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OPTIMAL CONTROL OF BRANCHING DIFFUSION PROCESSES: A FINITE HORIZON PROBLEM

BY JULIEN CLAISSE[1]

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In this paper, we aim to develop the stochastic control theory of branching diffusion processes where both the movement and the reproduction of the particles depend on the control. More precisely, we study the problem of minimizing the expected value of the product of individual costs penalizing the final position of each particle. In this setting, we show that the value function is the unique viscosity solution of a nonlinear parabolic PDE, that is, the Hamilton–Jacobi–Bellman equation corresponding to the problem. To this end, we extend the dynamic programming approach initiated by Nisio [*J. Math. Kyoto Univ.* **25** (1985) 549–575] to deal with the lack of independence between the particles as well as between the reproduction and the movement of each particle. In particular, we exploit the particular form of the optimization criterion to derive a weak form of the branching property. In addition, we provide a precise formulation and a detailed justification of the adequate dynamic programming principle.

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CHANGE POINT DETECTION IN NETWORK MODELS: PREFERENTIAL ATTACHMENT AND LONG RANGE DEPENDENCE

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Inspired by empirical data on real world complex networks, the last few years have seen an explosion in proposed generative models to understand and explain observed properties of real world networks, including power law degree distribution and "small world" distance scaling. In this context, a natural question is how to understand the effect of *change points*—how abrupt changes in parameters driving the network model change structural properties of the network. We study this phenomenon in one popular class of dynamically evolving networks: preferential attachment models. We derive asymptotic properties of various functionals of the network including the degree distribution as well as maximal degree asymptotics, in essence showing that the change point does effect the degree distribution but does *not* change the degree exponent. This provides evidence for long range dependence and sensitive dependence of the evolution of the network on the initial evolution of the process. We propose an estimator for the change point and prove consistency properties of this estimator. The methodology developed highlights the effect of the nonergodic nature of the evolution of the network on classical change point estimators.

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ERGODIC THEORY FOR CONTROLLED MARKOV CHAINS WITH STATIONARY INPUTS[1]

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Consider a stochastic process $\boldsymbol{X}$ on a finite state space $\mathsf{X} = \{1, \dots, d\}$. It is conditionally Markov, given a real-valued "input process" $\boldsymbol{\zeta}$. This is assumed to be small, which is modeled through the scaling,

$$\zeta_t = \varepsilon \zeta_t^1, \qquad 0 \le \varepsilon \le 1,$$

where $\boldsymbol{\zeta}^1$ is a bounded stationary process. The following conclusions are obtained, subject to smoothness assumptions on the controlled transition matrix and a mixing condition on $\boldsymbol{\zeta}$:

(i) A stationary version of the process is constructed, that is coupled with a stationary version of the Markov chain $\boldsymbol{X}^\bullet$ obtained with $\boldsymbol{\zeta} \equiv 0$. The triple $(\boldsymbol{X}, \boldsymbol{X}^\bullet, \boldsymbol{\zeta})$ is a jointly stationary process satisfying

$$\mathsf{P}\{X(t) \ne X^\bullet(t)\} = O(\varepsilon).$$

Moreover, a second-order Taylor-series approximation is obtained:

$$\mathsf{P}\{X(t) = i\} = \mathsf{P}\{X^\bullet(t) = i\} + \varepsilon^2 \pi^{(2)}(i) + o(\varepsilon^2), \qquad 1 \le i \le d,$$

with an explicit formula for the vector $\pi^{(2)} \in \mathbb{R}^d$.

(ii) For any $m \ge 1$ and any function $f : \{1, \dots, d\} \times \mathbb{R} \to \mathbb{R}^m$, the stationary stochastic process $Y(t) = f(X(t), \zeta(t))$ has a power spectral density S_f that admits a second-order Taylor series expansion: A function $\mathrm{S}_f^{(2)} : [-\pi, \pi] \to \mathbb{C}^{m \times m}$ is constructed such that

$$\mathrm{S}_f(\theta) = \mathrm{S}_f^\bullet(\theta) + \varepsilon^2 \mathrm{S}_f^{(2)}(\theta) + o(\varepsilon^2), \qquad \theta \in [-\pi, \pi]$$

in which the first term is the power spectral density obtained with $\varepsilon = 0$. An explicit formula for the function $\mathrm{S}_f^{(2)}$ is obtained, based in part on the bounds in (i).

The results are illustrated with two general examples: mean field games, and a version of the timing channel of Anantharam and Verdu.

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NASH EQUILIBRIA OF THRESHOLD TYPE FOR TWO-PLAYER NONZERO-SUM GAMES OF STOPPING

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This paper analyses two-player nonzero-sum games of optimal stopping on a class of linear regular diffusions with *not nonsingular* boundary behaviour [in the sense of Itô and McKean (*Diffusion Processes and Their Sample Paths* (1974) Springer, page 108)]. We provide sufficient conditions under which Nash equilibria are realised by each player stopping the diffusion at one of the two boundary points of an interval. The boundaries of this interval solve a system of algebraic equations. We also provide conditions sufficient for the uniqueness of the equilibrium in this class.

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LOCAL INHOMOGENEOUS CIRCULAR LAW[1]

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We consider large random matrices X with centered, independent entries, which have comparable but not necessarily identical variances. Girko's circular law asserts that the spectrum is supported in a disk and in case of identical variances, the limiting density is uniform. In this special case, the *local circular law* by Bourgade et al. [*Probab. Theory Related Fields* **159** (2014) 545–595; *Probab. Theory Related Fields* **159** (2014) 619–660] shows that the empirical density converges even locally on scales slightly above the typical eigenvalue spacing. In the general case, the limiting density is typically inhomogeneous and it is obtained via solving a system of deterministic equations. Our main result is the local *inhomogeneous* circular law in the bulk spectrum on the optimal scale for a general variance profile of the entries of X.

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Key words and phrases. Circular law, local law, variance profile.

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DIFFUSION APPROXIMATIONS FOR CONTROLLED WEAKLY INTERACTING LARGE FINITE STATE SYSTEMS WITH SIMULTANEOUS JUMPS[1]

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We consider a rate control problem for an N-particle weakly interacting finite state Markov process. The process models the state evolution of a large collection of particles and allows for multiple particles to change state simultaneously. Such models have been proposed for large communication systems (e.g., ad hoc wireless networks) but are also suitable for other settings such as chemical-reaction networks. An associated diffusion control problem is presented and we show that the value function of the N-particle controlled system converges to the value function of the limit diffusion control problem as $N \to \infty$. The diffusion coefficient in the limit model is typically degenerate; however, under suitable conditions there is an equivalent formulation in terms of a controlled diffusion with a uniformly nondegenerate diffusion coefficient. Using this equivalence, we show that near optimal continuous feedback controls exist for the diffusion control problem. We then construct near asymptotically optimal control policies for the N-particle system based on such continuous feedback controls. Results from some numerical experiments are presented.

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DUALITY AND FIXATION IN Ξ-WRIGHT–FISHER PROCESSES WITH FREQUENCY-DEPENDENT SELECTION

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A two-types, discrete-time population model with finite, constant size is constructed, allowing for a general form of frequency-dependent selection and skewed offspring distribution. Selection is defined based on the idea that individuals first choose a (random) number of *potential* parents from the previous generation and then, from the selected pool, they inherit the type of the fittest parent. The probability distribution function of the number of potential parents per individual thus parametrises entirely the selection mechanism. Using sampling- and moment-duality, weak convergence is then proved both for the allele frequency process of the selectively weak type and for the population's ancestral process. The scaling limits are, respectively, a two-types Ξ-Fleming–Viot jump-diffusion process with frequency-dependent selection, and a branching-coalescing process with general branching and simultaneous multiple collisions. Duality also leads to a characterisation of the probability of extinction of the selectively weak allele, in terms of the ancestral process' ergodic properties.

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COMBINATORIAL LÉVY PROCESSES[1]

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Combinatorial Lévy processes evolve on general state spaces of combinatorial structures, of which standard examples include processes on sets, graphs and n-ary relations and more general possibilities are given by processes on graphs with community structure and multilayer networks. In this setting, the usual Lévy process properties of stationary, independent increments are defined in an unconventional way in terms of the symmetric difference operation on sets. The main theorems characterize both finite and infinite state space combinatorial Lévy processes by a unique σ-finite measure. Under the additional assumption of exchangeability, I prove a more explicit characterization by which every exchangeable combinatorial Lévy process corresponds to a Poisson point process on the same state space. Associated behavior of the projection into a space of limiting objects reflects certain structural features of the covering process.

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EIGENVALUE VERSUS PERIMETER IN A SHAPE THEOREM FOR SELF-INTERACTING RANDOM WALKS[1]

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We study paths of time-length t of a continuous-time random walk on $\mathbb{Z}^2$ subject to self-interaction that depends on the geometry of the walk range and a collection of random, uniformly positive and finite edge weights. The interaction enters through a Gibbs weight at inverse temperature β; the "energy" is the total sum of the edge weights for edges on the outer boundary of the range. For edge weights sampled from a translation-invariant, ergodic law, we prove that the range boundary condensates around an asymptotic shape in the limit $t \to \infty$ followed by $\beta \to \infty$. The limit shape is a minimizer (unique, modulo translates) of the sum of the principal harmonic frequency of the domain and the perimeter with respect to the first-passage percolation norm derived from (the law of) the edge weights. A dense subset of all norms in $\mathbb{R}^2$, and thus a large variety of shapes, arise from the class of weight distributions to which our proofs apply.

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VOLATILITY AND ARBITRAGE

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The capitalization-weighted cumulative variation

$$\sum_{i=1}^{d} \int_0^{\cdot} \mu_i(t)\, \mathrm{d}\langle \log \mu_i \rangle(t)$$

in an equity market consisting of a fixed number d of assets with capitalization weights $\mu_i(\cdot)$, is an observable and a nondecreasing function of time. If this observable of the market is not just nondecreasing but actually grows at a rate bounded away from zero, then strong arbitrage can be constructed relative to the market over sufficiently long time horizons. It has been an open issue for more than ten years, whether such strong outperformance of the market is possible also over arbitrary time horizons under the stated condition. We show that this is not possible in general, thus settling this long-open question. We also show that, under appropriate additional conditions, outperformance over any time horizon indeed becomes possible, and exhibit investment strategies that effect it.

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A SKOROKHOD MAP ON MEASURE-VALUED PATHS WITH APPLICATIONS TO PRIORITY QUEUES

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The Skorokhod map on the half-line has proved to be a useful tool for studying processes with nonnegativity constraints. In this work, we introduce a measure-valued analog of this map that transforms each element ζ of a certain class of càdlàg paths that take values in the space of signed measures on $[0, \infty)$ to a càdlàg path that takes values in the space of nonnegative measures on $[0, \infty)$ in such a way that for each $x > 0$, the path $t \mapsto \zeta_t[0, x]$ is transformed via a Skorokhod map on the half-line, and the regulating functions for different $x > 0$ are coupled. We establish regularity properties of this map and show that the map provides a convenient tool for studying queueing systems in which tasks are prioritized according to a continuous parameter. Three such well-known models are the *earliest-deadline-first*, the *shortest-job-first* and the *shortest-remaining-processing-time* scheduling policies. For these applications, we show how the map provides a unified framework within which to form fluid model equations, prove uniqueness of solutions to these equations and establish convergence of scaled state processes to the fluid model. In particular, for these models, we obtain new convergence results in time-inhomogeneous settings, which appear to fall outside the purview of existing approaches.

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BSDES WITH MEAN REFLECTION

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In this paper, we study a new type of BSDE, where the distribution of the Y-component of the solution is required to satisfy an additional constraint, written in terms of the expectation of a loss function. This constraint is imposed at any deterministic time t and is typically weaker than the classical pointwise one associated to reflected BSDEs. Focusing on solutions (Y, Z, K) with deterministic K, we obtain the well-posedness of such equation, in the presence of a natural Skorokhod-type condition. Such condition indeed ensures the minimality of the enhanced solution, under an additional structural condition on the driver. Our results extend to the more general framework where the constraint is written in terms of a static risk measure on Y. In particular, we provide an application to the super-hedging of claims under running risk management constraint.

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LIMIT THEOREMS FOR INTEGRATED LOCAL EMPIRICAL CHARACTERISTIC EXPONENTS FROM NOISY HIGH-FREQUENCY DATA WITH APPLICATION TO VOLATILITY AND JUMP ACTIVITY ESTIMATION

BY JEAN JACOD AND VIKTOR TODOROV[1]

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We derive limit theorems for functionals of local empirical characteristic functions constructed from high-frequency observations of Itô semimartingales contaminated with noise. In a first step, we average locally the data to mitigate the effect of the noise, and then in a second step, we form local empirical characteristic functions from the pre-averaged data. The final statistics are formed by summing the local empirical characteristic exponents over the observation interval. The limit behavior of the statistics is governed by the observation noise, the diffusion coefficient of the Itô semimartingale and the behavior of its jump compensator around zero. Different choices for the block sizes for pre-averaging and formation of the local empirical characteristic function as well as for the argument of the characteristic function make the asymptotic role of the diffusion, the jumps and the noise differ. The derived limit results can be used in a wide range of applications and in particular for doing the following in a noisy setting: (1) efficient estimation of the time-integrated diffusion coefficient in presence of jumps of arbitrary activity, and (2) efficient estimation of the jump activity (Blumenthal–Getoor) index.

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Key words and phrases. Blumenthal–Getoor index, central limit theorem, empirical characteristic function, integrated volatility, irregular sampling, Itô semimartingale, jumps, jump activity, microstructure noise, quadratic variation, stable process.

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DISORDER AND WETTING TRANSITION: THE PINNED HARMONIC CRYSTAL IN DIMENSION THREE OR LARGER

BY GIAMBATTISTA GIACOMIN[1] AND HUBERT LACOIN[2]

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We consider the lattice Gaussian free field in $d+1$ dimensions, $d=3$ or larger, on a large box (linear size N) with boundary conditions zero. On this field, two potentials are acting: one, that models the presence of a wall, penalizes the field when it enters the lower half space and one, the *pinning potential*, rewards visits to the proximity of the wall. The wall can be soft, that is, the field has a finite penalty to enter the lower half-plane, or hard, when the penalty is infinite. In general, the pinning potential is disordered and it gives on average a reward $h \in \mathbb{R}$ (a negative reward is a penalty): the energetic contribution when the field at site x visits the pinning region is $\beta\omega_x + h$, $\{\omega_x\}_{x\in\mathbb{Z}^d}$ are i.i.d. centered and exponentially integrable random variables of unit variance and $\beta \geq 0$. In [*J. Math. Phys.* **41** (2000) 1211–1223], it is shown that, when $\beta = 0$ (i.e., in the nondisordered model), a delocalization-localization transition happens at $h=0$, in particular the free energy of the system is zero for $h \leq 0$ and positive for $h > 0$. We show that, for $\beta \neq 0$, the transition happens at $h = h_c(\beta) := -\log \mathbb{E}\exp(\beta\omega_x)$, and we find the precise asymptotic behavior of the logarithm of the free energy density of the system when $h \searrow h_c(\beta)$. In particular, we show that the transition is of infinite order in the sense that the free energy is smaller than any power of $h - h_c(\beta)$ in the neighborhood of the critical point and that disorder does not modify at all the nature of the transition. We also provide results on the behavior of the paths of the random field in the limit $N \to \infty$.

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LAW OF LARGE NUMBERS FOR THE LARGEST COMPONENT IN A HYPERBOLIC MODEL OF COMPLEX NETWORKS

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We consider the component structure of a recent model of random graphs on the hyperbolic plane that was introduced by Krioukov et al. The model exhibits a power law degree sequence, small distances and clustering, features that are associated with so-called complex networks. The model is controlled by two parameters α and ν where, roughly speaking, α controls the exponent of the power law and ν controls the average degree. Refining earlier results, we are able to show a law of large numbers for the largest component. That is, we show that the fraction of points in the largest component tends in probability to a constant c that depends only on α, ν, while all other components are sublinear. We also study how c depends on α, ν. To deduce our results, we introduce a local approximation of the random graph by a continuum percolation model on $\mathbb{R}^2$ that may be of independent interest.

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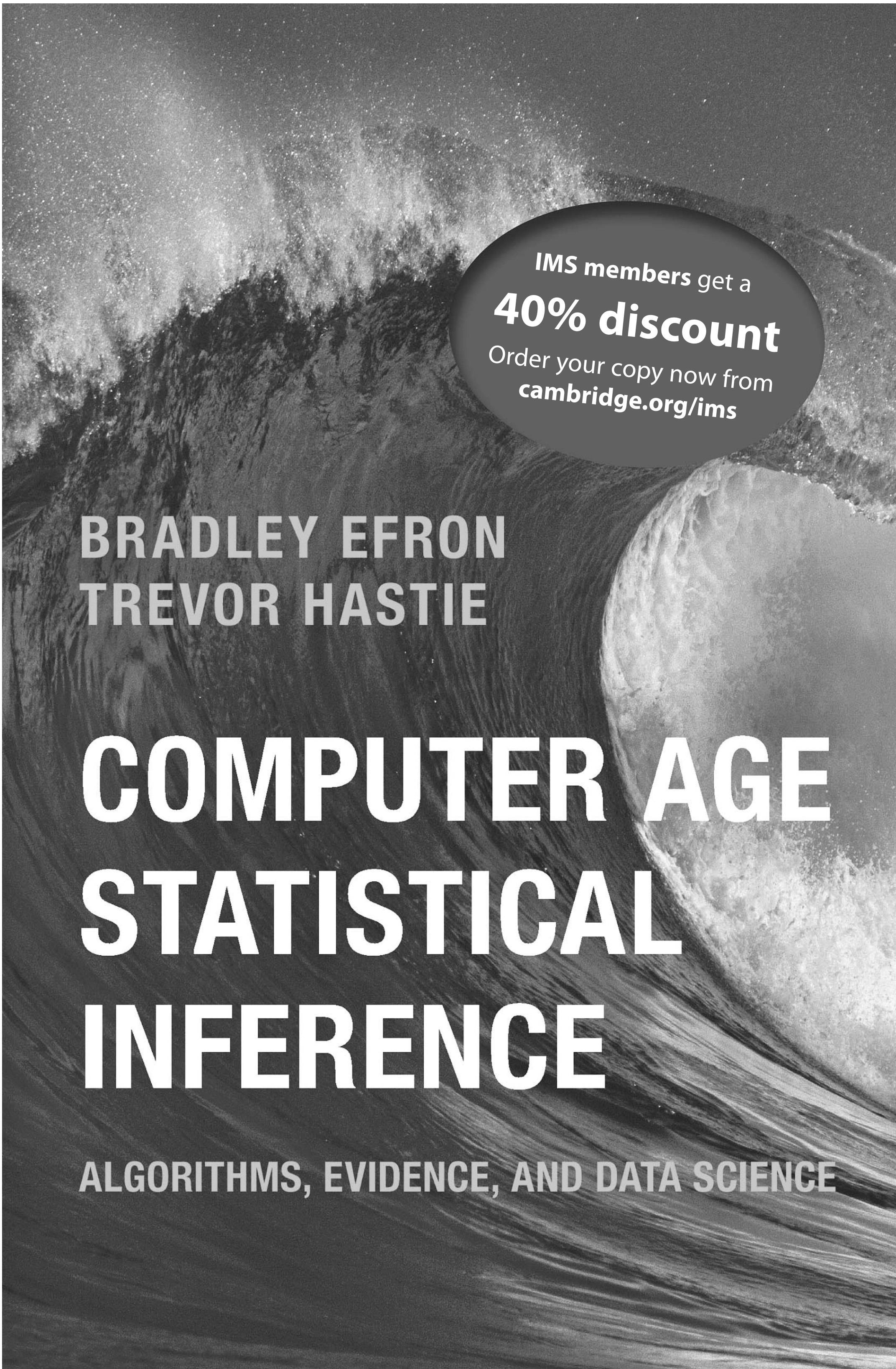
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