ISSN 1050-5164 (print)
ISSN 2168-8737 (online)

THE ANNALS *of* APPLIED PROBABILITY

AN OFFICIAL JOURNAL OF THE
INSTITUTE OF MATHEMATICAL STATISTICS

Articles

continued

Vol. 27, No. 5—October 2017

ISSN 1050-5164 (print)
ISSN 2168-8737 (online)

THE ANNALS
of
APPLIED PROBABILITY

AN OFFICIAL JOURNAL OF THE
INSTITUTE OF MATHEMATICAL STATISTICS

Articles—*Continued from front cover*

Erratum

Vol. 27, No. 5—October 2017

The Annals of Applied Probability
2017, Vol. 27, No. 5, 2605–2645
DOI: 10.1214/16-AAP1245
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BRANCHING BROWNIAN MOTION AND SELECTION IN THE SPATIAL Λ-FLEMING–VIOT PROCESS

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We ask the question "when will natural selection on a gene in a spatially structured population cause a detectable trace in the patterns of genetic variation observed in the contemporary population?" We focus on the situation in which "neighbourhood size", that is the effective local population density, is small. The genealogy relating individuals in a sample from the population is embedded in a spatial version of the ancestral selection graph and through applying a diffusive scaling to this object we show that whereas in dimensions at least three, selection is barely impeded by the spatial structure, in the most relevant dimension, $d = 2$, selection must be stronger (by a factor of $\log(1/\mu)$ where μ is the neutral mutation rate) if we are to have a chance of detecting it. The case $d = 1$ was handled in Etheridge, Freeman and Straulino (The Brownian net and selection in the spatial Lambda-Fleming–Viot. Preprint).

The mathematical interest is that although the system of branching and coalescing lineages that forms the ancestral selection graph converges to a branching Brownian motion, this reflects a delicate balance of a branching rate that grows to infinity and the instant annullation of almost all branches through coalescence caused by the strong local competition in the population.

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MSC2010 subject classifications. Primary 60G99; secondary 92B05.

Key words and phrases. Spatial Lambda–Fleming–Viot process, branching, coalescing, natural selection, branching Brownian motion, population genetics.

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The Annals of Applied Probability
2017, Vol. 27, No. 5, 2646–2674
DOI: 10.1214/16-AAP1253
© Institute of Mathematical Statistics, 2017

THE GLAUBER DYNAMICS OF COLORINGS ON TREES IS RAPIDLY MIXING THROUGHOUT THE NONRECONSTRUCTION REGIME

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The mixing time of the Glauber dynamics for spin systems on trees is closely related to the reconstruction problem. Martinelli, Sinclair and Weitz established this correspondence for a class of spin systems with soft constraints bounding the log-Sobolev constant by a comparison with the block dynamics [*Comm. Math. Phys.* **250** (2004) 301–334; *Random Structures Algorithms* **31** (2007) 134–172]. However, when there are hard constraints, the dynamics inside blocks may be reducible.

We introduce a variant of the block dynamics extending these results to a wide class of spin systems with hard constraints. This applies to essentially any spin system that has nonreconstruction provided that on average the root is not locally frozen in a large neighborhood. In particular, we prove that the mixing time of the Glauber dynamics for colorings on the regular tree is $O(n \log n)$ in the entire nonreconstruction regime.

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MSC2010 subject classifications. 60J10.

Key words and phrases. Mixing time, Glauber dynamics, graph colorings, reconstruction problem.

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The Annals of Applied Probability
2017, Vol. 27, No. 5, 2675–2697
DOI: 10.1214/16-AAP1254
© Institute of Mathematical Statistics, 2017

CONTAGIOUS SETS IN RANDOM GRAPHS

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We consider the following activation process in undirected graphs: a vertex is active either if it belongs to a set of initially activated vertices or if at some point it has at least r active neighbors. A *contagious set* is a set whose activation results with the entire graph being active. Given a graph G, let $m(G, r)$ be the minimal size of a contagious set.

We study this process on the binomial random graph $G := G(n, p)$ with $p := \frac{d}{n}$ and $1 \ll d \ll (\frac{n \log\log n}{\log^2 n})^{\frac{r-1}{r}}$. Assuming $r > 1$ to be a constant that does not depend on n, we prove that

$$m(G, r) = \Theta\left(\frac{n}{d^{\frac{r}{r-1}} \log d}\right),$$

with high probability. We also show that the threshold probability for $m(G, r) = r$ to hold is $p^* = \Theta(\frac{1}{(n \log^{r-1} n)^{1/r}})$.

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MSC2010 subject classifications. 05C80, 60K35, 60C05.
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The Annals of Applied Probability
2017, Vol. 27, No. 5, 2698–2752
DOI: 10.1214/16-AAP1256
© Institute of Mathematical Statistics, 2017

A STOCHASTIC MCKEAN–VLASOV EQUATION FOR ABSORBING DIFFUSIONS ON THE HALF-LINE

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We study a finite system of diffusions on the half-line, absorbed when they hit zero, with a correlation effect that is controlled by the proportion of the processes that have been absorbed. As the number of processes in the system becomes large, the empirical measure of the population converges to the solution of a nonlinear stochastic heat equation with Dirichlet boundary condition. The diffusion coefficients are allowed to have finitely many discontinuities (piecewise Lipschitz) and we prove pathwise uniqueness of solutions to the limiting stochastic PDE. As a corollary, we obtain a representation of the limit as the unique solution to a stochastic McKean–Vlasov problem. Our techniques involve energy estimation in the dual of the first Sobolev space, which connects the regularity of solutions to their boundary behaviour, and tightness calculations in the Skorokhod M1 topology defined for distribution-valued processes, which exploits the monotonicity of the loss process L. The motivation for this model comes from the analysis of large portfolio credit problems in finance.

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Key words and phrases. McKean–Vlasov problem, nonlinear SPDE, Skorokhod M1 topology.

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The Annals of Applied Probability
2017, Vol. 27, No. 5, 2753–2806
DOI: 10.1214/16-AAP1265
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A FUNCTIONAL LIMIT THEOREM FOR LIMIT ORDER BOOKS WITH STATE DEPENDENT PRICE DYNAMICS[1]

BY CHRISTIAN BAYER, ULRICH HORST AND JINNIAO QIU

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We consider a stochastic model for the dynamics of the two-sided limit order book (LOB). Our model is flexible enough to allow for a dependence of the price dynamics on volumes. For the joint dynamics of best bid and ask prices and the standing buy and sell volume densities, we derive a functional limit theorem, which states that our LOB model converges in distribution to a fully coupled SDE-SPDE system when the order arrival rates tend to infinity and the impact of an individual order arrival on the book as well as the tick size tends to zero. The SDE describes the bid/ask price dynamics while the SPDE describes the volume dynamics.

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MSC2010 subject classifications. Primary 60B11, 90B22; secondary 91B70.

Key words and phrases. Limit order book, functional limit theorem, stochastic partial differential equation.

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The Annals of Applied Probability
2017, Vol. 27, No. 5, 2807–2861
DOI: 10.1214/16-AAP1267
© Institute of Mathematical Statistics, 2017

STOCHASTIC PARTICLE APPROXIMATION OF THE KELLER–SEGEL EQUATION AND TWO-DIMENSIONAL GENERALIZATION OF BESSEL PROCESSES

BY NICOLAS FOURNIER AND BENJAMIN JOURDAIN

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We are interested in the two-dimensional Keller–Segel partial differential equation. This equation is a model for chemotaxis (and for Newtonian gravitational interaction). When the total mass of the initial density is one, it is known to exhibit blow-up in finite time as soon as the sensitivity χ of bacteria to the chemo-attractant is larger than 8π. We investigate its approximation by a system of N two-dimensional Brownian particles interacting through a singular attractive kernel in the drift term.

In the very subcritical case $\chi < 2\pi$, the diffusion strongly dominates this singular drift: we obtain existence for the particle system and prove that its flow of empirical measures converges, as $N \to \infty$ and up to extraction of a subsequence, to a weak solution of the Keller–Segel equation.

We also show that for any $N \geq 2$ and any value of $\chi > 0$, pairs of particles do collide with positive probability: the singularity of the drift is indeed visited. Nevertheless, when $\chi < 2\pi N$, it is possible to control the drift and obtain existence of the particle system until the first time when at least three particles collide. We check that this time is a.s. infinite, so that global existence holds for the particle system, if and only if $\chi \leq 8\pi(N-2)/(N-1)$.

Finally, we remark that in the system with $N = 2$ particles, the difference between the two positions provides a natural two-dimensional generalization of Bessel processes, which we study in details.

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MSC2010 subject classifications. 65C35, 35K55, 60H10.
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The Annals of Applied Probability
2017, Vol. 27, No. 5, 2862–2906
DOI: 10.1214/16-AAP1269
© Institute of Mathematical Statistics, 2017

ASYMPTOTICALLY OPTIMAL CONTROL FOR A MULTICLASS QUEUEING MODEL IN THE MODERATE DEVIATION HEAVY TRAFFIC REGIME[1]

BY RAMI ATAR AND ASAF COHEN

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A multi-class single-server queueing model with finite buffers, in which scheduling and admission of customers are subject to control, is studied in the moderate deviation heavy traffic regime. A risk-sensitive cost set over a finite time horizon $[0, T]$ is considered. The main result is the asymptotic optimality of a control policy derived via an underlying differential game. The result is the first to address a queueing control problem at the moderate deviation regime that goes beyond models having the so-called pathwise minimality property. Moreover, despite the well-known fact that an optimal control over a finite time interval is generically of a nonstationary feedback type, the proposed policy forms a stationary feedback, provided T is sufficiently large.

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MSC2010 subject classifications. 49N70, 60F10, 60K25, 93E20.
Key words and phrases. Moderate deviations, heavy traffic, risk-sensitive cost, differential games.

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The Annals of Applied Probability
2017, Vol. 27, No. 5, 2907–2925
DOI: 10.1214/16-AAP1271
© Institute of Mathematical Statistics, 2017

REINFORCEMENT LEARNING FROM COMPARISONS: THREE ALTERNATIVES ARE ENOUGH, TWO ARE NOT

BY BENOÎT LASLIER[1] AND JEAN-FRANÇOIS LASLIER

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This paper deals with two generalizations of the Polya urn model where, instead of sampling one ball from the urn at each time, we sample two or three balls. The processes are defined on the basis of the problem of finding the best alternative using pairwise comparisons which are not necessarily transitive: they can be thought of as evolutionary processes that tend to reinforce currently efficient alternatives. The two processes exhibit different behaviors: with three balls sampled, we prove almost sure convergence towards the unique optimal solution of the comparisons problem while, in some cases, the process with two balls sampled has almost surely no limit. This is an example of a natural reinforcement model with no exchangeability whose asymptotic behavior can be precisely characterized.

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MSC2010 subject classifications. 60J20, 91A22, 91E40.
Key words and phrases. Learning, tournament, urn process.

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The Annals of Applied Probability
2017, Vol. 27, No. 5, 2926–2955
DOI: 10.1214/16-AAP1273
© Institute of Mathematical Statistics, 2017

DISTANCE-BASED SPECIES TREE ESTIMATION UNDER THE COALESCENT: INFORMATION-THEORETIC TRADE-OFF BETWEEN NUMBER OF LOCI AND SEQUENCE LENGTH

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We consider the reconstruction of a phylogeny from multiple genes under the multispecies coalescent. We establish a connection with the sparse signal detection problem, where one seeks to distinguish between a distribution and a mixture of the distribution and a sparse signal. Using this connection, we derive an information-theoretic trade-off between the number of genes, m, needed for an accurate reconstruction and the sequence length, k, of the genes. Specifically, we show that to detect a branch of length f, one needs $m = \Theta(1/[f^2\sqrt{k}])$ genes.

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MSC2010 subject classifications. 60K35, 92D15.
Key words and phrases. Phylogenetics, coalescent theory, sequence-length requirement.

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The Annals of Applied Probability
2017, Vol. 27, No. 5, 2956–3003
DOI: 10.1214/16-AAP1274
© Institute of Mathematical Statistics, 2017

CENTRAL LIMIT THEOREM FOR AN ADAPTIVE RANDOMLY REINFORCED URN MODEL

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The generalized Pólya urn (GPU) models and their variants have been investigated in several disciplines. However, typical assumptions made with respect to the GPU do not include urn models with a diagonal replacement matrix, which arise in several applications, specifically in clinical trials. To facilitate mathematical analyses of models in these applications, we introduce an adaptive randomly reinforced urn model that uses accruing statistical information to adaptively skew the urn proportion toward specific targets. We study several probabilistic aspects that are important in implementing the urn model in practice. Specifically, we establish the law of large numbers and a central limit theorem for the number of sampled balls. To establish these results, we develop new techniques involving last exit times and crossing time analyses of the proportion of balls in the urn. To obtain precise estimates in these techniques, we establish results on the harmonic moments of the total number of balls in the urn. Finally, we describe our main results in the context of an application to response-adaptive randomization in clinical trials. Our simulation experiments in this context demonstrate the ease and scope of our model.

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MSC2010 subject classifications. Primary 60F15, 60F05, 60E20, 60G99; secondary 68Q87, 97K50.

Key words and phrases. Clinical trials, crossing times, harmonic moments, last exit times, generalized Pólya urn, target allocation.

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The Annals of Applied Probability
2017, Vol. 27, No. 5, 3004–3049
DOI: 10.1214/17-AAP1275
© Institute of Mathematical Statistics, 2017

ERGODICITY OF INHOMOGENEOUS MARKOV CHAINS THROUGH ASYMPTOTIC PSEUDOTRAJECTORIES[1]

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In this work, we consider an inhomogeneous (discrete time) Markov chain and are interested in its long time behavior. We provide sufficient conditions to ensure that some of its asymptotic properties can be related to the ones of a homogeneous (continuous time) Markov process. Renowned examples such as a bandit algorithms, weighted random walks or decreasing step Euler schemes are included in our framework. Our results are related to functional limit theorems, but the approach differs from the standard "Tightness/Identification" argument; our method is unified and based on the notion of pseudotrajectories on the space of probability measures.

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MSC2010 subject classifications. Primary 60J10; secondary 60J25, 60B10.
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The Annals of Applied Probability
2017, Vol. 27, No. 5, 3050–3112
DOI: 10.1214/17-AAP1276
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UNIVERSALITY IN MARGINALLY RELEVANT DISORDERED SYSTEMS

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We consider disordered systems of a directed polymer type, for which disorder is so-called marginally relevant. These include the usual (short-range) directed polymer model in dimension $(2+1)$, the long-range directed polymer model with Cauchy tails in dimension $(1+1)$ and the disordered pinning model with tail exponent $1/2$. We show that in a suitable weak disorder and continuum limit, the partition functions of these different models converge to a universal limit: a log-normal random field with a multi-scale correlation structure, which undergoes a phase transition as the disorder strength varies. As a by-product, we show that the solution of the two-dimensional stochastic heat equation, suitably regularized, converges to the same limit. The proof, which uses the celebrated fourth moment theorem, reveals an interesting chaos structure shared by all models in the above class.

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MSC2010 subject classifications. Primary 82B44; secondary 82D60, 60K35.
Key words and phrases. Directed polymer, pinning model, polynomial chaos, disordered system, fourth moment theorem, marginal disorder relevance, stochastic heat equation.

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The Annals of Applied Probability
2017, Vol. 27, No. 5, 3113–3152
DOI: 10.1214/17-AAP1277
© Institute of Mathematical Statistics, 2017

FINITE SYSTEM SCHEME FOR MUTUALLY CATALYTIC BRANCHING WITH INFINITE BRANCHING RATE

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For many stochastic diffusion processes with mean field interaction, convergence of the rescaled total mass processes towards a diffusion process is known.

Here, we show convergence of the so-called finite system scheme for interacting jump-type processes known as mutually catalytic branching processes with infinite branching rate. Due to the lack of second moments, the rescaling of time is different from the finite rate mutually catalytic case. The limit of rescaled total mass processes is identified as the finite rate mutually catalytic branching diffusion. The convergence of rescaled processes holds jointly with convergence of coordinate processes, where the latter converge at a different time scale.

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MSC2010 subject classifications. Primary 60K35; secondary 60J80, 60J60, 60J75, 60F05, 60H20.

Key words and phrases. Finite systems scheme, interacting diffusions, meanfield limit, mutually catalytic branching.

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The Annals of Applied Probability
2017, Vol. 27, No. 5, 3153–3188
DOI: 10.1214/17-AAP1278
© Institute of Mathematical Statistics, 2017

REFLECTED BSDES WHEN THE OBSTACLE IS NOT RIGHT-CONTINUOUS AND OPTIMAL STOPPING

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In the first part of the paper, we study reflected backward stochastic differential equations (RBSDEs) with lower obstacle which is assumed to be right upper-semicontinuous but not necessarily right-continuous. We prove existence and uniqueness of the solutions to such RBSDEs in appropriate Banach spaces. The result is established by using some results from optimal stopping theory, some tools from the general theory of processes such as Mertens' decomposition of optional strong supermartingales, as well as an appropriate generalization of Itô's formula due to Gal'chouk and Lenglart. In the second part of the paper, we provide some links between the RBSDE studied in the first part and an optimal stopping problem in which the risk of a financial position ξ is assessed by an f-conditional expectation $\mathcal{E}^f(\cdot)$ (where f is a Lipschitz driver). We characterize the "value function" of the problem in terms of the solution to our RBSDE. Under an additional assumption of left upper-semicontinuity along stopping times on ξ, we show the existence of an optimal stopping time. We also provide a generalization of Mertens' decomposition to the case of strong $\mathcal{E}^f$-supermartingales.

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MSC2010 subject classifications. Primary 60G40, 93E20, 60H30; secondary 60G07, 47N10.

Key words and phrases. Backward stochastic differential equation, reflected backward stochastic differential equation, optimal stopping, f-expectation, strong optional supermartingale, Mertens' decomposition, dynamic risk measure, strong $\mathcal{E}^f$-supermartingale.

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The Annals of Applied Probability
2017, Vol. 27, No. 5, 3189–3200
DOI: 10.1214/17-AAP1279
© Institute of Mathematical Statistics, 2017

BALLISTIC AND SUB-BALLISTIC MOTION OF INTERFACES IN A FIELD OF RANDOM OBSTACLES

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We consider a discretized version of the quenched Edwards–Wilkinson model for the propagation of a driven interface through a random field of obstacles. Our model consists of a system of ordinary differential equations on a d-dimensional lattice coupled by the discrete Laplacian. At each lattice point, the system is subject to a constant driving force and a random obstacle force impeding free propagation. The obstacle force depends on the current state of the solution, and thus renders the problem nonlinear. For independent and identically distributed obstacle strengths with an exponential moment, we prove ballistic propagation (i.e., propagation with a positive velocity) of the interface if the driving force is large enough. For a specific case of dependent obstacles, we show that no stationary solution exists, but still the propagation of the front is not ballistic.

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MSC2010 subject classifications. 34F05, 60H10.
Key words and phrases. Interfaces, heterogeneous media, random media, asymptotic behavior of nonnegative solutions.

The Annals of Applied Probability
2017, Vol. 27, No. 5, 3201–3245
DOI: 10.1214/17-AAP1280
© Institute of Mathematical Statistics, 2017

A MEAN-FIELD STOCHASTIC CONTROL PROBLEM WITH PARTIAL OBSERVATIONS

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The authors would like to dedicate this paper to Professor Hans-Jürgen Engelbert, on the occasion of his 70th birthday, for his generous guidance and inspirational discussions throughout the past decades.

In this paper, we are interested in a new type of *mean-field*, non-Markovian stochastic control problems with partial observations. More precisely, we assume that the coefficients of the controlled dynamics depend not only on the paths of the state, but also on the conditional law of the state, given the observation to date. Our problem is strongly motivated by the recent study of the mean field games and the related McKean–Vlasov stochastic control problem, but with added aspects of path-dependence and partial observation. We shall first investigate the well-posedness of the state-observation dynamics, with combined reference probability measure arguments in nonlinear filtering theory and the Schauder fixed-point theorem. We then study the stochastic control problem with a partially observable system in which the conditional law appears nonlinearly in both the coefficients of the system and cost function. As a consequence, the control problem is intrinsically "time-inconsistent", and we prove that the Pontryagin stochastic maximum principle holds in this case and characterize the adjoint equations, which turn out to be a new form of mean-field type BSDEs.

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MSC2010 subject classifications. 60H10, 60H30, 93E03, 93E11, 93E20.
Key words and phrases. Conditional mean-field SDEs, non-Markovian stochastic control system, nonlinear filtering, stochastic maximum principle, mean-field backward SDEs.

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The Annals of Applied Probability
2017, Vol. 27, No. 5, 3246–3253
DOI: 10.1214/17-AAP1291
Main article DOI: 10.1214/09-AAP666
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ERRATUM: FIRST PASSAGE PERCOLATION ON RANDOM GRAPHS WITH FINITE MEAN DEGREES [*Ann. Appl. Probab.* 20(5) (2010) 1907–1965]

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In this erratum, we correct a mistake in the above paper, where we were using an exchangeability result that is obviously false.

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MSC2010 subject classifications. Primary 60C05; secondary 05C80, 90B15.
Key words and phrases. Flows, random graph, first passage percolation, hopcount, central limit theorem, configuration model.