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CAN ONE MAKE A LASER OUT OF CARDBOARD?

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We consider two-dimensional and three-dimensional semi-infinite tubes made of "Lambertian" material, so that the distribution of the direction of a reflected light ray has the density proportional to the cosine of the angle with the normal vector. If the light source is far away from the opening of the tube then the exiting rays are (approximately) collimated in two dimensions but are not collimated in three dimensions. An observer looking into the three-dimensional tube will see "infinitely bright" spot at the center of vision. In other words, in three dimensions, the light brightness grows to infinity near the center as the light source moves away.

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NEW BERRY–ESSEEN BOUNDS FOR FUNCTIONALS OF BINOMIAL POINT PROCESSES

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We obtain explicit Berry–Esseen bounds in the Kolmogorov distance for the normal approximation of nonlinear functionals of vectors of independent random variables. Our results are based on the use of Stein's method and of random difference operators, and generalise the bounds obtained by Chatterjee (2008), concerning normal approximations in the Wasserstein distance. In order to obtain lower bounds for variances, we also revisit the classical Hoeffding decompositions, for which we provide a new proof and a new representation. Several applications are discussed in detail: in particular, new Berry–Esseen bounds are obtained for set approximations with random tessellations, as well as for functionals of coverage processes.

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MAXIMALLY PERSISTENT CYCLES IN RANDOM GEOMETRIC COMPLEXES

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We initiate the study of persistent homology of random geometric simplicial complexes. Our main interest is in maximally persistent cycles of degree-k in persistent homology, for a either the Čech or the Vietoris–Rips filtration built on a uniform Poisson process of intensity n in the unit cube $[0, 1]^d$. This is a natural way of measuring the largest "k-dimensional hole" in a random point set. This problem is in the intersection of geometric probability and algebraic topology, and is naturally motivated by a probabilistic view of topological inference.

We show that for all $d \geq 2$ and $1 \leq k \leq d-1$ the maximally persistent cycle has (multiplicative) persistence of order

$$\Theta\left(\left(\frac{\log n}{\log\log n}\right)^{1/k}\right),$$

with high probability, characterizing its rate of growth as $n \to \infty$. The implied constants depend on k, d and on whether we consider the Vietoris–Rips or Čech filtration.

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Key words and phrases. Persistent homology, topological inference, stochastic topology.

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CONTACT PROCESSES ON RANDOM REGULAR GRAPHS

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We show that the contact process on a random d-regular graph initiated by a single infected vertex obeys the "cutoff phenomenon" in its supercritical phase. In particular, we prove that, when the infection rate is larger than the lower critical value of the contact process on the infinite d-regular tree, there are positive constants C, p depending on the infection rate such that for any $\varepsilon > 0$, when the number n of vertices is large then (a) at times $t < (C - \varepsilon)\log n$ the fraction of infected vertices is vanishingly small, but (b) at time $(C + \varepsilon)\log n$ the fraction of infected vertices is within ε of p, with probability p.

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PHASE TRANSITION IN A SEQUENTIAL ASSIGNMENT PROBLEM ON GRAPHS

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We study the following sequential assignment problem on a finite graph $G = (V, E)$. Each edge $e \in E$ starts with an integer value $n_e \geq 0$, and we write $n = \sum_{e \in E} n_e$. At time t, $1 \leq t \leq n$, a uniformly random vertex $v \in V$ is generated, and one of the edges f incident with v must be selected. The value of f is then decreased by 1. There is a unit final reward if the configuration $(0, \dots, 0)$ is reached. Our main result is that there is a *phase transition*: as $n \to \infty$, the expected reward under the optimal policy approaches a constant $c_G > 0$ when $(n_e/n : e \in E)$ converges to a point in the interior of a certain convex set $\mathcal{R}_G$, and goes to 0 exponentially when $(n_e/n : e \in E)$ is bounded away from $\mathcal{R}_G$. We also obtain estimates in the near-critical region, that is when $(n_e/n : e \in E)$ lies close to $\partial \mathcal{R}_G$. We supply quantitative error bounds in our arguments.

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METASTABILITY FOR GLAUBER DYNAMICS ON RANDOM GRAPHS[1]

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In this paper, we study metastable behaviour at low temperature of Glauber spin–flip dynamics on random graphs. We fix a large number of vertices and randomly allocate edges according to the configuration model with a prescribed degree distribution. Each vertex carries a spin that can point either up or down. Each spin interacts with a positive magnetic field, while spins at vertices that are connected by edges also interact with each other via a ferromagnetic pair potential. We start from the configuration where all spins point down, and allow spins to flip up or down according to a Metropolis dynamics at positive temperature. We are interested in the time it takes the system to reach the configuration where all spins point up. In order to achieve this transition, the system needs to create a sufficiently large droplet of up-spins, called critical droplet, which triggers the crossover.

In the limit as the temperature tends to zero, and subject to a certain *key hypothesis* implying metastable behaviour, the average crossover time follows the classical *Arrhenius law*, with an exponent and a prefactor that are controlled by the *energy* and the *entropy* of the critical droplet. The crossover time divided by its average is exponentially distributed. We study the scaling behaviour of the exponent as the number of vertices tends to infinity, deriving upper and lower bounds. We also identify a regime for the magnetic field and the pair potential in which the key hypothesis is satisfied. The critical droplets, representing the saddle points for the crossover, have a size that is of the order of the number of vertices. This is because the random graphs generated by the configuration model are expander graphs.

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RANDOMIZED HAMILTONIAN MONTE CARLO

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Tuning the durations of the Hamiltonian flow in Hamiltonian Monte Carlo (also called Hybrid Monte Carlo) (HMC) involves a tradeoff between computational cost and sampling quality, which is typically challenging to resolve in a satisfactory way. In this article, we present and analyze a randomized HMC method (RHMC), in which these durations are i.i.d. exponential random variables whose mean is a free parameter. We focus on the small time step size limit, where the algorithm is rejection-free and the computational cost is proportional to the mean duration. In this limit, we prove that RHMC is geometrically ergodic under the same conditions that imply geometric ergodicity of the solution to underdamped Langevin equations. Moreover, in the context of a multidimensional Gaussian distribution, we prove that the sampling efficiency of RHMC, unlike that of constant duration HMC, behaves in a regular way. This regularity is also verified numerically in non-Gaussian target distributions. Finally, we suggest variants of RHMC for which the time step size is not required to be small.

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Key words and phrases. Randomization, Markov chain Monte Carlo, Hamiltonian Monte Carlo, hybrid Monte Carlo, Lyapunov functions, geometric ergodicity, integrated autocorrelation time, equilibrium mean squared displacement.

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FAST LANGEVIN BASED ALGORITHM FOR MCMC IN HIGH DIMENSIONS

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We introduce new Gaussian proposals to improve the efficiency of the standard Hastings–Metropolis algorithm in Markov chain Monte Carlo (MCMC) methods, used for the sampling from a target distribution in large dimension d. The improved complexity is $\mathcal{O}(d^{1/5})$ compared to the complexity $\mathcal{O}(d^{1/3})$ of the standard approach. We prove an asymptotic diffusion limit theorem and show that the relative efficiency of the algorithm can be characterised by its overall acceptance rate (with asymptotical value 0.704), independently of the target distribution. Numerical experiments confirm our theoretical findings.

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DISCRETE BECKNER INEQUALITIES VIA THE BOCHNER–BAKRY–EMERY APPROACH FOR MARKOV CHAINS[1]

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Discrete convex Sobolev inequalities and Beckner inequalities are derived for time-continuous Markov chains on finite state spaces. Beckner inequalities interpolate between the modified logarithmic Sobolev inequality and the Poincaré inequality. Their proof is based on the Bakry–Emery approach and on discrete Bochner-type inequalities established by Caputo, Dai Pra and Posta and recently extended by Fathi and Maas for logarithmic entropies. The abstract result for convex entropies is applied to several Markov chains, including birth-death processes, zero-range processes, Bernoulli–Laplace models, and random transposition models, and to a finite-volume discretization of a one-dimensional Fokker–Planck equation, applying results by Mielke.

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Key words and phrases. Time-continuous Markov chain, functional inequality, entropy decay, discrete Beckner inequality, stochastic particle systems.

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SEMI-STATIC COMPLETENESS AND ROBUST PRICING BY INFORMED INVESTORS

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We consider a continuous-time financial market that consists of securities available for dynamic trading, and securities only available for static trading. We work in a robust framework where a set of non-dominated models is given. The concept of semi-static completeness is introduced: it corresponds to having exact replication by means of semi-static strategies. We show that semi-static completeness is equivalent to an extremality property, and give a characterization of the induced filtration structure. Furthermore, we consider investors with additional information and, for specific types of extra information, we characterize the models that are semi-statically complete for the informed investors. Finally, we provide some examples where robust pricing for informed and uninformed agents can be done over semi-statically complete models.

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THE L^2-CUTOFFS FOR REVERSIBLE MARKOV CHAINS

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In this article, we considers reversible Markov chains of which L^2-distances can be expressed in terms of Laplace transforms. The cutoff of Laplace transforms was first discussed by Chen and Saloff-Coste in [*J. Funct. Anal.* **258** (2010) 2246–2315], while we provide here a completely different pathway to analyze the L^2-distance. Consequently, we obtain several considerably simplified criteria and this allows us to proceed advanced theoretical studies, including the comparison of cutoffs between discrete time lazy chains and continuous time chains. For an illustration, we consider product chains, a rather complicated model which could be involved to analyze using the method in [*J. Funct. Anal.* **258** (2010) 2246–2315], and derive the equivalence of their L^2-cutoffs.

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Key words and phrases. Product chains, cutoff phenomenon.

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HIGH ORDER EXPANSIONS FOR RENEWAL FUNCTIONS AND APPLICATIONS TO RUIN THEORY[1]

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A high order expansion of the renewal function is provided under the assumption that the inter-renewal time distribution is light tailed with finite moment generating function g on a neighborhood of 0. This expansion relies on complex analysis and is expressed in terms of the residues of the function $1/(1-g)$. Under the assumption that g can be extended into a meromorphic function on the complex plane and some technical conditions, we obtain even an exact expansion of the renewal function. An application to risk theory is given where we consider high order expansion of the ruin probability for the standard compound Poisson risk model. This precises the well- known Crámer-Lundberg approximation of the ruin probability when the initial reserve is large.

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A STRONG ORDER 1/2 METHOD FOR MULTIDIMENSIONAL SDES WITH DISCONTINUOUS DRIFT

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In this paper, we consider multidimensional stochastic differential equations (SDEs) with discontinuous drift and possibly degenerate diffusion coefficient. We prove an existence and uniqueness result for this class of SDEs and we present a numerical method that converges with strong order $1/2$. Our result is the first one that shows existence and uniqueness as well as strong convergence for such a general class of SDEs.

The proof is based on a transformation technique that removes the discontinuity from the drift such that the coefficients of the transformed SDE are Lipschitz continuous. Thus the Euler–Maruyama method can be applied to this transformed SDE. The approximation can be transformed back, giving an approximation to the solution of the original SDE.

As an illustration, we apply our result to an SDE the drift of which has a discontinuity along the unit circle and we present an application from stochastic optimal control.

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Key words and phrases. Stochastic differential equations, discontinuous drift, degenerate diffusion, existence and uniqueness of solutions, numerical methods for stochastic differential equations, strong convergence rate.

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AN APPLICATION OF THE KMT CONSTRUCTION TO THE PATHWISE WEAK ERROR IN THE EULER APPROXIMATION OF ONE-DIMENSIONAL DIFFUSION PROCESS WITH LINEAR DIFFUSION COEFFICIENT

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It is well known that the strong error approximation in the space of continuous paths equipped with the supremum norm between a diffusion process, with smooth coefficients, and its Euler approximation with step $1/n$ is $O(n^{-1/2})$ and that the weak error estimation between the marginal laws at the terminal time T is $O(n^{-1})$. An analysis of the weak trajectorial error has been developed by Alfonsi, Jourdain and Kohatsu-Higa [*Ann. Appl. Probab.* **24** (2014) 1049–1080], through the study of the p-Wasserstein distance between the two processes. For a one-dimensional diffusion, they obtained an intermediate rate for the pathwise Wasserstein distance of order $n^{-2/3+\varepsilon}$. Using the Komlós, Major and Tusnády construction, we improve this bound assuming that the diffusion coefficient is linear and we obtain a rate of order $\log n/n$.

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ASYMPTOTIC LOWER BOUNDS FOR OPTIMAL TRACKING: A LINEAR PROGRAMMING APPROACH

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We consider the problem of tracking a target whose dynamics is modeled by a continuous Itô semi-martingale. The aim is to minimize both deviation from the target and tracking efforts. We establish the existence of asymptotic lower bounds for this problem, depending on the cost structure. These lower bounds can be related to the time-average control of Brownian motion, which is characterized as a deterministic linear programming problem. A comprehensive list of examples with explicit expressions for the lower bounds is provided.

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Key words and phrases. Optimal tracking, asymptotic lower bound, occupation measure, linear programming, singular control, impulse control, regular control.

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MINIMAX OPTIMALITY IN ROBUST DETECTION OF A DISORDER TIME IN DOUBLY-STOCHASTIC POISSON PROCESSES

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We consider the minimax quickest detection problem of an unobservable time of proportional change in the intensity of a doubly-stochastic Poisson process. We seek a stopping rule that minimizes the robust Lorden criterion, formulated in terms of the number of events until detection, both for the worst-case delay and the false alarm constraint. This problem, introduced by Page [*Biometrika* **41** (1954) 100–115], has received more attention in the continuous path framework (for Wiener processes) than for point processes, where optimality results only concern the Bayesian framework [In *Advances in Finance and Stochastics* (2002) 295–312, Springer, Berlin]. We prove the CUSUM optimality conjectured but not solved for the Poisson case of the CUSUM strategy in the general setting of the stochastic intensity framework. We use finite variation calculus and elementary martingale properties to characterize the performance functions of the CUSUM stopping rule in terms of scale functions. These are solutions of some delayed differential equations that can be solved simply. The case of detecting a decline in intensity is easier to study, because the performance functions are continuous. In the case of a rise where the performance functions are not continuous, differential calculus requires using a discontinuous local time at the discontinuity level, difficult to estimate. The conjecture was considered proven by the community, but the proof was still lacking for this reason. Some numerical considerations are provided at the end of the article.

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NONEXTENSIVE CONDENSATION IN REINFORCED BRANCHING PROCESSES[1]

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We study a class of branching processes in which a population consists of immortal individuals equipped with a fitness value. Individuals produce offspring with a rate given by their fitness, and offspring may either belong to the same family, sharing the fitness of their parent, or be founders of new families, with a fitness sampled from a fitness distribution μ. Examples that can be embedded in this class are stochastic house-of-cards models, urn models with reinforcement and the preferential attachment tree of Bianconi and Barabási. Our focus is on the case when the fitness distribution μ has bounded support and regularly varying tail at the essential supremum. In this case, there exists a condensation phase, in which asymptotically a proportion of mass in the empirical fitness distribution of the overall population condenses in the maximal fitness value. Our main results describe the asymptotic behaviour of the size and fitness of the largest family at a given time. In particular, we show that as time goes to infinity the size of the largest family is always negligible compared to the overall population size. This implies that condensation, when it arises, is nonextensive and emerges as a collective effort of several families none of which can create a condensate on its own. Our result disproves claims made in the physics literature in the context of preferential attachment trees.

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STRUCTURES IN SUPERCRITICAL SCALE-FREE PERCOLATION

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Scale-free percolation is a percolation model on $\mathbb{Z}^d$ which can be used to model real-world networks. We prove bounds for the graph distance in the regime where vertices have infinite degrees. We fully characterize transience versus recurrence for dimension 1 and 2 and give sufficient conditions for transience in dimension 3 and higher. Finally, we show the existence of a hierarchical structure for parameters where vertices have degrees with infinite variance and obtain bounds on the cluster density.

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