistics*: *Probability and Statistics*. Wiley, New York. MR1373653

[28] SHAKED, M. and SHANTHIKUMAR, J. G. (2007). *Stochastic Orders*. *Springer Series in Statistics*. Springer, New York. MR2265633

[29] STEIN, C. (1965). Approximation of improper prior measures by prior probability measures. In *Proc*. *Internat*. *Res*. *Sem*., *Statist*. *Lab*., *Univ*. *California*, *Berkeley*, *Calif*., 1963 217–240. Springer, New York. MR0199937

[30] STEIN, C. (1986). *Approximate Computation of Expectations*. *Institute of Mathematical Statistics Lecture Notes—Monograph Series* **7**. IMS, Hayward, CA. MR0882007

[31] STEIN, C., DIACONIS, P., HOLMES, S. and REINERT, G. (2004). Use of exchangeable pairs in the analysis of simulations. In *Stein's Method*: *Expository Lectures and Applications*. *Institute of Mathematical Statistics Lecture Notes—Monograph Series* **46** 1–26. IMS, Beachwood, OH. MR2118600

[32] VALLENDER, S. (1974). Calculation of the Wasserstein distance between probability distributions on the line. *Theory Probab*. *Appl*. **18** 784–786.

[33] VILLANI, C. (2009). *Optimal Transport*: *Old and New*. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **338**. Springer, Berlin. MR2459454

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AN EPIDEMIC IN A DYNAMIC POPULATION WITH IMPORTATION OF INFECTIVES

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Consider a large uniformly mixing dynamic population, which has constant birth rate and exponentially distributed lifetimes, with mean population size n. A Markovian SIR (susceptible → infective → recovered) infectious disease, having importation of infectives, taking place in this population is analysed. The main situation treated is where $n \to \infty$, keeping the basic reproduction number R_0 as well as the importation rate of infectives fixed, but assuming that the quotient of the average infectious period and the average lifetime tends to 0 faster than $1/\log n$. It is shown that, as $n \to \infty$, the behaviour of the 3-dimensional process describing the evolution of the fraction of the population that are susceptible, infective and recovered, is encapsulated in a 1-dimensional regenerative process $S = \{S(t); t \geq 0\}$ describing the limiting fraction of the population that are susceptible. The process S grows deterministically, except at one random time point per regenerative cycle, where it jumps down by a size that is completely determined by the waiting time since the start of the regenerative cycle. Properties of the process S, including the jump size and stationary distributions, are determined.

REFERENCES

ANDERSSON, H. and BRITTON, T. (2000a). *Stochastic Epidemic Models and Their Statistical Analysis*. Springer, New York. MR1784822

ANDERSSON, H. and BRITTON, T. (2000b). Stochastic epidemics in dynamic populations: quasi-stationarity and extinction. *J. Math. Biol.* **41** 559–580. MR1804526

ASMUSSEN, S. (1987). *Applied Probability and Queues*. Wiley, Chichester. MR0889893

BALL, F., MOLLISON, D. and SCALIA-TOMBA, G. (1997). Epidemics with two levels of mixing. *Ann. Appl. Probab.* **7** 46–89. MR1428749

BALL, F., BRITTON, T., HOUSE, T., ISHAM, V., MOLLISON, D., PELLIS, L. and SCALIA TOMBA, G. (2015). Seven challenges for metapopulation models of epidemics, including households models. *Epidemics* **10** 63–67.

BARBOUR, A. D. (1975). The duration of the closed stochastic epidemic. *Biometrika* **62** 477–482. MR0392031

BARTLETT, M. S. (1956). Deterministic and stochastic models for recurrent epidemics. In *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability*, 1954–1955, *Vol. IV* 81–109. Univ. California Press, Berkeley and Los Angeles. MR0084932

BILLINGSLEY, P. (1968). *Convergence of Probability Measures*. Wiley, New York. MR0233396

DIEKMANN, O., HEESTERBEEK, H. and BRITTON, T. (2013). *Mathematical Tools for Understanding Infectious Disease Dynamics*. Princeton Univ. Press, Princeton, NJ. MR3015083

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Key words and phrases. Branching process, regenerative process, SIR epidemic, Skorohod metric, weak convergence.

ETHIER, S. N. and KURTZ, T. G. (1986). *Markov Processes*: *Characterization and Convergence*. Wiley, New York. MR0838085

HAMER, W. H. (1906). Epidemic disease in England—the evidence of variability and of persistence of type. *Lancet* **1** 733–739.

NÅSELL, I. (1999). On the time to extinction in recurrent epidemics. *J. R. Stat. Soc. Ser. B. Stat. Methodol.* **61** 309–330. MR1680338

NERMAN, O. (1981). On the convergence of supercritical general (C-M-J) branching processes. *Z. Wahrsch. Verw. Gebiete* **57** 365–395. MR0629532

NEWMAN, M. E. J. (2002). Spread of epidemic disease on networks. *Phys. Rev. E* (3) **66** 016128, 11. MR1919737

PELLIS, L., BALL, F., BANSAL, S., EAMES, K., HOUSE, T., ISHAM, V. and TRAPMAN, P. (2015). Eight challenges for network epidemic models. *Epidemics* **10** 58–62.

POLLARD, D. (1984). *Convergence of Stochastic Processes*. Springer, New York. MR0762984

POLLETT, P. K. (1990). On a model for interference between searching insect parasites. *J. Aust. Math. Soc. B* **32** 133–150. MR1070004

SKOROHOD, A. V. (1956). Limit theorems for stochastic processes. *Theory Probab. Appl.* **1** 261–290.

SOPER, H. E. (1929). The interpretation of periodicity in disease prevalence (with discussion). *J. R. Statist. Soc.* **92** 34–73.

WHITT, W. (2002). *Stochastic-Process Limits*: *An Introduction to Stochastic-Process Limits and Their Application to Queues*. Springer, New York. MR1876437

WHITTLE, P. (1955). The outcome of a stochastic epidemic—A note on Bailey's paper. *Biometrika* **42** 116–122. MR0070099

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ε-STRONG SIMULATION FOR MULTIDIMENSIONAL STOCHASTIC DIFFERENTIAL EQUATIONS VIA ROUGH PATH ANALYSIS[1]

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Consider a multidimensional diffusion process $X = \{X(t) : t \in [0, 1]\}$. Let $\varepsilon > 0$ be a *deterministic*, user defined, tolerance error parameter. Under standard regularity conditions on the drift and diffusion coefficients of X, we construct a probability space, supporting both X and an explicit, piecewise constant, fully simulatable process X_ε such that

$$\sup_{0 \le t \le 1} \|X_\varepsilon(t) - X(t)\|_\infty < \varepsilon$$

with probability one. Moreover, the user can adaptively choose $\varepsilon' \in (0, \varepsilon)$ so that $X_{\varepsilon'}$ (also piecewise constant and fully simulatable) can be constructed conditional on X_ε to ensure an error smaller than ε' with probability one. Our construction requires a detailed study of continuity estimates of the Itô map using Lyons' theory of rough paths. We approximate the underlying Brownian motion, jointly with the Lévy areas with a deterministic ε error in the underlying rough path metric.

REFERENCES

[1] BAYER, C., FRIZ, P. K., RIEDEL, S. and SCHOENMAKERS, J. (2016). From rough path estimates to multilevel Monte Carlo. *SIAM J. Numer. Anal.* **54** 1449–1483. MR3501366

[2] BESKOS, A., PAPASPILIOPOULOS, O. and ROBERTS, G. O. (2006). Retrospective exact simulation of diffusion sample paths with applications. *Bernoulli* **12** 1077–1098. MR2274855

[3] BESKOS, A., PELUCHETTI, S. and ROBERTS, G. (2012). ε-strong simulation of the Brownian path. *Bernoulli* **18** 1223–1248. MR2995793

[4] BESKOS, A. and ROBERTS, G. O. (2005). Exact simulation of diffusions. *Ann. Appl. Probab.* **15** 2422–2444. MR2187299

[5] BLANCHET, J. and CHEN, X. (2015). Steady-state simulation of reflected Brownian motion and related stochastic networks. *Ann. Appl. Probab.* **25** 3209–3250. MR3404635

[6] CHEN, N. and HUANG, Z. (2013). Localization and exact simulation of Brownian motion-driven stochastic differential equations. *Math. Oper. Res.* **38** 591–616. MR3092549

[7] DAVIE, A. M. (2007). Differential equations driven by rough paths: An approach via discrete approximation. *Appl. Math. Res. Express. AMRX* 2 Art. ID abm009, 40. MR2387018

[8] FRIZ, P. K. and VICTOIR, N. B. (2010). *Multidimensional Stochastic Processes as Rough Paths*: *Theory and Applications*. *Cambridge Studies in Advanced Mathematics* **120**. Cambridge Univ. Press, Cambridge. MR2604669

[9] GAINES, J. G. and LYONS, T. J. (1994). Random generation of stochastic area integrals. *SIAM J. Appl. Math.* **54** 1132–1146. MR1284705

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[10] LYONS, T. J. (1998). Differential equations driven by rough signals. *Rev. Mat. Iberoam.* **14** 215–310. MR1654527

[11] POLLOCK, M., JOHANSEN, A. M. and ROBERTS, G. O. (2016). On the exact and ε-strong simulation of (jump) diffusions. *Bernoulli* **22** 794–856. MR3449801

[12] RHEE, C. and GLYNN, P. (2012). A new approach to unbiased estimation for SDE's. In *Proceedings of the Winter Simulation Conference*. Winter Simulation Conference.

[13] STEELE, J. M. (2001). *Stochastic Calculus and Financial Applications*. *Applications of Mathematics* (*New York*) **45**. Springer, New York. MR1783083

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UNIVERSAL LIMIT THEOREMS IN GRAPH COLORING PROBLEMS WITH CONNECTIONS TO EXTREMAL COMBINATORICS

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This paper proves limit theorems for the number of monochromatic edges in uniform random colorings of general random graphs. These can be seen as generalizations of the birthday problem (what is the chance that there are two friends with the same birthday?). It is shown that if the number of colors grows to infinity, the asymptotic distribution is either a Poisson mixture or a Normal depending solely on the limiting behavior of the ratio of the number of edges in the graph and the number of colors. This result holds for any graph sequence, deterministic or random. On the other hand, when the number of colors is fixed, a necessary and sufficient condition for asymptotic normality is determined. Finally, using some results from the emerging theory of dense graph limits, the asymptotic (nonnormal) distribution is characterized for any converging sequence of dense graphs. The proofs are based on moment calculations which relate to the results of Erdős and Alon on extremal subgraph counts. As a consequence, a simpler proof of a result of Alon, estimating the number of isomorphic copies of a cycle of given length in graphs with a fixed number of edges, is presented.

REFERENCES

[1] ALDOUS, D. (1989). *Probability Approximations via the Poisson Clumping Heuristic*. *Applied Mathematical Sciences* **77**. Springer, New York. MR0969362

[2] ALON, N. (1981). On the number of subgraphs of prescribed type of graphs with a given number of edges. *Israel J. Math.* **38** 116–130. MR0599482

[3] ALON, N. (1986). On the number of certain subgraphs contained in graphs with a given number of edges. *Israel J. Math.* **53** 97–120. MR0861901

[4] ARRATIA, R., GOLDSTEIN, L. and GORDON, L. (1990). Poisson approximation and the Chen–Stein method. *Statist. Sci.* **5** 403–434. MR1092983

[5] BARBOUR, A. D., HOLST, L. and JANSON, S. (1992). *Poisson Approximation*. *Oxford Studies in Probability* **2**. Oxford Univ. Press, New York. MR1163825

[6] BERAN, R. (1975). Tail probabilities of noncentral quadratic forms. *Ann. Statist.* **3** 969–974. MR0381122

[7] BOLLOBÁS, B., JANSON, S. and RIORDAN, O. (2007). The phase transition in inhomogeneous random graphs. *Random Structures Algorithms* **31** 3–122. MR2337396

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[8] BORGS, C., CHAYES, J. T., LOVÁSZ, L., SÓS, V. T. and VESZTERGOMBI, K. (2008). Convergent sequences of dense graphs. I. Subgraph frequencies, metric properties and testing. *Adv. Math.* **219** 1801–1851. MR2455626

[9] BORGS, C., CHAYES, J. T., LOVÁSZ, L., SÓS, V. T. and VESZTERGOMBI, K. (2012). Convergent sequences of dense graphs II. Multiway cuts and statistical physics. *Ann. of Math.* (2) **176** 151–219. MR2925382

[10] BURDA, M., HARDING, M. and HAUSMAN, J. (2012). A Poisson mixture model of discrete choice. *J. Econometrics* **166** 184–203. MR2862959

[11] CERQUETTI, A. and FORTINI, S. (2006). A Poisson approximation for coloured graphs under exchangeability. *Sankhyā* **68** 183–197. MR2303080

[12] CHATTERJEE, S. (2008). A new method of normal approximation. *Ann. Probab.* **36** 1584–1610. MR2435859

[13] CHATTERJEE, S. and DIACONIS, P. (2013). Estimating and understanding exponential random graph models. *Ann. Statist.* **41** 2428–2461. MR3127871

[14] CHATTERJEE, S., DIACONIS, P. and MECKES, E. (2005). Exchangeable pairs and Poisson approximation. *Probab. Surv.* **2** 64–106. MR2121796

[15] CHUNG, F. R. K., GRAHAM, R. L. and WILSON, R. M. (1989). Quasi-random graphs. *Combinatorica* **9** 345–362. MR1054011

[16] CHURCH, K. and GALE, W. A. (1995). Poisson mixtures. *Nat. Lang. Eng.* **1** 163–190.

[17] CONLON, D., FOX, J. and ZHAO, Y. (2014). Extremal results in sparse pseudorandom graphs. *Adv. Math.* **256** 206–290. MR3177293

[18] DASGUPTA, A. (2005). The matching, birthday and the strong birthday problem: A contemporary review. *J. Statist. Plann. Inference* **130** 377–389. MR2128015

[19] DE JONG, P. (1987). A central limit theorem for generalized quadratic forms. *Probab. Theory Related Fields* **75** 261–277. MR0885466

[20] DIACONIS, P. and HOLMES, S. (2002). A Bayesian peek into Feller volume I. *Sankhyā Ser. A* **64** 820–841. Special issue in memory of D. Basu. MR1981513

[21] DIACONIS, P. and MOSTELLER, F. (1989). Methods for studying coincidences. *J. Amer. Statist. Assoc.* **84** 853–861. MR1134485

[22] DONG, F. M., KOH, K. M. and TEO, K. L. (2005). *Chromatic Polynomials and Chromaticity of Graphs*. World Scientific Co. Pte. Ltd., Hackensack, NJ. MR2159409

[23] DURRETT, R. (2010). *Probability: Theory and Examples*, 4th ed. Cambridge Univ. Press, Cambridge. MR2722836

[24] ERDŐS, P. (1962). On the number of complete subgraphs contained in certain graphs. *Magyar Tud. Akad. Mat. Kutató Int. Közl.* **7** 459–464. MR0151956

[25] FADNAVIS, S. (2011). A generalization of the birthday problem and the chromatic polynomial. Preprint. Available at arXiv:1105.0698v2.

[26] FANG, X. (2015). A universal error bound in the CLT for counting monochromatic edges in uniformly colored graphs. *Electron. Commun. Probab.* **20** 21. MR3320409

[27] FRIEDGUT, E. and KAHN, J. (1998). On the number of copies of one hypergraph in another. *Israel J. Math.* **105** 251–256. MR1639767

[28] GELMAN, A. (2013). 600: The average American knows how many people? *New York Times* February 19, page D7.

[29] GÖTZE, F. and TIKHOMIROV, A. (2002). Asymptotic distribution of quadratic forms and applications. *J. Theoret. Probab.* **15** 423–475. MR1898815

[30] GÖTZE, F. and TIKHOMIROV, A. N. (1999). Asymptotic distribution of quadratic forms. *Ann. Probab.* **27** 1072–1098. MR1699003

[31] GREENWOOD, M. and YULE, G. V. (1920). An inquiry into the nature of frequency distributions representative of multiple happenings with particular reference to the occurrence of multiple attack of disease or of repeated accidents. *J. Roy. Statist. Soc. Ser. A* **83** 255–279.

[32] HALL, P. (1984). Central limit theorem for integrated square error of multivariate nonparametric density estimators. *J. Multivariate Anal.* **14** 1–16. MR0734096

[33] JANSON, S., OLESZKIEWICZ, K. and RUCIŃSKI, A. (2004). Upper tails for subgraph counts in random graphs. *Israel J. Math.* **142** 61–92. MR2085711

[34] JENSEN, T. R. and TOFT, B. (1995). *Graph Coloring Problems*. Wiley, New York. MR1304254

[35] JENSEN, T. R. and TOFT, B. (2013). Unsolved graph coloring problems. Manuscript in preparation.

[36] KILLWORTH, P. and BERNARD, H. (1979). The reversal small-world experiment. *Social Networks* **1** 159–192.

[37] LE CAM, L. and TRAXLER, R. (1978). On the asymptotic behavior of mixtures of Poisson distributions. *Z. Wahrsch. Verw. Gebiete* **44** 1–45. MR0503463

[38] LOVÁSZ, L. (2012). *Large Networks and Graph Limits*. *American Mathematical Society Colloquium Publications* **60**. Amer. Math. Soc., Providence, RI. MR3012035

[39] NEMHAUSER, G. L. and TROTTER, L. E. JR. (1974). Properties of vertex packing and independence system polyhedra. *Math. Program.* **6** 48–61. MR0379294

[40] NOURDIN, I., PECCATI, G., POLY, G. and SIMONE, R. (2016). Classical and free fourth moment theorems: Universality and thresholds. *J. Theoret. Probab.* **29** 653–680. MR3500415

[41] NOURDIN, I., PECCATI, G. and REINERT, G. (2010). Stein's method and stochastic analysis of Rademacher functionals. *Electron. J. Probab.* **15** 1703–1742. MR2735379

[42] NOURDIN, I., PECCATI, G. and REINERT, G. (2010). Invariance principles for homogeneous sums: Universality of Gaussian Wiener chaos. *Ann. Probab.* **38** 1947–1985. MR2722791

[43] ROTAR, V. I. (1973). Some limit theorems for polynomials of second degree. *Theory Probab. Appl.* **18** 499–507.

[44] SCHRIJVER, A. (2003). *Combinatorial Optimization: Polyhedra and Efficiency*. *Algorithms and Combinatorics* **24**. Springer, Berlin.

[45] SETHURAMAN, J. (1961). Some limit theorems for joint distributions. *Sankhyā Ser. A* **23** 379–386. MR0139190

[46] STANLEY, R. P. (1995). A symmetric function generalization of the chromatic polynomial of a graph. *Adv. Math.* **111** 166–194. MR1317387

[47] SWEETING, T. J. (1989). On conditional weak convergence. *J. Theoret. Probab.* **2** 461–474. MR1011199

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NUCLEATION SCALING IN JIGSAW PERCOLATION

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Jigsaw percolation is a nonlocal process that iteratively merges connected clusters in a deterministic "puzzle graph" by using connectivity properties of a random "people graph" on the same set of vertices. We presume the Erdős–Rényi people graph with edge probability p and investigate the probability that the puzzle is solved, that is, that the process eventually produces a single cluster. In some generality, for puzzle graphs with N vertices of degrees about D (in the appropriate sense), this probability is close to 1 or small depending on whether $pD \log N$ is large or small. The one dimensional ring and two dimensional torus puzzles are studied in more detail and in many cases the exact scaling of the critical probability is obtained. The paper strengthens several results of Brummitt, Chatterjee, Dey, and Sivakoff who introduced this model.

REFERENCES

[1] AIZENMAN, M. and LEBOWITZ, J. L. (1988). Metastability effects in bootstrap percolation. *J. Phys. A* **21** 3801–3813. MR0968311

[2] BALOGH, J., BOLLOBÁS, B., DUMINIL-COPIN, H. and MORRIS, R. (2012). The sharp threshold for bootstrap percolation in all dimensions. *Trans. Amer. Math. Soc.* **364** 2667–2701. MR2888224

[3] BARBOUR, A. D., HOLST, L. and JANSON, S. (1992). *Poisson Approximation. Oxford Studies in Probability* **2**. Oxford Univ. Press, New York. MR1163825

[4] BOLLOBÁS, B., KOHAYAKAWA, Y. and ŁUCZAK, T. (1994). On the evolution of random Boolean functions. In *Extremal Problems for Finite Sets* (P. Frankl, Z. Füredi, G. Katona and D. Miklós, eds.). *Bolyai Society Mathematical Studies* **3**. János Bolyai Mathematical Society, Budapest. MR1319153

[5] BOLLOBÁS, B., RIORDAN, O., SLIVKEN, E. and SMITH, P. (2015). The threshold for jigsaw percolation on random graphs. Preprint. Available at arXiv:1503.05186.

[6] BRUMMITT, C. D., CHATTERJEE, S., DEY, P. S. and SIVAKOFF, D. (2015). Jigsaw percolation: What social networks can collaboratively solve a puzzle? *Ann. Appl. Probab.* **25** 2013–2038. MR3349000

[7] FINCH, S. R. (1999). Several constants arising in statistical mechanics. *Ann. Comb.* **3** 323–335. MR1772353

[8] FISCH, R., GRAVNER, J. and GRIFFEATH, D. (1993). Metastability in the Greenberg–Hastings model. *Ann. Appl. Probab.* **3** 935–967. MR1241030

[9] FRIEDGUT, E. and KALAI, G. (1996). Every monotone graph property has a sharp threshold. *Proc. Amer. Math. Soc.* **124** 2993–3002. MR1371123

[10] GRAVNER, J. (1996). Percolation times in two-dimensional models for excitable media. *Electron. J. Probab.* **1** 19 pp. (electronic). MR1423465

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[11] GRAVNER, J. and HOLROYD, A. E. (2009). Local bootstrap percolation. *Electron. J. Probab.* **14** 385–399. MR2480546

[12] GRAVNER, J., HOLROYD, A. E. and MORRIS, R. (2012). A sharper threshold for bootstrap percolation in two dimensions. *Probab. Theory Related Fields* **153** 1–23. MR2925568

[13] HOLROYD, A. E. (2003). Sharp metastability threshold for two-dimensional bootstrap percolation. *Probab. Theory Related Fields* **125** 195–224. MR1961342

[14] KESTEN, H. (1982). *Percolation Theory for Mathematicians*. *Progress in Probability and Statistics* **2**. Birkhäuser, Boston, MA. MR0692943

[15] KESTEN, H. (1990). Asymptotics in high dimensions for percolation. In *Disorder in Physical Systems* (G. Grimmett and D. J. A. Welsh, eds.). *Oxford Sci. Publ.* 219–240. Oxford Univ. Press, New York. MR1064563

[16] O'CONNELL, N. (1998). Some large deviation results for sparse random graphs. *Probab. Theory Related Fields* **110** 277–285. MR1616567

[17] RUSSO, L. (1981). On the critical percolation probabilities. *Z. Wahrsch. Verw. Gebiete* **56** 229–237. MR0618273

[18] SEDGEWICK, R. (1997). *Algorithms in C*, 3rd ed. Addison-Wesley, Reading, MA.

[19] SIVAKOFF, D. (2014). Site percolation on the d-dimensional Hamming torus. *Combin. Probab. Comput.* **23** 290–315. MR3166471

[20] WIERMAN, J. C. (1995). Substitution method critical probability bounds for the square lattice site percolation model. *Combin. Probab. Comput.* **4** 181–188. MR1342860

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DEGREE SEQUENCE OF RANDOM PERMUTATION GRAPHS

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In this paper, we study the asymptotics of the degree sequence of permutation graphs associated with a sequence of random permutations. The limiting finite-dimensional distributions of the degree proportions are established using results from graph and permutation limit theories. In particular, we show that for a uniform random permutation, the joint distribution of the degree proportions of the vertices labeled $\lceil nr_1 \rceil, \lceil nr_2 \rceil, \ldots, \lceil nr_s \rceil$ in the associated permutation graph converges to independent random variables $D(r_1), D(r_2), \ldots, D(r_s)$, where $D(r_i) \sim \text{Unif}(r_i, 1 - r_i)$, for $r_i \in [0, 1]$ and $i \in \{1, 2, \ldots, s\}$. Moreover, the degree proportion of the mid-vertex (the vertex labeled $n/2$) has a central limit theorem, and the minimum degree converges to a Rayleigh distribution after an appropriate scaling. Finally, the asymptotic finite-dimensional distributions of the permutation graph associated with a Mallows random permutation is determined, and interesting phase transitions are observed. Our results extend to other nonuniform measures on permutations as well.

REFERENCES

[1] ACAN, H. and PITTEL, B. (2013). On the connected components of a random permutation graph with a given number of edges. *J. Combin. Theory Ser. A* **120** 1947–1975. MR3102170

[2] AWASTHI, P., BLUM, A., SHEFFET, O. and VIJAYARAGHAVAN, A. (2014). Learning mixtures of ranking models. Available at arXiv:1410.8750.

[3] BAFNA, V. and PEVZNER, P. A. (1996). Genome rearrangements and sorting by reversals. *SIAM J. Comput.* **25** 272–289. MR1379301

[4] BAIK, J., DEIFT, P. and JOHANSSON, K. (1999). On the distribution of the length of the longest increasing subsequence of random permutations. *J. Amer. Math. Soc.* **12** 1119–1178. MR1682248

[5] BEŔARD, S., BERGERON, A., CHAUVE, C. and PAUL, C. (2007). Perfect sorting by reversals is not always difficult. *IEEE/ACM Transactions on Computational Biology and Bioinformatics* **4** 4–16.

[6] BHATNAGAR, N. and PELED, R. (2015). Lengths of monotone subsequences in a Mallows permutation. *Probab. Theory Related Fields* **161** 719–780. MR3334280

[7] BILLINGSLEY, P. (1999). *Convergence of Probability Measures*, 2nd ed. Wiley, New York. MR1700749

[8] BÓNA, M. (2004). *Combinatorics of Permutations*. Chapman & Hall/CRC, Boca Raton, FL. MR2078910

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[9] BÓNA, M. (2007). The copies of any permutation pattern are asymptotically normal. Available at arXiv:0712.2792.

[10] BORGS, C., CHAYES, J. T., LOVÁSZ, L., SÓS, V. T. and VESZTERGOMBI, K. (2008). Convergent sequences of dense graphs. I. Subgraph frequencies, metric properties and testing. *Adv. Math.* **219** 1801–1851. MR2455626

[11] BORGS, C., CHAYES, J. T., LOVÁSZ, L., SÓS, V. T. and VESZTERGOMBI, K. (2012). Convergent sequences of dense graphs II. Multiway cuts and statistical physics. *Ann. of Math.* (2) **176** 151–219. MR2925382

[12] BRAVERMAN, M. and MOSSEL, E. (2009). Sorting from noisy information. Available at arXiv:0910.1191.

[13] CHEN, H., BRANAVAN, S. R. K., BARZILAY, R. and KARGER, D. R. (2009). Content modeling using latent permutations. *J. Artificial Intelligence Res.* **36** 129–163. MR2602357

[14] DIACONIS, P. (1988). *Group Representations in Probability and Statistics*. *Institute of Mathematical Statistics Lecture Notes—Monograph Series* **11**. IMS, Hayward, CA. MR0964069

[15] DIACONIS, P., GRAHAM, R. and HOLMES, S. P. (2001). Statistical problems involving permutations with restricted positions. In *State of the Art in Probability and Statistics* (*Leiden*, 1999). *Institute of Mathematical Statistics Lecture Notes—Monograph Series* **36** 195–222. IMS, Beachwood, OH. MR1836562

[16] DIACONIS, P., HOLMES, S. and JANSON, S. (2008). Threshold graph limits and random threshold graphs. *Internet Math.* **5** 267–320 (2009). MR2573956

[17] DIACONIS, P., HOLMES, S. and JANSON, S. (2013). Interval graph limits. *Ann. Comb.* **17** 27–52. MR3027572

[18] DIACONIS, P. and RAM, A. (2000). Analysis of systematic scan Metropolis algorithms using Iwahori–Hecke algebra techniques. *Michigan Math. J.* **48** 157–190. MR1786485

[19] EVEN, S., PNUELI, A. and LEMPEL, A. (1972). Permutation graphs and transitive graphs. *J. Assoc. Comput. Mach.* **19** 400–410. MR0313120

[20] FEIGIN, P. and COHEN, A. (1978). On a model for concordance between judges. *J. Roy. Statist. Soc. Ser. B* **40** 203–213.

[21] FLAJOLET, P. and SEDGEWICK, R. (2009). *Analytic Combinatorics*. Cambridge Univ. Press, Cambridge. MR2483235

[22] FULMAN, J. (2004). Stein's method and non-reversible Markov chains. In *Stein's Method*: *Expository Lectures and Applications*. *Institute of Mathematical Statistics Lecture Notes—Monograph Series* **46** 69–77. IMS, Beachwood, OH. MR2118603

[23] GLEBOV, R., GRZESIK, A., KLIMOŠOVÁ, T. and KRÁL', D. (2015). Finitely forcible graphons and permutons. *J. Combin. Theory Ser. B* **110** 112–135. MR3279390

[24] GLEBOV, R., HOPPEN, C., KLIMOŠOVÁ, T., KOHAYAKAWA, Y., KRÁL', D. and LIU, H. (2014). Large permutations and parameter testing. Available at arXiv:1412.5622.

[25] GOLUMBIC, M. C. (1980). *Algorithmic Graph Theory and Perfect Graphs*. Academic Press, New York. MR0562306

[26] HOPPEN, C., KOHAYAKAWA, Y., MOREIRA, C. G., RÁTH, B. and MENEZES SAMPAIO, R. (2013). Limits of permutation sequences. *J. Combin. Theory Ser. B* **103** 93–113. MR2995721

[27] HOPPEN, C., KOHAYAKAWA, Y., MOREIRA, C. G. and SAMPAIO, R. M. (2011). Testing permutation properties through subpermutations. *Theoret. Comput. Sci.* **412** 3555–3567. MR2839700

[28] HUANG, J., GUESTRIN, C. and GUIBAS, L. (2009). Fourier theoretic probabilistic inference over permutations. *J. Mach. Learn. Res.* **10** 997–1070. MR2520800

[29] JANSON, S., NAKAMURA, B. and ZEILBERGER, D. (2015). On the asymptotic statistics of the number of occurrences of multiple permutation patterns. *J. Comb.* **6** 117–143. MR3338847

[30] KNUTH, D. E. (1998). *The Art of Computer Programming*. *Sorting and Searching*. *Vol*. 3. Addison-Wesley, Reading, MA. 2nd ed. [of MR0445948]. MR3077154

[31] KNUTH, D. E. (2005). *The Art of Computer Programming*: *Generating All Tuples and Permutations*. *Vol*. 4, *Fasc*. 2. Addison-Wesley, Upper Saddle River, NJ. MR2251595

[32] KONDOR, R., HOWARD, A. and JEBARA, T. (2007). Multi-object tracking with representations of the symmetric group. In *Proceedings of the Eleventh International Conference on Artificial Intelligence and Statistics* (*AISTATS*) 211–218.

[33] KRÁL', D. and PIKHURKO, O. (2013). Quasirandom permutations are characterized by 4-point densities. *Geom*. *Funct*. *Anal*. **23** 570–579. MR3053756

[34] LAHIRI, S. N. and CHATTERJEE, A. (2007). A Berry–Esseen theorem for hypergeometric probabilities under minimal conditions. *Proc*. *Amer. Math*. *Soc*. **135** 1535–1545. MR2276664

[35] LEBANON, G. and LAFFERTY, J. (2002). Cranking: Combining rankings using conditional probability models on permutations. In *Proceedings of the* 19*th International Conference on Machine Learning* 363–370.

[36] LEBANON, G. and MAO, Y. (2008). Non-parametric modeling of partially ranked data. *J*. *Mach*. *Learn*. *Res*. **9** 2401–2429. MR2452632

[37] LOVÁSZ, L. (2012). *Large Networks and Graph Limits*. *American Mathematical Society Colloquium Publications* **60**. Amer. Math. Soc., Providence, RI. MR3012035

[38] MALLOWS, C. L. (1957). Non-null ranking models. I. *Biometrika* **44** 114–130. MR0087267

[39] MEILA, M. and BAO, L. (2008). Estimation and clustering with infinite rankings. In *Proceedings of the 24th Conference in Uncertainty in Artificial Intelligence* 393–402.

[40] MEILĂ, M. and BAO, L. (2010). An exponential model for infinite rankings. *J*. *Mach*. *Learn*. *Res*. **11** 3481–3518. MR2756191

[41] MEILA, M., PHADNIS, K., PATTERSON, A. and BLIMES, J. (2007). Consensus ranking under the exponential model. Statistics Technical Report 515, Univ. Washington, Seattle, WA.

[42] MUELLER, C. and STARR, S. (2013). The length of the longest increasing subsequence of a random Mallows permutation. *J*. *Theoret*. *Probab*. **26** 514–540. MR3055815

[43] MUKHERJEE, S. (2015). Estimation of parameters in non-uniform models on permutations. *Ann*. *Statist*. To appear. Available at arXiv:1307.0978.

[44] PNUELI, A., LEMPEL, A., EVEN, S. and PNUELI, A. (1971). Transitive orientation of graphs and identification of permutation graphs. *Canad*. *J*. *Math*. **23** 160–175. MR0292717

[45] SHERWANI, N. A. (1999). *Algorithms for VLSI Physical Design Automation*, 3rd ed. Kluwer Academic, Boston, MA.

[46] SKALA, M. (2013). Hypergeometric tail inequalities: Ending the insanity. Available at arXiv:1311.5939.

[47] STARR, S. (2009). Thermodynamic limit for the Mallows model on S_n. *J*. *Math*. *Phys*. **50** 095208, 15. MR2566888

[48] STARR, S. and WALTERS, M. (2015). Phase uniqueness for the Mallows measure on permutations. Available at arXiv:1502.03727.

[49] SWEETING, T. J. (1989). On conditional weak convergence. *J*. *Theoret*. *Probab*. **2** 461–474. MR1011199

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CONVEX DUALITY FOR STOCHASTIC SINGULAR CONTROL PROBLEMS

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We develop a general theory of convex duality for certain singular control problems, taking the abstract results by Kramkov and Schachermayer [*Ann. Appl. Probab.* **9** (1999) 904–950] for optimal expected utility from nonnegative random variables to the level of optimal expected utility from increasing, adapted controls. The main contributions are the formulation of a suitable duality framework, the identification of the problem's dual functional as well as the full duality for the primal and dual value functions and their optimizers. The scope of our results is illustrated by an irreversible investment problem and the Hindy–Huang–Kreps utility maximization problem for incomplete financial markets.

REFERENCES

[1] ALVAREZ, L. H. R. (2011). Optimal capital accumulation under price uncertainty and costly reversibility. *J. Econom. Dynam. Control* **35** 1769–1788. MR2833253

[2] BANK, P. and BAUMGARTEN, C. (2010). Parameter-dependent optimal stopping problems for one-dimensional diffusions. *Electron. J. Probab.* **15** 1971–1993. MR2745722

[3] BANK, P. and EL KAROUI, N. (2004). A stochastic representation theorem with applications to optimization and obstacle problems. *Ann. Probab.* **32** 1030–1067. MR2044673

[4] BANK, P. and FÖLLMER, H. (2003). American options, multi-armed bandits, and optimal consumption plans: A unifying view. In *Paris-Princeton Lectures on Mathematical Finance*, 2002. *Lecture Notes in Math.* **1814** 1–42. Springer, Berlin. MR2021789

[5] BANK, P. and RIEDEL, F. (2001). Optimal consumption choice with intertemporal substitution. *Ann. Appl. Probab.* **11** 750–788. MR1865023

[6] BENEŠ, V. E., SHEPP, L. A. and WITSENHAUSEN, H. S. (1980/81). Some solvable stochastic control problems. *Stochastics* **4** 39–83. MR0587428

[7] BENTH, F. E., KARLSEN, K. H. and REIKVAM, K. (2001). Optimal portfolio management rules in a non-Gaussian market with durability and intertemporal substitution. *Finance Stoch.* **5** 447–467. MR1861996

[8] BOUCHARD, B. and PHAM, H. (2004). Wealth-path dependent utility maximization in incomplete markets. *Finance Stoch.* **8** 579–603. MR2212119

[9] CADENILLAS, A. and HAUSSMANN, U. G. (1994). The stochastic maximum principle for a singular control problem. *Stoch. Stoch. Rep.* **49** 211–237. MR1785006

[10] CHIAROLLA, M. B. and FERRARI, G. (2014). Identifying the free boundary of a stochastic, irreversible investment problem via the Bank–El Karoui representation theorem. *SIAM J. Control Optim.* **52** 1048–1070. MR3181698

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[11] DELBAEN, F. and SCHACHERMAYER, W. (1994). A general version of the fundamental theorem of asset pricing. *Math. Ann.* **300** 463–520. MR1304434

[12] DELBAEN, F. and SCHACHERMAYER, W. (1998). The fundamental theorem of asset pricing for unbounded stochastic processes. *Math. Ann.* **312** 215–250. MR1671792

[13] FERRARI, G. (2015). On an integral equation for the free-boundary of stochastic, irreversible investment problems. *Ann. Appl. Probab.* **25** 150–176. MR3297769

[14] HINDY, A. and HUANG, C. (1992). Intertemporal preferences for uncertain consumption: A continuous time approach. *Econometrica* **60** 781–801. MR1168738

[15] HINDY, A. and HUANG, C. (1993). Optimal consumption and portfolio rules with durability and local substitution. *Econometrica* **61** 85–121. MR1201704

[16] HINDY, A., HUANG, C. and KREPS, D. (1992). On intertemporal preferences in continuous time: The case of certainty. *J. Math. Econom.* **21** 401–440. MR1183611

[17] KARATZAS, I. and SHREVE, S. E. (1984). Connections between optimal stopping and singular stochastic control. I. Monotone follower problems. *SIAM J. Control Optim.* **22** 856–877. MR0762624

[18] KARATZAS, I. and ŽITKOVIĆ, G. (2003). Optimal consumption from investment and random endowment in incomplete semimartingale markets. *Ann. Probab.* **31** 1821–1858. MR2016601

[19] KAUPPILA, H. (2010). Convex duality in singular control—optimal consumption choice with intertemporal substitution and optimal investment in incomplete markets. Ph.D. thesis, Columbia Univ.

[20] KIISKI, M. and PERKKIÖ, A.-P. (2015). Optional and predictable projections of normal integrands and convex-valued processes. Preprint. Available at arXiv:1508.02176.

[21] KRAMKOV, D. and SCHACHERMAYER, W. (1999). The asymptotic elasticity of utility functions and optimal investment in incomplete markets. *Ann. Appl. Probab.* **9** 904–950. MR1722287

[22] KRAMKOV, D. and SCHACHERMAYER, W. (2003). Necessary and sufficient conditions in the problem of optimal investment in incomplete markets. *Ann. Appl. Probab.* **13** 1504–1516. MR2023886

[23] MERTON, R. C. (1971). Optimum consumption and portfolio rules in a continuous-time model. *J. Econom. Theory* **3** 373–413. MR0456373

[24] MOSTOVYI, O. (2015). Necessary and sufficient conditions in the problem of optimal investment with intermediate consumption. *Finance Stoch.* **19** 135–159. MR3292127

[25] ROCKAFELLAR, R. T. (1970). *Convex Analysis. Princeton Mathematical Series, No.* 28. Princeton Univ. Press, Princeton, NJ. MR0274683

[26] SIMONS, S. (1998). *Minimax and Monotonicity. Lecture Notes in Math.* **1693**. Springer, Berlin. MR1723737

[27] ŽITKOVIĆ, G. (2005). Utility maximization with a stochastic clock and an unbounded random endowment. *Ann. Appl. Probab.* **15** 748–777. MR2114989

[28] ŽITKOVIĆ, G. (2010). Convex compactness and its applications. *Math. Financ. Econ.* **3** 1–12. MR2651515

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STATIONARY EDEN MODEL ON CAYLEY GRAPHS

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We consider two stationary versions of the Eden model, on the upper half planar lattice, resulting in an infinite forest covering the half plane. Under weak assumptions on the weight distribution and by relying on ergodic theorems, we prove that almost surely all trees are finite. Using the mass transport principle, we generalize the result to Eden model in graphs of the form $G \times \mathbb{Z}_+$, where G is a Cayley graph. This generalizes certain known results on the two-type Richardson model, in particular of Deijfen and Häggström in 2007 [*Ann. Appl. Probab.* **17** (2007) 1639–1656].

REFERENCES

[1] AHLBERG, D. (2013). Convergence towards an asymptotic shape in first-passage percolation on cone-like subgraphs of the integer lattice. *J. Theoret. Probab.* 1–25.

[2] AHLBERG, D. (2014). A temporal perspective on the rate of convergence in first-passage percolation under a moment condition. Preprint. Available at arXiv:1408.0833.

[3] BENJAMINI, I., LYONS, R., PERES, Y. and SCHRAMM, O. (1999). Group-invariant percolation on graphs. *Geom. Funct. Anal.* **9** 29–66. MR1675890

[4] BERGER, N., KAGAN, J. J. and PROCACCIA, E. B. (2014). Stretched IDLA. *ALEA* **11** 471–481.

[5] BISKUP, M. and PROCACCIA, E. B. (2016). Eigenvalue vs perimeter in a shape theorem for self-interacting random walks. Preprint. Available at arXiv:1603.03817.

[6] CERF, R. and THÉRET, M. (2016). Weak shape theorem in first passage percolation with infinite passage times. *Ann. Inst. Henri Poincaré Probab. Stat.* **52** 1351–1381. MR3531712

[7] DEIJFEN, M. and HÄGGSTRÖM, O. (2007). The two-type Richardson model with unbounded initial configurations. *Ann. Appl. Probab.* **17** 1639–1656. MR2358637

[8] EDEN, M. (1961). A two-dimensional growth process. In *Proc. 4th Berkeley Sympos. Math. Statist. and Prob., Vol. IV* 223–239. Univ. California Press, Berkeley, CA. MR0136460

[9] GARET, O., MARCHAND, R., PROCACCIA, E. B. and THÉRET, M. (2015). Continuity of the time and isoperimetric constants in supercritical percolation. Preprint. Available at arXiv:1512.00742.

[10] IMBRIE, J. Z. and SPENCER, T. (1988). Diffusion of directed polymers in a random environment. *J. Stat. Phys.* **52** 609–626. MR0968950

[11] KARDAR, M. and ZHANG, Y. C. (1987). Scaling of directed polymers in random media. *Phys. Rev. Lett.* **58** 2087.

[12] KESTEN, H. (1986). Aspects of first passage percolation. In *École D'été de Probabilités de Saint-Flour, XIV—1984. Lecture Notes in Math.* **1180** 125–264. Springer, Berlin. MR0876084

[13] KESTEN, H. (1993). On the speed of convergence in first-passage percolation. *Ann. Appl. Probab.* **3** 296–338. MR1221154

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Key words and phrases. First passage percolation, Eden model, tree.

[14] KIM, J. M., MOORE, M. A. and BRAY, A. J. (1991). Zero-temperature directed polymers in a random potential. *Phys. Rev. A* **44** 2345.

[15] LAWLER, G. F., BRAMSON, M. and GRIFFEATH, D. (1992). Internal diffusion limited aggregation. *Ann. Probab.* **20** 2117–2140. MR1188055

[16] LYONS, R. and PERES, Y. (2016). *Probability on Trees and Networks*. Cambridge University Press, New York. Available at http://pages.iu.edu/~rdlyons/.

[17] RICHARDSON, D. (1973). Random growth in a tessellation. *Proc. Cambridge Philos. Soc.* **74** 515–528. MR0329079

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STEIN'S METHOD FOR STEADY-STATE DIFFUSION APPROXIMATIONS OF $M/Ph/n+M$ SYSTEMS[1]

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We consider $M/Ph/n+M$ queueing systems in steady state. We prove that the Wasserstein distance between the stationary distribution of the normalized system size process and that of a piecewise Ornstein–Uhlenbeck (OU) process is bounded by $C/\sqrt{\lambda}$, where the constant C is independent of the arrival rate λ and the number of servers n as long as they are in the Halfin-Whitt parameter regime. For each integer $m > 0$, we also establish a similar bound for the difference of the mth steady-state moments. For the proofs, we develop a modular framework that is based on Stein's method. The framework has three components: Poisson equation, generator coupling, and state space collapse. The framework, with further refinement, is likely applicable to steady-state diffusion approximations for other stochastic systems.

REFERENCES

[1] AKSIN, Z., ARMONY, M. and MEHROTRA, V. (2007). The modern call center: A multi-disciplinary perspective on operations management research. *Prod. Oper. Manag.* **16** 665–688.

[2] ARMONY, M., ISRAELIT, S., MANDELBAUM, A., MARMOR, Y. N., TSEYTLIN, Y. and YOM-TOV, G. B. (2015). On patient flow in hospitals: A data-based queueing-science perspective. *Stoch. Syst.* **5** 146–194. MR3442392

[3] ASMUSSEN, S. (2003). *Applied Probability and Queues. Stochastic Modelling and Applied Probability*, 2nd ed. *Applications of Mathematics* (*New York*) **51**. Springer, New York. MR1978607

[4] BARBOUR, A. D. (1988). Stein's method and Poisson process convergence. *J. Appl. Probab.* **25** 175–184.

[5] BARBOUR, A. D. (1990). Stein's method for diffusion approximations. *Probab. Theory Related Fields* **84** 297–322. MR1035659

[6] BELL, S. L. and WILLIAMS, R. J. (2005). Dynamic scheduling of a parallel server system in heavy traffic with complete resource pooling: Asymptotic optimality of a threshold policy. *Electron. J. Probab.* **10** 1044–1115. MR2164040

[7] BOROVKOV, A. (1964). Some limit theorems in the theory of mass service, I. *Theory Probab. Appl.* **9** 550–565.

[8] BOROVKOV, A. (1965). Some limit theorems in the theory of mass service, II. *Theory Probab. Appl.* **10** 375–400.

[9] BRAMSON, M. (1998). State space collapse with application to heavy traffic limits for multiclass queueing networks. *Queueing Syst. Theory Appl.* **30** 89–148. MR1663763

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[10] BUDHIRAJA, A. and LEE, C. (2009). Stationary distribution convergence for generalized Jackson networks in heavy traffic. *Math*. *Oper*. *Res*. **34** 45–56. MR2542988

[11] CHATTERJEE, S. (2014). A short survey of Stein's method. Proceedings of ICM 2014. To appear.

[12] CHEN, L. H. Y., GOLDSTEIN, L. and SHAO, Q.-M. (2011). *Normal Approximation by Stein's Method*. Springer, Heidelberg. MR2732624

[13] DAI, J. G., DIEKER, A. B. and GAO, X. (2014). Validity of heavy-traffic steady-state approximations in many-server queues with abandonment. *Queueing Syst*. **78** 1–29. MR3238006

[14] DAI, J. G. and HE, S. (2013). Many-server queues with customer abandonment: Numerical analysis of their diffusion model. *Stoch*. *Syst*. **3** 96–146. MR3353469

[15] DAI, J. G., HE, S. and TEZCAN, T. (2010). Many-server diffusion limits for $G/Ph/n + GI$ queues. *Ann*. *Appl*. *Probab*. **20** 1854–1890. MR2724423

[16] DAI, J. G. and TEZCAN, T. (2011). State space collapse in many-server diffusion limits of parallel server systems. *Math*. *Oper*. *Res*. **36** 271–320. MR2828761

[17] DIEKER, A. B. and GAO, X. (2013). Positive recurrence of piecewise Ornstein–Uhlenbeck processes and common quadratic Lyapunov functions. *Ann*. *Appl*. *Probab*. **23** 1291–1317. MR3098433

[18] ERYILMAZ, A. and SRIKANT, R. (2012). Asymptotically tight steady-state queue length bounds implied by drift conditions. *Queueing Syst*. **72** 311–359. MR2989493

[19] ETHIER, S. N. and KURTZ, T. G. (1986). *Markov Processes*. *Characterization and Convergence*. Wiley, New York. MR0838085

[20] FOSCHINI, G. J. and SALZ, J. (1978). A basic dynamic routing problem and diffusion. *IEEE Trans*. *Commun*. **26** 320–327.

[21] GAMARNIK, D. and STOLYAR, A. L. (2012). Multiclass multiserver queueing system in the Halfin–Whitt heavy traffic regime: Asymptotics of the stationary distribution. *Queueing Syst*. **71** 25–51. MR2925789

[22] GAMARNIK, D. and ZEEVI, A. (2006). Validity of heavy traffic steady-state approximation in generalized Jackson networks. *Ann*. *Appl*. *Probab*. **16** 56–90. MR2209336

[23] GANS, N., KOOLE, G. and MANDELBAUM, A. (2003). Telephone call centers: Tutorial, review, and research prospects. *Manuf*. *Serv*. *Oper*. *Manag*. **5** 79–141.

[24] GIBBS, A. L. and SU, F. E. (2002). On choosing and bounding probability metrics. *Int*. *Stat*. *Rev*. **70** 419–435.

[25] GILBARG, D. and TRUDINGER, N. S. (1983). *Elliptic Partial Differential Equations of Second Order*, 2nd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **224**. Springer, Berlin. MR0737190

[26] GLYNN, P. W. and ZEEVI, A. (2008). Bounding stationary expectations of Markov processes. In *Markov Processes and Related Topics*: *A Festschrift for Thomas G*. *Kurtz*. *Inst*. *Math*. *Stat*. *Collect*. **4** 195–214. IMS, Beachwood, OH. MR2574232

[27] GURVICH, I. (2014). Diffusion models and steady-state approximations for exponentially ergodic Markovian queues. *Ann*. *Appl*. *Probab*. **24** 2527–2559. MR3262510

[28] GURVICH, I. (2014). Validity of heavy-traffic steady-state approximations in multiclass queueing networks: The case of queue-ratio disciplines. *Math*. *Oper*. *Res*. **39** 121–162. MR3173006

[29] GURVICH, I., HUANG, J. and MANDELBAUM, A. (2014). Excursion-based universal approximations for the Erlang—A queue in steady-state. *Math*. *Oper*. *Res*. **39** 325–373. MR3205550

[30] HALFIN, S. and WHITT, W. (1981). Heavy-traffic limits for queues with many exponential servers. *Oper*. *Res*. **29** 567–588. MR0629195

[31] HARRISON, J. M. (1978). The diffusion approximation for tandem queues in heavy traffic. *Adv*. *in Appl*. *Probab*. **10** 886–905. MR0509222

[32] HARRISON, J. M. (1998). Heavy traffic analysis of a system with parallel servers: Asymptotic optimality of discrete-review policies. *Ann. Appl. Probab.* **8** 822–848. MR1627791

[33] HARRISON, J. M. and LÓPEZ, M. J. (1999). Heavy traffic resource pooling in parallel-server systems. *Queueing Syst. Theory Appl.* **33** 339–368. MR1742575

[34] HARRISON, J. M. and REIMAN, M. I. (1981). Reflected Brownian motion on an orthant. *Ann. Probab.* **9** 302–308. MR0606992

[35] HENDERSON, S. G. (1997). Variance reduction via an approximating Markov process. Ph.D. thesis, Dept. Operations Research, Stanford Univ. Available at http://people.orie.cornell.edu/shane/pubs/thesis.pdf.

[36] IGLEHART, D. L. and WHITT, W. (1970). Multiple channel queues in heavy traffic. I. *Adv. in Appl. Probab.* **2** 150–177. MR0266331

[37] IGLEHART, D. L. and WHITT, W. (1970). Multiple channel queues in heavy traffic. II. Sequences, networks, and batches. *Adv. in Appl. Probab.* **2** 355–369. MR0282443

[38] KANG, W. N., KELLY, F. P., LEE, N. H. and WILLIAMS, R. J. (2009). State space collapse and diffusion approximation for a network operating under a fair bandwidth sharing policy. *Ann. Appl. Probab.* **19** 1719–1780. MR2569806

[39] KATSUDA, T. (2010). State-space collapse in stationarity and its application to a multiclass single-server queue in heavy traffic. *Queueing Syst.* **65** 237–273. MR2652044

[40] KNOBLAUCH, A. (2008). Closed-form expressions for the moments of the binomial probability distribution. *SIAM J. Appl. Math.* **69** 197–204. MR2447945

[41] LUK, H. M. (1994). Stein's method for the gamma distribution and related statistical applications. Ph.D. thesis, Univ. Southern California, Los Angeles, CA.

[42] MEYN, S. P. and TWEEDIE, R. L. (1993). Stability of Markovian processes. III. Foster–Lyapunov criteria for continuous-time processes. *Adv. in Appl. Probab.* **25** 518–548. MR1234295

[43] PETERSON, W. P. (1991). A heavy traffic limit theorem for networks of queues with multiple customer types. *Math. Oper. Res.* **16** 90–118. MR1106792

[44] REED, J. (2009). The $G/GI/N$ queue in the Halfin–Whitt regime. *Ann. Appl. Probab.* **19** 2211–2269. MR2588244

[45] REIMAN, M. I. (1984). Some diffusion approximations with state space collapse. In *Modelling and Performance Evaluation Methodology* (*Paris*, 1983) (F. Baccelli and G. Fayolle, eds.). *Lecture Notes in Control and Inform. Sci.* **60** 209–240. Springer, Berlin. MR0893658

[46] REIMAN, M. I. (1984). Open queueing networks in heavy traffic. *Math. Oper. Res.* **9** 441–458. MR0757317

[47] ROSS, N. (2011). Fundamentals of Stein's method. *Probab. Surv.* **8** 210–293. MR2861132

[48] SHI, P., CHOU, M., DAI, J. G., DING, D. and SIM, J. (2016). Models and insights for hospital inpatient operations: Time-dependent ED boarding time. *Manage. Sci.* **62** 1–28. DOI:10.1287/mnsc.2014.2112.

[49] STEIN, C. (1972). A bound for the error in the normal approximation to the distribution of a sum of dependent random variables. In *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability* (*Univ. California*, *Berkeley*, *Calif.*, 1970/1971). *Probability Theory* **II** 583–602. Univ. California Press, Berkeley, CA. MR0402873

[50] STEIN, C. (1986). *Approximate Computation of Expectations*. *Institute of Mathematical Statistics Lecture Notes—Monograph Series* **7**. IMS, Hayward, CA. MR0882007

[51] TEZCAN, T. (2008). Optimal control of distributed parallel server systems under the Halfin and Whitt regime. *Math. Oper. Res.* **33** 51–90. MR2393541

[52] WHITT, W. (1971). Weak convergence theorems for priority queues: Preemptive-resume discipline. *J. Appl. Probab.* **8** 74–94. MR0307389

[53] WHITT, W. (2002). *Stochastic-Process Limits*. *An Introduction to Stochastic-Process Limits and Their Application to Queues*. Springer, New York. MR1876437

[54] WILLIAMS, R. J. (1998). Diffusion approximations for open multiclass queueing networks: Sufficient conditions involving state space collapse. *Queueing Syst. Theory Appl.* **30** 27–88. MR1663759

[55] YE, H.-Q. and YAO, D. D. (2016). Diffusion limit of fair resource control—Stationarity and interchange of limits. *Math. Oper. Res.* **41** 1161–1207.

[56] ZHANG, J. and ZWART, B. (2008). Steady state approximations of limited processor sharing queues in heavy traffic. *Queueing Syst.* **60** 227–246. MR2461617

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LOOKING FOR VERTEX NUMBER ONE

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Given an instance of the preferential attachment graph $G_n = ([n], E_n)$, we would like to find vertex 1, using only "local" information about the graph; that is, by exploring the neighborhoods of small sets of vertices. Borgs et al. gave an algorithm which runs in time $O(\log^4 n)$, which is local in the sense that at each step, it needs only to search the neighborhood of a set of vertices of size $O(\log^4 n)$. We give an algorithm to find vertex 1, which w.h.p. runs in time $O(\omega \log n)$ and which is local in the strongest sense of operating only on neighborhoods of single vertices. Here $\omega = \omega(n)$ is any function that goes to infinity with n.

REFERENCES

[1] ALELIUNAS, R., KARP, R. M., LIPTON, R. J., LOVÁSZ, L. and RACKOFF, C. (1979). Random walks, universal traversal sequences, and the complexity of maze problems. In 20*th Annual Symposium on Foundations of Computer Science* (*San Juan*, *Puerto Rico*, 1979) 218–223. IEEE, New York. MR0598110

[2] BARABÁSI, A.-L. and ALBERT, R. (1999). Emergence of scaling in random networks. *Science* **286** 509–512. MR2091634

[3] BOLLOBÁS, B. and RIORDAN, O. (2004). The diameter of a scale-free random graph. *Combinatorica* **24** 5–34. MR2057681

[4] BOLLOBÁS, B., RIORDAN, O., SPENCER, J. and TUSNÁDY, G. (2001). The degree sequence of a scale-free random graph process. *Random Structures Algorithms* **18** 279–290. MR1824277

[5] BORGS, C., BRAUTBAR, M., CHAYES, J., KHANNA, S. and LUCIER, B. (2012). The power of local information in social networks. In *International Workshop on Internet and Network Economics* 406–419. Springer, Berlin.

[6] BRAUTBAR, M. and KEARNS, M. (2010). Local algorithms for finding intersting individuals in large networks. *Innovations in Computer Science* (*ITCS*) 188–199.

[7] COOPER, C. and FRIEZE, A. (2003). A general model of web graphs. *Random Structures Algorithms* **22** 311–335. MR1966545

[8] GRIMMETT, G. R. and STIRZAKER, D. R. (2001). *Probability and Random Processes*, 3rd ed. Oxford Univ. Press, New York. MR2059709

[9] HOEFFDING, W. (1963). Probability inequalities for sums of bounded random variables. *J. Amer. Statist. Assoc.* **58** 13–30. MR0144363

[10] MIHAIL, M., PAPADIMITRIOU, C. and SABERI, A. (2006). On certain connectivity properties of the Internet topology. *J. Comput. System Sci.* **72** 239–251. MR2205286

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KAC'S WALK ON n-SPHERE MIXES IN $n \log n$ STEPS

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Determining the mixing time of Kac's random walk on the sphere S^{n-1} is a long-standing open problem. We show that the total variation mixing time of Kac's walk on S^{n-1} is between $\frac{1}{2}n\log(n)$ and $200n\log(n)$ for all n sufficiently large. Our bound is thus optimal up to a constant factor, improving on the best-known upper bound of $O(n^5\log(n)^2)$ due to Jiang [*Ann. Appl. Probab.* **22** (2012) 1712–1727]. Our main tool is a "non-Markovian" coupling recently introduced by the second author in [*Ann. Appl. Probab.* **24** (2014) 114–130] for obtaining the convergence rates of certain high dimensional Gibbs samplers in continuous state spaces.

REFERENCES

[1] BOLLOBÁS, B. (2001). *Random Graphs*, 2nd ed. *Cambridge Studies in Advanced Mathematics* **73**. Cambridge Univ. Press, Cambridge. MR1864966

[2] BORMASHENKO, O. (2011). A coupling argument for the random transposition walk. Preprint. Available at arXiv:1109.3915.

[3] CAPUTO, P. (2008). On the spectral gap of the Kac walk and other binary collision processes. *ALEA Lat. Am. J. Probab. Math. Stat.* **4** 205–222. MR2429910

[4] CARLEN, E. A., CARVALHO, M. C. and LOSS, M. (2003). Determination of the spectral gap for Kac's master equation and related stochastic evolution. *Acta Math.* **191** 1–54. MR2020418

[5] CARLEN, E. A., CARVALHO, M. C., ROUX, J. L., LOSS, M. and VILLANI, C. (2008). Entropy and chaos in the Kac model. Preprint. Available at arXiv:0808.3192.

[6] CONNOR, S. and JACKA, S. (2008). Optimal co-adapted coupling for the symmetric random walk on the hypercube. *J. Appl. Probab.* **45** 703–713. MR2455179

[7] ERDŐS, P. and RÉNYI, A. (1961). On a classical problem of probability theory. *Magyar Tud. Akad. Mat. Kutató Int. Közl.* **6** 215–220. MR0150807

[8] HAURAY, M. and MISCHLER, S. (2014). On Kac's chaos and related problems. *J. Funct. Anal.* **266** 6055–6157. MR3188710

[9] HAYES, T. P. and VIGODA, E. (2003). A non-Markovian coupling for randomly sampling colorings. In *Foundations of Computer Science*, 2003. *Proceedings*. 44*th Annual IEEE Symposium on* 618–627. IEEE, Cambridge, MA.

[10] HAYES, T. P. and VIGODA, E. (2007). Variable length path coupling. *Random Structures & Algorithms* **31** 251–272. MR2352176

[11] JANVRESSE, E. (2001). Spectral gap for Kac's model of Boltzmann equation. *Ann. Probab.* **29** 288–304. MR1825150

[12] JIANG, Y. (2012). Total variation bound for Kac's random walk. *Ann. Appl. Probab.* **22** 1712–1727. MR2985175

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[13] KAC, M. (1956). Foundations of kinetic theory. In *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability*, 1954–1955, *Vol. III* 171–197. Univ. California Press, Berkeley and Los Angeles. MR0084985

[14] KENDALL, W. S. (2015). Coupling, local times, immersions. *Bernoulli* **21** 1014–1046. MR3338655

[15] LEVIN, D. A., PERES, Y. and WILMER, E. L. (2009). *Markov Chains and Mixing Times*. Amer. Math. Soc., Providence, RI. MR2466937

[16] MADRAS, N. and SEZER, D. (2010). Quantitative bounds for Markov chain convergence: Wasserstein and total variation distances. *Bernoulli* **16** 882–908. MR2730652

[17] MASLEN, D. K. (2003). The eigenvalues of Kac's master equation. *Math*. *Z*. **243** 291–331. MR1961868

[18] MISCHLER, S. and MOUHOT, C. (2013). Kac's program in kinetic theory. *Invent*. *Math*. **193** 1–147. MR3069113

[19] OLIVEIRA, R. I. (2009). On the convergence to equilibrium of Kac's random walk on matrices. *Ann*. *Appl*. *Probab*. 1200–1231.

[20] OLLIVIER, Y. (2009). Ricci curvature of Markov chains on metric spaces. *J*. *Funct*. *Anal*. **256** 810–864. MR2484937

[21] ROBERTS, G. O. and ROSENTHAL, J. S. (2002). One-shot coupling for certain stochastic recursive sequences. *Stochastic Process*. *Appl*. **99** 195–208. MR1901153

[22] SMITH, A. (2014). A Gibbs sampler on the n-simplex. *Ann*. *Appl*. *Probab*. **24** 114–130. MR3161643