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HYDRODYNAMIC LIMITS AND PROPAGATION OF CHAOS FOR INTERACTING RANDOM WALKS IN DOMAINS[1]

BY ZHEN-QING CHEN AND WAI-TONG (LOUIS) FAN

University of Washington and University of Wisconsin

A new non-conservative stochastic reaction–diffusion system in which two families of random walks in two adjacent domains interact near the interface is introduced and studied in this paper. Such a system can be used to model the transport of positive and negative charges in a solar cell or the population dynamics of two segregated species under competition. We show that in the macroscopic limit, the particle densities converge to the solution of a coupled nonlinear heat equations. For this, we first prove that propagation of chaos holds by establishing the uniqueness of a new BBGKY hierarchy. A local central limit theorem for reflected diffusions in bounded Lipschitz domains is also established as a crucial tool.

REFERENCES

[1] BASS, R. F. and HSU, P. (1991). Some potential theory for reflecting Brownian motion in Hölder and Lipschitz domains. *Ann. Probab.* **19** 486–508. MR1106272

[2] BASS, R. F. and KUMAGAI, T. (2008). Symmetric Markov chains on $\mathbb{Z}^d$ with unbounded range. *Trans. Amer. Math. Soc.* **360** 2041–2075. MR2366974

[3] BOLDRIGHINI, C., DE MASI, A. and PELLEGRINOTTI, A. (1992). Nonequilibrium fluctuations in particle systems modelling reaction–diffusion equations. *Stochastic Process. Appl.* **42** 1–30. MR1172504

[4] BOLDRIGHINI, C., DE MASI, A., PELLEGRINOTTI, A. and PRESUTTI, E. (1987). Collective phenomena in interacting particle systems. *Stochastic Process. Appl.* **25** 137–152. MR0904269

[5] BURDZY, K. and CHEN, Z.-Q. (2008). Discrete approximations to reflected Brownian motion. *Ann. Probab.* **36** 698–727. MR2393994

[6] BURDZY, K. and CHEN, Z.-Q. (2013). Reflecting random walk in fractal domains. *Ann. Probab.* **41** 2791–2819. MR3112932

[7] BURDZY, K., HOŁYST, R. and MARCH, P. (2000). A Fleming–Viot particle representation of the Dirichlet Laplacian. *Comm. Math. Phys.* **214** 679–703. MR1800866

[8] BURDZY, K. and QUASTEL, J. (2006). An annihilating-branching particle model for the heat equation with average temperature zero. *Ann. Probab.* **34** 2382–2405. MR2294987

[9] CARLEN, E. A., KUSUOKA, S. and STROOCK, D. W. (1987). Upper bounds for symmetric Markov transition functions. *Ann. Inst. Henri Poincaré Probab. Stat.* **23** 245–287. MR0898496

[10] CHEN, M.-F. (2004). *From Markov Chains to Non-equilibrium Particle Systems*, 2nd ed. World Scientific, River Edge, NJ. MR2091955

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[11] CHEN, Z. Q. (1993). On reflecting diffusion processes and Skorokhod decompositions. *Probab. Theory Related Fields* **94** 281–315. MR1198650

[12] CHEN, Z.-Q. and FAN, W.-T. (2017). Systems of interacting diffusions with partial annihilations through membranes. *Ann. Probab.* **45** 100–146.

[13] CHEN, Z.-Q. and FUKUSHIMA, M. (2012). *Symmetric Markov Processes, Time Change, and Boundary Theory*. Princeton Univ. Press, Princeton, NJ. MR2849840

[14] CHEN, Z. Q., WILLIAMS, R. J. and ZHAO, Z. (1995). On the existence of positive solutions for semilinear elliptic equations with Neumann boundary conditions. *Probab. Theory Related Fields* **101** 251–276. MR1318196

[15] COX, J. T., DURRETT, R. and PERKINS, E. A. (2013). Voter model perturbations and reaction diffusion equations. *Astérisque* **349** vi+113. MR3075759

[16] DAVID, G. (1988). Morceaux de graphes lipschitziens et intégrales singulières sur une surface. *Rev. Mat. Iberoam.* **4** 73–114. MR1009120

[17] DAVID, G. and SEMMES, S. (1991). Singular integrals and rectifiable sets in $\mathbf{R}^n$: Beyond Lipschitz graphs. *Astérisque* **193** 152. MR1113517

[18] DELMOTTE, T. (1999). Parabolic Harnack inequality and estimates of Markov chains on graphs. *Rev. Mat. Iberoam.* **15** 181–232. MR1681641

[19] DITTRICH, P. (1988). A stochastic model of a chemical reaction with diffusion. *Probab. Theory Related Fields* **79** 115–128. MR0952998

[20] DITTRICH, P. (1988). A stochastic particle system: Fluctuations around a nonlinear reaction–diffusion equation. *Stochastic Process. Appl.* **30** 149–164. MR0968171

[21] DURRETT, R. and LEVIN, S. (1994). The importance of being discrete (and spatial). *Theor. Popul. Biol.* **46** 363–394.

[22] ERDÖS, L., SCHLEIN, B. and YAU, H.-T. (2007). Derivation of the cubic non-linear Schrödinger equation from quantum dynamics of many-body systems. *Invent. Math.* **167** 515–614. MR2276262

[23] ETHIER, S. N. and KURTZ, T. G. (1986). *Markov Processes: Characterization and Convergence. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics*. Wiley, New York. MR0838085

[24] FAN, W.-T. (2014). Systems of reflected diffusions with interactions through membranes. Ph.D. thesis.

[25] FUNAKI, T. (1999). Free boundary problem from stochastic lattice gas model. *Ann. Inst. Henri Poincaré Probab. Stat.* **35** 573–603. MR1705681

[26] GOLSE, F. (2005). Hydrodynamic limits. In *European Congress of Mathematics* 699–717. Eur. Math. Soc., Zürich. MR2185776

[27] GUO, M. Z., PAPANICOLAOU, G. C. and VARADHAN, S. R. S. (1988). Nonlinear diffusion limit for a system with nearest neighbor interactions. *Comm. Math. Phys.* **118** 31–59. MR0954674

[28] GYRYA, P. and SALOFF-COSTE, L. (2011). Neumann and Dirichlet heat kernels in inner uniform domains. *Astérisque* **336** viii+144. MR2807275

[29] KIPNIS, C. and LANDIM, C. (1999). *Scaling Limits of Interacting Particle Systems*. Springer, Berlin. MR1707314

[30] KIPNIS, C., OLLA, S. and VARADHAN, S. R. S. (1989). Hydrodynamics and large deviation for simple exclusion processes. *Comm. Pure Appl. Math.* **42** 115–137. MR0978701

[31] KOTELENEZ, P. (1986). Law of large numbers and central limit theorem for linear chemical reactions with diffusion. *Ann. Probab.* **14** 173–193. MR0815964

[32] KOTELENEZ, P. (1988). High density limit theorems for nonlinear chemical reactions with diffusion. *Probab. Theory Related Fields* **78** 11–37. MR0940864

[33] KURTZ, T. G. (1971). Limit theorems for sequences of jump Markov processes approximating ordinary differential processes. *J. Appl. Probab.* **8** 344–356. MR0287609

[34] KURTZ, T. G. (1981). *Approximation of Population Processes*. *CBMS-NSF Regional Conference Series in Applied Mathematics* **36**. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA. MR0610982

[35] LANG, R. and NGUYEN, X.-X. (1980). Smoluchowski's theory of coagulation in colloids holds rigorously in the Boltzmann–Grad-limit. *Z. Wahrsch. Verw. Gebiete* **54** 227–280. MR0602510

[36] MAY, R. M. and NOWAK, M. A. (1992). Evolutionary games and spatial chaos. *Nature* **359** 826–829.

[37] MAZ'JA, V. G. (1985). *Sobolev Spaces*. Springer, Berlin. MR0817985

[38] SALOFF-COSTE, L. (1997). Lectures on finite Markov chains. In *Lectures on Probability Theory and Statistics* (*Saint-Flour*, 1996). *Lecture Notes in Math*. **1665** 301–413. Springer, Berlin. MR1490046

[39] STANLEY, R. P. (1999). *Enumerative Combinatorics*. *Vol*. 2. Cambridge Univ. Press, Cambridge. MR1676282

[40] STROOCK, D. W. (1988). Diffusion semigroups corresponding to uniformly elliptic divergence form operators. In *Séminaire de Probabilités*, *XXII*. *Lecture Notes in Math*. **1321** 316–347. Springer, Berlin. MR0960535

[41] STROOCK, D. W. and ZHENG, W. (1997). Markov chain approximations to symmetric diffusions. *Ann. Inst. Henri Poincaré Probab. Stat*. **33** 619–649. MR1473568

[42] YAU, H.-T. (1991). Relative entropy and hydrodynamics of Ginzburg–Landau models. *Lett. Math. Phys*. **22** 63–80. MR1121850

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THE NUMBER OF POTENTIAL WINNERS IN BRADLEY–TERRY MODEL IN RANDOM ENVIRONMENT

BY RAPHAEL CHETRITE[*,†], ROLAND DIEL[*] AND MATTHIEU LERASLE[*]

Université Côte d'Azur[*] and Ton Duc Thang University[†]

We consider a Bradley–Terry model in random environment where each player faces each other once. More precisely, the strengths of the players are assumed to be random and we study the influence of their distributions on the asymptotic number of potential winners. First, we prove that under moment and convexity conditions, the asymptotic probability that the best player wins is 1. The convexity condition is natural when the distribution of strengths is unbounded and, in the bounded case, when this convexity condition fails the number of potential winners grows at a rate depending on the tail of the distribution. We also study the minimal strength required for an additional player to win in this last case.

REFERENCES

BEN-NAIM, E. and HENGARTNER, N. W. (2007). Efficiency of competitions. *Phys. Rev. E* **76** 026106.

BRADLEY, R. A. (1976). Science, statistics, and paired comparisons. *Biometrics* **32** 213–232. MR0408132

BRADLEY, R. A. and TERRY, M. E. (1952). Rank analysis of incomplete block designs. I. The method of paired comparisons. *Biometrika* **39** 324–345. MR0070925

CATTELAN, M. (2012). Models for paired comparison data: A review with emphasis on dependent data. *Statist. Sci.* **27** 412–433. MR3012434

DAVID, H. A. (1988). *The Method of Paired Comparisons*, 2nd ed. *Griffin's Statistical Monographs & Courses* **41**. Charles Griffin & Co., London. MR0947340

DAVIDSON, R. (1970). On extending the Bradley–Terry model to accomodate ties in paired comparison experiments. *J. Amer. Statist. Assoc.* **65** 317–328.

DAVIDSON, R. R. and BEAVER, R. J. (1977). On extending the Bradley–Terry model to incorporate within-pair order effects. *Biometrics* **33** 693–702. MR0488520

DE GENNES, P. G. (1979). *Scaling Concept in Polymer Physics*. Cornell Univ. Press, Ithaca, NY.

DEMBO, A. and ZEITOUNI, O. (2002). Large deviations and applications. In *Handbook of Stochastic Analysis and Applications*. *Statist. Textbooks Monogr.* **163** 361–416. Dekker, New York. MR1882715

DREWITZ, A. and RAMÍREZ, A. F. (2014). Selected topics in random walks in random environment. In *Topics in Percolative and Disordered Systems*. *Springer Proc. Math. Stat.* **69** 23–83. Springer, New York. MR3229286

FORD, L. R. JR. (1957). Solution of a ranking problem from binary comparisons. *Amer. Math. Monthly* **64** 28–33. MR0097876

MSC2010 subject classifications. 60K37, 60G70, 60K40.

Key words and phrases. Bradley–Terry model, random environment, paired comparisons, extreme values.

GLICKMAN, M. E. and JENSEN, S. T. (2005). Adaptive paired comparison design. *J. Statist. Plann. Inference* **127** 279–293. MR2103038

HASTIE, T. and TIBSHIRANI, R. (1998). Classification by pairwise coupling. *Ann. Statist.* **26** 451–471. MR1626055

HOEFFDING, W. (1963). Probability inequalities for sums of bounded random variables. *J. Amer. Statist. Assoc.* **58** 13–30. MR0144363

HUNTER, D. R. (2004). MM algorithms for generalized Bradley–Terry models. *Ann. Statist.* **32** 384–406. MR2051012

MCDIARMID, C. (1989). On the method of bounded differences. In *Surveys in Combinatorics*, 1989 (*Norwich*, 1989). *London Mathematical Society Lecture Note Series* **141** 148–188. Cambridge Univ. Press, Cambridge. MR1036755

PISIER, G. (1983). Some applications of the metric entropy condition to harmonic analysis. In *Banach Spaces*, *Harmonic Analysis*, *and Probability Theory* (*Storrs*, *Conn.*, 1980/1981). *Lecture Notes in Math.* **995** 123–154. Springer, Berlin. MR0717231

RAO, P. V. and KUPPER, L. L. (1967). Ties in paired-comparison experiments: A generalization of the Bradley–Terry model. *J. Amer. Statist. Assoc.* **62** 194–204. MR0217963

SIMONS, G. and YAO, Y.-C. (1999). Asymptotics when the number of parameters tends to infinity in the Bradley–Terry model for paired comparisons. *Ann. Statist.* **27** 1041–1060. MR1724040

SIRE, C. and REDNER, S. (2009). Understanding baseball team standings and streaks. *Eur. Phys. J. B* **67** 473–481.

YAN, T., YANG, Y. and XU, J. (2012). Sparse paired comparisons in the Bradley–Terry model. *Statist. Sinica* **22** 1305–1318. MR2987494

ZEITOUNI, O. (2012). Random walks in random environment. In *Computational Complexity. Vols.* 1–6 2564–2577. Springer, New York. MR3074632

ZERMELO, E. (1929). Die Berechnung der Turnier-Ergebnisse als ein Maximumproblem der Wahrscheinlichkeitsrechnung. *Math. Z.* **29** 436–460. MR1545015

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ON THE DISTRIBUTION OF THE LARGEST REAL EIGENVALUE FOR THE REAL GINIBRE ENSEMBLE

BY MIHAIL POPLAVSKYI[1], ROGER TRIBE[1] AND OLEG ZABORONSKI[2]

University of Warwick

Let $\sqrt{N} + \lambda_{\max}$ be the largest real eigenvalue of a random $N \times N$ matrix with independent $N(0, 1)$ entries (the "real Ginibre matrix"). We study the large deviations behaviour of the limiting $N \to \infty$ distribution $\mathbb{P}[\lambda_{\max} < t]$ of the shifted maximal real eigenvalue $\lambda_{\max}$. In particular, we prove that the right tail of this distribution is Gaussian: for $t > 0$,

$$\mathbb{P}[\lambda_{\max} < t] = 1 - \frac{1}{4}\operatorname{erfc}(t) + O(e^{-2t^2}).$$

This is a rigorous confirmation of the corresponding result of [*Phys. Rev. Lett.* **99** (2007) 050603]. We also prove that the left tail is exponential, with correct asymptotics up to $O(1)$: for $t < 0$,

$$\mathbb{P}[\lambda_{\max} < t] = e^{\frac{1}{2\sqrt{2\pi}}\zeta(\frac{3}{2})t + O(1)},$$

where ζ is the Riemann zeta-function.

Our results have implications for interacting particle systems. The edge scaling limit of the law of real eigenvalues for the real Ginibre ensemble is a rescaling of a fixed time distribution of annihilating Brownian motions (ABMs) with the step initial condition; see [Garrod, Poplavskyi, Tribe and Zaboronski (2015)]. Therefore, the tail behaviour of the distribution of $X_s^{(\max)}$—the position of the rightmost annihilating particle at fixed time $s > 0$—can be read off from the corresponding answers for $\lambda_{\max}$ using $X_s^{(\max)} \overset{D}{=} \sqrt{4s}\lambda_{\max}$.

REFERENCES

[1] BORODIN, A. and GORIN, V. (2012). Lectures on integrable probability. Preprint. Available at arXiv:1212.3351v2.

[2] BORODIN, A. and SINCLAIR, C. D. (2009). The Ginibre ensemble of real random matrices and its scaling limits. *Comm. Math. Phys.* **291** 177–224. MR2530159

[3] DEIFT, P. and GIOEV, D. (2007). Universality at the edge of the spectrum for unitary, orthogonal, and symplectic ensembles of random matrices. *Comm. Pure Appl. Math.* **60** 867–910. MR2306224

[4] DEIFT, P., KRIECHERBAUER, T., MCLAUGHLIN, K. T.-R., VENAKIDES, S. and ZHOU, X. (1999). Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory. *Comm. Pure Appl. Math.* **52** 1335–1425. MR1702716

MSC2010 subject classifications. Primary 60B20; secondary 60F10.
Key words and phrases. Real Ginibre ensemble, Fredholm determinant.

[5] DERRIDA, B. and ZEITAK, R. (1996). Distribution of domain sizes in the zero temperature Glauber dynamics of the one-dimensional Potts model. *Phys. Rev. E* **54** 2513–2525.

[6] FELLER, W. (1971). *An Introduction to Probability Theory and Its Applications. Vol. II*, 2nd ed. Wiley, New York. MR0270403

[7] FORRESTER, P. J. (2015). Diffusion processes and the asymptotic bulk gap probability for the real Ginibre ensemble. *J. Phys. A* **48** 324001. MR3376016

[8] FORRESTER, P. J. and NAGAO, T. (2007). Eigenvalue statistics of the real Ginibre ensemble. *Phys. Rev. Lett.* **99** 050603.

[9] GARROD, B., POPLAVSKYI, M., TRIBE, R. and ZABORONSKI, O. (2015). Interacting particle systems on **Z** as Pfaffian point processes I—Annihilating and coalescing random walks. Preprint. Available at arXiv:1507.01843.

[10] GINIBRE, J. (1965). Statistical ensembles of complex, quaternion, and real matrices. *J. Math. Phys.* **6** 440–449. MR0173726

[11] KRAPIVSKY, P. L. and BEN-NAIM, E. (2015). Irreversible reactions and diffusive escape: Stationary properties. *J. Stat. Mech. Theory Exp.* P05003. MR3360299

[12] KRAPIVSKY, P. L., MALLICK, K. and SADHU, T. (2015). Melting of an Ising quadrant. *J. Phys. A* **48** 015005. MR3291423

[13] LOTOV, V. I. (1996). On some boundary crossing problems for Gaussian random walks. *Ann. Probab.* **24** 2154–2171. MR1415246

[14] RIDER, B. and SINCLAIR, C. D. (2014). Extremal laws for the real Ginibre ensemble. *Ann. Appl. Probab.* **24** 1621–1651. MR3211006

[15] SOMMERS, H.-J. (2007). Symplectic structure of the real Ginibre ensemble. *J. Phys. A* **40** F671–F676. MR2371225

[16] SOSHNIKOV, A. (1999). Universality at the edge of the spectrum in Wigner random matrices. *Comm. Math. Phys.* **207** 697–733. MR1727234

[17] TAO, T. and VU, V. (2015). Random matrices: Universality of local spectral statistics of non-Hermitian matrices. *Ann. Probab.* **43** 782–874. MR3306005

[18] TRACY, C. A. and WIDOM, H. (1994). Level-spacing distributions and the Airy kernel. *Comm. Math. Phys.* **159** 151–174. MR1257246

[19] TRACY, C. A. and WIDOM, H. (1996). On orthogonal and symplectic matrix ensembles. *Comm. Math. Phys.* **177** 727–754. MR1385083

[20] TRIBE, R. and ZABORONSKI, O. (2011). Pfaffian formulae for one dimensional coalescing and annihilating systems. *Electron. J. Probab.* **16** 2080–2103. MR2851057

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PORTFOLIO OPTIMISATION BEYOND SEMIMARTINGALES: SHADOW PRICES AND FRACTIONAL BROWNIAN MOTION

BY CHRISTOPH CZICHOWSKY[1] AND WALTER SCHACHERMAYER[2]

London School of Economics and Political Science and Universität Wien

While absence of arbitrage in frictionless financial markets requires price processes to be semimartingales, non-semimartingales can be used to model prices in an arbitrage-free way, if proportional transaction costs are taken into account. In this paper we show, for a class of price processes which are not necessarily semimartingales, the existence of an optimal trading strategy for utility maximisation under transaction costs by establishing the existence of a so-called shadow price. This is a semimartingale price process, taking values in the bid ask spread, such that frictionless trading for that price process leads to the same optimal strategy and utility as the original problem under transaction costs. Our results combine arguments from convex duality with the stickiness condition introduced by P. Guasoni. They apply in particular to exponential utility and geometric fractional Brownian motion. In this case, the shadow price is an Itô process. As a consequence, we obtain a rather surprising result on the pathwise behaviour of fractional Brownian motion: the trajectories may touch an Itô process in a one-sided manner without reflection.

REFERENCES

[1] ACCIAIO, B. (2005). Absolutely continuous optimal martingale measures. *Statist. Decisions* **23** 81–100. MR2207692

[2] ANKIRCHNER, S. and IMKELLER, P. (2005). Finite utility on financial markets with asymmetric information and structure properties of the price dynamics. *Ann. Inst. Henri Poincaré Probab. Stat.* **41** 479–503. MR2139030

[3] BAYRAKTAR, E. and SAYIT, H. (2010). On the stickiness property. *Quant. Finance* **10** 1109–1112. MR2739089

[4] BAYRAKTAR, E. and YU, X. (2013). On the market viability under proportional transaction costs. Preprint.

[5] BENDER, C. (2012). Simple arbitrage. *Ann. Appl. Probab.* **22** 2067–2085. MR3025689

[6] BENEDETTI, G., CAMPI, L., KALLSEN, J. and MUHLE-KARBE, J. (2013). On the existence of shadow prices. *Finance Stoch.* **17** 801–818. MR3105934

[7] BIAGINI, S. and FRITTELLI, M. (2008). A unified framework for utility maximization problems: An Orlicz space approach. *Ann. Appl. Probab.* **18** 929–966. MR2418234

[8] BOUCHARD, B. (2002). Utility maximization on the real line under proportional transaction costs. *Finance Stoch.* **6** 495–516. MR1932382

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[9] BOUCHARD, B. and MAZLIAK, L. (2003). A multidimensional bipolar theorem in $L^0(\mathbb{R}^d; \Omega, \mathcal{F}, P)$. *Stochastic Process. Appl.* **107** 213–231. MR1999789
[10] BRANNATH, W. and SCHACHERMAYER, W. (1999). A bipolar theorem for $L^0_+(\Omega, \mathcal{F}, \mathbb{P})$. In *Séminaire de Probabilités XXXIII* 349–354. Springer, Berlin. MR1768009
[11] CAMPI, L. and OWEN, M. P. (2011). Multivariate utility maximization with proportional transaction costs. *Finance Stoch.* **15** 461–499. MR2833096
[12] CAMPI, L. and SCHACHERMAYER, W. (2006). A super-replication theorem in Kabanov's model of transaction costs. *Finance Stoch.* **10** 579–596. MR2276320
[13] CHERIDITO, P. (2003). Arbitrage in fractional Brownian motion models. *Finance Stoch.* **7** 533–553. MR2014249
[14] CHERNY, A. (2008). Brownian moving averages have conditional full support. *Ann. Appl. Probab.* **18** 1825–1830. MR2432181
[15] CSISZÁR, I. (1975). I-divergence geometry of probability distributions and minimization problems. *Ann. Probab.* **3** 146–158. MR0365798
[16] CVITANIĆ, J. and KARATZAS, I. (1996). Hedging and portfolio optimization under transaction costs: A martingale approach. *Math. Finance* **6** 133–165. MR1384221
[17] CVITANIĆ, J. and WANG, H. (2001). On optimal terminal wealth under transaction costs. *J. Math. Econom.* **35** 223–231. MR1822345
[18] CZICHOWSKY, C., MUHLE-KARBE, J. and SCHACHERMAYER, W. (2014). Transaction costs, shadow prices, and duality in discrete time. *SIAM J. Financial Math.* **5** 258–277. MR3194656
[19] CZICHOWSKY, C., PEYRE, R., SCHACHERMAYER, W. and YANG, J. (2016). Shadow prices, fractional Brownian motion and portfolio optimisation under transaction costs. Preprint. Available at arXiv:1608.01415.
[20] CZICHOWSKY, C. and SCHACHERMAYER, W. (2016). Duality theory for portfolio optimisation under transaction costs. *Ann. Appl. Probab.* **26** 1888–1941. MR3513609
[21] CZICHOWSKY, C. and SCHACHERMAYER, W. (2016). Strong supermartingales and limits of nonnegative martingales. *Ann. Probab.* **44** 171–205. MR3456335
[22] CZICHOWSKY, C., SCHACHERMAYER, W. and YANG, J. (2017). Shadow prices for continuous price processes. *Math. Finance* **27** 623–658.
[23] DAVIS, M. H. A. and NORMAN, A. R. (1990). Portfolio selection with transaction costs. *Math. Oper. Res.* **15** 676–713. MR1080472
[24] DEELSTRA, G., PHAM, H. and TOUZI, N. (2001). Dual formulation of the utility maximization problem under transaction costs. *Ann. Appl. Probab.* **11** 1353–1383. MR1878301
[25] DELBAEN, F., GRANDITS, P., RHEINLÄNDER, T., SAMPERI, D., SCHWEIZER, M. and STRICKER, C. (2002). Exponential hedging and entropic penalties. *Math. Finance* **12** 99–123. MR1891730
[26] DELBAEN, F. and SCHACHERMAYER, W. (1994). A general version of the fundamental theorem of asset pricing. *Math. Ann.* **300** 463–520. MR1304434
[27] FÖLLMER, H. and SCHIED, A. (2011). *Stochastic Finance: An Introduction in Discrete Time*. de Gruyter, Berlin. MR2779313
[28] GASBARRA, D., SOTTINEN, T. and VAN ZANTEN, H. (2011). Conditional full support of Gaussian processes with stationary increments. *J. Appl. Probab.* **48** 561–568. MR2840316
[29] GUASONI, P. (2002). Optimal investment with transaction costs and without semimartingales. *Ann. Appl. Probab.* **12** 1227–1246. MR1936591
[30] GUASONI, P. (2006). No arbitrage under transaction costs, with fractional Brownian motion and beyond. *Math. Finance* **16** 569–582. MR2239592
[31] GUASONI, P. and RÁSONYI, M. (2015). Fragility of arbitrage and bubbles in local martingale diffusion models. *Finance Stoch.* **19** 215–231. MR3320320
[32] GUASONI, P., RÁSONYI, M. and SCHACHERMAYER, W. (2008). Consistent price systems and face-lifting pricing under transaction costs. *Ann. Appl. Probab.* **18** 491–520. MR2398764

[33] GUASONI, P., RÁSONYI, M. and SCHACHERMAYER, W. (2010). The fundamental theorem of asset pricing for continuous processes under small transaction costs. *Ann. Finance* **6** 157–191.

[34] HE, H. and PEARSON, N. D. (1991). Consumption and portfolio policies with incomplete markets and short-sale constraints: The infinite-dimensional case. *J. Econom. Theory* **54** 259–304. MR1122311

[35] HERCZEGH, A., PROKAJ, V. and RÁSONYI, M. (2014). Diversity and no arbitrage. *Stoch. Anal. Appl.* **32** 876–888. MR3254711

[36] KABANOV, Y. M. and STRICKER, C. (2002). On the optimal portfolio for the exponential utility maximization: Remarks to the six-author paper "Exponential hedging and entropic penalties" [Math. Finance **12** (2002), no. 2, 99–123; MR1891730 (2003b:91046)] by F. Delbaen, P. Grandits, T. Rheinländer, D. Samperi, M. Schweizer and C. Stricker. *Math. Finance* **12** 125–134. MR1891731

[37] KALLSEN, J. and MUHLE-KARBE, J. (2011). Existence of shadow prices in finite probability spaces. *Math. Methods Oper. Res.* **73** 251–262. MR2776563

[38] KARATZAS, I., LEHOCZKY, J. P. and SHREVE, S. E. (1987). Optimal portfolio and consumption decisions for a "small investor" on a finite horizon. *SIAM J. Control Optim.* **25** 1557–1586. MR0912456

[39] KARATZAS, I., LEHOCZKY, J. P., SHREVE, S. E. and XU, G.-L. (1991). Martingale and duality methods for utility maximization in an incomplete market. *SIAM J. Control Optim.* **29** 702–730. MR1089152

[40] KARDARAS, C. and PLATEN, E. (2011). On the semimartingale property of discounted asset-price processes. *Stochastic Process. Appl.* **121** 2678–2691. MR2832419

[41] KRAMKOV, D. and SCHACHERMAYER, W. (1999). The asymptotic elasticity of utility functions and optimal investment in incomplete markets. *Ann. Appl. Probab.* **9** 904–950. MR1722287

[42] LARSEN, K. and ŽITKOVIĆ, G. (2008). On the semimartingale property via bounded logarithmic utility. *Ann. Finance* **4** 255–268.

[43] MANDELBROT, B. (1967). The variation of some other speculative prices. *J. Bus.* **40** 393–413.

[44] NUALART, D. (2003). Stochastic integration with respect to fractional Brownian motion and applications. In *Stochastic Models* (*Mexico City*, 2002). *Contemp. Math.* **336** 3–39. Amer. Math. Soc., Providence, RI. MR2037156

[45] PAKKANEN, M. S. (2010). Stochastic integrals and conditional full support. *J. Appl. Probab.* **47** 650–667. MR2731340

[46] PAKKANEN, M. S. (2011). Brownian semistationary processes and conditional full support. *Int. J. Theor. Appl. Finance* **14** 579–586. MR2819842

[47] PEYRE, R. (2017). No simple arbitrage for fractional Brownian motion. *Bernoulli*. To appear.

[48] ROGERS, L. C. G. (1997). Arbitrage with fractional Brownian motion. *Math. Finance* **7** 95–105. MR1434408

[49] ROUGE, R. and EL KAROUI, N. (2000). Pricing via utility maximization and entropy. *Math. Finance* **10** 259–276. MR1802922

[50] SCHACHERMAYER, W. (2001). Optimal investment in incomplete markets when wealth may become negative. *Ann. Appl. Probab.* **11** 694–734. MR1865021

[51] SCHACHERMAYER, W. (2003). A super-martingale property of the optimal portfolio process. *Finance Stoch.* **7** 433–456. MR2014244

[52] SCHACHERMAYER, W. (2014). Admissible trading strategies under transaction costs. In *Séminaire de Probabilités XLVI. Lecture Notes in Math.* **2123** 317–331. Springer, Cham. MR3330822

[53] SHREVE, S. E. and SONER, H. M. (1994). Optimal investment and consumption with transaction costs. *Ann. Appl. Probab.* **4** 609–692. MR1284980

[54] SONER, H. M., SHREVE, S. E. and CVITANIĆ, J. (1995). There is no nontrivial hedging portfolio for option pricing with transaction costs. *Ann. Appl. Probab.* **5** 327–355. MR1336872

[55] TAKSAR, M., KLASS, M. J. and ASSAF, D. (1988). A diffusion model for optimal portfolio selection in the presence of brokerage fees. *Math. Oper. Res.* **13** 277–294. MR0942619

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MODEL-FREE SUPERHEDGING DUALITY

BY MATTEO BURZONI*, MARCO FRITTELLI† AND MARCO MAGGIS†

ETH Zurich and University of Milan†*

In a model-free discrete time financial market, we prove the superhedging duality theorem, where trading is allowed with dynamic and semistatic strategies. We also show that the initial cost of the cheapest portfolio that dominates a contingent claim on every possible path $\omega \in \Omega$, might be strictly greater than the upper bound of the no-arbitrage prices. We therefore characterize the subset of trajectories on which this duality gap disappears and prove that it is an analytic set.

REFERENCES

[1] ACCIAIO, B., BEIGLBÖCK, M., PENKNER, F. and SCHACHERMAYER, W. (2016). A model-free version of the fundamental theorem of asset pricing and the super-replication theorem. *Math. Finance* **26** 233–251. MR3481303

[2] ALIPRANTIS, C. D. and BORDER, K. C. (2006). *Infinite Dimensional Analysis: A Hitchhiker's Guide*, 3rd ed. Springer, Berlin. MR2378491

[3] BEIGLBÖCK, M., HENRY-LABORDÈRE, P. and PENKNER, F. (2013). Model-independent bounds for option prices—A mass transport approach. *Finance Stoch.* **17** 477–501. MR3066985

[4] BERTSEKAS, D. P. and SHREVE, S. E. (1978). *Stochastic Optimal Control: The Discrete Time Case. Mathematics in Science and Engineering* **139**. Academic Press, New York. MR0511544

[5] BOUCHARD, B. and NUTZ, M. (2015). Arbitrage and duality in nondominated discrete-time models. *Ann. Appl. Probab.* **25** 823–859. MR3313756

[6] BROWN, H., HOBSON, D. and ROGERS, L. C. G. (2001). Robust hedging of barrier options. *Math. Finance* **11** 285–314. MR1839367

[7] BURZONI, M., FRITTELLI, M. and MAGGIS, M. (2016). Universal arbitrage aggregator in discrete-time markets under uncertainty. *Finance Stoch.* **20** 1–50. MR3441285

[8] COX, A. M. G. and OBŁÓJ, J. (2011). Robust pricing and hedging of double no-touch options. *Finance Stoch.* **15** 573–605. MR2833100

[9] DELBAEN, F. and SCHACHERMAYER, W. (1994). A general version of the fundamental theorem of asset pricing. *Math. Ann.* **300** 463–520. MR1304434

[10] DELLACHERIE, C. and MEYER, P.-A. (1982). *Probabilities and Potential B. Theory of Martingales. North-Holland Mathematics Studies* **72**. North-Holland, Amsterdam. MR0745449

[11] DOLINSKY, Y. and SONER, H. M. (2014). Martingale optimal transport and robust hedging in continuous time. *Probab. Theory Related Fields* **160** 391–427. MR3256817

[12] DOLINSKY, Y. and SONER, H. M. (2015). Martingale optimal transport in the Skorokhod space. *Stochastic Process. Appl.* **125** 3893–3931. MR3373308

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[13] EL KAROUI, N. and QUENEZ, M.-C. (1995). Dynamic programming and pricing of contingent claims in an incomplete market. *SIAM J. Control Optim.* **33** 29–66. MR1311659

[14] GALICHON, A., HENRY-LABORDÈRE, P. and TOUZI, N. (2014). A stochastic control approach to no-arbitrage bounds given marginals, with an application to lookback options. *Ann. Appl. Probab.* **24** 312–336. MR3161649

[15] HENRY-LABORDÈRE, P., OBŁÓJ, J., SPOIDA, P. and TOUZI, N. (2016). The maximum maximum of a martingale with given n marginals. *Ann. Appl. Probab.* **26** 1–44. MR3449312

[16] HOBSON, D. (2011). The Skorokhod embedding problem and model-independent bounds for option prices. In *Paris-Princeton Lectures on Mathematical Finance* 2010. *Lecture Notes in Math.* **2003** 267–318. Springer, Berlin. MR2762363

[17] HOBSON, D. G. (1998). Robust hedging of the lookback option. *Finance Stoch.* **2** 329–347.

[18] KABANOV, Y. and STRICKER, C. (2001). A teachers' note on no-arbitrage criteria. In *Séminaire de Probabilités, XXXV. Lecture Notes in Math.* **1755** 149–152. Springer, Berlin. MR1837282

[19] KARATZAS, I. (1997). *Lectures on the Mathematics of Finance. CRM Monograph Series* **8**. Amer. Math. Soc., Providence, RI. MR1421066

[20] OBŁOJ, J. and HOU, Z. (2015). On robust pricing-hedging duality in continuous time. Preprint.

[21] RIEDEL, F. (2015). Financial economics without probabilistic prior assumptions. *Decis. Econ. Finance* **38** 75–91. MR3325302

[22] ROCKAFELLAR, R. T. and WETS, R. J.-B. (1998). *Variational Analysis. Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **317**. Springer, Berlin. MR1491362

[23] TAN, X. and TOUZI, N. (2013). Optimal transportation under controlled stochastic dynamics. *Ann. Probab.* **41** 3201–3240. MR3127880

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EXACT SIMULATION OF THE WRIGHT–FISHER DIFFUSION

BY PAUL A. JENKINS[1] AND DARIO SPANÒ

University of Warwick

The Wright–Fisher family of diffusion processes is a widely used class of evolutionary models. However, simulation is difficult because there is no known closed-form formula for its transition function. In this article, we demonstrate that it is in fact possible to simulate *exactly* from a broad class of Wright–Fisher diffusion processes and their bridges. For those diffusions corresponding to reversible, neutral evolution, our key idea is to exploit an eigenfunction expansion of the transition function; this approach even applies to its infinite-dimensional analogue, the Fleming–Viot process. We then develop an exact rejection algorithm for processes with more general drift functions, including those modelling natural selection, using ideas from retrospective simulation. Our approach also yields methods for exact simulation of the moment dual of the Wright–Fisher diffusion, the ancestral process of an infinite-leaf Kingman coalescent tree. We believe our new perspective on diffusion simulation holds promise for other models admitting a transition eigenfunction expansion.

REFERENCES

BARBOUR, A. D., ETHIER, S. N. and GRIFFITHS, R. C. (2000). A transition function expansion for a diffusion model with selection. *Ann. Appl. Probab.* **10** 123–162. MR1765206

BESKOS, A., PAPASPILIOPOULOS, O. and ROBERTS, G. O. (2006). Retrospective exact simulation of diffusion sample paths with applications. *Bernoulli* **12** 1077–1098. MR2274855

BESKOS, A., PAPASPILIOPOULOS, O. and ROBERTS, G. O. (2008). A factorisation of diffusion measure and finite sample path constructions. *Methodol. Comput. Appl. Probab.* **10** 85–104. MR2394037

BESKOS, A. and ROBERTS, G. O. (2005). Exact simulation of diffusions. *Ann. Appl. Probab.* **15** 2422–2444. MR2187299

BLADT, M., FINCH, S. and SØRENSEN, M. (2016). Simulation of multivariate diffusion bridges. *J. R. Stat. Soc. Ser. B. Stat. Methodol.* **78** 343–369. MR3454200

BLADT, M. and SØRENSEN, M. (2014). Simple simulation of diffusion bridges with application to likelihood inference for diffusions. *Bernoulli* **20** 645–675. MR3178513

BOLLBACK, J. P., YORK, T. L. and NIELSEN, R. (2008). Estimation of $2N_e s$ from temporal allele frequency data. *Genetics* **179** 497–502.

CHALEYAT-MAUREL, M. and GENON-CATALOT, V. (2009). Filtering the Wright–Fisher diffusion. *ESAIM Probab. Stat.* **13** 197–217. MR2518546

CHIB, S., PITT, M. and SHEPHARD, N. (2010). Likelihood based inference for diffusion driven state space models. Working Paper.

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COOP, G. and GRIFFITHS, R. C. (2004). Ancestral inference on gene trees under selection. *Theor. Popul. Biol.* **66** 219–232.

DANGERFIELD, C. E., KAY, D. and BURRAGE, K. (2010). Stochastic models and simulation of ion channel dynamics. *Proc. Comput. Sci.* **1** 1581–1590.

DANGERFIELD, C. E., KAY, D., MACNAMARA, S. and BURRAGE, K. (2012). A boundary preserving numerical algorithm for the Wright–Fisher model with mutation. *BIT* **52** 283–304. MR2931351

DAWSON, D. A. (1978). Geostochastic calculus. *Canad. J. Statist.* **6** 143–168. MR0532855

DELBAEN, F. and SHIRAKAWA, H. (2002). An interest rate model with upper and lower bounds. *Asia-Pac. Financ. Mark.* **9** 191–209.

DEVROYE, L. (1986). *Nonuniform Random Variate Generation*. Springer, New York. MR0836973

ETHIER, S. N. and GRIFFITHS, R. C. (1993). The transition function of a Fleming-Viot process. *Ann. Probab.* **21** 1571–1590. MR1235429

ETHIER, S. N. and KURTZ, T. G. (1993). Fleming–Viot processes in population genetics. *SIAM J. Control Optim.* **31** 345–386. MR1205982

FAVARO, S., RUGGIERO, M. and WALKER, S. G. (2009). On a Gibbs sampler based random process in Bayesian nonparametrics. *Electron. J. Stat.* **3** 1556–1566. MR2578838

FITZSIMMONS, P., PITMAN, J. and YOR, M. (1993). Markovian bridges: Construction, Palm interpretation, and splicing. In *Seminar on Stochastic Processes*, 1992 (*Seattle*, *WA*, 1992). *Progress in Probability* **33** 101–134. Birkhäuser, Boston, MA. MR1278079

GOLIGHTLY, A. and WILKINSON, D. J. (2006). Bayesian sequential inference for nonlinear multivariate diffusions. *Stat. Comput.* **16** 323–338. MR2297534

GOLIGHTLY, A. and WILKINSON, D. J. (2008). Bayesian inference for nonlinear multivariate diffusion models observed with error. *Comput. Statist. Data Anal.* **52** 1674–1693. MR2422763

GOURIEROUX, C. and JASIAK, J. (2006). Multivariate Jacobi process with application to smooth transitions. *J. Econometrics* **131** 475–505. MR2276008

GRIFFITHS, R. C. (1979). A transition density expansion for a multi-allele diffusion model. *Adv. in Appl. Probab.* **11** 310–325. MR0526415

GRIFFITHS, R. C. (1980). Lines of descent in the diffusion approximation of neutral Wright–Fisher models. *Theor. Popul. Biol.* **17** 37–50. MR0568666

GRIFFITHS, R. C. (1984). Asymptotic line-of-descent distributions. *J. Math. Biol.* **21** 67–75. MR0770713

GRIFFITHS, R. C. (2006). Coalescent lineage distributions. *Adv. in Appl. Probab.* **38** 405–429. MR2264950

GRIFFITHS, R. C. and LI, W. H. (1983). Simulating allele frequencies in a population and the genetic differentiation of populations under mutation pressure. *Theor. Popul. Biol.* **23** 19–33.

GRIFFITHS, R. C. and SPANÒ, D. (2010). Diffusion processes and coalescent trees. In *Probability and Mathematical Genetics*, *Papers in Honour of Sir John Kingman*. (N. H. Bingham and C. M. Goldie, eds.). *London Mathematical Society Lecture Note Series* **378** 358–379. Cambridge Univ. Press, Cambridge. MR2744247

GRIFFITHS, R. C. and SPANÒ, D. (2013). Orthogonal polynomial kernels and canonical correlations for Dirichlet measures. *Bernoulli* **19** 548–598. MR3037164

GUTENKUNST, R. N., HERNANDEZ, R. D., WILLIAMSON, S. H. and BUSTAMANTE, C. D. (2009). Inferring the joint demographic history of multiple populations from multidimensional SNP frequency data. *PLoS Genet.* **5** e1000695.

JENKINS, P. A. (2013). Exact simulation of the sample paths of a diffusion with a finite entrance boundary. Preprint. Available at arXiv:1311.5777.

JEWETT, E. M. and ROSENBERG, N. A. (2014). Theory and applications of a deterministic approximation to the coalescent model. *Theor. Popul. Biol.* **93** 14–29.

KARLIN, S. and TAYLOR, H. M. (1981). *A Second Course in Stochastic Processes*. Academic Press, New York. MR0611513

KLOEDEN, P. E. and PLATEN, E. (1999). *Numerical Solution of Stochastic Differential Equations*, 3rd printing. Springer, Berlin.

LINETSKY, V. (2005). On the transition densities for reflected diffusions. *Adv. in Appl. Probab.* **37** 435–460. MR2144561

LUKIĆ, S., HEY, J. and CHEN, K. (2011). Non-equilibrium allele frequency spectra via spectral methods. *Theor. Popul. Biol.* **79** 203–219.

MALASPINAS, A. S., MALASPINAS, O., EVANS, S. N. and SLATKIN, M. (2012). Estimating allele age and selection coefficient from time-serial data. *Genetics* **192** 599–607.

MENA, R. H. and RUGGIERO, M. (2016). Dynamic density estimation with diffusive Dirichlet mixtures. *Bernoulli* **22** 901–926. MR3449803

NEUENKIRCH, A. and SZPRUCH, L. (2014). First order strong approximations of scalar SDEs defined in a domain. *Numer. Math.* **128** 103–136. MR3248050

ØKSENDAL, B. (2003). *Stochastic Differential Equations. An Introduction with Applications*, 6th ed. Springer, Berlin. MR2001996

PAPASPILIOPOULOS, O. and ROBERTS, G. O. (2008). Retrospective Markov chain Monte Carlo methods for Dirichlet process hierarchical models. *Biometrika* **95** 169–186. MR2409721

PAPASPILIOPOULOS, O. and RUGGIERO, M. (2014). Optimal filtering and the dual process. *Bernoulli* **20** 1999–2019. MR3263096

PAPASPILIOPOULOS, O., RUGGIERO, M. and SPANÒ, D. (2014). Filtering hidden Markov measures. Preprint. Available at arXiv:1411.4944.

PARDOUX, E. (2009). Probabilistic models of population genetics. Lecture Notes, available from http://www.latp.univ-mrs.fr/~pardoux/enseignement/cours_genpop.pdf.

PITMAN, J. and YOR, M. (1981). Bessel processes and infinitely divisible laws. In *Stochastic Integrals* (*Proc. Sympos., Univ. Durham, Durham*, 1980) (D. Williams, ed.). *Lecture Notes in Math.* **851** 285–370. Springer, Berlin. MR0620995

POLLOCK, M., JOHANSEN, A. M. and ROBERTS, G. O. (2016). On the exact and ε-strong simulation of (jump) diffusions. *Bernoulli* **22** 794–856. MR3449801

SCHRAIBER, J., GRIFFITHS, R. C. and EVANS, S. N. (2013). Analysis and rejection sampling of Wright-Fisher diffusion bridges. *Theor. Popul. Biol.* **89** 64–74.

SCHURZ, H. (1996). Numerical regularization for SDEs: Construction of nonnegative solutions. *Dynam. Systems Appl.* **5** 323–351. MR1410392

SONG, Y. S. and STEINRÜCKEN, M. (2012). A simple method for finding explicit analytic transition densities of diffusion processes with general diploid selection. *Genetics* **190** 1117–1129.

STEINRÜCKEN, M., WANG, Y. X. R. and SONG, Y. S. (2013). An explicit transition density expansion for a multi-allelic Wright-Fisher diffusion with general diploid selection. *Theor. Popul. Biol.* **83** 1–14.

TAVARÉ, S. (1984). Line-of-descent and genealogical processes, and their applications in population genetics models. *Theor. Popul. Biol.* **26** 119–164. MR0770050

WALKER, S. G. (2007). Sampling the Dirichlet mixture model with slices. *Comm. Statist. Simulation Comput.* **36** 45–54. MR2370888

WALKER, S. G., HATJISPYROS, S. J. and NICOLERIS, T. (2007). A Fleming–Viot process and Bayesian nonparametrics. *Ann. Appl. Probab.* **17** 67–80. MR2292580

WILLIAMSON, S. H., HERNANDEZ, R., FLEDEL-ALON, A., ZHU, L., NIELSEN, R. and BUSTAMANTE, C. D. (2005). Simultaneous inference of selection and population growth from patterns of variation in the human genome. *Proc. Natl. Acad. Sci. USA* **102** 7882–7887.

ZHAO, L., LASCOUX, M., OVERALL, A. D. J. and WAXMAN, D. (2013). The characteristic trajectory of a fixing allele: A consequence of fictitious selection that arises from conditioning. *Genetics* **195** 993–1006.

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MESOSCOPIC EIGENVALUE STATISTICS OF WIGNER MATRICES[1]

BY YUKUN HE AND ANTTI KNOWLES

ETH Zürich

We prove that the linear statistics of the eigenvalues of a Wigner matrix converge to a universal Gaussian process on all mesoscopic spectral scales, that is, scales larger than the typical eigenvalue spacing and smaller than the global extent of the spectrum.

REFERENCES

[1] ALTSHULER, B. L. and SHKLOVSKII, B. I. (1986). Repulsion of energy levels and the conductance of small metallic samples. *Zh. Eksp. Teor. Fiz.* (*Sov. Phys. JETP*) **91** 220.
[2] BAI, Z. D. and YAO, J. (2005). On the convergence of the spectral empirical process of Wigner matrices. *Bernoulli* **11** 1059–1092. MR2189081
[3] BENAYCH-GEORGES, F. and KNOWLES, A. Lectures on the local semicircle law for Wigner matrices. Preprint, arXiv:1601.04055.
[4] BOUTET DE MONVEL, A. and KHORUNZHY, A. (1999). Asymtotic distribution of smoothed eigenvalue density. I. Gaussian random matrices. *Random Oper. Stoch. Equ.* **7** 1–22.
[5] BOUTET DE MONVEL, A. and KHORUNZHY, A. (1999). Asymptotic distribution of smoothed eigenvalue density. II. Wigner random matrices. *Random Oper. Stoch. Equ.* **7** 149–168.
[6] BREUER, J. and DUITS, M. (2016). Universality of mesoscopic fluctuations for orthogonal polynomial ensembles. *Comm. Math. Phys.* **342** 491–531. MR3459158
[7] DAVIES, E. B. (1995). The functional calculus. *J. Lond. Math. Soc.* (2) **52** 166–176. MR1345723
[8] DUITS, M. and JOHANSSON, K. On mesoscopic equilibrium for linear statistics in Dyson's Brownian motion. Preprint, arXiv:1312.4295.
[9] ERDŐS, L. (2011). Universality of Wigner random matrices: A survey of recent results. *Russian Math. Surveys* **66** 67–198.
[10] ERDŐS, L. and KNOWLES, A. (2015). The Altshuler–Shklovskii formulas for random band matrices I: The unimodular case. *Comm. Math. Phys.* **333** 1365–1416.
[11] ERDŐS, L. and KNOWLES, A. (2015). The Altshuler–Shklovskii formulas for random band matrices II: The general case. *Ann. Henri Poincaré* **16** 709–799. MR3311888
[12] ERDŐS, L., KNOWLES, A., YAU, H.-T. and YIN, J. (2012). Spectral statistics of Erdős–Rényi graphs II: Eigenvalue spacing and the extreme eigenvalues. *Comm. Math. Phys.* **314** 587–640.
[13] ERDŐS, L., KNOWLES, A., YAU, H.-T. and YIN, J. (2013). The local semicircle law for a general class of random matrices. *Electron. J. Probab.* **18** no. 59, 58. MR3068390
[14] ERDŐS, L. and YAU, H.-T. (2012). Universality of local spectral statistics of random matrices. *Bull. Amer. Math. Soc.* **49** 377–414.

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Key words and phrases. Wigner matrices, mesoscopic eigenvalue distribution, linear statistics, universality.

[15] ERDŐS, L., YAU, H.-T. and YIN, J. (2011). Universality for generalized Wigner matrices with Bernoulli distribution. *J. Comb.* **2** 15–81. MR2847916

[16] ERDŐS, L., YAU, H.-T. and YIN, J. (2012). Rigidity of eigenvalues of generalized Wigner matrices. *Adv. Math*. **229** 1435–1515.

[17] LODHIA, A. and SIMM, N. J. Mesoscopic linear statistics of Wigner matrices. Preprint, arXiv:1503.03533.

[18] LYTOVA, A. and PASTUR, L. (2009). Central limit theorem for linear eigenvalue statistics of random matrices with independent entries. *Ann*. *Probab.* **37** 1778–1840.

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NONASYMPTOTIC CONVERGENCE ANALYSIS FOR THE UNADJUSTED LANGEVIN ALGORITHM[1]

BY ALAIN DURMUS AND ÉRIC MOULINES

LTCI, Telecom ParisTech & CNRS and Ecole Polytechnique

In this paper, we study a method to sample from a target distribution π over $\mathbb{R}^d$ having a positive density with respect to the Lebesgue measure, known up to a normalisation factor. This method is based on the Euler discretization of the overdamped Langevin stochastic differential equation associated with π. For both constant and decreasing step sizes in the Euler discretization, we obtain nonasymptotic bounds for the convergence to the target distribution π in total variation distance. A particular attention is paid to the dependency on the dimension d, to demonstrate the applicability of this method in the high-dimensional setting. These bounds improve and extend the results of Dalalyan [*J. R. Stat. Soc. Ser. B. Stat. Methodol.* (2017) **79** 651–676].

REFERENCES

[1] ANDRIEU, C., DE FREITAS, N., DOUCET, A. and JORDAN, M. I. (2003). An introduction to MCMC for machine learning. *Mach. Learn.* **50** 5–43.

[2] BAKRY, D., BARTHE, F., CATTIAUX, P. and GUILLIN, A. (2008). A simple proof of the Poincaré inequality for a large class of probability measures. *Electronic Communications in Probability* [*electronic Only*] **13** 60–66.

[3] BAKRY, D., CATTIAUX, P. and GUILLIN, A. (2008). Rate of convergence for ergodic continuous Markov processes: Lyapunov versus Poincaré. *J. Funct. Anal.* **254** 727–759.

[4] BAKRY, D., GENTIL, I. and LEDOUX, M. (2014). *Analysis and Geometry of Markov Diffusion Operators*. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **348**. Springer, Cham. MR3155209

[5] BOBKOV, S. G. (1999). Isoperimetric and analytic inequalities for log-concave probability measures. *Ann. Probab.* **27** 1903–1921. MR1742893

[6] BOLLEY, F., GENTIL, I. and GUILLIN, A. (2012). Convergence to equilibrium in Wasserstein distance for Fokker–Planck equations. *J. Funct. Anal.* **263** 2430–2457. MR2964689

[7] BOUCHERON, S., LUGOSI, G. and MASSART, P. (2013). Concentration inequalities. In *A Nonasymptotic Theory of Independence*. Oxford Univ. Press, Oxford.

[8] BUBLEY, R., DYER, M. and JERRUM, M. (1998). An elementary analysis of a procedure for sampling points in a convex body. *Random Structures Algorithms* **12** 213–235. MR1635248

[9] CATTIAUX, P. and GUILLIN, A. (2009). Trends to equilibrium in total variation distance. *Ann. Inst. Henri Poincaré Probab. Stat.* **45** 117–145. MR2500231

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[10] CHEN, M. F. and LI, S. F. (1989). Coupling methods for multidimensional diffusion processes. *Ann. Probab.* **17** 151–177. MR0972776

[11] COTTER, S. L., ROBERTS, G. O., STUART, A. M. and WHITE, D. (2013). MCMC methods for functions: Modifying old algorithms to make them faster. *Statist. Sci.* **28** 424–446. MR3135540

[12] DALALYAN, A. S. (2017). Theoretical guarantees for approximate sampling from smooth and log-concave densities. *J. R. Stat. Soc. Ser. B. Stat. Methodol.* **79** 651–676.

[13] DALALYAN, A. S. and TSYBAKOV, A. B. (2012). Sparse regression learning by aggregation and Langevin Monte-Carlo. *J. Comput. System Sci.* **78** 1423–1443.

[14] DURMUS, A., MOULINES, É. and PEREYRA, M. Sampling from convex non continuously differentiable functions, when Moreau meets Langevin. In preparation.

[15] EBERLE, A. (2015). Reflection couplings and contraction rates for diffusions. *Probab. Theory Related Fields* 1–36. MR3568041

[16] ETHIER, S. N. and KURTZ, T. G. (1986). *Markov Processes: Characterization and Convergence*. Wiley, New York. MR0838085

[17] GRENANDER, U. (1983). *Tutorial in Pattern Theory*. Brown Univ., Providence, RI.

[18] GRENANDER, U. and MILLER, M. I. (1994). Representations of knowledge in complex systems. *J. Roy. Statist. Soc. Ser. B* **56** 549–603. MR1293234

[19] HOLLEY, R. and STROOCK, D. (1987). Logarithmic Sobolev inequalities and stochastic Ising models. *J. Stat. Phys.* **46** 1159–1194. MR0893137

[20] IKEDA, N. and WATANABE, S. (1989). *Stochastic Differential Equations and Diffusion Processes. North-Holland Mathematical Library*. Elsevier, Amsterdam.

[21] KARATZAS, I. and SHREVE, S. E. (1991). *Brownian Motion and Stochastic Calculus*. Springer, New York.

[22] KULLBACK, S. (1997). *Information Theory and Statistics*. Dover, Mineola, NY.

[23] LAMBERTON, D. and PAGÈS, G. (2002). Recursive computation of the invariant distribution of a diffusion. *Bernoulli* **8** 367–405. MR1913112

[24] LAMBERTON, D. and PAGÈS, G. (2003). Recursive computation of the invariant distribution of a diffusion: The case of a weakly mean reverting drift. *Stoch. Dyn.* **3** 435–451.

[25] LEMAIRE, V. (2005). Estimation de la mesure invariante d'un processus de diffusion. Ph.D. thesis, Univ. Paris-Est.

[26] LEMAIRE, V. and MENOZZI, S. (2010). On some non asymptotic bounds for the Euler scheme. *Electron. J. Probab.* **15** 1645–1681. MR2735377

[27] LINDVALL, T. and ROGERS, L. C. G. (1986). Coupling of multidimensional diffusions by reflection. *Ann. Probab.* **14** 860–872. MR0841588

[28] LOVÁSZ, L. and VEMPALA, S. (2007). The geometry of logconcave functions and sampling algorithms. *Random Structures Algorithms* **30** 307–358. MR2309621

[29] MATTINGLY, J. C., STUART, A. M. and HIGHAM, D. J. (2002). Ergodicity for SDEs and approximations: Locally Lipschitz vector fields and degenerate noise. *Stochastic Process. Appl.* **101** 185–232.

[30] MEYN, S. and TWEEDIE, R. (2009). *Markov Chains and Stochastic Stability*, 2nd ed. Cambridge Univ. Press, New York.

[31] MEYN, S. P. and TWEEDIE, R. L. (1993). Stability of Markovian processes III: Foster-Lyapunov criteria for continuous-time processes. *Adv. in Appl. Probab.* **25** 518–548. MR1234295

[32] MEYN, S. P. and TWEEDIE, R. L. (1993). Stability of Markovian processes. III. Foster–Lyapunov criteria for continuous-time processes. *Adv. in Appl. Probab.* **25** 518–548. MR1234295

[33] NESTEROV, Y. (2004). *Introductory Lectures on Convex Optimization: A Basic Course*. Kluwer, Boston, MA. MR142598

[34] PARISI, G. (1981). Correlation functions and computer simulations. *Nuclear Phys*. *B* **180** 378–384. MR0615176

[35] ROBERTS, G. O. and TWEEDIE, R. L. (1996). Exponential convergence of Langevin distributions and their discrete approximations. *Bernoulli* **2** 341–363. MR1440273

[36] ROBERTS, G. O. and TWEEDIE, R. L. (2000). Rates of convergence of stochastically monotone and continuous time Markov models. *J*. *Appl*. *Probab*. **37** 359–373. MR1780996

[37] TALAY, D. and TUBARO, L. (1991). Expansion of the global error for numerical schemes solving stochastic differential equations. *Stoch*. *Anal*. *Appl*. **8** 483–509. MR1091544

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MINIMAL SPANNING TREES AND STEIN'S METHOD

BY SOURAV CHATTERJEE[1] AND SANCHAYAN SEN[2]

Stanford University and McGill University

Kesten and Lee [*Ann. Appl. Probab.* **6** (1996) 495–527] proved that the total length of a minimal spanning tree on certain random point configurations in $\mathbb{R}^d$ satisfies a central limit theorem. They also raised the question: how to make these results quantitative? Error estimates in central limit theorems satisfied by many other standard functionals studied in geometric probability are known, but techniques employed to tackle the problem for those functionals do not apply directly to the minimal spanning tree. Thus, the problem of determining the convergence rate in the central limit theorem for Euclidean minimal spanning trees has remained open. In this work, we establish bounds on the convergence rate for the Poissonized version of this problem by using a variation of Stein's method. We also derive bounds on the convergence rate for the analogous problem in the setup of the lattice $\mathbb{Z}^d$.

The contribution of this paper is twofold. First, we develop a general technique to compute convergence rates in central limit theorems satisfied by minimal spanning trees on sequences of weighted graphs, including minimal spanning trees on Poisson points inside a sequence of growing cubes. Second, we present a way of quantifying the Burton–Keane argument for the uniqueness of the infinite open cluster. The latter is interesting in its own right and based on a generalization of our technique, Duminil-Copin, Ioffe and Velenik [*Ann. Probab.* **44** (2016) 3335–3356] have recently obtained bounds on probability of two-arm events in a broad class of translation-invariant percolation models.

REFERENCES

[1] ADDARIO-BERRY, L., BROUTIN, N., GOLDSCHMIDT, C. and MIERMONT, G. (2013). The scaling limit of the minimum spanning tree of the complete graph. Preprint. Available at http://arxiv.org/abs/1301.1664.

[2] AIZENMAN, M., BURCHARD, A., NEWMAN, C. M. and WILSON, D. B. (1999). Scaling limits for minimal and random spanning trees in two dimensions. *Random Structures Algorithms* **15** 319–367. MR1716768

[3] AIZENMAN, M., KESTEN, H. and NEWMAN, C. M. (1987). Uniqueness of the infinite cluster and continuity of connectivity functions for short and long range percolation. *Comm. Math. Phys.* **111** 505–531. MR0901151

[4] ALDOUS, D. (1990). A random tree model associated with random graphs. *Random Structures Algorithms* **1** 383–402. MR1138431

[5] ALDOUS, D. and STEELE, J. M. (1992). Asymptotics for Euclidean minimal spanning trees on random points. *Probab. Theory Related Fields* **92** 247–258. MR1161188

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[6] ALEXANDER, K. S. (1994). Rates of convergence of means for distance-minimizing subadditive Euclidean functionals. *Ann. Appl. Probab.* **4** 902–922.
[7] ALEXANDER, K. S. (1995). Percolation and minimal spanning forests in infinite graphs. *Ann. Probab.* **23** 87–104.
[8] ALEXANDER, K. S. (1996). The RSW theorem for continuum percolation and the CLT for Euclidean minimal spanning trees. *Ann. Appl. Probab.* **6** 466–494. MR1398054
[9] ALEXANDER, K. S. and MOLCHANOV, S. A. (1994). Percolation of level sets for two-dimensional random fields with lattice symmetry. *J. Stat. Phys.* **77** 627–643. MR1301459
[10] AVRAM, F. and BERTSIMAS, D. (1992). The minimum spanning tree constant in geometrical probability and under the independent model: A unified approach. *Ann. Appl. Probab.* **2** 113–130. MR1143395
[11] AVRAM, F. and BERTSIMAS, D. (1993). On central limit theorems in geometrical probability. *Ann. Appl. Probab.* **3** 1033–1046. MR1241033
[12] BAI, Z. D., LEE, S. and PENROSE, M. D. (2006). Rooted edges of a minimal directed spanning tree on random points. *Adv. in Appl. Probab.* **38** 1–30. MR2213961
[13] BALDI, P. and RINOTT, Y. (1989). On normal approximations of distributions in terms of dependency graphs. *Ann. Probab.* **17** 1646–1650. MR1048950
[14] BALDI, P., RINOTT, Y. and STEIN, C. (1989). A normal approximation for the number of local maxima of a random function on a graph. In *Probability, Statistics, and Mathematics* 59–81. Academic Press, Boston, MA. MR1031278
[15] BARBOUR, A. D. (1990). Stein's method for diffusion approximations. *Probab. Theory Related Fields* **84** 297–322.
[16] BARBOUR, A. D., KAROŃSKI, M. and RUCIŃSKI, A. (1989). A central limit theorem for decomposable random variables with applications to random graphs. *J. Combin. Theory Ser. B* **47** 125–145. MR1047781
[17] BEARDWOOD, J., HALTON, J. H. and HAMMERSLEY, J. M. (1959). The shortest path through many points. *Math. Proc. Cambridge Philos. Soc.* **55** 299–327.
[18] BHATT, A. G. and ROY, R. (2004). On a random directed spanning tree. *Adv. in Appl. Probab.* **36** 19–42. MR2035772
[19] BOLLOBÁS, B. and RIORDAN, O. (2006). *Percolation*. Cambridge Univ. Press, Cambridge. MR2283880
[20] BOLTHAUSEN, E. (1984). An estimate of the remainder in a combinatorial central limit theorem. *Probab. Theory Related Fields* **66** 379–386. MR0751577
[21] BURTON, R. M. and KEANE, M. (1989). Density and uniqueness in percolation. *Comm. Math. Phys.* **121** 501–505. MR0990777
[22] CAMIA, F., FONTES, L. R. and NEWMAN, C. M. (2006). Two-dimensional scaling limits via marked nonsimple loops. *Bull. Braz. Math. Soc.* (*N.S.*) **37** 537–559.
[23] CERF, R. (2013). A lower bound on the two-arms exponent for critical percolation on the lattice. *Ann. Probab.* **43** 2458–2480.
[24] CHATTERJEE, S. (2008). A new method of normal approximation. *Ann. Probab.* **36** 1584–1610.
[25] CHATTERJEE, S. (2009). Fluctuations of eigenvalues and second order Poincaré inequalities. *Probab. Theory Related Fields* **143** 1–40.
[26] CHATTERJEE, S. and SOUNDARARAJAN, K. (2012). Random multiplicative functions in short intervals. *Int. Math. Res. Not. IMRN* **2012** 479–492.
[27] CHEN, L. H. Y. and SHAO, Q.-M. (2004). Normal approximation under local dependence. *Ann. Probab.* **32** 1985–2028. MR2073183
[28] DUMINIL-COPIN, H., IOFFE, D. and VELENIK, Y. (2016). A quantitative Burton–Keane estimate under strong FKG condition. *Ann. Probab.* **44** 3335–3356. MR3551198
[29] FRIEZE, A. M. (1985). On the value of a random minimum spanning tree problem. *Discrete Appl. Math.* **10** 47–56. MR0770868

[30] GANDOLFI, A., GRIMMETT, G. and RUSSO, L. (1988). On the uniqueness of the infinite cluster in the percolation model. *Comm. Math. Phys.* **114** 549–552. MR0929129
[31] GOLDSTEIN, L. and REINERT, G. (1997). Stein's method and the zero bias transformation with application to simple random sampling. *Ann. Appl. Probab.* **7** 935–952. MR1484792
[32] GOLDSTEIN, L. and RINOTT, Y. (1996). Multivariate normal approximations by Stein's method and size bias couplings. *J. Appl. Probab.* **33** 1–17. MR1371949
[33] GRIMMETT, G. (1999). *Percolation*, 2nd ed. *Grundlehren der Mathematischen Wissenschaften* **321**. Springer, Berlin. MR1707339
[34] HÄGGSTRÖM, O. (1995). Random-cluster measures and uniform spanning trees. *Stochastic Process. Appl.* **59** 1–75. MR1357655
[35] JANSON, S. (1995). The minimal spanning tree in a complete graph and a functional limit theorem for trees in a random graph. *Random Structures Algorithms* **7** 337–355. MR1369071
[36] KESTEN, H. and LEE, S. (1996). The central limit theorem for weighted minimal spanning trees on random points. *Ann. Appl. Probab.* **6** 495–527. MR1398055
[37] KOZMA, G. and NACHMIAS, A. (2010). Arm exponents in high dimensional percolation. *J. Amer. Math. Soc.* **24** 375–409. MR2748397
[38] LACHIÉZE-REY, R. and PECCATI, G. (2015). New Kolmogorov bounds for functionals of binomial point processes. Preprint. Available at http://arxiv.org/abs/1505.04640.
[39] LAST, G., PECCATI, G. and SCHULTE, M. (2014). Normal approximation on Poisson spaces: Mehler's formula, second order Poincaré inequalities and stabilization. Preprint. Available at http://arxiv.org/abs/1401.7568.
[40] LEE, S. (1997). The central limit theorem for Euclidean minimal spanning trees. I. *Ann. Appl. Probab.* **7** 996–1020. MR1484795
[41] LEE, S. (1999). The central limit theorem for Euclidean minimal spanning trees II. *Adv. in Appl. Probab.* **31** 969–984. MR1747451
[42] LYONS, R., PERES, Y. and SCHRAMM, O. (2006). Minimal spanning forests. *Ann. Probab.* **34** 1665–1692. MR2271476
[43] MEESTER, R. and ROY, R. (1996). *Continuum Percolation*. Cambridge Univ. Press, Cambridge. MR1409145
[44] PENROSE, M. D. (1996). The random minimal spanning tree in high dimensions. *Ann. Probab.* **24** 1903–1925. MR1415233
[45] PENROSE, M. D. (1997). The longest edge of the random minimal spanning tree. *Ann. Appl. Probab.* **7** 340–361. MR1442317
[46] PENROSE, M. D. (1998). Random minimal spanning tree and percolation on the N-cube. *Random Structures Algorithms* **12** 63–82. MR1637395
[47] PENROSE, M. D. (2003). *Random Geometric Graphs*. Oxford University Press, Oxford. MR1986198
[48] PENROSE, M. D. and WADE, A. R. (2004). Random minimal directed spanning trees and Dickman-type distributions. *Adv. in Appl. Probab.* **36** 691–714. MR2079909
[49] PENROSE, M. D. and YUKICH, J. E. (2003). Weak laws of large numbers in geometric probability. *Ann. Appl. Probab.* **13** 277–303. MR1952000
[50] PENROSE, M. D. and YUKICH, J. E. (2005). Normal approximation in geometric probability. In *Stein's Method and Applications* (A. D. Barbour and L. H. Y. Chen, eds.). *Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap.* **5** 37–58. Singapore Univ. Press, Singapore. MR2201885
[51] PETE, G., GARBAN, C. and SCHRAMM, O. (2013). The scaling limits of the Minimal Spanning Tree and Invasion Percolation in the plane. Preprint. Available at http://arxiv.org/abs/1309.0269.
[52] RINOTT, Y. and ROTAR, V. (1997). On coupling constructions and rates in the CLT for dependent summands with applications to the antivoter model and weighted U-statistics. *Ann. Appl. Probab.* **7** 1080–1105. MR1484798

[53] ROY, R. (1990). The Russo–Seymour–Welsh theorem and the equality of critical densities and the "dual" critical densities for continuum percolation on $\mathbb{R}^2$. *Ann*. *Probab*. **18** 1563–1575. MR1071809

[54] SMIRNOV, S. and WERNER, W. (2001). Critical exponents for two-dimensional percolation. *Math*. *Res*. *Lett*. **8** 729–744. MR1879816

[55] STEELE, J. M. (1981). Subadditive Euclidean functionals and nonlinear growth in geometric probability. *Ann*. *Probab*. **9** 365–376. MR0626571

[56] STEELE, J. M. (1987). On Frieze's $\zeta(3)$ limit for lengths of minimal spanning trees. *Discrete Appl*. *Math*. **18** 99–103. MR0905183

[57] STEELE, J. M. (1988). Growth rates of Euclidean minimal spanning trees with power weighted edges. *Ann*. *Probab*. **16** 1767–1787. MR0958215

[58] STEIN, C. (1972). A bound for the error in the normal approximation to the distribution of a sum of dependent random variables. In *Proc. of the Sixth Berkeley Sympos. Math. Statist. Probab., Vol. II: Probability Theory* 583–602. Univ. California Press, Berkeley, CA. MR0402873

[59] STEIN, C. (1986). *Approximate Computation of Expectations*. *Institute of Mathematical Statistics Lecture Notes—Monograph Series* **7**. IMS, Hayward, CA. MR0882007

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THE SHAPE OF THE ONE-DIMENSIONAL PHYLOGENETIC LIKELIHOOD FUNCTION[1]

BY VU DINH AND FREDERICK A. MATSEN IV

Fred Hutchinson Cancer Research Center

By fixing all parameters in a phylogenetic likelihood model except for one branch length, one obtains a one-dimensional likelihood function. In this work, we introduce a mathematical framework to characterize the shapes of such one-dimensional phylogenetic likelihood functions. This framework is based on analyses of algebraic structures on the space of all frequency patterns with respect to a polynomial representation of the likelihood functions. Using this framework, we provide conditions under which the one-dimensional phylogenetic likelihood functions are guaranteed to have at most one stationary point, and this point is the maximum likelihood branch length. These conditions are satisfied by common simple models including all binary models, the Jukes–Cantor model and the Felsenstein 1981 model.

We then prove that for the simplest model that does not satisfy our conditions, namely, the Kimura 2-parameter model, the one-dimensional likelihood functions may have multiple stationary points. As a proof of concept, we construct a nondegenerate example in which the phylogenetic likelihood function has two local maxima and a local minimum. To construct such examples, we derive a general method of constructing a tree and sequence data with a specified frequency pattern at the root. We then extend the result to prove that the space of all rescaled and translated one-dimensional phylogenetic likelihood functions under the Kimura 2-parameter model is dense in the space of all nonnegative continuous functions on $[0, \infty)$ with finite limits. These results indicate that one-dimensional likelihood functions under advanced evolutionary models can be more complex than it is typically assumed by phylogenetic inference algorithms; however, these complexities can be effectively captured by the Kimura 2-parameter model.

REFERENCES

[1] BEN-ISRAEL, A. and MOND, B. (1986). What is invexity? *J. Aust. Math. Soc. A* **28** 1–9. MR0846778

[2] BRYANT, D., GALTIER, N. and POURSAT, M.-A. (2005). Likelihood calculation in molecular phylogenetics. In *Mathematics of Evolution and Phylogeny* 33–62.

[3] CHOR, B., HENDY, M. D., HOLLAND, B. R. and PENNY, D. (2000). Multiple maxima of likelihood in phylogenetic trees: An analytic approach. *Mol. Biol. Evol.* **17** 1529–1541.

[4] FAROUKI, R. T. (2012). The Bernstein polynomial basis: A centennial retrospective. *Comput. Aided Geom. Design* **29** 379–419. MR2921860

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[5] FELSENSTEIN, J. (1981). Evolutionary trees from DNA sequences: A maximum likelihood approach. *J. Mol. Evol.* **17** 368–376.
[6] FELSENSTEIN, J. (2004). *Inferring Phylogenies*. Sinauer Associates, Sunderland.
[7] FUKAMI, K. and TATENO, Y. (1989). On the maximum likelihood method for estimating molecular trees: Uniqueness of the likelihood point. *J. Mol. Evol.* **28** 460–464.
[8] HANSON, M. A. (1981). On sufficiency of the Kuhn–Tucker conditions. *J. Math. Anal. Appl.* **80** 545–550. MR0614849
[9] HAYASHIDA, T. (1949). Arc-wise connected subgroup of a vector group. *Kodai Math. Semin. Rep.* **1** 16–16.
[10] JUKES, T. H. and CANTOR, C. R. (1969). Evolution of protein molecules. *Mammalian Protein Metabolism* **3** 21–132.
[11] KIMURA, M. (1980). A simple method for estimating evolutionary rates of base substitutions through comparative studies of nucleotide sequences. *J. Mol. Evol.* **16** 111–120.
[12] ROGERS, J. S. and SWOFFORD, D. L. (1999). Multiple local maxima for likelihoods of phylogenetic trees: A simulation study. *Mol. Biol. Evol.* **16** 1079–1085.
[13] STEEL, M. (1994). The maximum likelihood point for a phylogenetic tree is not unique. *Syst. Biol.* **43** 560–564.

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ON THE CAPACITY FUNCTIONAL OF THE INFINITE CLUSTER OF A BOOLEAN MODEL

BY GÜNTER LAST[*,1], MATHEW D. PENROSE[†,1] AND SERGEI ZUYEV[‡,§]

Karlsruhe Institute of Technology[*], *University of Bath*[†], *Chalmers University of Technology*[‡] *and University of Gothenburg*[§]

Consider a Boolean model in $\mathbb{R}^d$ with balls of random, bounded radii with distribution F_0, centered at the points of a Poisson process of intensity $t > 0$. The capacity functional of the infinite cluster Z_∞ is given by $\theta_L(t) = \mathbb{P}\{Z_\infty \cap L \neq \varnothing\}$, defined for each compact $L \subset \mathbb{R}^d$.

We prove for any fixed L and F_0 that $\theta_L(t)$ is infinitely differentiable in t, except at the critical value t_c; we give a Margulis–Russo-type formula for the derivatives. More generally, allowing the distribution F_0 to vary and viewing θ_L as a function of the measure $F := tF_0$, we show that it is infinitely differentiable in all directions with respect to the measure F in the supercritical region of the cone of positive measures on a bounded interval.

We also prove that $\theta_L(\cdot)$ grows at least linearly at the critical value. This implies that the critical exponent known as β is at most 1 (if it exists) for this model. Along the way, we extend a result of Tanemura [*J. Appl. Probab.* **30** (1993) 382–396], on regularity of the supercritical Boolean model in $d \geq 3$ with fixed-radius balls, to the case with bounded random radii.

REFERENCES

BOLLOBÁS, B. (1979). *Graph Theory*: *An Introductory Course*. Springer, New York. MR0536131

CHAYES, J. T. and CHAYES, L. (1986). Inequality for the infinite-cluster density in Bernoulli percolation. *Phys*. *Rev*. *Lett*. **56** 1619–1622. MR0864316

DUMINIL-COPIN, H. and TASSION, V. (2015). A new proof of sharpness of the phase transition for Bernoulli percolation on $\mathbb{Z}^d$. *Enseign*. *Math*. **62** 199–206. MR3605816

GOUÉRÉ, J. B. (2008). Subcritical regimes in the Poisson Boolean model of continuum percolation. *Ann*. *Probab*. **36** 1209–1220. MR2435847

GOUÉRÉ, J. B. and MARCHAND, R. (2011). Continuum percolation in high dimensions. Technical report, available at arXiv:1108.6133.

GRIMMETT, G. (1999). *Percolation*, 2nd ed. Springer, Berlin. MR1707339

GRIMMETT, G. R. and MARSTRAND, J. M. (1990). The supercritical phase of percolation is well behaved. *Proc*. *Roy*. *Soc*. *London Ser*. *A* **430** 439–457. MR1068308

GUPTA, J. and RAO, B. (1999). Van den Berg–Kesten inequality for the Poisson Boolean model for continuum percolation. *Sankhya*, *Ser*. *A* **61** 337–346.

HALL, P. (1988). *Introduction to the Theory of Coverage Processes*. Wiley, New York. MR0973404

JIANG, J., ZHANG, S. and GUO, T. (2011). Russo's formula, uniqueness of the infinite cluster, and continuous differentiability of free energy for continuum percolation. *J*. *Appl*. *Probab*. **48** 597–610.

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KALLENBERG, O. (2002). *Foundations of Modern Probability*, 2nd ed. Springer, New York. MR1876169

LAST, G. (2014). Perturbation analysis of Poisson processes. *Bernoulli* **20** 486–513. MR3178507

LAST, G. and PENROSE, M. (2017). *Lectures on the Poisson Process*. Cambridge Univ. Press, Cambridge. To appear.

MEESTER, R. and ROY, R. (1996). *Continuum Percolation*. Cambridge Univ. Press, Cambridge. MR1409145

MEESTER, R., ROY, R. and SARKAR, A. (1994). Nonuniversality and continuity of the critical covered volume fraction in continuum percolation. *J. Stat. Phys.* **75** 123–134. MR1273055

MOLCHANOV, I. (2005). *Theory of Random Sets*. Springer, London. MR2132405

MOLCHANOV, I. and ZUYEV, S. (2000). Variational analysis of functionals of Poisson processes. *Math. Oper. Res.* **25** 485–508.

PENROSE, M. (2003). *Random Geometric Graphs*. Oxford Univ. Press, London.

PENROSE, M. D. (2007). Gaussian limits for random geometric measures. *Electron. J. Probab.* **12** 989–1035. MR2336596

SCHNEIDER, R. and WEIL, W. (2008). *Stochastic and Integral Geometry*. Springer, Berlin. MR2455326

STOYAN, D., KENDALL, W. S. and MECKE, J. (1987). *Stochastic Geometry and Its Applications*. Wiley, Chichester. MR0895588

TANEMURA, H. (1993). Behavior of the supercritical phase of a continuum percolation model on $\mathbf{R}^d$. *J. Appl. Probab.* **30** 382–396. MR1212670

ZUEV, S. A. (1992). Russo's formula for the Poisson point processes and its applications. *Diskret. Mat.* **4** 149–160.

ZUEV, S. A. and SIDORENKO, A. F. (1985). Continuous models of percolation theory II. *Teotretich. i Matematich. Fizika* **62** 253–262.

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ON THE ROBUST DYNKIN GAME

BY ERHAN BAYRAKTAR[1] AND SONG YAO[2]

University of Michigan and University of Pittsburgh

We analyze a robust version of the Dynkin game over a set $\mathcal{P}$ of mutually singular probabilities. We first prove that conservative player's lower and upper value coincide (let us denote the value by V). Such a result connects the robust Dynkin game with second-order doubly reflected backward stochastic differential equations. Also, we show that the value process V is a submartingale under an appropriately defined nonlinear expectation $\underline{\mathscr{E}}$ up to the first time τ_* when V meets the lower payoff process L. If the probability set $\mathcal{P}$ is weakly compact, one can even find an optimal triplet $(\mathbb{P}_*, \tau_*, \gamma_*)$ for the value V_0.

The mutual singularity of probabilities in $\mathcal{P}$ causes major technical difficulties. To deal with them, we use some new methods including two approximations with respect to the set of stopping times.

REFERENCES

[1] ALARIO-NAZARET, M., LEPELTIER, J.-P. and MARCHAL, B. (1982). Dynkin games. In *Stochastic Differential Systems* (*Bad Honnef*, 1982). *Lect. Notes Control Inf. Sci.* **43** 23–32. Springer, Berlin. MR0814103

[2] ALVAREZ, L. H. R. (2008). A class of solvable stopping games. *Appl. Math. Optim.* **58** 291–314. MR2456849

[3] BAYRAKTAR, E., KARATZAS, I. and YAO, S. (2010). Optimal stopping for dynamic convex risk measures. *Illinois J. Math.* **54** 1025–1067. MR2928345

[4] BAYRAKTAR, E. and SÎRBU, M. (2014). Stochastic Perron's method and verification without smoothness using viscosity comparison: Obstacle problems and Dynkin games. *Proc. Amer. Math. Soc.* **142** 1399–1412. MR3162260

[5] BAYRAKTAR, E. and YAO, S. (2014). On the robust optimal stopping problem. *SIAM J. Control Optim.* **52** 3135–3175. MR3267150

[6] BAYRAKTAR, E. and YAO, S. (2015). Doubly reflected BSDEs with integrable parameters and related Dynkin games. *Stochastic Process. Appl.* **125** 4489–4542. MR3406594

[7] BAYRAKTAR, E. and YAO, S. (2015). Optimal stopping with random maturity under nonlinear expectations. Available at http://arxiv.org/abs/1505.07533.

[8] BENSOUSSAN, A. and FRIEDMAN, A. (1974). Nonlinear variational inequalities and differential games with stopping times. *J. Funct. Anal.* **16** 305–352. MR0354049

[9] BENSOUSSAN, A. and FRIEDMAN, A. (1977). Nonzero-sum stochastic differential games with stopping times and free boundary problems. *Trans. Amer. Math. Soc.* **231** 275–327. MR0453082

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[10] BISMUT, J.-M. (1977). Sur un problème de Dynkin. *Z. Wahrsch. Verw. Gebiete* **39** 31–53. MR0444218

[11] BISMUT, J.-M. (1979). Contrôle de processus alternants et applications. *Z. Wahrsch. Verw. Gebiete* **47** 241–288. MR0525309

[12] BOETIUS, F. (2005). Bounded variation singular stochastic control and Dynkin game. *SIAM J. Control Optim*. **44** 1289–1321 (electronic). MR2178033

[13] BUCKDAHN, R. and LI, J. (2009). Probabilistic interpretation for systems of Isaacs equations with two reflecting barriers. *NoDEA Nonlinear Differential Equations Appl*. **16** 381–420.

[14] CATTIAUX, P. and LEPELTIER, J.-P. (1990). Existence of a quasi-Markov Nash equilibrium for nonzero sum Markov stopping games. *Stoch. Stoch. Rep*. **30** 85–103. MR1054193

[15] COSSO, A. (2013). Stochastic differential games involving impulse controls and double-obstacle quasi-variational inequalities. *SIAM J. Control Optim*. **51** 2102–2131. MR3053571

[16] CVITANIĆ, J. and KARATZAS, I. (1996). Backward stochastic differential equations with reflection and Dynkin games. *Ann. Probab*. **24** 2024–2056. MR1415239

[17] DOLINSKY, Y. (2014). Hedging of game options under model uncertainty in discrete time. *Electron. Commun. Probab*. **19** no. 19, 11. MR3183572

[18] DYNKIN, E. (1967). Game variant of a problem on optimal stopping. *Sov. Math., Dokl*. **10** 270–274.

[19] EKREN, I., TOUZI, N. and ZHANG, J. (2014). Optimal stopping under nonlinear expectation. *Stochastic Process. Appl*. **124** 3277–3311. MR3231620

[20] EKREN, I. and ZHANG, J. (2016). Pseudo Markovian viscosity solutions of fully nonlinear degenerate PPDEs. Available at https://arxiv.org/abs/1604.02239.

[21] EKSTRÖM, E. (2006). Properties of game options. *Math. Methods Oper. Res*. **63** 221–238. MR2264747

[22] EL ASRI, B., HAMADÈNE, S. and WANG, H. (2011). L^p-solutions for doubly reflected backward stochastic differential equations. *Stoch. Anal. Appl*. **29** 907–932. MR2847329

[23] FRIEDMAN, A. (1973). Stochastic games and variational inequalities. *Arch. Ration. Mech. Anal*. **51** 321–346. MR0351571

[24] FUKUSHIMA, M. and TAKSAR, M. (2002). Dynkin games via Dirichlet forms and singular control of one-dimensional diffusions. *SIAM J. Control Optim*. **41** 682–699. MR1939866

[25] HAMADÈNE, S. (2006). Mixed zero-sum stochastic differential game and American game options. *SIAM J. Control Optim*. **45** 496–518. MR2246087

[26] HAMADÈNE, S. and HASSANI, M. (2005). BSDEs with two reflecting barriers: The general result. *Probab. Theory Related Fields* **132** 237–264. MR2199292

[27] HAMADÈNE, S. and HASSANI, M. (2014). The multi-player nonzero-sum Dynkin game in discrete time. *Math. Methods Oper. Res*. **79** 179–194. MR3191211

[28] HAMADÈNE, S. and HDHIRI, I. (2006). Backward stochastic differential equations with two distinct reflecting barriers and quadratic growth generator. *J. Appl. Math. Stoch. Anal*. Art. ID 95818, 28. MR2212582

[29] HAMADÈNE, S. and LEPELTIER, J.-P. (2000). Reflected BSDEs and mixed game problem. *Stochastic Process. Appl*. **85** 177–188. MR1731020

[30] HAMADÈNE, S., LEPELTIER, J.-P. and WU, Z. (1999). Infinite horizon reflected backward stochastic differential equations and applications in mixed control and game problems. *Probab. Math. Statist*. **19** 211–234. MR1750901

[31] HAMADÈNE, S. and MOHAMMED, H. (2014). The multiplayer nonzero-sum Dynkin game in continuous time. *SIAM J. Control Optim*. **52** 821–835. MR3176316

[32] HAMADÈNE, S., ROTENSTEIN, E. and ZĂLINESCU, A. (2009). A generalized mixed zero-sum stochastic differential game and double barrier reflected BSDEs with quadratic growth coefficient. *An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.)* **55** 419–444.

[33] HAMADÈNE, S. and ZHANG, J. (2009/10). The continuous time nonzero-sum Dynkin game problem and application in game options. *SIAM J. Control Optim.* **48** 3659–3669.

[34] KALLSEN, J. and KÜHN, C. (2004). Pricing derivatives of American and game type in incomplete markets. *Finance Stoch.* **8** 261–284. MR2048831

[35] KARATZAS, I. and SHREVE, S. E. (1991). *Brownian Motion and Stochastic Calculus*, 2nd ed. *Graduate Texts in Mathematics* **113**. Springer, New York. MR1121940

[36] KARATZAS, I. and WANG, H. (2001). Connections between bounded variation control and Dynkin games. In *Optimal Control and Partial Differential Equations* (*Volume in honor of A. Bensoussan*) (Menaldi, J. L., Rofman, E. and Sulem, A., eds.) 363–373. IOS Press, Amsterdam.

[37] KARATZAS, I. and ZAMFIRESCU, I. M. (2008). Martingale approach to stochastic differential games of control and stopping. *Ann. Probab.* **36** 1495–1527. MR2435857

[38] KIFER, Y. (2000). Game options. *Finance Stoch.* **4** 443–463. MR1779588

[39] KIFER, Y. (2013). Dynkin games and Israeli options. *ISRN Probability and Statistics* **2013** Article ID 856458, 17 pages.

[40] KOBYLANSKI, M., QUENEZ, M.-C. and DE CAMPAGNOLLE, M. R. (2014). Dynkin games in a general framework. *Stochastics* **86** 304–329. MR3180039

[41] LARAKI, R. and SOLAN, E. (2005). The value of zero-sum stopping games in continuous time. *SIAM J. Control Optim.* **43** 1913–1922 (electronic).

[42] LEPELTIER, J.-P. and MAINGUENEAU, M. A. (1984). Le jeu de Dynkin en théorie générale sans l'hypothèse de Mokobodski. *Stochastics* **13** 25–44. MR0752475

[43] MA, J. and CVITANIĆ, J. (2001). Reflected forward-backward SDEs and obstacle problems with boundary conditions. *J. Appl. Math. Stoch. Anal.* **14** 113–138. MR1838341

[44] MATOUSSI, A., PIOZIN, L. and POSSAMAÏ, D. (2014). Second-order BSDEs with general reflection and game options under uncertainty. *Stochastic Process. Appl.* **124** 2281–2321. MR3192498

[45] MORIMOTO, H. (1984). Dynkin games and martingale methods. *Stochastics* **13** 213–228. MR0762817

[46] NAGAI, H. (1987). Non-zero-sum stopping games of symmetric Markov processes. *Probab. Theory Related Fields* **75** 487–497. MR0894901

[47] NEVEU, J. (1975). *Discrete-Parameter Martingales*, Revised ed. *North-Holland Mathematical Library* **10**. North-Holland, Amsterdam. MR0402915

[48] NUTZ, M. (2012). A quasi-sure approach to the control of non-Markovian stochastic differential equations. *Electron. J. Probab.* **17** 1–23. MR2900464

[49] NUTZ, M. and ZHANG, J. (2015). Optimal stopping under adverse nonlinear expectation and related games. *Ann. Appl. Probab.* **25** 2503–2534. MR3375882

[50] OHTSUBO, Y. (1987). A nonzero-sum extension of Dynkin's stopping problem. *Math. Oper. Res.* **12** 277–296. MR0888977

[51] PENG, S. (2007). G-Brownian motion and dynamic risk measure under volatility uncertainty. Available at http://lanl.arxiv.org/abs/0711.2834.

[52] ROGERS, L. C. G. and WILLIAMS, D. (2000). *Diffusions, Markov Processes, and Martingales. Vol. 2: Itô Calculus*. Cambridge Univ. Press, Cambridge. Reprint of the second (1994) edition. MR1780932

[53] ROSENBERG, D., SOLAN, E. and VIEILLE, N. (2001). Stopping games with randomized strategies. *Probab. Theory Related Fields* **119** 433–451. MR1821142

[54] SNELL, J. L. (1952). Applications of martingale system theorems. *Trans. Amer. Math. Soc.* **73** 293–312. MR0050209

[55] SONER, H., TOUZI, N. and ZHANG, J. (2013). Dual formulation of second order target problems. *Ann. Appl. Probab.* **23** 308–347. MR3059237

[56] STETTNER, Ł. (1982/83). Zero-sum Markov games with stopping and impulsive strategies. *Appl. Math. Optim.* **9** 1–24. MR0674679

[57] STROOCK, D. W. and VARADHAN, S. R. S. (2006). *Multidimensional Diffusion Processes*. Springer, Berlin. Reprint of the 1997 edition. MR2190038

[58] TAKSAR, M. I. (1985). Average optimal singular control and a related stopping problem. *Math*. *Oper*. *Res*. **10** 63–81. MR0787007

[59] TOUZI, N. and VIEILLE, N. (2002). Continuous-time Dynkin games with mixed strategies. *SIAM J*. *Control Optim*. **41** 1073–1088. MR1972503

[60] XU, M. (2007). Reflected backward SDEs with two barriers under monotonicity and general increasing conditions. *J*. *Theoret*. *Probab*. **20** 1005–1039. MR2359066

[61] YASUDA, M. (1985). On a randomized strategy in Neveu's stopping problem. *Stochastic Process*. *Appl*. **21** 159–166. MR0834995

[62] YIN, J. (2012). Reflected backward stochastic differential equations with two barriers and Dynkin games under Knightian uncertainty. *Bull*. *Sci*. *Math*. **136** 709–729. MR2959781

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EXTENDED CONVERGENCE OF THE EXTREMAL PROCESS OF BRANCHING BROWNIAN MOTION

BY ANTON BOVIER[1,*] AND LISA HARTUNG[2,*,†]

Bonn University and New York University†*

We extend the results of Arguin et al. [*Probab. Theory Related Fields* **157** (2013) 535–574] and Aïdékon et al. [*Probab. Theory Related Fields* **157** (2013) 405–451] on the convergence of the extremal process of branching Brownian motion by adding an extra dimension that encodes the "location" of the particle in the underlying Galton–Watson tree. We show that the limit is a cluster point process on $\mathbb{R}_+ \times \mathbb{R}$ where each cluster is the atom of a Poisson point process on $\mathbb{R}_+ \times \mathbb{R}$ with a random intensity measure $Z(dz) \times C\mathrm{e}^{-\sqrt{2}x}\,dx$, where the random measure is explicitly constructed from the derivative martingale. This work is motivated by an analogous result for the Gaussian free field by Biskup and Louidor [Full extremal process, cluster law and freezing for two-dimensional discrete Gaussian free field (2016)].

REFERENCES

[1] AÏDÉKON, E., BERESTYCKI, J., BRUNET, É. and SHI, Z. (2013). Branching Brownian motion seen from its tip. *Probab. Theory Related Fields* **157** 405–451. MR3101852

[2] ARGUIN, L.-P., BOVIER, A. and KISTLER, N. (2011). Genealogy of extremal particles of branching Brownian motion. *Comm. Pure Appl. Math.* **64** 1647–1676. MR2838339

[3] ARGUIN, L.-P., BOVIER, A. and KISTLER, N. (2012). Poissonian statistics in the extremal process of branching Brownian motion. *Ann. Appl. Probab.* **22** 1693–1711. MR2985174

[4] ARGUIN, L.-P., BOVIER, A. and KISTLER, N. (2013). The extremal process of branching Brownian motion. *Probab. Theory Related Fields* **157** 535–574. MR3129797

[5] ATHREYA, K. B. and NEY, P. E. (1972). *Branching Processes. Die Grundlehren der Mathematischen Wissenschaften* Springer, New York. MR0373040

[6] BARRAL, J., KUPIAINEN, A., NIKULA, M., SAKSMAN, E. and WEBB, C. (2014). Critical Mandelbrot cascades. *Comm. Math. Phys.* **325** 685–711. MR3148099

[7] BARRAL, J., RHODES, R. and VARGAS, V. (2012). Limiting laws of supercritical branching random walks. *C. R. Math. Acad. Sci. Paris* **350** 535–538. MR2929063

[8] BISKUP, M. and LOUIDOR, O. (2014). Conformal symmetries in the extremal process of two-dimensional discrete Gaussian free field. Available at arXiv:1410.4676.

[9] BISKUP, M. and LOUIDOR, O. (2016). Extreme local extrema of two-dimensional discrete Gaussian free field. *Comm. Math. Phys.* **345** 271–304. MR3509015

[10] BISKUP, M. and LOUIDOR, O. (2016). Full extremal process, cluster law and freezing for two-dimensional discrete Gaussian free field. Preprint. Available at arXiv:1606.00510.

[11] BOVIER, A. (2016). *Gaussian Processes on Trees: From Spin-Glasses to Branching Brownian Motion. Cambridge Studies in Advanced Mathematics* **163**. Cambridge Univ. Press, Cambridge.

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[12] BRAMSON, M., DING, J. and ZEITOUNI, O. (2016). Convergence in law of the maximum of the two-dimensional discrete Gaussian free field. *Comm. Pure Appl. Math.* **69** 62–123. MR3433630

[13] BRAMSON, M. D. (1978). Maximal displacement of branching Brownian motion. *Comm. Pure Appl. Math.* **31** 531–581. MR0494541

[14] COX, D. R. (1955). Some statistical methods connected with series of events. *J. R. Stat. Soc. Ser. B. Stat. Methodol.* **17** 129–157. MR0092301

[15] DUPLANTIER, B., RHODES, R., SHEFFIELD, S. and VARGAS, V. (2014). Critical Gaussian multiplicative chaos: Convergence of the derivative martingale. *Ann. Probab.* **42** 1769–1808. MR3262492

[16] DUPLANTIER, B., RHODES, R., SHEFFIELD, S. and VARGAS, V. (2014). Renormalization of critical Gaussian multiplicative chaos and KPZ relation. *Comm. Math. Phys.* **330** 283–330. MR3215583

[17] HARDY, R. and HARRIS, S. C. (2006). A conceptual approach to a path result for branching Brownian motion. *Stochastic Process. Appl.* **116** 1992–2013. MR2307069

[18] HARDY, R. and HARRIS, S. C. (2009). A spine approach to branching diffusions with applications to L^p-convergence of martingales. In *Séminaire de Probabilités XLII*. *Lecture Notes in Math.* **1979** 281–330. Springer, Berlin. MR2599214

[19] KAHANE, J.-P. (1985). Sur le chaos multiplicatif. *Ann. Sci. Math. Québec* **9** 105–150. MR0829798

[20] KINGMAN, J. F. C. (1993). *Poisson Processes*. *Oxford Studies in Probability* **3**. Clarendon Press, New York. MR1207584

[21] LALLEY, S. P. and SELLKE, T. (1987). A conditional limit theorem for the frontier of a branching Brownian motion. *Ann. Probab.* **15** 1052–1061. MR0893913

[22] LEADBETTER, M., LINDGREN, G. and ROOTZÉN, H. (1983). *Extremes and Related Properties of Random Sequences and Processes*. Springer, New York. MR0691492

[23] MADAULE, T. (2011). Convergence in law for the branching random walk seen from its tip. Preprint. Available at arXiv:1107.2543.

[24] RHODES, R. and VARGAS, V. (2014). Gaussian multiplicative chaos and applications: A review. *Probab. Surv.* **11** 315–392. MR3274356

[25] SHI, Z. (2015). *Branching Random Walks*. *Lecture Notes in Math.* **2151**. Springer, Cham. MR3444654

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THE PRICING OF CONTINGENT CLAIMS AND OPTIMAL POSITIONS IN ASYMPTOTICALLY COMPLETE MARKETS

BY MICHAIL ANTHROPELOS[1,*],
SCOTT ROBERTSON[2,†] AND KONSTANTINOS SPILIOPOULOS[3,†]

University of Piraeus and Boston University†*

We study utility indifference prices and optimal purchasing quantities for a contingent claim, in an incomplete semimartingale market, in the presence of vanishing hedging errors and/or risk aversion. Assuming that the average indifference price converges to a well-defined limit, we prove that optimally taken positions become large in absolute value at a specific rate. We draw motivation from and make connections to large deviations theory, and in particular, the celebrated Gärtner–Ellis theorem. We analyze a series of well studied examples where this limiting behavior occurs, such as fixed markets with vanishing risk aversion, the basis risk model with high correlation, models of large markets with vanishing trading restrictions and the Black–Scholes–Merton model with either vanishing default probabilities or vanishing transaction costs. Lastly, we show that the large claim regime could naturally arise in partial equilibrium models.

REFERENCES

[1] ANTHROPELOS, M. and ŽITKOVIĆ, G. (2010). On agent's agreement and partial-equilibrium pricing in incomplete markets. *Math. Finance* **20** 411–446. MR2667897

[2] ATKESON, G. A., EISFELDT, L. A. and WEILL, P.-O. (2013). The market of OTC derivatives. *National Bureau of Economics Reseach*. To appear.

[3] BARLES, G. and PERTHAME, B. (1988). Exit time problems in optimal control and vanishing viscosity method. *SIAM J. Control Optim.* **26** 1133–1148. MR0957658

[4] BARLES, G. and SONER, H. M. (1998). Option pricing with transaction costs and a nonlinear Black–Scholes equation. *Finance Stoch.* **2** 369–397. MR1809526

[5] BARRIEU, P. and EL KAROUI, N. (2005). Inf-convolution of risk measures and optimal risk transfer. *Finance Stoch.* **9** 269–298. MR2211128

[6] BICHUCH, M. (2012). Asymptotic analysis for optimal investment in finite time with transaction costs. *SIAM J. Financial Math.* **3** 433–458. MR2968041

[7] BIS (2014). Amounts outstanding of over-the-counter derivatives by risk category and instrument, Bank for International Settlements (BIS). Available at http://www.bis.org/about/index.htm..

[8] BOUCHARD, B., ELIE, R. and MOREAU, L. (2012). A note on utility based pricing and asymptotic risk diversification. *Math. Financ. Econ.* **6** 59–74. MR2924152

[9] CARMONA, R. (2009). *Indifference Pricing.Theory and Applications*. Princeton Univ. Press, Princeton, NJ. MR2547456

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[10] CONSTANTINIDES, G. M. (1986). Capital market equilibrium with transaction costs. *J. Polit. Econ*. 842–862.

[11] CZICHOWSKY, C. and SCHACHERMAYER, W. (2016). Duality theory for portfolio optimisation under transaction costs. *Ann. Appl. Probab.* **26** 1888–1941. MR3513609

[12] DAVIS, M. (1997). Option pricing in incomplete markets. In *Mathematics of Derivative Securities* 216-226. MR1491376

[13] DAVIS, M. H. A., PANAS, V. G. and ZARIPHOPOULOU, T. (1993). European option pricing with transaction costs. *SIAM J. Control Optim*. **31** 470–493. MR1205985

[14] DELBAEN, F., GRANDITS, P., RHEINLÄNDER, T., SAMPERI, D., SCHWEIZER, M. and STRICKER, C. (2002). Exponential hedging and entropic penalties. *Math. Finance* **12** 99–123. MR1891730

[15] DEMBO, A. and ZEITOUNI, O. (1998). *Large Deviations Techniques and Applications*, 2nd ed. *Applications of Mathematics* (*New York*) **38**. Springer, New York. MR1619036

[16] FLEMING, W. H. and SONER, H. M. (2006). *Controlled Markov Processes and Viscosity Solutions*, 2nd ed. *Stochastic Modelling and Applied Probability* **25**. Springer, New York. MR2179357

[17] FÖLLMER, H. and SCHIED, A. (2004). *Stochastic Finance*, extended ed. *de Gruyter Studies in Mathematics* **27**. de Gruyter, Berlin. MR2169807

[18] FRITTELLI, M. (2000). The minimal entropy martingale measure and the valuation problem in incomplete markets. *Math. Finance* **10** 39–52. MR1743972

[19] GRANDITS, P. and RHEINLÄNDER, T. (2002). On the minimal entropy martingale measure. *Ann. Probab.* **30** 1003–1038. MR1920099

[20] GUASONI, P. and MUHLE-KARBE, J. (2013). Portfolio choice with transaction costs: A user's guide. In *Paris-Princeton Lectures on Mathematical Finance* 2013. *Lecture Notes in Math*. **2081** 169–201. Springer, Cham. MR3183924

[21] HENDERSON, V. (2002). Valuation of claims on nontraded assets using utility maximization. *Math. Finance* **12** 351–373. MR1926237

[22] HODGES, S. D. and NEUBERGER, A. (1989). Optimal replication of contingent claims under transactions costs. *Review of Futures Markets* **8** 222–239.

[23] HYND, R. (2014). Option pricing in the large risk aversion, small transaction cost limit. *Comm. Partial Differential Equations* **39** 1998–2027. MR3251862

[24] İLHAN, A., JONSSON, M. and SIRCAR, R. (2005). Optimal investment with derivative securities. *Finance Stoch*. **9** 585–595. MR2212897

[25] ISHIKAWA, T. and ROBERTSON, S. (2015). Contingent claim pricing in markets with defaultable assets. Working Paper.

[26] KABANOV, Y. M. and STRICKER, C. (2002). On the optimal portfolio for the exponential utility maximization: Remarks to the six-author paper "Exponential hedging and entropic penalties" [Math. Finance **12** (2002), no. 2, 99–123; MR1891730 (2003b:91046)] by F. Delbaen, P. Grandits, T. Rheinländer, D. Samperi, M. Schweizer and C. Stricker. *Math. Finance* **12** 125–134. MR1891731

[27] KALLSEN, J. and MUHLE-KARBE, J. (2010). On using shadow prices in portfolio optimization with transaction costs. *Ann. Appl. Probab.* **20** 1341–1358. MR2676941

[28] KALLSEN, J. and MUHLE-KARBE, J. (2015). Option pricing and hedging with small transaction costs. *Math. Finance* **25** 702–723. MR3402459

[29] LEE, Y. and RHEINLÄNDER, T. (2012). Optimal martingale measures for defaultable assets. *Stochastic Process. Appl.* **122** 2870–2884. MR2931345

[30] MANIA, M. and SCHWEIZER, M. (2005). Dynamic exponential utility indifference valuation. *Ann. Appl. Probab.* **15** 2113–2143. MR2152255

[31] MUSIELA, M. and ZARIPHOPOULOU, T. (2004). An example of indifference prices under exponential preferences. *Finance Stoch*. **8** 229–239. MR2048829

[32] OWEN, M. P. and ŽITKOVIĆ, G. (2009). Optimal investment with an unbounded random endowment and utility-based pricing. *Math*. *Finance* **19** 129–159. MR2489605
[33] ROBERTSON, S. (2012). Pricing for large positions in contingent claims. *Math*. *Finance*. To appear.
[34] ROBERTSON, S. and SPILIOPOULOS, K. Indifference pricing for contingent claims: Large deviations effects. *Mathematical Finance*. To appear.
[35] SHREVE, S. E. and SONER, H. M. (1994). Optimal investment and consumption with transaction costs. *Ann*. *Appl*. *Probab*. **4** 609–692. MR1284980
[36] SIORPAES, P. (2015). Optimal investment and price dependence in a semi-static market. *Finance Stoch*. **19** 161–187. MR3292128
[37] STOIKOV, S. and ZARIPHOPOULOU, T. (2005). Optimal investments in the presence of unhedgeable risks and under cara preferences. *IMA*. To appear.
[38] TEHRANCHI, M. (2004). Explicit solutions of some utility maximization problems in incomplete markets. *Stochastic Process*. *Appl*. **114** 109–125. MR2094149
[39] VAN DER VAART, (1998). Asymptotic statistics. In *Cambridge Series in Statistical and Probabilistic Mathematics*. Cambridge Univ. Press, New York. MR1652247

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CONTINUOUS INVENTORY MODELS OF DIFFUSION TYPE: LONG-TERM AVERAGE COST CRITERION[1]

BY KURT L. HELMES*, RICHARD H. STOCKBRIDGE† AND CHAO ZHU†

Humboldt University of Berlin and University of Wisconsin–Milwaukee†*

This paper establishes conditions for optimality of an (s, S) ordering policy for the minimization of the long-term average cost of one-dimensional diffusion inventory models. The class of such models under consideration have general drift and diffusion coefficients and boundary points that are consistent with the notion that demand should tend to reduce the inventory level. Characterization of the cost of a general (s, S) policy as a function F of two variables naturally leads to a nonlinear optimization problem over the ordering levels s and S. Existence of an optimizing pair (s_*, S_*) is established for these models. Using the minimal value F_* of F, along with (s_*, S_*), a function G is identified which is proven to be a solution of a quasi-variational inequality provided a simple condition holds. At this level of generality, optimality of the (s_*, S_*) ordering policy is established within a large class of ordering policies for which local martingale and transversality conditions involving G hold. For specific models, optimality of an (s, S) policy in the general class of admissible policies can be established using comparison results. This most general optimality result is shown for the classical drifted Brownian motion inventory model with holding and fixed plus proportional ordering costs. Optimality of an (s, S) ordering policy is also extended to the general class of admissible policies for a geometric Brownian motion inventory model with fixed plus level-dependent ordering costs. However, for a drifted Brownian motion process with reflection at $\{0\}$, a new class of non-Markovian policies is introduced which have lower costs than the (s, S) policies. In addition, interpreting reflection at $\{0\}$ as "just-in-time" ordering, a necessary and sufficient condition is given that determines when just-in-time ordering is better than traditional (s, S) policies.

REFERENCES

ARROW, K. J., KARLIN, S. and SCARF, H. (1958). *Studies in the Mathematical Theory of Inventory and Production*. *Stanford Mathematical Studies in the Social Sciences*, *I*. Stanford Univ. Press, Stanford, Calif. MR0094248

BAKKER, M., RIEZEBOS, J. and TEUNTER, R. H. (2012). Review of inventory systems with deterioration since 2001. *European J. Oper. Res.* **221** 275–284. MR2923587

BARON, O., BERMAN, O. and PERRY, D. (2011). Shelf space management when demand depends on the inventory level. *Prod. Oper. Manag.* **20** 714–726.

BATHER, J. A. (1966). A continuous time inventory model. *J. Appl. Probab.* **3** 538–549. MR0213148

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BENKHEROUF, L. and BENSOUSSAN, A. (2009). Optimality of an (s, S) policy with compound Poisson and diffusion demands: A quasi-variational inequalities approach. *SIAM J. Control Optim.* **48** 756–762. MR2486092

BENSOUSSAN, A. (2011). *Dynamic Programming and Inventory Control*. *Studies in Probability, Optimization and Statistics* **3**. IOS Press, Amsterdam. MR2883170

BENSOUSSAN, A., LIU, R. H. and SETHI, S. P. (2005). Optimality of an (s, S) policy with compound Poisson and diffusion demands: A quasi-variational inequalities approach. *SIAM J. Control Optim.* **44** 1650–1676. MR2193500

BENSOUSSAN, A., MOUSSAWI-HAIDAR, L. and ÇAKANYILDIRIM, M. (2010). Inventory control with an order-time constraint: Optimality, uniqueness and significance. *Ann. Oper. Res.* **181** 603–640. MR2741011

BENSOUSSAN, A., ÇAKANYILDIRIM, M., LI, M. and SETHI, S. P. (2013). Optimal inventory control with shrinkage and observed sales. *Stochastics* **85** 589–603. MR3176476

BERMAN, O. and PERRY, D. (2006). An EOQ model with state-dependent demand rate. *European J. Oper. Res.* **171** 255–272. MR2181232

BORODIN, A. N. and SALMINEN, P. (2002). *Handbook of Brownian Motion—Facts and Formulae*, 2nd ed. *Probability and Its Applications*. Birkhäuser, Basel. MR1912205

CADENILLAS, A., LAKNER, P. and PINEDO, M. (2010). Optimal control of a mean-reverting inventory. *Oper. Res.* **58** 1697–1710. MR2752714

DAI, J. G. and YAO, D. (2013a). Brownian inventory models with convex holding cost, Part 1: Average-optimal controls. *Stoch. Syst.* **3** 442–499. MR3353209

DAI, J. G. and YAO, D. (2013b). Brownian inventory models with convex holding cost, Part 2: Discount-optimal controls. *Stoch. Syst.* **3** 500–573. MR3353210

FLEISCHMANN, M. and KUIK, R. (2003). On optimal inventory control with independent stochastic item returns. *European J. Oper. Res.* **151** 25–37. MR1997006

GIRLICH, H.-J. and CHIKÁN, A. (2001). The origins of dynamic inventory modelling under uncertainty: (the men, their work and connection with the Stanford studies). *Int. J. Prod. Econ.* **71** 351–363.

HARRISON, J. M. (1985). *Brownian Motion and Stochastic Flow Systems*. *Wiley Series in Probability and Mathematical Statistics*: *Probability and Mathematical Statistics*. Wiley, New York. MR0798279

HARRISON, J. M., SELLKE, T. M. and TAYLOR, A. J. (1983). Impulse control of Brownian motion. *Math. Oper. Res.* **8** 454–466. MR0716124

HE, S., YAO, D. and ZHANG, H. (2015). Optimal ordering policy for inventory systems with quantity-dependent setup costs. ArXiv E-prints.

HELMES, K. L., STOCKBRIDGE, R. H. and ZHU, C. (2015). A measure approach for continuous inventory models: Discounted cost criterion. *SIAM J. Control Optim.* **53** 2100–2140. MR3374634

JAIN, S., KUMAR, M. and ADVANI, P. (2008). An inventory model with inventory level-dependent demand rate, deterioration, partial backlogging and decrease in demand. *Int. J. Oper. Res.* **5** 154–159.

KARLIN, S. and TAYLOR, H. M. (1981). *A Second Course in Stochastic Processes*. Academic Press, New York. MR0611513

KURTZ, T. G. and STOCKBRIDGE, R. H. (2001). Stationary solutions and forward equations for controlled and singular martingale problems. *Electron. J. Probab.* **6** no. 17, 52. MR1873294

MANDL, P. (1968). *Analytical Treatment of One-Dimensional Markov Processes*. *Die Grundlehren der Mathematischen Wissenschaften*, *Band* 151. Academia Publishing House of the Czechoslovak Academy of Sciences, Prague. MR0247667

PRESMAN, E. and SETHI, S. P. (2006). Inventory models with continuous and Poisson demands and discounted and average costs. *Prod. Oper. Manag.* **15** 279–293.

SULEM, A. (1986). A solvable one-dimensional model of a diffusion inventory system. *Math. Oper. Res.* **11** 125–133. MR0830113

URBAN, T. L. (2005). Inventory models with inventory-level-dependent demand: A comprehensive review and unifying theory. *European J. Oper. Res.* **162** 792–804.

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ONE-DIMENSIONAL LONG-RANGE DIFFUSION LIMITED AGGREGATION II: THE TRANSIENT CASE

BY GIDEON AMIR[1], OMER ANGEL[2] AND GADY KOZMA[1]

Bar Ilan University, University of British Columbia and The Weizmann Institute of Science

We examine diffusion-limited aggregation for a one-dimensional random walk with long jumps. We achieve upper and lower bounds on the growth rate of the aggregate as a function of the number of moments a single step of the walk has. In this paper, we handle the case of transient walks.

REFERENCES

[1] AMIR, G. (2009). One-dimensional long-range diffusion-limited aggregation III—The limit aggregate. *Ann. Inst. Henri Poincaré*. To appear. Available at arXiv:0911.0122.
[2] AMIR, G., ANGEL, O., BENJAMINI, I. and KOZMA, G. (2016). One-dimensional long-range diffusion-limited aggregation I. *Ann. Probab.* **44** 3546–3579. MR3551204
[3] BASS, R. F. and LEVIN, D. A. (2002). Transition probabilities for symmetric jump processes. *Trans. Amer. Math. Soc.* **354** 2933–2953. MR1895210
[4] BERTILSSON, D. (1999). On Brennan's Conjecture in Conformal Mapping. Ph.D. thesis, Dept. Mathematics, Royal Institute of Technology, Stockholm, Sweden. MR1728890
[5] CARLESON, L. (1985). On the support of harmonic measure for sets of Cantor type. *Ann. Acad. Sci. Fenn. Ser. A I Math.* **10** 113–123. MR0802473
[6] CHOQUET, G. (1953–1954). Theory of capacities. *Ann. Inst. Fourier* (*Grenoble*) **5** 131–295. MR0080760
[7] FITZSIMMONS, P. J. (1999). Markov processes with equal capacities. *J. Theoret. Probab.* **12** 271–292. MR1675005
[8] HORN, R. A. and JOHNSON, C. R. (1990). *Matrix Analysis. Corrected Reprint of the* 1985 *Original*. Cambridge Univ. Press, Cambridge. MR1084815
[9] JONES, P. W. and WOLFF, T. H. (1988). Hausdorff dimension of harmonic measures in the plane. *Acta Math.* **161** 131–144. MR0962097
[10] LE GALL, J.-F. and ROSEN, J. (1991). The range of stable random walks. *Ann. Probab.* **19** 650–705. MR1106281
[11] MAKAROV, N. G. (1989). Вероятностные методы втеории конформных отображений. [Russian: Probability methods in the theory of conformal mappings.] *Algebra i Analiz* **1** 3–59. English translation in *Leningrad Math. J.* **1** (1990) 1–56.
[12] MAKAROV, N. G. (1998). Fine structure of harmonic measure. *Algebra i Analiz* **10** 1–62.
[13] MEAKIN, P. (1998). *Fractals, Scaling and Growth Far from Equilibrium. Cambridge Nonlinear Science Series* **5**. Cambridge Univ. Press, Cambridge. MR1489739
[14] SPITZER, F. (1976). *Principles of Random Walks*, 2nd edn. *Graduate Texts in Mathematics* **34**. Springer, New York. MR0388547
[15] WILLIAMSON, J. A. (1968). Random walks and Riesz kernels. *Pacific J. Math.* **25** 393–415. MR0226741

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WRIGHT–FISHER CONSTRUCTION OF THE TWO-PARAMETER POISSON–DIRICHLET DIFFUSION

BY CRISTINA COSTANTINI[*,1], PIERPAOLO DE BLASI[†,2,3], STEWART N. ETHIER[‡,4], MATTEO RUGGIERO[†,2,3] AND DARIO SPANÒ[§]

University of Chieti-Pescara[*], *University of Torino*[†], *University of Utah*[‡] *and University of Warwick*[§]

The two-parameter Poisson–Dirichlet diffusion, introduced in 2009 by Petrov, extends the infinitely-many-neutral-alleles diffusion model, related to Kingman's one-parameter Poisson–Dirichlet distribution and to certain Fleming–Viot processes. The additional parameter has been shown to regulate the clustering structure of the population, but is yet to be fully understood in the way it governs the reproductive process. Here, we shed some light on these dynamics by formulating a K-allele Wright–Fisher model for a population of size N, involving a uniform mutation pattern and a specific state-dependent migration mechanism. Suitably scaled, this process converges in distribution to a K-dimensional diffusion process as $N \to \infty$. Moreover, the descending order statistics of the K-dimensional diffusion converge in distribution to the two-parameter Poisson–Dirichlet diffusion as $K \to \infty$. The choice of the migration mechanism depends on a delicate balance between reinforcement and redistributive effects. The proof of convergence to the infinite-dimensional diffusion is nontrivial because the generators do not converge on a core. Our strategy for overcoming this complication is to prove a priori that in the limit there is no "loss of mass", that is, that, for each limit point of the sequence of finite-dimensional diffusions (after a reordering of components by size), allele frequencies sum to one.

REFERENCES

BERTOIN, J. (2006). *Random Fragmentation and Coagulation Processes*. *Cambridge Studies in Advanced Mathematics* **102**. Cambridge Univ. Press, Cambridge. MR2253162

BILLINGSLEY, P. (1968). *Convergence of Probability Measures*. Wiley, New York. MR0233396

COSTANTINI, C. and KURTZ, T. G. (2015). Viscosity methods giving uniqueness for martingale problems. *Electron*. *J*. *Probab*. **20** no. 67, 1–27. MR3361255

ETHIER, S. N. (1976). A class of degenerate diffusion processes occurring in population genetics. *Comm*. *Pure Appl*. *Math*. **29** 483–493. MR0428488

ETHIER, S. N. (2014). A property of Petrov's diffusion. *Electron*. *Commun*. *Probab*. **19** no. 65, 1–4. MR3262071

ETHIER, S. N. and KURTZ, T. G. (1981). The infinitely-many-neutral-alleles diffusion model. *Adv*. *in Appl*. *Probab*. **13** 429–452. MR0615945

ETHIER, S. N. and KURTZ, T. G. (1986). *Markov Processes*: *Characterization and Convergence*. Wiley, New York. MR0838085

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ETHIER, S. N. and KURTZ, T. G. (1993). Fleming–Viot processes in population genetics. *SIAM J. Control Optim*. **31** 345–386. MR1205982

FENG, S. (2010). *The Poisson–Dirichlet Distribution and Related Topics*. Springer, Heidelberg. MR2663265

FENG, S. and SUN, W. (2010). Some diffusion processes associated with two parameter Poisson–Dirichlet distribution and Dirichlet process. *Probab. Theory Related Fields* **148** 501–525. MR2678897

FENG, S., SUN, W., WANG, F.-Y. and XU, F. (2011). Functional inequalities for the two-parameter extension of the infinitely-many-neutral-alleles diffusion. *J. Funct. Anal*. **260** 399–413. MR2737405

KINGMAN, J. F. C. (1975). Random discrete distributions. *J. R. Stat. Soc. Ser. B* **37** 1–22. MR0368264

LIJOI, A. and PRÜNSTER, I. (2010). Models beyond the Dirichlet process. In *Bayesian Nonparametrics* (N. L. Hjort, C. C. Holmes, P. Müller and S. G. Walker, eds.) 80–136. Cambridge Univ. Press, Cambridge. MR2730661

NAGYLAKI, T. (1992). *Introduction to Theoretical Population Genetics*. Springer, Berlin. MR1224677

PERMAN, M., PITMAN, J. and YOR, M. (1992). Size-biased sampling of Poisson point processes and excursions. *Probab. Theory Related Fields* **92** 21–39. MR1156448

PETROV, L. A. (2009). A two-parameter family of infinite-dimensional diffusions on the Kingman simplex. *Funct. Anal. Appl*. **43** 279–296. MR2596654

PITMAN, J. (1995). Exchangeable and partially exchangeable random partitions. *Probab. Theory Related Fields* **102** 145–158. MR1337249

PITMAN, J. (1996). Some developments of the Blackwell–MacQueen urn scheme. In *Statistics, Probability and Game Theory* (T. S. Ferguson, L. S. Shapley and J. B. MacQueen, eds.). *Institute of Mathematical Statistics Lecture Notes—Monograph Series* **30**. Inst. Math. Statist., Hayward, CA. MR1481784

PITMAN, J. (2006). *Combinatorial Stochastic Processes*. *École D'été de Probabilités de Saint-Fleur XXXII*. *Lecture Notes in Math*. **1875**. Springer, Berlin. MR2245368

PITMAN, J. and YOR, M. (1997). The two-parameter Poisson–Dirichlet distribution derived from a stable subordinator. *Ann. Probab.* **25** 855–900. MR1434129

RUGGIERO, M. (2014). Species dynamics in the two-parameter Poisson–Dirichlet diffusion model. *J. Appl. Probab.* **51** 174–190. MR3189450

RUGGIERO, M. and WALKER, S. G. (2009). Countable representation for infinite-dimensional diffusions derived from the two-parameter Poisson–Dirichlet process. *Electron. Commun. Probab.* **14** 501–517. MR2564485

RUGGIERO, M., WALKER, S. G. and FAVARO, S. (2013). Alpha-diversity processes and normalized inverse-Gaussian diffusions. *Ann. Appl. Probab.* **23** 386–425. MR3059239

SATO, K. (1978). Diffusion operators in population genetics and convergence of Markov chains. In *Measure Theory Applications to Stochastic Analysis* (*Proc. Conf., Res. Inst. Math., Oberwolfach*, 1977). *Lecture Notes in Math*. **695** 127–137. Springer, Berlin. MR0527079

SHIGA, T. (1981). Diffusion processes in population genetics. *J. Math. Kyoto Univ.* **21** 133–151. MR0606316

TEH, Y. W. and JORDAN, M. I. (2010). Hierarchical Bayesian nonparametric models with applications. In *Bayesian Nonparametrics* (N. L. Hjort, C. C. Holmes, P. Müller and S. G. Walker, eds.) 158–207 Cambridge Univ. Press, Cambridge. MR2730663

ZHOU, Y. (2015). Ergodic inequality of a two-parameter infinitely-many-alleles diffusion model. *J. Appl. Probab.* **52** 238–246. MR3336858