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THE SHAPE OF MULTIDIMENSIONAL BRUNET–DERRIDA PARTICLE SYSTEMS

BY NATHANAËL BERESTYCKI[1] AND LEE ZHUO ZHAO

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We introduce particle systems in one or more dimensions in which particles perform branching Brownian motion and the population size is kept constant equal to $N > 1$, through the following selection mechanism: at all times only the N fittest particles survive, while all the other particles are removed. Fitness is measured with respect to some given score function $s : \mathbb{R}^d \to \mathbb{R}$. For some choices of the function s, it is proved that the cloud of particles travels at positive speed in some possibly random direction. In the case where s is linear, we show under some mild assumptions that the shape of the cloud scales like $\log N$ in the direction parallel to motion but at least $(\log N)^{3/2}$ in the orthogonal direction. We conjecture that the exponent $3/2$ is sharp. In order to prove this, we obtain the following result of independent interest: in one-dimensional systems, the genealogical time is greater than $c(\log N)^3$. We discuss several open problems and explain how our results can be viewed as a rigorous justification in our setting of empirical observations made by Burt [*Evolution* **54** (2000) 337–351] in support of Weismann's arguments for the role of recombination in population genetics.

REFERENCES

[1] BELL, G. (1982). *The Masterpiece of Nature*. Univ. California Press, Berkeley, CA.

[2] BÉRARD, J. and GOUÉRÉ, J.-B. (2010). Brunet–Derrida behavior of branching-selection particle systems on the line. *Comm. Math. Phys.* **298** 323–342. MR2669438

[3] BERESTYCKI, J., BERESTYCKI, N. and SCHWEINSBERG, J. (2011). Survival of near-critical branching Brownian motion. *J. Stat. Phys.* **143** 833–854. MR2811463

[4] BERESTYCKI, J., BERESTYCKI, N. and SCHWEINSBERG, J. (2013). The genealogy of branching Brownian motion with absorption. *Ann. Probab.* **41** 527–618. MR3077519

[5] BERESTYCKI, J. and YU, F. Unpublished work.

[6] BERESTYCKI, N. (2009). *Recent Progress in Coalescent Theory*. *Ensaios Matemáticos* [*Mathematical Surveys*] **16**. Sociedade Brasileira de Matemática, Rio de Janeiro. MR2574323

[7] BRUNET, E. and DERRIDA, B. (1997). Shift in the velocity of a front due to a cutoff. *Phys. Rev. E* (3) **56** 2597–2604. MR1473413

[8] BRUNET, E. and DERRIDA, B. (1999). Microscopic models of traveling wave equations. *Comput. Phys. Commun.* **121–122** 376–381.

[9] BRUNET, É. and DERRIDA, B. (2001). Effect of microscopic noise on front propagation. *J. Stat. Phys.* **103** 269–282. MR1828730

[10] BRUNET, E., DERRIDA, B., MUELLER, A. H. and MUNIER, S. (2006). Noisy traveling waves: Effect of selection on genealogies. *Europhys. Lett.* **76** 1–7. MR2299937

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[11] BRUNET, É., DERRIDA, B., MUELLER, A. H. and MUNIER, S. (2007). Effect of selection on ancestry: An exactly soluble case and its phenomenological generalization. *Phys. Rev. E* (3) **76** 041104, 20. MR2365627

[12] BURT, A. (2000). Perspective: Sex, recombination and the efficacy of selection—Was Weismann right? *Evolution* **54** 337–351.

[13] DURRETT, R. and REMENIK, D. (2011). Brunet–Derrida particle systems, free boundary problems and Wiener–Hopf equations. *Ann. Probab.* **39** 2043–2078. MR2932664

[14] ETHERIDGE, A. M. (2000). *An Introduction to Superprocesses*. *University Lecture Series* **20**. Amer. Math. Soc., Providence, RI. MR1779100

[15] GOLDSCHMIDT, C. and MARTIN, J. B. (2005). Random recursive trees and the Bolthausen–Sznitman coalescent. *Electron. J. Probab.* **10** 718–745. MR2164028

[16] GROISMAN, P. and JONCKHEERE, M. (2013) Front propagation and quasi-stationary distributions: The same selection principle? Available at arXiv:1304.4847.

[17] HARRIS, J. W. and HARRIS, S. C. (2007). Survival probabilities for branching Brownian motion with absorption. *Electron. Commun. Probab.* **12** 81–92. MR2300218

[18] ITÔ, K. and MCKEAN, H. P. JR. (1965). *Diffusion Processes and Their Sample Paths*. *Die Grundlehren der Mathematischen Wissenschaften* **Band 125**. Springer, New York. MR0199891

[19] MAILLARD, P. (2012). Branching Brownian motion with selection. Ph.D. thesis, Univ. Pierre et Marie Curie. Available at arXiv:1210.3500.

[20] MAILLARD, P. (2016). Speed and fluctuations of N-particle branching Brownian motion with spatial selection. *Probab. Theory Related Fields* **166** 1061–1173. MR3568046

[21] ROYNETTE, B., VALLOIS, P. and YOR, M. (2009). Penalisations of multidimensional Brownian motion. VI. *ESAIM Probab. Stat.* **13** 152–180. MR2518544

[22] WEISMANN, A. (1889). The significance of sexual reproduction in the theory of natural selection. In *Essays upon Heredity and Kindred Biological Problems* (E. B. Poulton, S. Schönland and A. E. Shipley, eds.) 251–332. Clarendon Press, Oxford.

[23] WILLIAMS, G. C. (1966). *Adaptation and Natural Selection*. Princeton Univ. Press, Princeton, NJ.

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ON DIRECTIONAL DERIVATIVES OF SKOROKHOD MAPS IN CONVEX POLYHEDRAL DOMAINS

BY DAVID LIPSHUTZ[1] AND KAVITA RAMANAN[2]

Brown University

The study of both sensitivity analysis and differentiability of the stochastic flow of a reflected process in a convex polyhedral domain is challenging due to the abrupt change in the nature of the dynamics at the boundary and is further complicated because the boundary is not smooth. These difficulties can be addressed by studying directional derivatives of an associated extended Skorokhod map, which is a deterministic mapping that takes an unconstrained path to a suitably reflected or constrained version. In this work, we develop an axiomatic framework for the analysis of directional derivatives of a large class of Lipschitz continuous extended Skorokhod maps in convex polyhedral domains with oblique directions of reflection. We establish existence of directional derivatives at a path whose reflected version satisfies a certain boundary jitter property, and also show that the right-continuous regularization of such a directional derivative can be characterized as the unique solution to a Skorokhod-type problem, where both the domain and directions of reflection vary (discontinuously) depending on the state of the reflected path. A key step in the analysis is the proof of certain contraction properties for a family of (oblique) derivative projection operators. The results of this paper are used in subsequent work to study differentiability of stochastic flows and sensitivity analysis for a large class of reflected diffusions in convex polyhedral domains.

REFERENCES

[1] ANDRES, S. (2009). Pathwise differentiability for SDEs in a convex polyhedron with oblique reflection. *Ann. Inst. Henri Poincaré Probab. Stat.* **45** 104–116. MR2500230

[2] ANDRES, S. (2011). Pathwise differentiability for SDEs in a smooth domain with reflection. *Electron. J. Probab.* **16** 845–879. MR2793243

[3] ATAR, R., BUDHIRAJA, A. and RAMANAN, K. (2008). Deterministic and stochastic differential inclusions with multiple surfaces of discontinuity. *Probab. Theory Related Fields* **142** 249–283. MR2413272

[4] BANNER, A. D., FERNHOLZ, R. and KARATZAS, I. (2005). Atlas models of equity markets. *Ann. Appl. Probab.* **15** 2296–2330. MR2187296

[5] BILLINGSLEY, P. (1999). *Convergence of Probability Measures*, 2nd ed. Wiley, New York. MR1700749

[6] BISMUT, J.-M. (1981). A generalized formula of Itô and some other properties of stochastic flows. *Z. Wahrsch. Verw. Gebiete* **55** 331–350. MR0608026

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[7] BURDZY, K. (2009). Differentiability of stochastic flow of reflected Brownian motions. *Electron. J. Probab.* **14** 2182–2240. MR2550297

[8] BURDZY, K., CHEN, Z.-Q. and SYLVESTER, J. (2004). The heat equation and reflected Brownian motion in time-dependent domains. *Ann. Probab.* **32** 775–804. MR2039943

[9] BURDZY, K., KANG, W. and RAMANAN, K. (2009). The Skorokhod problem in a time-dependent interval. *Stochastic Process. Appl.* **119** 428–452. MR2493998

[10] BURDZY, K. and NUALART, D. (2002). Brownian motion reflected on Brownian motion. *Probab. Theory Related Fields* **122** 471–493. MR1902187

[11] CHEN, H. and MANDELBAUM, A. (1991). Stochastic discrete flow networks: Diffusion approximations and bottlenecks. *Ann. Probab.* **19** 1463–1519. MR1127712

[12] CHEN, X. (2014). Exact gradient simulation for stochastic fluid networks in steady state. In *Proceedings of the Winter Simulation Conference* 2014 586–594.

[13] COSTANTINI, C. (1992). The Skorohod oblique reflection problem in domains with corners and application to stochastic differential equations. *Probab. Theory Related Fields* **91** 43–70. MR1142761

[14] CUDINA, M. and RAMANAN, K. (2011). Asymptotically optimal controls for time-inhomogeneous networks. *SIAM J. Control Optim.* **49** 611–645. MR2784703

[15] DEUSCHEL, J.-D. and ZAMBOTTI, L. (2005). Bismut–Elworthy's formula and random walk representation for SDEs with reflection. *Stochastic Process. Appl.* **115** 907–925. MR2134484

[16] DIEKER, A. B. and GAO, X. (2014). Sensitivity analysis for diffusion processes constrained to an orthant. *Ann. Appl. Probab.* **24** 1918–1945. MR3226168

[17] DUPUIS, P. and ISHII, H. (1991). On Lipschitz continuity of the solution mapping to the Skorokhod problem, with applications. *Stoch. Stoch. Rep.* **35** 31–62. MR1110990

[18] DUPUIS, P. and RAMANAN, K. (1998). A Skorokhod problem formulation and large deviation analysis of a processor sharing model. *Queueing Syst.* **28** 109–124. MR1628485

[19] DUPUIS, P. and RAMANAN, K. (1999). Convex duality and the Skorokhod problem. I. *Probab. Theory Related Fields* **115** 153–195. MR1720348

[20] DUPUIS, P. and RAMANAN, K. (1999). Convex duality and the Skorokhod problem. II. *Probab. Theory Related Fields* **115** 197–236. MR1720348

[21] DUPUIS, P. and RAMANAN, K. (2000). A multiclass feedback queueing network with a regular Skorokhod problem. *Queueing Syst.* **36** 327–349. MR1823974

[22] ELWORTHY, K. D. (1978). Stochastic dynamical systems and their flows. In *Stochastic Analysis* (*Proc. Internat. Conf.*, *Northwestern Univ.*, *Evanston*, *Ill.*, 1978) (A. Friedman and M. Pinsky, eds.) 79–95. Academic Press, New York. MR0517235

[23] HARRISON, J. M. and REIMAN, M. I. (1981). Reflected Brownian motion on an orthant. *Ann. Probab.* **9** 302–308. MR0606992

[24] HONNAPPA, H., JAIN, R. and WARD, A. R. (2015). A queueing model with independent arrivals, and its fluid and diffusion limits. *Queueing Syst.* **80** 71–103. MR3341683

[25] IKEDA, N. and WATANABE, S. (1981). *Stochastic Differential Equations and Diffusion Processes*. *North-Holland Mathematical Library* **24**. North Holland–Kodansha, Tokyo. MR0637061

[26] KANG, W. and WILLIAMS, R. J. (2007). An invariance principle for semimartingale reflecting Brownian motions in domains with piecewise smooth boundaries. *Ann. Appl. Probab.* **17** 741–779. MR2308342

[27] KUNITA, H. (1981). On the decomposition of solutions of stochastic differential equations. In *Stochastic Integrals* (*Proc. Sympos.*, *Univ. Durham*, *Durham*, 1980) (D. Williams, ed.). *Lecture Notes in Math.* **851** 213–255. Springer, Berlin. MR0620992

[28] KUNITA, H. (1997). *Stochastic Flows and Stochastic Differential Equations*. *Cambridge Studies in Advanced Mathematics* **24**. Cambridge Univ. Press, Cambridge. MR1472487

[29] LIONS, P.-L. and SZNITMAN, A.-S. (1984). Stochastic differential equations with reflecting boundary conditions. *Comm. Pure Appl. Math.* **37** 511–537. MR0745330

[30] LIPSHUTZ, D. and RAMANAN, K. (2016). On directional derivatives of Skorokhod maps in convex polyhedral domains. Preprint. Available at arXiv:1602.01860v1.

[31] LIPSHUTZ, D. and RAMANAN, K. (2017). Pathwise differentiability of reflected diffusions in convex polyhedral domains. Preprint. Available at arXiv:1705.02278v1.

[32] LIPSHUTZ, D. and RAMANAN, K. (2017). A Monte Carlo method for estimating sensitivities of reflected diffusions in convex polyhedral domains. Preprint. Available at arXiv:1711.11506v1.

[33] MANDELBAUM, A. and MASSEY, W. A. (1995). Strong approximations for time-dependent queues. *Math. Oper. Res.* **20** 33–64. MR1320446

[34] MANDELBAUM, A. and RAMANAN, K. (2010). Directional derivatives of oblique reflection maps. *Math. Oper. Res.* **35** 527–558. MR2724063

[35] MÉTIVIER, M. (1982). Pathwise differentiability with respect to a parameter of solutions of stochastic differential equations. In *Seminar on Probability, XVI. Lecture Notes in Math.* **920** 490–502. Springer, Berlin. MR0658709

[36] NYSTRÖM, K. and ÖNSKOG, T. (2010). The Skorohod oblique reflection problem in time-dependent domains. *Ann. Probab.* **38** 2170–2223. MR2683628

[37] PILIPENKO, A. (2013). Differentiability of stochastic reflecting flow with respect to starting point. *Commun. Stoch. Anal.* **7** 17–37. MR3080985

[38] RAMANAN, K. (2006). Reflected diffusions defined via the extended Skorokhod map. *Electron. J. Probab.* **11** 934–992. MR2261058

[39] RAMANAN, K. and REIMAN, M. I. (2008). The heavy traffic limit of an unbalanced generalized processor sharing model. *Ann. Appl. Probab.* **18** 22–58. MR2380890

[40] REIMAN, M. I. (1984). Open queueing networks in heavy traffic. *Math. Oper. Res.* **9** 441–458. MR0757317

[41] SAISHO, Y. (1987). Stochastic differential equations for multidimensional domain with reflecting boundary. *Probab. Theory Related Fields* **74** 455–477. MR0873889

[42] SKOROHOD, A. V. (1961). Stochastic equations for diffusion processes in a bounded region. *Theory Probab. Appl.* **6** 264–274.

[43] TANAKA, H. (1979). Stochastic differential equations with reflecting boundary condition in convex regions. *Hiroshima Math. J.* **9** 163–177. MR0529332

[44] WARREN, J. (2007). Dyson's Brownian motions, intertwining and interlacing. *Electron. J. Probab.* **12** 573–590. MR2299928

[45] WHITT, W. (2002). An Introduction to Stochastic-Process Limits and their Application to Queues. Internet supplement. Available at http://www.columbia.edu/~ww2040/supplement.html.

[46] ZAMBOTTI, L. (2004). Fluctuations for a $\nabla\phi$ interface model with repulsion from a wall. *Probab. Theory Related Fields* **129** 315–339. MR2128236

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SPATIAL GIBBS RANDOM GRAPHS

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Many real-world networks of interest are embedded in physical space. We present a new random graph model aiming to reflect the interplay between the geometries of the graph and of the underlying space. The model favors configurations with small average graph distance between vertices, but adding an edge comes at a cost measured according to the geometry of the ambient physical space. In most cases, we identify the order of magnitude of the average graph distance as a function of the parameters of the model. As the proofs reveal, hierarchical structures naturally emerge from our simple modeling assumptions. Moreover, a critical regime exhibits an infinite number of discontinuous phase transitions.

REFERENCES

[1] ACHLIOPTAS, D. and SIMINELAKIS, P.. Product measure approximation of symmetric graph properties. Preprint. Available at arXiv:1502.07787.

[2] AIELLO, W., BONATO, A., COOPER, C., JANSSEN, J. and PRAŁAT, P. (2008). A spatial web graph model with local influence regions. *Internet Math*. **5** 175–196. MR2560268

[3] AIZENMAN, M., KESTEN, H. and NEWMAN, C. M. (1987). Uniqueness of the infinite cluster and continuity of connectivity functions for short and long range percolation. *Comm. Math. Phys*. **111** 505–531. MR0901151

[4] AIZENMAN, M. and NEWMAN, C. M. (1986). Discontinuity of the percolation density in one-dimensional $1/|x-y|^2$ percolation models. *Comm. Math. Phys*. **107** 611–647. MR0868738

[5] ALDOUS, D. J. (2008). Optimal spatial transportation networks where link costs are sublinear in link capacity. *J. Stat. Mech. Theory Exp*. **2008** P03006.

[6] ALDOUS, D. J. and KENDALL, W. S. (2008). Short-length routes in low-cost networks via Poisson line patterns. *Adv. in Appl. Probab*. **40** 1–21. MR2411811

[7] ALDOUS, D. J. and SHUN, J. (2010). Connected spatial networks over random points and a route-length statistic. *Statist. Sci*. **25** 275–288. MR2791668

[8] BARABÁSI, A.-L. and ALBERT, R. (1999). Emergence of scaling in random networks. *Science* **286** 509–512. MR2091634

[9] BENJAMINI, I. and BERGER, N. (2001). The diameter of long-range percolation clusters on finite cycles. *Random Structures Algorithms* **19** 102–111. MR1848786

[10] BISKUP, M. (2004). On the scaling of the chemical distance in long-range percolation models. *Ann. Probab*. **32** 2938–2977. MR2094435

[11] BISKUP, M. (2011). Graph diameter in long-range percolation. *Random Structures Algorithms* **39** 210–227. MR2850269

[12] CHATTERJEE, S. and DIACONIS, P. (2013). Estimating and understanding exponential random graph models. *Ann. Statist*. **41** 2428–2461. MR3127871

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[13] COOPER, C., FRIEZE, A. and PRAŁAT, P. (2014). Some typical properties of the spatial preferred attachment model. *Internet Math*. **10** 116–136. MR3274542
[14] COPPERSMITH, D., GAMARNIK, D. and SVIRIDENKO, M. (2002). The diameter of a long-range percolation graph. *Random Structures Algorithms* **21** 1–13. MR1913075
[15] DEIJFEN, M., VAN DER HOFSTAD, R. and HOOGHIEMSTRA, G. (2013). Scale-free percolation. *Ann. Inst. Henri Poincaré Probab. Stat.* **49** 817–838. MR3112435
[16] DING, J. and SLY, A. Distances in critical long range percolation. Preprint. Available at arXiv:1303.3995.
[17] FLAXMAN, A. D., FRIEZE, A. M. and VERA, J. (2006). A geometric preferential attachment model of networks. *Internet Math*. **3** 187–205. MR2321829
[18] GANDOLFI, A., KEANE, M. S. and NEWMAN, C. M. (1992). Uniqueness of the infinite component in a random graph with applications to percolation and spin glasses. *Probab. Theory Related Fields* **92** 511–527. MR1169017
[19] GANTERT, N., RAMANAN, K. and REMBART, F. (2014). Large deviations for weighted sums of stretched exponential random variables. *Electron. Commun. Probab.* **19** no. 41, 14. MR3233203
[20] JACOB, E. and MÖRTERS, P. (2015). Spatial preferential attachment networks: Power laws and clustering coefficients. *Ann. Appl. Probab.* **25** 632–662. MR3313751
[21] JACOB, E. and MÖRTERS, P. (2017). Robustness of scale-free spatial networks. *Ann. Probab.* **45** 1680–1722. MR3650412
[22] JANSSEN, J., PRAŁAT, P. and WILSON, R. (2013). Geometric graph properties of the spatial preferred attachment model. *Adv. in Appl. Math*. **50** 243–267. MR3003346
[23] JORDAN, J. (2010). Degree sequences of geometric preferential attachment graphs. *Adv. in Appl. Probab.* **42** 319–330. MR2675104
[24] KELLER, E. F. (2005). Revisiting "scale-free" networks. *BioEssays* **27** 1060–1068.
[25] LI, L., ALDERSON, D., WILLINGER, W. and DOYLE, J. (2004). A first-principles approach to understanding the Internet's router-level topology. *Comput. Commun. Rev.* **34** 3–14.
[26] LOUF, R., JENSEN, P. and BARTHELEMY, M. (2013). Emergence of hierarchy in cost-driven growth of spatial networks. *Proc. Natl. Acad. Sci. USA* **110** 8824–8829. MR3082275
[27] MENGISTU, H., HUIZINGA, J., MOURET, J.-B. and CLUNE, J. (2016). The evolutionary origins of hierarchy. *PLoS Comput. Biol.* **12** e1004829.
[28] MOUNTFORD, T. and MOURRAT, J.-C. (2013). Lyapunov exponents of random walks in small random potential: The lower bound. *Comm. Math. Phys.* **323** 1071–1120. MR3106503
[29] NEWMAN, C. M. and SCHULMAN, L. S. (1986). One-dimensional $1/|j-i|^s$ percolation models: The existence of a transition for $s \leq 2$. *Comm. Math. Phys.* **104** 547–571. MR0841669
[30] SCHULMAN, L. S. (1983). Long range percolation in one dimension. *J. Phys. A* **16** L639–L641. MR0723249
[31] SIMON, H. A. (1955). On a class of skew distribution functions. *Biometrika* **42** 425–440. MR0073085
[32] SIMON, H. A. (1960). Some further notes on a class of skew distribution functions. *Inf. Control* **3** 80–88. MR0130733
[33] SPORNS, O. and KÖTTER, R. (2004). Motifs in brain networks. *PLoS Biology* **2** e369.
[34] SPORNS, O. and TONONI, G. (2001). Classes of network connectivity and dynamics. *Complexity* **7** 28–38. MR1889195
[35] SPORNS, O., TONONI, G. and EDELMAN, G. M. (2000). Connectivity and complexity: The relationship between neuroanatomy and brain dynamics. *Neural Netw.* **13** 909–922.
[36] SPORNS, O., TONONI, G. and EDELMAN, G. M. (2000). Theoretical neuroanatomy: Relating anatomical and functional connectivity in graphs and cortical connection matrices. *Cereb. Cortex* **10** 127–141.
[37] VAN DER HOFSTAD, R. Random graphs and complex networks. Available at http://www.win.tue.nl/~rhofstad/NotesRGCN.html.

[38] VERMA, T., RUSSMANN, F., ARAÚJO, N., NAGLER, J. and HERRMANN, H. (2016). Emergence of core-peripheries in networks. *Nat. Commun.* **7** 10441.
[39] YULE, G. U. (1925). A mathematical theory of evolution, based on the conclusions of Dr. JC Willis, FRS. *Philos. Trans. R. Soc. Lond. B, Biol. Sci.* **213** 21–87.

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ON THE STABILITY AND THE UNIFORM PROPAGATION OF CHAOS PROPERTIES OF ENSEMBLE KALMAN–BUCY FILTERS

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The ensemble Kalman filter is a sophisticated and powerful data assimilation method for filtering high dimensional problems arising in fluid mechanics and geophysical sciences. This Monte Carlo method can be interpreted as a mean-field McKean–Vlasov-type particle interpretation of the Kalman–Bucy diffusions. In contrast to more conventional particle filters and nonlinear Markov processes, these models are designed in terms of a diffusion process with a diffusion matrix that depends on particle covariance matrices.

Besides some recent advances on the stability of nonlinear Langevin-type diffusions with drift interactions, the long-time behaviour of models with interacting diffusion matrices and conditional distribution interaction functions has never been discussed in the literature. One of the main contributions of the article is to initiate the study of this new class of models. The article presents a series of new functional inequalities to quantify the stability of these nonlinear diffusion processes.

In the same vein, despite some recent contributions on the convergence of the ensemble Kalman filter when the number of sample tends to infinity very little is known on stability and the long-time behaviour of these mean-field interacting type particle filters. The second contribution of this article is to provide uniform propagation of chaos properties as well as $\mathbb{L}_n$-mean error estimates w.r.t. to the time horizon. Our regularity condition is also shown to be sufficient and necessary for the uniform convergence of the ensemble Kalman filter.

The stochastic analysis developed in this article is based on an original combination of functional inequalities and Foster–Lyapunov techniques with coupling, martingale techniques, random matrices and spectral analysis theory.

REFERENCES

[1] ABOU-KANDIL, H., FREILING, G., IONESCU, V. and JANK, G. (2003). *Matrix Riccati Equations in Control and Systems Theory*. *Systems & Control*: *Foundations & Applications*. Birkhäuser, Basel. MR1997753

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[2] ALLEN, J. I., EKNES, M. and EVENSEN, G. (2002). An Ensemble Kalman filter with a complex marine ecosystem model: Hindcasting phytoplankton in the Cretan Sea. *Ann. Geophys.* **20** 1–13.

[3] ANDERSON, B. D. O. (1971). Stability properties of Kalman–Bucy filters. *J. Franklin Inst.* **291** 137–144. MR0281515

[4] ANDERSON, J. L. (2001). An ensemble adjustment Kalman filter for data assimilation. *Mon. Weather Rev.* **129** 2884–2903.

[5] ANDERSON, J. L. (2003). A local least squares framework for ensemble filtering. *Mon. Weather Rev.* **131** 634–642.

[6] BARAS, J. S., BENSOUSSAN, A. and JAMES, M. R. (1988). Dynamic observers as asymptotic limits of recursive filters: Special cases. *SIAM J. Appl. Math.* **48** 1147–1158. MR0960476

[7] BERNSTEIN, D. S. (1988). Inequalities for the trace of matrix exponentials. *SIAM J. Matrix Anal. Appl.* **9** 156–158. MR0938494

[8] BERRY, T. and HARLIM, J. (2014). Linear theory for filtering nonlinear multiscale systems with model error. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **470** 20140168, 25. MR3209215

[9] BISHOP, A. N. and DEL MORAL, P. (2016). On the stability of Kalman–Bucy diffusion processes. Available at arXiv:1610.04686.

[10] BITTANTI, S., LAUB, A. J. and WILLEMS, J. C. (1991). *The Riccati Equation. Communications and Control Engineering Series*. Springer, Berlin.

[11] BOLLEY, F., GUILLIN, A. and MALRIEU, F. (2010). Trend to equilibrium and particle approximation for a weakly selfconsistent Vlasov–Fokker–Planck equation. *M2AN Math. Model. Numer. Anal.* **44** 867–884. MR2731396

[12] BOUGEROL, P. and FAKHFAKH, S. (1996). A note on the stability of the Kalman–Bucy filter with randomly time-varying parameters. *J. Math. Sci.* **78** 28–33. MR1381032

[13] BROCKETT, R. W. (2015). *Finite Dimensional Linear Systems. Classics in Applied Mathematics* **74**. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA. Reprint of the 1970 original. MR3486166

[14] BUCY, R. S. (1965). Nonlinear filtering theory. *IEEE Trans. Automat. Control* **10** 198–198.

[15] BUCY, R. S. (1967). Global theory of the Riccati equation. *J. Comput. System Sci.* **1** 349–361. MR0229916

[16] BURGERS, G., VAN LEEUWEN, P. J. and EVENSEN, G. (1998). Analysis scheme in the ensemble Kalman filter. *Mon. Weather Rev.* **126** 1719–1724.

[17] CATTIAUX, P., GUILLIN, A. and MALRIEU, F. (2008). Probabilistic approach for granular media equations in the non-uniformly convex case. *Probab. Theory Related Fields* **140** 19–40. MR2357669

[18] COPPEL, W. A. (1978). *Dichotomies in Stability Theory. Lecture Notes in Mathematics* **629**. Springer, Berlin–New York. MR0481196

[19] COSTA, E. F. (2008). On the stability of the recursive Kalman filter for linear time-invariant systems. In *Proceedings of the IEEE* 2008 *American Control Conference* 1286–1291, Seattle, WA.

[20] DEL MORAL, P. (2004). *Feynman–Kac Formulae. Genealogical and Interacting Particle Systems with Applications. Probability and Its Applications* (*New York*). Springer, New York. MR2044973

[21] DEL MORAL, P. (2013). *Mean Field Simulation for Monte Carlo Integration. Monographs on Statistics and Applied Probability* **126**. CRC Press, Boca Raton, FL. MR3060209

[22] DEL MORAL, P. and TUGAUT, J. (2013). Uniform propagation of chaos and creation of chaos for a class of nonlinear diffusions. https://hal.archives-ouvertes.fr/hal-00798813.

[23] DYDA, B. and TUGAUT, J. (2017). Exponential rate of convergence independent of the dimension in a mean-field system of particles. *Probab. Math. Statist.* **37** 145–161. MR3652205

[24] EINICKE, G. (2012). Continuous-time minimum-variance filtering. In *Smoothing*, *Filtering and Prediction—Estimating the Past*, *Present and Future*. InTech, London.

[25] EKNES, M. and EVENSEN, G. (2002). An Ensemble Kalman filter with a 1-D marine ecosystem model. *JMS* **36** 75–100.

[26] EVENSEN, G. (1994). Sequential data assimilation with a non-linear quasi-geostrophic model using Monte Carlo methods to forecast error statistics. *J. Geophys. Res.* **99** 10143–10162.

[27] EVENSEN, G., HOVE, J., MEISINGSET, H. C., REISO, E. and SEIM, K. S. (2007). Using the EnKF for assisted history matching of a North Sea reservoir model. *SPE* **106184**.

[28] GIVENS, C. R. and SHORTT, R. M. (1984). A class of Wasserstein metrics for probability distributions. *Michigan Math. J.* **31** 231–240. MR0752258

[29] GOTTWALD, G. and MAJDA, A. J. (2013). A mechanism for catastrophic filter divergence in data assimilation for sparse observation networks. *Nonlinear Process. Geophys.* **20** 705–712.

[30] HARLIM, J. and HUNT, B. (2005). Local Ensemble Transform Kalman Filter: An Efficient Scheme for Assimilating Atmospheric Data. Preprint. Available at https://www.atmos.umd.edu/ekalnay/pubs/harlim_hunt05.pdf.

[31] HARLIM, J. and MAJDA, A. J. (2010). Catastrophic filter divergence in filtering nonlinear dissipative systems. *Commun. Math. Sci.* **8** 27–43. MR2655912

[32] HOUTEKAMER, P. and MITCHELL, H. L. (1998). Data assimilation using an ensemble Kalman filter technique. *Mon. Weather Rev.* **126** 796–811.

[33] ILCHMANN, A., OWENS, D. H. and PRÄTZEL-WOLTERS, D. (1987). Sufficient conditions for stability of linear time-varying systems. *Systems Control Lett.* **9** 157–163. MR0906235

[34] JOHNS, C. J. and MANDEL, J. (2008). A two-stage ensemble Kalman filter for smooth data assimilation. *Environ. Ecol. Stat.* **15** 101–110. MR2412683

[35] KALNAY, E. (2003). *Atmospheric Modeling*, *Data Assimilation*, *and Predictability*. Cambridge Univ. Press, Cambridge.

[36] KELLY, D., MAJDA, A. J. and TONG, X. T. (2015). Concrete ensemble Kalman filters with rigorous catastrophic filter divergence. *Proc. Natl. Acad. Sci. USA* **112** 10589–10594.

[37] KRAUSE, J. M. and KUMAR, K. S. P. (1986). An alternate stability analysis framework for adaptive control. *Systems Control Lett.* **7** 19–24. MR0833061

[38] KREISSELMEIER, G. (1985). An approach to stable indirect adaptive control. *Automatica J. IFAC* **21** 425–431. MR0798187

[39] KUSHNER, H. J. (1964). On the differential equations satisfied by conditional probablitity densities of Markov processes, with applications. *J. Soc. Indust. Appl. Math. Ser. A Control* **2** 106–119. MR0180407

[40] KWAKERNAAK, H. and SIVAN, R. (1972). *Linear Optimal Control Systems*. Wiley, New York. MR0406607

[41] LANCASTER, P. and RODMAN, L. (1995). *Algebraic Riccati Equations*. *Oxford Science Publications*. The Clarendon Press, New York. MR1367089

[42] LAWSON, J. and LIM, Y. (2007). A Birkhoff contraction formula with applications to Riccati equations. *SIAM J. Control Optim.* **46** 930–951. MR2338433

[43] LE GLAND, F., MONBET, V. and TRAN, V.-D. (2011). Large sample asymptotics for the ensemble Kalman filter. In *The Oxford Handbook of Nonlinear Filtering* 598–631. Oxford Univ. Press, Oxford. MR2884610

[44] LISAETER, K. A., ROSANOVA, J. and EVENSEN, G. (2003). Assimilation of ice concentration in a coupled ice-ocean model, using the Ensemble Kalman filter. *Ocean Dyn.* **53** 368–388.

[45] LUENBERGER, D. G. (1966). Observers for multivariable systems. *IEEE Trans. Automat. Control* **AC-11** 190–199.

[46] MAJDA, A. J. and HARLIM, J. (2012). *Filtering Complex Turbulent Systems*. Cambridge Univ. Press, Cambridge. MR2934167

[47] MALRIEU, F. (2001). Logarithmic Sobolev inequalities for some nonlinear PDE's. *Stochastic Process. Appl.* **95** 109–132. MR1847094

[48] MANDEL, J., COBB, L. and BEEZLEY, J. D. (2011). On the convergence of the ensemble Kalman filter. *Appl. Math.* **56** 533–541. MR2886236

[49] MCKEAN, H. P. JR. (1966). A class of Markov processes associated with nonlinear parabolic equations. *Proc. Natl. Acad. Sci. USA* **56** 1907–1911. MR0221595

[50] MÉLÉARD, S. (1996). Asymptotic behaviour of some interacting particle systems; McKean–Vlasov and Boltzmann models. In *Probabilistic Models for Nonlinear Partial Differential Equations* (*Montecatini Terme*, 1995). *Lecture Notes in Math.* **1627** 42–95. Springer, Berlin. MR1431299

[51] MOLER, C. and VAN LOAN, C. (2003). Nineteen dubious ways to compute the exponential of a matrix, twenty-five years later. *SIAM Rev.* **45** 3–49. MR1981253

[52] NAEVDAL, G., JOHNSEN, L. M., AANONSEN, S. I. and VEFRING, E. H. (2005). Reservoir monitoring and continuous model updating using ensemble Kalman filter. *SPE Journal* **10** 66–74.

[53] OCONE, D. and PARDOUX, E. (1996). Asymptotic stability of the optimal filter with respect to its initial condition. *SIAM J. Control Optim.* **34** 226–243. MR1372912

[54] OTT, E., HUNT, B. R., SZUNYOGH, I., ZIMIN, A. V., KOSTELICH, E. J., CORAZZA, M., KALNAY, E., PATIL, D. and YORKE, J. A. (2004). A local ensemble Kalman filter for atmospheric data assimilation. *Tellus, Ser. A Dyn. Meteorol. Oceanogr.* **56** 415–428.

[55] POUBELLE, M.-A., PETERSEN, I. R., GEVERS, M. R. and BITMEAD, R. R. (1986). A miscellany of results on an equation of Count J. F. Riccati. *IEEE Trans. Automat. Control* **31** 651–654. MR0844922

[56] ROSENBROCK, H. H. (1963). The stability of linear time-dependent control systems. *J. Electron. Control* **15** 73–80. MR0159085

[57] SEILER, A., EVENSEN, G., SKJERVHEIM, J.-A., HOVE, J. and VABØ, J. G. (2011). Using the ensemble Kalman filter for history matching and uncertainty quantification of complex reservoir models. In *Large-Scale Inverse Problems and Quantification of Uncertainty. Wiley Ser. Comput. Stat.* 247–271. Wiley, Chichester. MR2856659

[58] SKJERVHEIM, J.-A., EVENSEN, G., AANONSEN, S. I., RUUD, B. O. and JOHANSEN, T. A. (2005). Incorporating 4D seismic data in reservoir simulation models using ensemble Kalman filter. *SPE* **12** 95789.

[59] SNYDER, D. A. (2016). On the Relation of Schatten Norms and the Thompson Metric. Available at ArXiv:1608.03301.

[60] SONTAG, E. D. (1998). *Mathematical Control Theory: Deterministic Finite-Dimensional Systems*, 2nd ed. *Texts in Applied Mathematics* **6**. Springer, New York. MR1640001

[61] STRATONOVIČ, R. L. (1960). Conditional Markov processes. *Teor. Verojatnost. i Primenen.* **5** 172–195. MR0137157

[62] SU, J., LI, B. and CHEN, W.-H. (2015). On existence, optimality and asymptotic stability of the Kalman filter with partially observed inputs. *Automatica J. IFAC* **53** 149–154. MR3318583

[63] SZNITMAN, A.-S. (1991). Topics in propagation of chaos. In *École D'Été de Probabilités de Saint-Flour XIX—1989. Lecture Notes in Math.* **1464** 165–251. Springer, Berlin. MR1108185

[64] TONG, X. T., MAJDA, A. J. and KELLY, D. (2016). Nonlinear stability and ergodicity of ensemble based Kalman filters. *Nonlinearity* **29** 657–691. MR3461611

[65] VAN LOAN, C. (1977). The sensitivity of the matrix exponential. *SIAM J. Numer. Anal.* **14** 971–981. MR0468137

[66] VESELIĆ, K. (2011). *Damped Oscillations of Linear Systems: A Mathematical Introduction. Lecture Notes in Math.* **2023**. Springer, Heidelberg. MR2798348

[67] WEN, X.-H. and CHEN, W. H. (2005). Real-time reservoir model updating using ensemble Kalman filter SPE-92991-MS. In *SPE Reservoir Simulation Symposium*, *the Woodlands*, *Texas*.
[68] WONHAM, W. M. (1968). On a matrix Riccati equation of stochastic control. *SIAM J*. *Control* **6** 681–697. MR0239161
[69] YANG, X. (2000). A matrix trace inequality. *J*. *Math*. *Anal*. *Appl*. **250** 372–374. MR1893896
[70] YANG, X. M., YANG, X. Q. and TEO, K. L. (2001). A matrix trace inequality. *J*. *Math*. *Anal*. *Appl*. **263** 327–331. MR1865285
[71] ZAKAI, M. (1969). On the optimal filtering of diffusion processes. *Z. Wahrsch*. *Verw. Gebiete* **11** 230.
[72] ZELIKIN, M. I. (1991). On the theory of the matrix Riccati equation *Mat*. *Sb.* **182** 970–984. Available at http://www.mathnet.ru/links/5052aa2d9c921f105d61c94a204912c1/sm1336.pdf.
[73] ZIEBUR, A. D. (1970). On determining the structure of A by analyzing e^{At}. *SIAM Rev.* **12** 98–102. MR0254074

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A LARGE SCALE ANALYSIS OF UNRELIABLE STOCHASTIC NETWORKS

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The problem of reliability of a large distributed system is analyzed via a new mathematical model. A typical framework is a system where a set of files are duplicated on several data servers. When one of these servers breaks down, all copies of files stored on it are lost. In this way, repeated failures may lead to losses of files. The efficiency of such a network is directly related to the performances of the mechanism used to duplicate files on servers. In this paper, we study the evolution of the network using a natural duplication policy giving priority to the files with the least number of copies.

We investigate the asymptotic behavior of the network when the number N of servers is large. The analysis is complicated by the large dimension of the state space of the empirical distribution of the state of the network. A stochastic model of the evolution of the network which has values in state space whose dimension does not depend on N is introduced. Despite this description does not have the Markov property, it turns out that it is converging in distribution, when the number of nodes goes to infinity, to a nonlinear Markov process. The rate of decay of the network, which is the key characteristic of interest of these systems, can be expressed in terms of this asymptotic process. The corresponding mean-field convergence results are established. A lower bound on the exponential decay, with respect to time, of the fraction of the number of initial files with at least one copy is obtained.

REFERENCES

[1] ALDOUS, D. and DIACONIS, P. (1987). Strong uniform times and finite random walks. *Adv. in Appl. Math.* **8** 69–97. MR0876954

[2] ALFONSI, A., CORBETTA, J. and JOURDAIN, B. (2016). Evolution of the Wasserstein distance between the marginals of two Markov processes. Available at arXiv:1606.02994.

[3] BILLINGSLEY, P. (1999). *Convergence of Probability Measures*, 2nd ed. Wiley, New York. MR1700749

[4] BRÉMAUD, P. (1981). *Point Processes and Queues*: *Martingale Dynamics*. Springer, New York. MR0636252

[5] BROWN, T. (1978). A martingale approach to the Poisson convergence of simple point processes. *Ann. Probab.* **6** 615–628. MR0482991

[6] CAPUTO, P., DAI PRA, P. and POSTA, G. (2009). Convex entropy decay via the Bochner–Bakry–Emery approach. *Ann. Inst. Henri Poincaré Probab. Stat.* **45** 734–753. MR2548501

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[7] CARRILLO, J. A., MCCANN, R. J. and VILLANI, C. (2003). Kinetic equilibration rates for granular media and related equations: Entropy dissipation and mass transportation estimates. *Rev. Mat. Iberoam.* **19** 971–1018. MR2053570

[8] CATTIAUX, P., GUILLIN, A. and MALRIEU, F. (2008). Probabilistic approach for granular media equations in the non-uniformly convex case. *Probab. Theory Related Fields* **140** 19–40. MR2357669

[9] CHUN, B.-G., DABEK, F., HAEBERLEN, A., SIT, E., WEATHERSPOON, H., KAASHOEK, M. F., KUBIATOWICZ, J. and MORRIS, R. (2006). Efficient replica maintenance for distributed storage systems. *NSDI* 45–58.

[10] DAWSON, D. A. (1993). Measure-valued Markov processes. In *École D'Été de Probabilités de Saint-Flour XXI*—1991. *Lecture Notes in Math.* **1541** 1–260. Springer, Berlin. MR1242575

[11] DESVILLETTES, L. and VILLANI, C. (2001). On the trend to global equilibrium in spatially inhomogeneous entropy-dissipating systems: The linear Fokker–Planck equation. *Comm. Pure Appl. Math.* **54** 1–42. MR1787105

[12] FALLAT, S. M. and JOHNSON, C. R. (2011). *Totally Nonnegative Matrices*. Princeton Univ. Press, Princeton, NJ. MR2791531

[13] FEUILLET, M. and ROBERT, P. (2014). A scaling analysis of a transient stochastic network. *Adv. in Appl. Probab.* **46** 516–535. MR3215544

[14] HORN, R. A. and JOHNSON, C. R. (1990). *Matrix Analysis*. Cambridge Univ. Press, Cambridge. MR1084815

[15] JACOD, J. (1979). *Calcul Stochastique et Problèmes de Martingales*. *Lecture Notes in Math.* **714**. Springer, Berlin. MR0542115

[16] JOULIN, A. (2009). A new Poisson-type deviation inequality for Markov jump processes with positive Wasserstein curvature. *Bernoulli* **15** 532–549. MR2543873

[17] KASAHARA, Y. and WATANABE, S. (1986). Limit theorems for point processes and their functionals. *J. Math. Soc. Japan* **38** 543–574. MR0845720

[18] KINGMAN, J. F. C. (1993). *Poisson Processes*. *Oxford Studies in Probability* **3**. Oxford Univ. Press, New York. MR1207584

[19] MARKOWICH, P. A. and VILLANI, C. (2000). On the trend to equilibrium for the Fokker–Planck equation: An interplay between physics and functional analysis. *Mat. Contemp.* **19** 1–29. VI Workshop on Partial Differential Equations, Part II (Rio de Janeiro, 1999). MR1812873

[20] MEYN, S. P. and TWEEDIE, R. L. (1993). *Markov Chains and Stochastic Stability*. Springer, London. MR1287609

[21] NUMMELIN, E. (1984). *General Irreducible Markov Chains and Nonnegative Operators*. *Cambridge Tracts in Mathematics* **83**. Cambridge Univ. Press, Cambridge. MR0776608

[22] OLLIVIER, Y. (2009). Ricci curvature of Markov chains on metric spaces. *J. Funct. Anal.* **256** 810–864. MR2484937

[23] PICCONI, F., BAYNAT, B. and SENS, P. (2007). An analytical estimation of durability in DHTs. In *Distributed Computing and Internet Technology* (T. Janowski and H. Mohanty, eds.). *Lecture Notes in Computer Science* **4882** 184–196. Springer, Berlin.

[24] PINHEIRO, E., WEBER, W.-D. and BARROSO, L. A. (2007). Failure trends in a large disk drive population. In *5th USENIX Conference on File and Storage Technologies (FAST'07)* 17–29. Available at http://research.google.com/archive/disk_failures.pdf.

[25] RAMABHADRAN, S. and PASQUALE, J. (2006). Analysis of long-running replicated systems. In *Proceedings IEEE INFOCOM* 2006. 25*TH IEEE International Conference on Computer Communications* 1–9, IEEE, Barcelona.

[26] RHEA, S., GODFREY, B., KARP, B., KUBIATOWICZ, J., RATNASAMY, S., SHENKER, S., STOICA, I. and YU, H. (2005). OpenDHT: A public DHT service and its uses. In *Proceedings of SIGCOMM* 73–84.

[27] ROGERS, L. C. G. and WILLIAMS, D. (1987). *Diffusions, Markov Processes, and Martingales. Vol. 2: Itô Calculus*. Wiley, New York. MR0921238

[28] ROWSTRON, A. and DRUSCHEL, P. (2001). Storage management and caching in PAST, a large-scale, persistent peer-to-peer storage utility. In *Proceedings of SOSP* 188–201. ACM, New York.

[29] SUN, W., FEUILLET, M. and ROBERT, P. (2016). Analysis of large unreliable stochastic networks. *Ann. Appl. Probab.* **26** 2959–3000. MR3563199

[30] SUN, WEN, SIMON, V., MONNET, S., ROBERT, P. and SENS, P. (2017). Analysis of a stochastic model of replication in large distributed storage systems: A mean-field approach, ACM-Sigmetrics (Urbana-Champaign, IL). ACM, New York.

[31] SZNITMAN, A.-S. (1991). Topics in propagation of chaos. In *École D'Été de Probabilités de Saint-Flour XIX—1989*. *Lecture Notes in Math*. **1464** 165–251. Springer, Berlin. MR1108185

[32] THAI, M.-N. (2015). Birth and death process in mean field type interaction. Available at arXiv:1510.03238.

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REFLECTED BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS WITH RESISTANCE

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In this article, we study a class of reflected backward stochastic differential equations (introduced in El Karoui et al. [*Ann. Probab.* **25** (1997) 702–737], RBSDE for short) with nonlinear resistance by means of Skorohod's equation. The advantage of this approach lies in its pathwise nature and, therefore, provides additional information about solutions of RBSDE. As an application of our approach, we will consider reflected backward problems with resistance as well. This class of RBSDEs possess significance in the super-hedging with wealth constraint.

REFERENCES

[1] BANK, P. and EL KAROUI, N. (2004). A stochastic representation theorem with applications to optimization and obstacle problems. *Ann. Probab.* **32** 1030–1067. MR2044673

[2] BISMUT, J.-M. (1976). Théorie probabiliste du contrôle des diffusions. *Mem. Amer. Math. Soc.* **4** xiii+130. MR0453161

[3] EL KAROUI, N., KAPOUDJIAN, C., PARDOUX, E., PENG, S. and QUENEZ, M. C. (1997). Reflected solutions of backward SDE's, and related obstacle problems for PDE's. *Ann. Probab.* **25** 702–737. MR1434123

[4] EL KAROUI, N., PENG, S. and QUENEZ, M. C. (1997). Backward stochastic differential equations in finance. *Math. Finance* **7** 1–71. MR1434407

[5] EL KAROUI, N. and QUENEZ, M.-C. (1995). Dynamic programming and pricing of contingent claims in an incomplete market. *SIAM J. Control Optim.* **33** 29–66. MR1311659

[6] HAMADÈNE, S. (2002). Reflected BSDE's with discontinuous barrier and application. *Stoch. Stoch. Rep.* **74** 571–596. MR1943580

[7] HE, S. W., WANG, J. G. and YAN, J. A. (1992). *Semimartingale Theory and Stochastic Calculus*. Kexue Chubanshe (Science Press), Beijing; CRC Press, Boca Raton, FL. MR1219534

[8] KARATZAS, I. and SHREVE, S. E. (1991). *Brownian Motion and Stochastic Calculus*, 2nd ed. *Graduate Texts in Mathematics* **113**. Springer, New York. MR1121940

[9] KOBYLANSKI, M., LEPELTIER, J. P., QUENEZ, M. C. and TORRES, S. (2002). Reflected BSDE with superlinear quadratic coefficient. *Probab. Math. Statist.* **22** 51–83. MR1944142

[10] LEPELTIER, J.-P., MATOUSSI, A. and XU, M. (2005). Reflected backward stochastic differential equations under monotonicity and general increasing growth conditions. *Adv. in Appl. Probab.* **37** 134–159. MR2135157

[11] LEPELTIER, J.-P. and XU, M. (2005). Penalization method for reflected backward stochastic differential equations with one r.c.l.l. barrier. *Statist. Probab. Lett.* **75** 58–66. MR2185610

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[12] LEPELTIER, J.-P. and XU, M. (2007). Reflected BSDE with quadratic growth and unbounded terminal value. arXiv:0711.06191.

[13] MA, J. and WANG, Y. (2009). On variant reflected backward SDEs, with applications. *J. Appl. Math. Stoch. Anal.* Art. ID 854768, 26. MR2511615

[14] MATOUSSI, A. (1997). Reflected solutions of backward stochastic differential equations with continuous coefficient. *Statist. Probab. Lett.* **34** 347–354. MR1467440

[15] PARDOUX, É. and PENG, S. G. (1990). Adapted solution of a backward stochastic differential equation. *Systems Control Lett.* **14** 55–61. MR1037747

[16] PENG, S. and XU, M. (2005). The smallest g-supermartingale and reflected BSDE with single and double L^2 obstacles. *Ann. Inst. Henri Poincaré Probab. Stat.* **41** 605–630. MR2139035

[17] PENG, S. G. (1991). Probabilistic interpretation for systems of quasilinear parabolic partial differential equations. *Stoch. Stoch. Rep.* **37** 61–74. MR1149116

[18] XU, M. (2008). Backward stochastic differential equations with reflection and weak assumptions on the coefficients. *Stochastic Process. Appl.* **118** 968–980. MR2418253

[19] ZHANG, J. (2004). A numerical scheme for BSDEs. *Ann. Appl. Probab.* **14** 459–488. MR2023027

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ZOOMING IN ON A LÉVY PROCESS AT ITS SUPREMUM

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Let M and τ be the supremum and its time of a Lévy process X on some finite time interval. It is shown that zooming in on X at its supremum, that is, considering $((X_{\tau+t\varepsilon} - M)/a_\varepsilon)_{t\in\mathbb{R}}$ as $\varepsilon \downarrow 0$, results in $(\xi_t)_{t\in\mathbb{R}}$ constructed from two independent processes having the laws of some self-similar Lévy process $\widehat{X}$ conditioned to stay positive and negative. This holds when X is in the domain of attraction of $\widehat{X}$ under the zooming-in procedure as opposed to the classical zooming out [*Trans. Amer. Math. Soc.* **104** (1962) 62–78]. As an application of this result, we establish a limit theorem for the discretization errors in simulation of supremum and its time, which extends the result in [*Ann. Appl. Probab.* **5** (1995) 875–896] for a linear Brownian motion. Additionally, complete characterization of the domains of attraction when zooming in on a Lévy process is provided.

REFERENCES

ASMUSSEN, S., GLYNN, P. and PITMAN, J. (1995). Discretization error in simulation of one-dimensional reflecting Brownian motion. *Ann. Appl. Probab.* **5** 875–896. MR1384357

ASMUSSEN, S. and ROSIŃSKI, J. (2001). Approximations of small jumps of Lévy processes with a view towards simulation. *J. Appl. Probab.* **38** 482–493. MR1834755

AURZADA, F., DÖRING, L. and SAVOV, M. (2013). Small time Chung-type LIL for Lévy processes. *Bernoulli* **19** 115–136. MR3019488

BARCZY, M. and BERTOIN, J. (2011). Functional limit theorems for Lévy processes satisfying Cramér's condition. *Electron. J. Probab.* **16** 2020–2038. MR2851054

BERTOIN, J. (1993). Splitting at the infimum and excursions in half-lines for random walks and Lévy processes. *Stochastic Process. Appl.* **47** 17–35. MR1232850

BERTOIN, J. (1996). *Lévy Processes. Cambridge Tracts in Mathematics* **121**. Cambridge Univ. Press, Cambridge. MR1406564

BERTOIN, J., DONEY, R. A. and MALLER, R. A. (2008). Passage of Lévy processes across power law boundaries at small times. *Ann. Probab.* **36** 160–197. MR2370602

BINGHAM, N. H., GOLDIE, C. M. and TEUGELS, J. L. (1989). *Regular Variation. Encyclopedia of Mathematics and Its Applications* **27**. Cambridge Univ. Press, Cambridge. MR1015093

BLUMENTHAL, R. M. and GETOOR, R. K. (1961). Sample functions of stochastic processes with stationary independent increments. *J. Math. Mech.* **10** 493–516. MR0123362

BROADIE, M., GLASSERMAN, P. and KOU, S. (1997). A continuity correction for discrete barrier options. *Math. Finance* **7** 325–349. MR1482707

BROADIE, M., GLASSERMAN, P. and KOU, S. G. (1999). Connecting discrete and continuous path-dependent options. *Finance Stoch.* **3** 55–82. MR1805321

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CABALLERO, M. E. and CHAUMONT, L. (2006). Conditioned stable Lévy processes and the Lamperti representation. *J. Appl. Probab.* **43** 967–983. MR2274630

CARAVENNA, F. and CHAUMONT, L. (2008). Invariance principles for random walks conditioned to stay positive. *Ann. Inst. Henri Poincaré Probab. Stat.* **44** 170–190. MR2451576

CHAUMONT, L. (1996). Conditionings and path decompositions for Lévy processes. *Stochastic Process. Appl.* **64** 39–54. MR1419491

CHAUMONT, L. (2013). On the law of the supremum of Lévy processes. *Ann. Probab.* **41** 1191–1217. MR3098676

CHAUMONT, L. and DONEY, R. A. (2005). On Lévy processes conditioned to stay positive. *Electron. J. Probab.* **10** 948–961. MR2164035

CHAUMONT, L. and DONEY, R. A. (2010). Invariance principles for local times at the maximum of random walks and Lévy processes. *Ann. Probab.* **38** 1368–1389. MR2663630

CHEN, A. (2011). Sampling error of the supremum of a Lévy process. Ph.D. thesis, Univ. Illinois at Urbana-Champaign, Champaign, IL.

COVO, S. (2009). On approximations of small jumps of subordinators with particular emphasis on a Dickman-type limit. *J. Appl. Probab.* **46** 732–755. MR2562319

DIA, E. H. A. (2010). Exotic options under exponential Lévy model. Ph.D. thesis, Université Paris-Est, http://tel.archives-ouvertes.fr/INSMI/tel-00520583/fr.

DIA, E. H. A. and LAMBERTON, D. (2011). Connecting discrete and continuous lookback or hindsight options in exponential Lévy models. *Adv. in Appl. Probab.* **43** 1136–1165. MR2867949

DONEY, R. A. (2007). *Fluctuation Theory for Lévy Processes. Lecture Notes in Math.* **1897**. Springer, Berlin. Lectures from the 35th Summer School on Probability Theory held in Saint-Flour, July 6–23, 2005, Edited and with a foreword by Jean Picard. MR2320889

DONEY, R. A. and MALLER, R. A. (2002). Stability and attraction to normality for Lévy processes at zero and at infinity. *J. Theoret. Probab.* **15** 751–792. MR1922446

DUQUESNE, T. (2003). Path decompositions for real Levy processes. *Ann. Inst. Henri Poincaré Probab. Stat.* **39** 339–370. MR1962781

EMBRECHTS, P. and MAEJIMA, M. (2002). *Selfsimilar Processes*. Princeton Univ. Press, Princeton, NJ. MR1920153

ETHIER, S. N. and KURTZ, T. G. (1986). *Markov Processes: Characterization and Convergence*. Wiley, New York. MR0838085

FELLER, W. (1966). *An Introduction to Probability Theory and Its Applications. Vol. II*. Wiley, New York–London–Sydney. MR0210154

FERREIRO-CASTILLA, A., KYPRIANOU, A. E., SCHEICHL, R. and SURYANARAYANA, G. (2014). Multilevel Monte Carlo simulation for Lévy processes based on the Wiener–Hopf factorisation. *Stochastic Process. Appl.* **124** 985–1010. MR3138603

GNEDENKO, B. V. and KOLMOGOROV, A. N. (1954). *Limit Distributions for Sums of Independent Random Variables*. Addison-Wesley Company, Cambridge, MA. Translated and annotated by K. L. Chung. With an Appendix by J. L. Doob. MR0062975

GREENWOOD, P. and PITMAN, J. (1980). Fluctuation identities for Lévy processes and splitting at the maximum. *Adv. in Appl. Probab.* **12** 893–902. MR0588409

HIRANO, K. (2001). Lévy processes with negative drift conditioned to stay positive. *Tokyo J. Math.* **24** 291–308. MR1844435

JACOD, J. and PROTTER, P. (2012). *Discretization of Processes. Stochastic Modelling and Applied Probability* **67**. Springer, Heidelberg. MR2859096

JACOD, J. and SHIRYAEV, A. N. (1987). *Limit Theorems for Stochastic Processes. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]* **288**. Springer, Berlin. MR0959133

JANSSEN, A. J. E. M. and VAN LEEUWAARDEN, J. S. H. (2009). Equidistant sampling for the maximum of a Brownian motion with drift on a finite horizon. *Electron. Commun. Probab.* **14** 143–150. MR2491634

KALLENBERG, O. (2002). *Foundations of Modern Probability*, 2nd ed. Springer, New York. MR1876169

KOSULAJEFF, P. (1937). Sur la répartition de la partie fractionnaire d'une variable. *Math. Sbornik* **2** 1017–1019.

KUZNETSOV, A. (2010). Wiener–Hopf factorization and distribution of extrema for a family of Lévy processes. *Ann. Appl. Probab.* **20** 1801–1830. MR2724421

KWAŚNICKI, M., MAŁECKI, J. and RYZNAR, M. (2013a). First passage times for subordinate Brownian motions. *Stochastic Process. Appl.* **123** 1820–1850. MR3027903

KWAŚNICKI, M., MAŁECKI, J. and RYZNAR, M. (2013b). Suprema of Lévy processes. *Ann. Probab.* **41** 2047–2065. MR3098066

KYPRIANOU, A. E. (2006). *Introductory Lectures on Fluctuations of Lévy Processes with Applications*. Springer, Berlin. MR2250061

LAMPERTI, J. (1962). Semi-stable stochastic processes. *Trans. Amer. Math. Soc.* **104** 62–78. MR0138128

LEWIS, A. L. and MORDECKI, E. (2008). Wiener–Hopf factorization for Lévy processes having positive jumps with rational transforms. *J. Appl. Probab.* **45** 118–134. MR2409315

MALLER, R. A. (2009). Small-time versions of Strassen's law for Lévy processes. *Proc. Lond. Math. Soc.* (3) **98** 531–558. MR2481958

MALLER, R. A. (2015). Strong laws at zero for trimmed Lévy processes. *Electron. J. Probab.* **20** 88. MR3391871

MALLER, R. and MASON, D. M. (2008). Convergence in distribution of Lévy processes at small times with self-normalization. *Acta Sci. Math.* (*Szeged*) **74** 315–347. MR2431109

MICHNA, Z., PALMOWSKI, Z. and PISTORIUS, M. (2015). The distribution of the supremum for spectrally asymmetric Lévy processes. *Electron. Commun. Probab.* **20** 24. MR3327863

PITERBARG, V. I. (1996). *Asymptotic Methods in the Theory of Gaussian Processes and Fields. Translations of Mathematical Monographs* **148**. Amer. Math. Soc., Providence, RI. Translated from the Russian by V. V. Piterbarg, Revised by the author. MR1361884

RÉNYI, A. (1958). On mixing sequences of sets. *Acta Math. Acad. Sci. Hung.* **9** 215–228. MR0098161

RESNICK, S. I. (2007). *Heavy-Tail Phenomena*. Springer, New York. MR2271424

SAMORODNITSKY, G. and TAQQU, M. S. (1994). *Stable Non-Gaussian Random Processes*. Chapman & Hall, New York. MR1280932

SATO, K. (2013). *Lévy Processes and Infinitely Divisible Distributions. Cambridge Studies in Advanced Mathematics* **68**. Cambridge Univ. Press, Cambridge. Translated from the 1990 Japanese original, Revised edition of the 1999 English translation. MR3185174

SHIMURA, T. (1990). The strict domain of attraction of strictly stable law with index 1. *Jpn. J. Math.* **16** 351–363. MR1091168

SKOROHOD, A. V. (1957). Limit theorems for stochastic processes with independent increments. *Teor. Veroyatn. Primen.* **2** 145–177. MR0094842

VAN DER VAART, A. W. (1998). *Asymptotic Statistics. Cambridge Series in Statistical and Probabilistic Mathematics* **3**. Cambridge Univ. Press, Cambridge. MR1652247

WHITT, W. (1980). Some useful functions for functional limit theorems. *Math. Oper. Res.* **5** 67–85. MR0561155

WHITT, W. (2002). *Stochastic-Process Limits*. Springer, New York. An introduction to stochastic-process limits and their application to queues. MR1876437

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SPECTRAL GAP OF RANDOM HYPERBOLIC GRAPHS AND RELATED PARAMETERS

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Random hyperbolic graphs have been suggested as a promising model of social networks. A few of their fundamental parameters have been studied. However, none of them concerns their spectra. We consider the random hyperbolic graph model, as formalized by [*Automata, Languages, and Programming—39th International Colloquium—ICALP Part II*. (2012) 573–585 Springer], and essentially determine the spectral gap of their normalized Laplacian. Specifically, we establish that with high probability the second smallest eigenvalue of the normalized Laplacian of the giant component of an n-vertex random hyperbolic graph is at least $\Omega(n^{-(2\alpha-1)}/D)$, where $\frac{1}{2} < \alpha < 1$ is a model parameter and D is the network diameter (which is known to be at most polylogarithmic in n). We also show a matching (up to a polylogarithmic factor) upper bound of $n^{-(2\alpha-1)}(\log n)^{1+o(1)}$.

As a byproduct, we conclude that the conductance upper bound on the eigenvalue gap obtained via Cheeger's inequality is essentially tight. We also provide a more detailed picture of the collection of vertices on which the bound on the conductance is attained, in particular showing that for all subsets whose volume is $O(n^{\varepsilon})$ for $0 < \varepsilon < 1$ the obtained conductance is with high probability $\Omega(n^{-(2\alpha-1)\varepsilon+o(1)})$. Finally, we also show consequences of our result for the minimum and maximum bisection of the giant component.

REFERENCES

[1] ABDULLAH, M. A., BODE, M. and FOUNTOULAKIS, N. (2017). Typical distances in a geometric model for complex networks. *Internet Math*. DOI:10.24166/im.13.2017.

[2] ALON, N. and SPENCER, J. H. (2008). *The Probabilistic Method*, 3rd ed. Wiley, Hoboken, NJ. MR2437651

[3] BODE, M., FOUNTOULAKIS, N. and MÜLLER, T. (2015). On the largest component of a hyperbolic model of complex networks. *Electron. J. Combin*. **22** P3.24.

[4] BODE, M., FOUNTOULAKIS, N. and MÜLLER, T. (2016). The probability of connectivity in a hyperbolic model of complex networks. *Random Structures Algorithms* **49** 65–94. MR3521274

[5] BOGUÑÁ, M., PAPADOPOULOS, F. and KRIOUKOV, D. (2010). Sustaining the Internet with hyperbolic mapping. *Nat. Commun*. **1** 62.

[6] BRINGMANN, K., KEUSCH, R. and LENGLER, J. Average distance in a general class of scale-free networks with underlying geometry. Available at http://arxiv.org/pdf/1602.05712v1.pdf.

[7] BRINGMANN, K., KEUSCH, R. and LENGLER, J. Sampling Geometric Inhomogeneous Random Graphs in Linear Time. Preprint. Available at http://arxiv.org/pdf/1511.00576v3.pdf.

MSC2010 subject classifications. Primary 60C05; secondary 05C80.
Key words and phrases. Random hyperbolic graphs, spectral gap, conductance.

[8] CANDELLERO, E. and FOUNTOULAKIS, N. (2013). Clustering in random geometric graphs on the hyperbolic plane. Preprint. Available at http://arxiv.org/abs/1309.0459.

[9] CHUNG, F. R. K. (1997). *Spectral Graph Theory*. *CBMS Regional Conference Series in Mathematics* **92**. Amer. Math. Soc., Providence, RI. MR1421568

[10] DEIJFEN, M., VAN DER HOFSTAD, R. and HOOGHIEMSTRA, G. (2013). Scale-free percolation. *Ann. Inst. Henri Poincaré Probab. Stat.* **49** 817–838. MR3112435

[11] DIACONIS, P. and STROOCK, D. (1991). Geometric bounds for eigenvalues of Markov chains. *Ann. Appl. Probab.* **1** 36–61.

[12] FOUNTOULAKIS, N. (2012). On the evolution of random graphs on spaces with negative curvature. ArXiv E-prints.

[13] FOUNTOULAKIS, N. and MÜLLER, T. (2018). Law of large numbers for the largest component in a hyperbolic model of complex networks. *Ann. Appl. Probab.* **28** 607–650. MR3770885

[14] FRIEDRICH, T. and KROHMER, A. (2015). On the diameter of hyperbolic random graphs. In *Automata, Languages, and Programming. Part II. Lecture Notes in Computer Science* **9135** 614–625. Springer, Heidelberg. MR3382563

[15] GUGELMANN, L., PANAGIOTOU, K. and PETER, U. (2012). Random hyperbolic graphs: Degree sequence and clustering. In *Automata, Languages, and Programming—39th International Colloquium—ICALP Part II. LNCS* **7392** 573–585. Springer, Berlin.

[16] KIWI, M. and MITSCHE, D. (2015). A bound for the diameter of random hyperbolic graphs. In *Proceedings of the 12th Workshop on Analytic Algorithmics and Combinatorics—ANALCO* 26–39. SIAM, Philadelphia, PA.

[17] KOCH, C. and LENGLER, J. (2016). Bootstrap percolation on geometric inhomogeneous random graphs. In *Automata, Languages, and Programming—43rd International Colloquium—ICALP. LIPIcs* **55** 127:1–127:15.

[18] KRIOUKOV, D., PAPADOPOULOS, F., KITSAK, M., VAHDAT, A. and BOGUÑÁ, M. (2010). Hyperbolic geometry of complex networks. *Phys. Rev. E* **82** 036106.

[19] MOHAISEN, A., YUN, A. and KIM, Y. (2010). Measuring the mixing time of social graphs. In *Proceedings of the 10th ACM SIGCOMM Conference on Internet Measurement—ICM* 383–389. ACM, New York.

[20] SINCLAIR, A. (1992). Improved bounds for mixing rates of Markov chains and multicommodity flow. *Combin. Probab. Comput.* **1** 351–370.

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A PHASE TRANSITION REGARDING THE EVOLUTION OF BOOTSTRAP PROCESSES IN INHOMOGENEOUS RANDOM GRAPHS

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A bootstrap percolation process on a graph with infection threshold $r \geq 1$ is a dissemination process that evolves in time steps. The process begins with a subset of infected vertices and in each subsequent step every uninfected vertex that has at least r infected neighbours becomes infected and remains so forever.

Critical phenomena in bootstrap percolation processes were originally observed by Aizenman and Lebowitz in the late 1980s as finite-volume phase transitions in $\mathbb{Z}^d$ that are caused by the accumulation of small local islands of infected vertices. They were also observed in the case of dense (homogeneous) random graphs by Janson et al. [*Ann. Appl. Probab.* **22** (2012) 1989–2047]. In this paper, we consider the class of inhomogeneous random graphs known as the *Chung-Lu model*: each vertex is equipped with a positive weight and each pair of vertices appears as an edge with probability proportional to the product of the weights. In particular, we focus on the *sparse* regime, where the number of edges is proportional to the number of vertices.

The main results of this paper determine those weight sequences for which a critical phenomenon occurs: there is a critical density of vertices that are infected at the beginning of the process, above which a small (sublinear) set of infected vertices creates an avalanche of infections that in turn leads to an outbreak. We show that this occurs essentially only when the tail of the weight distribution dominates a power law with exponent 3 and we determine the critical density in this case.

REFERENCES

ABDULLAH, M. A. and FOUNTOULAKIS, N. (2014). A phase transition in the evolution of bootstrap percolation processes on preferential attachment graphs. *Random Structures Algorithms*. To appear. Available at http://arxiv.org/abs/1402.2815.

ADLER, J. and LEV, U. (2003). Bootstrap percolation: Visualizations and applications. *Braz. J. Phys*. **33** 641–644.

AIZENMAN, M. and LEBOWITZ, J. L. (1988). Metastability effects in bootstrap percolation. *J. Phys. A* **21** 3801–3813. MR0968311

AMINI, H. (2010a). Bootstrap percolation in living neural networks. *J. Stat. Phys*. **141** 459–475.

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AMINI, H. (2010b). Bootstrap percolation and diffusion in random graphs with given vertex degrees. *Electron. J. Combin.* **17** Research Paper 25. MR2595485

AMINI, H. and FOUNTOULAKIS, N. (2014). Bootstrap percolation in power-law random graphs. *J. Stat. Phys.* **155** 72–92. MR3180970

AMINI, H., FOUNTOULAKIS, N. and PANAGIOTOU, K. (2014). Bootstrap percolation in inhomogeneous random graphs. Preprint. Available at http://arxiv.org/abs/1402.2815.

BALOGH, J. and BOLLOBÁS, B. (2006). Bootstrap percolation on the hypercube. *Probab. Theory Related Fields* **134** 624–648.

BALOGH, J., BOLLOBÁS, B. and MORRIS, R. (2009). Bootstrap percolation in three dimensions. *Ann. Probab.* **37** 1329–1380. MR2546747

BALOGH, J., PERES, Y. and PETE, G. (2006). Bootstrap percolation on infinite trees and non-amenable groups. *Combin. Probab. Comput.* **15** 715–730. MR2248323

BALOGH, J. and PETE, G. (1998). Random disease on the square grid. *Proceedings of the Eighth International Conference "Random Structures and Algorithms" (Poznan, 1997). Random Structures Algorithms* **13** 409–422. MR1662792

BALOGH, J. and PITTEL, B. G. (2007). Bootstrap percolation on the random regular graph. *Random Structures Algorithms* **30** 257–286. MR2283230

BALOGH, J., BOLLOBÁS, B., DUMINIL-COPIN, H. and MORRIS, R. (2012). The sharp threshold for bootstrap percolation in all dimensions. *Trans. Amer. Math. Soc.* **36** 2667–2701.

BOLLOBÁS, B., JANSON, S. and RIORDAN, O. (2007). The phase transition in inhomogeneous random graphs. *Random Structures Algorithms* **31** 3–122. MR2337396

BOLLOBÁS, B., GUNDERSON, K., HOLMGREN, C., JANSON, S. and PRZYKUCKI, M. (2014). Bootstrap percolation on Galton–Watson trees. *Electron. J. Probab.* **19** 1–27.

CANDELLERO, E. and FOUNTOULAKIS, N. (2016). Bootstrap percolation and the geometry of complex networks. *Stochastic Process. Appl.* **126** 234–264.

CERF, R. and MANZO, F. (2002). The threshold regime of finite volume bootstrap percolation. *Stochastic Process. Appl.* **101** 69–82. MR1921442

CHALUPA, J., LEATH, P. L. and REICH, G. R. (1979). Bootstrap percolation on a Bethe lattice. *J. Phys. C, Solid State Phys.* **12** L31–L35.

CHUNG, F. and LU, L. (2002). Connected components in random graphs with given expected degree sequences. *Ann. Comb.* **6** 125–145.

CHUNG, F. and LU, L. (2003). The average distance in a random graph with given expected degrees. *Internet Math.* **1** 91–113.

CHUNG, F., LU, L. and VU, V. (2004). The spectra of random graphs with given expected degrees. *Internet Math.* **1** 257–275.

EBRAHIMI, R., GAO, J., GHASEMIESFEH, G. and SCHOENEBECK, G. (2017). How complex contagions spread quickly in the preferential attachment model and other time-evolving networks. *IEEE Trans. Netw. Sci. Eng.* **PP:99**.

ENTER, A. C. D. V. (1987). Proof of Straley's argument for bootstrap percolation. *J. Stat. Phys.* **48** 943–945.

FONTES, L. and SCHONMANN, R. (2008). Bootstrap percolation on homogeneous trees has 2 phase transitions. *J. Stat. Phys.* **132** 839–861.

FONTES, L. R., SCHONMANN, R. H. and SIDORAVICIUS, V. (2002). Stretched exponential fixation in stochastic Ising models at zero temperature. *Comm. Math. Phys.* **228** 495–518. MR1918786

HOLROYD, A. E. (2003). Sharp metastability threshold for two-dimensional bootstrap percolation. *Probab. Theory Related Fields* **125** 195–224.

JANSON, S., ŁUCZAK, T. and RUCIŃSKI, A. (2000). *Random Graphs*. Wiley-Interscience, New York.

JANSON, S., ŁUCZAK, T., TUROVA, T. and VALLIER, T. (2012). Bootstrap percolation on the random graph $G_{n,p}$. *Ann. Appl. Probab.* **22** 1989–2047. MR3025687

KARBASI, A., LENGLER, J. and STEGER, A. (2015). Normalization phenomena in asynchronous networks. In *Automata*, *Languages*, *and Programming*. *Part II*. *Lecture Notes in Computer Science* **9135** 688–700. Springer, Heidelberg. MR3382569

KOCH, C. and LENGLER, J. (2016). Bootstrap percolation on geometric inhomogeneous random graphs. Preprint. Available at http://arxiv.org/abs/1603.02057.

MCDIARMID, C. (1998). Concentration. In *Probabilistic Methods for Algorithmic Discrete Mathematics*. *Algorithms Combin*. **16** 195–248. Springer, Berlin. MR1678578

MORRIS, R. (2009). Zero-temperature Glauber dynamics on $\mathbb{Z}^d$. *Probab. Theory Related Fields* **149** 417–434.

PITTEL, B., SPENCER, J. and WORMALD, N. (1996). Sudden emergence of a giant k-core in a random graph. *J. Combin*. *Theory Ser. B* **67** 111–151.

SABHAPANDIT, S., DHAR, D. and SHUKLA, P. (2002). Hysteresis in the random-field Ising model and bootstrap percolation. *Phys*. *Rev. Lett*. **88** 197202.

SAUSSET, F., TONINELLI, C., BIROLI, G. and TARJUS, G. (2010). Bootstrap percolation and kinetically constrained models on hyperbolic lattices. *J. Stat*. *Phys*. **138** 411–430.

SCHONMANN, R. H. (1992). On the behavior of some cellular automata related to bootstrap percolation. *Ann*. *Probab.* **20** 174–193. MR1143417

SÖDERBERG, B. (2002). General formalism for inhomogeneous random graphs. *Phys*. *Rev. E* **66** 066121.

TLUSTY, T. and ECKMANN, J.-P. (2009). Remarks on bootstrap percolation in metric networks. *J*. *Phys*. *A* **42** 205004. MR2515591

TONINELLI, C., BIROLI, G. and FISHER, D. S. (2006). Jamming percolation and glass transitions in lattice models. *Phys*. *Rev. Lett*. **96** 035702.

WILLIAMS, D. (1991). *Probability with Martingales*. Cambridge Univ. Press, Cambridge. MR1155402

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SHARP THRESHOLDS FOR CONTAGIOUS SETS IN RANDOM GRAPHS

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For fixed $r \geq 2$, we consider bootstrap percolation with threshold r on the Erdős–Rényi graph $\mathcal{G}_{n,p}$. We identify a threshold for p above which there is with high probability a set of size r that can infect the entire graph. This improves a result of Feige, Krivelevich and Reichman, which gives bounds for this threshold, up to multiplicative constants.

As an application of our results, we obtain an upper bound for the threshold for K_4-percolation on $\mathcal{G}_{n,p}$, as studied by Balogh, Bollobás and Morris. This bound is shown to be asymptotically sharp in subsequent work.

These thresholds are closely related to the survival probabilities of certain time-varying branching processes, and we derive asymptotic formulae for these survival probabilities, which are of interest in their own right.

REFERENCES

[1] ADLER, J. and LEV, U. (2003). Bootstrap percolation: Visualizations and applications. *Braz. J. Phys.* **33** 641–644.

[2] AIZENMAN, M. and LEBOWITZ, J. L. (1988). Metastability effects in bootstrap percolation. *J. Phys. A* **21** 3801–3813. MR0968311

[3] AMINI, H. (2010). Bootstrap percolation and diffusion in random graphs with given vertex degrees. *Electron. J. Combin.* **17** Research Paper 25, 20. MR2595485

[4] AMINI, H. and FOUNTOULAKIS, N. (2012). What I tell you three times is true: Bootstrap percolation in small worlds. In *Internet and Network Economics*: 8*th International Workshop*, *WINE* 2012, *Liverpool*, *UK*, *December* 10–12, 2012. *Proceedings* (P. W. Goldberg, ed.) 462–474. Springer, Berlin.

[5] ANGEL, O. and KOLESNIK, B. (2017). Minimal contagious sets in random graphs. Preprint, available at arXiv:1705.06815.

[6] BALL, F. and BRITTON, T. (2005). An epidemic model with exposure-dependent severities. *J. Appl. Probab.* **42** 932–949. MR2203813

[7] BALL, F. and BRITTON, T. (2009). An epidemic model with infector and exposure dependent severity. *Math. Biosci.* **218** 105–120. MR2513676

[8] BALOGH, J. and BOLLOBÁS, B. (2006). Bootstrap percolation on the hypercube. *Probab. Theory Related Fields* **134** 624–648. MR2214907

[9] BALOGH, J., BOLLOBÁS, B., DUMINIL-COPIN, H. and MORRIS, R. (2012). The sharp threshold for bootstrap percolation in all dimensions. *Trans. Amer. Math. Soc.* **364** 2667–2701. MR2888224

[10] BALOGH, J., BOLLOBÁS, B. and MORRIS, R. (2009). Bootstrap percolation in three dimensions. *Ann. Probab.* **37** 1329–1380. MR2546747

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[11] BALOGH, J., BOLLOBÁS, B. and MORRIS, R. (2009). Majority bootstrap percolation on the hypercube. *Combin. Probab. Comput.* **18** 17–51. MR2497373

[12] BALOGH, J., BOLLOBÁS, B. and MORRIS, R. (2010). Bootstrap percolation in high dimensions. *Combin. Probab. Comput.* **19** 643–692. MR2726074

[13] BALOGH, J., BOLLOBÁS, B. and MORRIS, R. (2012). Graph bootstrap percolation. *Random Structures Algorithms* **41** 413–440. MR2993128

[14] BALOGH, J., PERES, Y. and PETE, G. (2006). Bootstrap percolation on infinite trees and non-amenable groups. *Combin. Probab. Comput.* **15** 715–730. MR2248323

[15] BALOGH, J. and PETE, G. (1998). Random disease on the square grid. In *Proceedings of the Eighth International Conference "Random Structures and Algorithms"* (*Poznan*, 1997) **13** 409–422. MR1662792

[16] BALOGH, J. and PITTEL, B. G. (2007). Bootstrap percolation on the random regular graph. *Random Structures Algorithms* **30** 257–286. MR2283230

[17] BOLLOBÁS, B. (1968). Weakly k-saturated graphs. In *Beiträge zur Graphentheorie* (*Kolloquium*, *Manebach*, 1967) 25–31. Teubner, Leipzig. MR0244077

[18] CERF, R. and CIRILLO, E. N. M. (1999). Finite size scaling in three-dimensional bootstrap percolation. *Ann. Probab.* **27** 1837–1850. MR1742890

[19] CERF, R. and MANZO, F. (2002). The threshold regime of finite volume bootstrap percolation. *Stochastic Process. Appl.* **101** 69–82. MR1921442

[20] CHALUPA, J., LEATH, P. L. and REICH, G. R. (1979). Bootstrap percolation on a Bethe lattice. *J. Phys. C* **21** L31–L35.

[21] DE GREGORIO, P., LAWLOR, A., BRADLEY, P. and DAWSON, K. A. (2005). Exact solution of a jamming transition: Closed equations for a bootstrap percolation problem. *Proc. Natl. Acad. Sci. USA* **102** 5669–5673 (electronic). MR2142892

[22] DREYER, P. A. JR. and ROBERTS, F. S. (2009). Irreversible k-threshold processes: Graph-theoretical threshold models of the spread of disease and of opinion. *Discrete Appl. Math.* **157** 1615–1627. MR2510242

[23] EINARSSON, H., LENGLER, J., PANAGIOTOU, K., MOUSSET, F. and STEGER, A. Bootstrap percolation with inhibition. Preprint available at arXiv:1410.3291.

[24] ERDŐS, P. and RÉNYI, A. (1959). On random graphs. I. *Publ. Math. Debrecen* **6** 290–297. MR0120167

[25] FEIGE, U., KRIVELEVICH, M. and REICHMAN, D. (2017). Contagious sets in random graphs. *Ann. Appl. Probab.* **27** 2675–2697. MR3719944

[26] FEY, A., LEVINE, L. and PERES, Y. (2010). Growth rates and explosions in sandpiles. *J. Stat. Phys.* **138** 143–159. MR2594895

[27] FLOCCHINI, P., LODI, E., LUCCIO, F., PAGLI, L. and SANTORO, N. (2004). Dynamic monopolies in tori. *Discrete Appl. Math.* **137** 197–212. MR2048030

[28] FONTES, L. R., SCHONMANN, R. H. and SIDORAVICIUS, V. (2002). Stretched exponential fixation in stochastic Ising models at zero temperature. *Comm. Math. Phys.* **228** 495–518. MR1918786

[29] FONTES, L. R. G. and SCHONMANN, R. H. (2008). Bootstrap percolation on homogeneous trees has 2 phase transitions. *J. Stat. Phys.* **132** 839–861. MR2430783

[30] FROBÖSE, K. (1989). Finite-size effects in a cellular automaton for diffusion. *J. Stat. Phys.* **55** 1285–1292. MR1002492

[31] GARRAHAN, J. P., SOLLICH, P. and TONINELLI, C. (2011). Kinetically constrained models. In *Dynamical Heterogeneities in Glasses*, *Colloids*, *and Granular Media* (L. Berthier, G. Biroli, J.-P. Bouchaud, L. Cipelletti and W. van Saarloos, eds.) 341–369. Oxford University Press, Oxford.

[32] GRANOVETTER, M. (1978). Threshold models of collective behavior. *Am. J. Sociol.* **83** 1420–1443.

[33] GRAVNER, J., HOLROYD, A. E. and MORRIS, R. (2012). A sharper threshold for bootstrap percolation in two dimensions. *Probab. Theory Related Fields* **153** 1–23. MR2925568
[34] GRAVNER, J. and MCDONALD, E. (1997). Bootstrap percolation in a polluted environment. *J. Stat. Phys.* **87** 915–927. MR1459046
[35] HOLMGREN, C., JUŠKEVIČIUS, T. and KETTLE, N. (2017). Majority bootstrap percolation on $G(n, p)$. *Electron. J. Combin.* **24** Paper 1.1, 32. MR3609171
[36] HOLROYD, A. E. (2003). Sharp metastability threshold for two-dimensional bootstrap percolation. *Probab. Theory Related Fields* **125** 195–224. MR1961342
[37] HOLROYD, A. E., LIGGETT, T. M. and ROMIK, D. (2004). Integrals, partitions, and cellular automata. *Trans. Amer. Math. Soc.* **356** 3349–3368. MR2052953
[38] JANSON, S. (2009). On percolation in random graphs with given vertex degrees. *Electron. J. Probab.* **14** 87–118. MR2471661
[39] JANSON, S., ŁUCZAK, T., TUROVA, T. and VALLIER, T. (2012). Bootstrap percolation on the random graph $G_{n,p}$. *Ann. Appl. Probab.* **22** 1989–2047. MR3025687
[40] JUŠKEVIČIUS, T. (2015). Probabilistic inequalities and bootstrap percolation. Ph.D. thesis, Univ. Memphis, Memphis, TN.
[41] KETTLE, N. (2014). Vertex disjoint subgraphs and non-repetitive sequences. Ph.D. thesis, Univ. Cambridge.
[42] KIRKPATRICK, S., WILCKE, W. W., GARNER, R. B. and HUELS, H. (2002). Percolation in dense storage arrays. *Phys. A* **314** 220–229. MR1961703
[43] KOLESNIK, B. (2017). Sharp threshold for K_4-percolation. Preprint, available at arXiv:1705.08882.
[44] KOMLÓS, J. and SZEMERÉDI, E. (1983). Limit distribution for the existence of Hamiltonian cycles in a random graph. *Discrete Math.* **43** 55–63. MR680304
[45] MANTEL, W. (1907). Problem 28. *Wiskundige Opgaven* **10** 60–61.
[46] MCCULLOCH, W. S. and PITTS, W. H. (1943). A logical calculus of ideas immanent in nervous activity. *Bull. Math. Biophys.* **7** 115–133.
[47] MORRIS, R. (2011). Zero-temperature Glauber dynamics on $\mathbb{Z}^d$. *Probab. Theory Related Fields* **149** 417–434. MR2776621
[48] POLLAK, M. and RIESS, I. (1975). Application of percolation theory to 2d–3d Heisenberg ferromagnets. *Phys. Status Solidi (b)* **69** K15–K18.
[49] SCALIA-TOMBA, G.-P. (1985). Asymptotic final-size distribution for some chain-binomial processes. *Adv. in Appl. Probab.* **17** 477–495. MR0798872
[50] SCHONMANN, R. H. (1992). On the behavior of some cellular automata related to bootstrap percolation. *Ann. Probab.* **20** 174–193. MR1143417
[51] SELLKE, T. (1983). On the asymptotic distribution of the size of a stochastic epidemic. *J. Appl. Probab.* **20** 390–394. MR698541
[52] STEFÁNSSON, S. Ö. and VALLIER, T. Majority bootstrap percolation on the random graph $G(n, p)$. Preprint available at arXiv:1503.07029.
[53] TLUSTY, T. and ECKMANN, J.-P. (2009). Remarks on bootstrap percolation in metric networks. *J. Phys. A* **42** 205004, 11. MR2515591
[54] TURÁN, P. (1941). Eine extremalaufgabe aus der graphentheorie. *Mat. Fiz Lapook* **48** 436–452.
[55] ULAM, S. (1952). Random processes and transformations. In *Proceedings of the International Congress of Mathematicians, Vol.* 2, *Cambridge, Mass.*, 1950 264–275. Amer. Math. Soc., Providence, RI. MR0045334
[56] VALLIER, T. (2007). Random graph models and their applications. Ph.D. thesis, Lund Univ.
[57] VAN ENTER, A. C. D. (1987). Proof of Straley's argument for bootstrap percolation. *J. Stat. Phys.* **48** 943–945. MR0914911
[58] VON NEUMANN, J. (1966). *Theory of Self-Reproducing Automata*. Univ. Illinois Press, Urbana, IL.

[59] WATTS, D. J. (2002). A simple model of global cascades on random networks. *Proc. Natl. Acad. Sci. USA* **99** 5766–5771 (electronic). MR1896072

[60] WOLFRAM, S., ed. (1986). *Theory and Applications of Cellular Automata. Including Selected Papers* 1983–1986. *Advanced Series on Complex Systems* **1**. World Scientific Publishing, Singapore. MR857608

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THE SAMPLE SIZE REQUIRED IN IMPORTANCE SAMPLING

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The goal of importance sampling is to estimate the expected value of a given function with respect to a probability measure ν using a random sample of size n drawn from a different probability measure μ. If the two measures μ and ν are nearly singular with respect to each other, which is often the case in practice, the sample size required for accurate estimation is large. In this article, it is shown that in a fairly general setting, a sample of size approximately $\exp(D(\nu \| \mu))$ is necessary and sufficient for accurate estimation by importance sampling, where $D(\nu \| \mu)$ is the Kullback–Leibler divergence of μ from ν. In particular, the required sample size exhibits a kind of cut-off in the logarithmic scale. The theory is applied to obtain a general formula for the sample size required in importance sampling for one-parameter exponential families (Gibbs measures).

REFERENCES

[1] AGAPIOU, S., PAPASPILIOPOULOS, O., SANZ-ALONSO, D. and STUART, A. M. (2017). Importance sampling: Computational complexity and intrinsic dimension. Preprint. Available at arXiv:1511.06196.

[2] ASMUSSEN, S. and GLYNN, P. W. (2007). *Stochastic Simulation*: *Algorithms and Analysis*. *Stochastic Modelling and Applied Probability* **57**. Springer, New York. MR2331321

[3] BAHADUR, R. R. (1960). Some approximations to the binomial distribution function. *Ann*. *Math*. *Stat*. **31** 43–54.

[4] BASSETTI, F. and DIACONIS, P. (2006). Examples comparing importance sampling and the Metropolis algorithm. *Illinois J*. *Math*. **50** 67–91. MR2247824

[5] BAXTER, R. J. (1982). *Exactly Solved Models in Statistical Mechanics*. Academic Press, London. MR0690578

[6] BHATTACHARYA, B. B., GANGULY, S., LUBETZKY, E. and ZHAO, Y. (2015). Upper tails and independence polynomials in random graphs. Preprint. Available at arXiv:1507.04074.

[7] BLANCHET, J. and GLYNN, P. (2008). Efficient rare-event simulation for the maximum of heavy-tailed random walks. *Ann*. *Appl*. *Probab*. **18** 1351–1378. MR2434174

[8] BLANCHET, J., GLYNN, P. and LEDER, K. (2012). On Lyapunov inequalities and subsolutions for efficient importance sampling. *ACM Trans*. *Model*. *Comput*. *Simul*. **22** 1104–1128.

[9] BLANCHET, J. and LIU, J. (2008). State-dependent importance sampling for regularly varying random walks. *Adv*. *Appl*. *Probab*. **40** 1104–1128. MR488534

[10] BLANCHET, J. and LIU, J. (2010). Efficient importance sampling in ruin problems for multidimensional regularly varying random walks. *J*. *Appl*. *Probab*. **47** 301–322. MR2668490

[11] BLITZSTEIN, J. and DIACONIS, P. (2010). A sequential importance sampling algorithm for generating random graphs with prescribed degrees. *Internet Math*. **6** 489–522.

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[12] BOUSQUET-MÉLOU, M. (2014). On the importance sampling of self-avoiding walks. *Combin. Probab. Comput.* **23** 725–748.

[13] CAPPÉ, O., MOULINES, E. and RYDÉN, T. (2005). *Inference in Hidden Markov Models*. Springer, New York. MR2159833

[14] CHAN, H. P. and LAI, T. L. (2007). Efficient importance sampling for Monte Carlo evaluation of exceedance probabilities. *Ann. Appl. Probab.* **17** 440–473. MR2308332

[15] CHAN, H. P. and LAI, T. L. (2011). A sequential Monte Carlo approach to computing tail probabilities in stochastic models. *Ann. Appl. Probab.* **21** 2315–2342. MR2895417

[16] CHATTERJEE, S. and DIACONIS, P. (2013). Estimating and understanding exponential random graph models. *Ann. Statist.* **41** 2428–2461. MR3127871

[17] CHEN, Y., DIACONIS, P., HOLMES, S. P. and LIU, J. S. (2005). Sequential Monte Carlo methods for statistical analysis of tables. *J. Amer. Statist. Assoc.* **100** 109–120.

[18] CHEN, Y. and LIU, J. S. (2007). Sequential Monte Carlo methods for permutation tests on truncated data. *Statist. Sinica* **17** 857–872. MR2397385

[19] DEL MORAL, P. (2004). *Feynman–Kac Formulae: Genealogical and Interacting Particle Systems with Applications*. Springer, New York.

[20] DEL MORAL, P. (2013). *Mean Field Simulation for Monte Carlo Integration*. CRC Press, Boca Raton, FL.

[21] DEL MORAL, P., KOHN, R. and PATRAS, F. (2015). A duality formula for Feynman–Kac path particle models. *C. R. Math. Acad. Sci. Paris* **353** 465–469.

[22] DIACONIS, P. and ZABELL, S. (1991). Closed form summation for classical distributions: Variations on a theme of de Moivre. *Statist. Sci.* **6** 284–302. MR1144242

[23] DOUCET, A., DE FREITAS, N. and GORDON, N., eds. (2001). *Sequential Monte Carlo Methods in Practice*. Springer, New York.

[24] DUPUIS, P., SPILIOPOULOS, K. and WANG, H. (2012). Importance sampling for multiscale diffusions. *Multiscale Model. Simul.* **10** 1–27.

[25] DUPUIS, P. and WANG, H. (2004). Importance sampling, large deviations, and differential games. *Stoch. Stoch. Rep.* **76** 481–508.

[26] EFRON, B. (2012). Bayesian inference and the parametric bootstrap. *Ann. Appl. Stat.* **6** 1971–1997. MR3058690

[27] FREER, C. E., MANSINGHKA, V. K. and ROY, D. M. (2010). When are probabilistic programs probably computationally tractable? Presented at the *NIPS Workshop on Monte Carlo Methods for Modern Applications, 2010*. Available at http://danroy.org/papers/FreerManRoy-NIPSMC-2010.pdf.

[28] GELMAN, A. and MENG, X.-L. (1998). Simulating normalizing constants: From importance sampling to bridge sampling to path sampling. *Statist. Sci.* **13** 163–185. MR1647507

[29] HAMMERSLEY, J. M. and HANDSCOMB, D. C. (1965). *Monte Carlo Methods*. Methuen & Co., Ltd., London.

[30] HESTERBERG, T. (1995). Weighted average importance sampling and defensive mixture distributions. *Technometrics* **37** 185–194.

[31] HUGGINS, J. H. and ROY, D. M. (2015). Convergence of sequential Monte Carlo-based sampling methods. Preprint. Available at arXiv:1503.00966.

[32] HULT, H. and NYQUIST, P. (2016). Large deviations for weighted empirical measures arising in importance sampling. *Stochastic Process. Appl.* **126** 138–170.

[33] KAHN, H. and MARSHALL, A. W. (1953). Methods of reducing sample size in Monte Carlo computations. *J. Oper. Res. Soc. Am.* **1** 263–278.

[34] KENYON, R., KRAL, D., RADIN, C. and WINKLER, P. (2015). A variational principle for permutations. Preprint. Available at arXiv:1506.02340.

[35] KENYON, R., RADIN, C., REN, K. and SADUN, L. (2014). Multipodal structure and phase transitions in large constrained graphs. Preprint. Available at arXiv:1405.0599.

[36] KENYON, R. and YIN, M. (2014). On the asymptotics of constrained exponential random graphs. Preprint. Available at arXiv:1406.3662.

[37] KNUTH, D. E. (1976). Mathematics and computer science: Coping with finiteness. *Science* **194** 1235–1242. MR0534161

[38] KNUTH, D. E. (1996). *Selected Papers on Computer Science*. *CSLI Lecture Notes* **59**. CSLI Publications, Stanford, CA; Cambridge University Press, Cambridge.

[39] LELIÈVRE, T., ROUSSET, M. and STOLTZ, G. (2010). *Free Energy Computations*: *A Mathematical Perspective*. World Scientific, Singapore.

[40] LIU, J. S. (2008). *Monte Carlo Strategies in Scientific Computing*. Springer, New York.

[41] LIU, J. S. and CHEN, R. (1995). Blind deconvolution via sequential imputations. *J. Amer. Statist. Assoc.* **90** 567–576. MR3363399

[42] MADRAS, N. (1998). Umbrella sampling and simulated tempering. In *Numerical Methods for Polymeric Systems* (*Minneapolis*, *MN*, 1996). *IMA Vol. Math. Appl.* **102** 19–32. Springer, New York. MR1655577

[43] MADRAS, N. and PICCIONI, M. (1999). Importance sampling for families of distributions. *Ann. Appl. Probab.* **9** 1202–1225. MR1728560

[44] MCCOY, B. M. (2010). *Advanced Statistical Mechanics*. *International Series of Monographs on Physics* **146**. Oxford Univ. Press, Oxford. MR2583103

[45] MUKHERJEE, S. (2013). Estimation in exponential families on permutations. Preprint. Available at arXiv:1307.0978.

[46] NAIMAN, D. Q. and WYNN, H. P. (1997). Abstract tubes, improved inclusion-exclusion identities and inequalities and importance sampling. *Ann. Statist.* **25** 1954–1983. MR1474076

[47] OWEN, A. and ZHOU, Y. (1999). Adaptive importance sampling by mixtures of products of beta distributions. Technical report No. 1999–25, Dept. Statistics, Stanford Univ., Stanford, CA.

[48] OWEN, A. and ZHOU, Y. (2000). Safe and effective importance sampling. *J. Amer. Statist. Assoc.* **95** 135–143.

[49] OWEN, A. B. (2005). Multidimensional variation for quasi-Monte Carlo. In *Contemporary Multivariate Analysis and Design of Experiments*. *Ser. Biostat.* **2** 49–74. World Sci. Publ., Hackensack, NJ.

[50] OWEN, A. B. (2006). Quasi-Monte Carlo for integrands with point singularities at unknown locations. In *Monte Carlo and Quasi-Monte Carlo Methods* 2004 403–417. Springer, Berlin. MR2208721

[51] ROBERT, C. P. and CASELLA, G. (2004). *Monte Carlo Statistical Methods*, 2nd ed. Springer, New York.

[52] ROSENBLUTH, M. N. and ROSENBLUTH, A. W. (1955). Monte Carlo calculation of the average extension of molecular chains. *J. Chem. Phys.* **23** 356–359.

[53] SHI, J., SIEGMUND, D. and YAKIR, B. (2007). Importance sampling for estimating p values in linkage analysis. *J. Amer. Statist. Assoc.* **102** 929–937.

[54] SIEGMUND, D. (1976). Importance sampling in the Monte Carlo study of sequential tests. *Ann. Statist.* **4** 673–684. MR0418369

[55] SRINIVASAN, R. (2002). *Importance Sampling*: *Applications in Communications and Detection*. Springer, Berlin. MR1949250

[56] STARR, S. (2009). Thermodynamic limit for the Mallows model on S_n. *J. Math. Phys.* **50** 095208.

[57] TORRIE, G. M. and VALLEAU, J. P. (1977). Nonphysical sampling distributions in Monte Carlo free-energy estimation: Umbrella sampling. *J. Comput. Phys.* **23** 187–199.

[58] WHITELEY, N., LEE, A. and HEINE, K. (2016). On the role of interaction in sequential Monte Carlo algorithms. *Bernoulli* **22** 494–529.

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UNIQUENESS AND PROPAGATION OF CHAOS FOR THE BOLTZMANN EQUATION WITH MODERATELY SOFT POTENTIALS

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We prove a strong/weak stability estimate for the 3D homogeneous Boltzmann equation with moderately soft potentials [$\gamma \in (-1, 0)$] using the Wasserstein distance with quadratic cost. This in particular implies the uniqueness in the class of all weak solutions, assuming only that the initial condition has a finite entropy and a finite moment of sufficiently high order. We also consider the Nanbu N-stochastic particle system, which approximates the weak solution. We use a probabilistic coupling method and give, under suitable assumptions on the initial condition, a rate of convergence of the empirical measure of the particle system to the solution of the Boltzmann equation for this singular interaction.

REFERENCES

[1] ALDOUS, D. (1978). Stopping times and tightness. *Ann. Probab.* **6** 335–340. MR0474446

[2] ALEXANDRE, R., DESVILLETTES, L., VILLANI, C. and WENNBERG, B. (2000). Entropy dissipation and long-range interactions. *Arch. Ration. Mech. Anal.* **152** 327–355.

[3] BHATT, A. G. and KARANDIKAR, R. L. (1993). Invariant measures and evolution equations for Markov processes characterized via martingale problems. *Ann. Probab.* **21** 2246–2268. MR1245309

[4] CORTEZ, R. and FONTBONA, J. (2015). Quantitative uniform propagation of chaos for Maxwell molecules. Arvix preprint. Available at arXiv:1512.09308.

[5] DESVILLETTES, L. and MOUHOT, C. (2009). Stability and uniqueness for the spatially homogeneous Boltzmann equation with long-range interactions. *Arch. Ration. Mech. Anal.* **193** 227–253.

[6] FIGALLI, A. (2008). Existence and uniqueness of martingale solutions for SDEs with rough or degenerate coefficients. *J. Funct. Anal.* **254** 109–153.

[7] FONTBONA, J., GUÉRIN, H. and MÉLÉARD, S. (2009). Measurability of optimal transportation and convergence rate for Landau type interacting particle systems. *Probab. Theory Related Fields* **143** 329–351.

[8] FOURNIER, N. (2015). Finiteness of entropy for the homogeneous Boltzmann equation with measure initial condition. *Ann. Appl. Probab.* **25** 860–897. MR3313757

[9] FOURNIER, N. and GUÉRIN, H. (2008). On the uniqueness for the spatially homogeneous Boltzmann equation with a strong angular singularity. *J. Stat. Phys.* **131** 749–781.

[10] FOURNIER, N. and GUILLIN, A. (2015). On the rate of convergence in Wasserstein distance of the empirical measure. *Probab. Theory Related Fields* **162** 707–738.

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[11] FOURNIER, N. and HAURAY, M. (2016). Propagation of chaos for the Landau equation with moderately soft potentials. *Ann. Probab.* **44** 3581–3660. MR3572320

[12] FOURNIER, N., HAURAY, M. and MISCHLER, S. (2014). Propagation of chaos for the 2D viscous vortex model. *J. Eur. Math. Soc.* (*JEMS*) **16** 1423–1466.

[13] FOURNIER, N. and MÉLÉARD, S. (2002). A stochastic particle numerical method for 3D Boltzmann equations without cutoff. *Math. Comp.* **71** 583–604. DOI:10.1090/S0025-5718-01-01339-4.

[14] FOURNIER, N. and MISCHLER, S. (2016). Rate of convergence of the Nanbu particle system for hard potentials and Maxwell molecules. *Ann. Probab.* **44** 589–627. MR3456347

[15] FOURNIER, N. and MOUHOT, C. (2009). On the well-posedness of the spatially homogeneous Boltzmann equation with a moderate angular singularity. *Comm. Math. Phys.* **289** 803–824. MR2511651

[16] GODINHO, D. and QUIÑINAO, C. (2015). Propagation of chaos for a subcritical Keller–Segel model. *Ann. Inst. Henri Poincaré Probab. Stat.* **51** 965–992. MR3365970

[17] GRAHAM, C. and MÉLÉARD, S. (1997). Stochastic particle approximations for generalized Boltzmann models and convergence estimates. *Ann. Probab.* **25** 115–132. MR1428502

[18] GRÜNBAUM, F. A. (1971). Propagation of chaos for the Boltzmann equation. *Arch. Ration. Mech. Anal.* **42** 323–345. MR0334788

[19] HAURAY, M. and JABIN, P.-E. (2015). Particle approximation of Vlasov equations with singular forces: Propagation of chaos. *Ann. Sci. Éc. Norm. Supér.* (4) **48** 891–940.

[20] HOROWITZ, J. and KARANDIKAR, R. L. (1990). Martingale problems associated with the Boltzmann equation. In *Seminar on Stochastic Processes*, 1989. *Progress in Probability* **18** 75–122. Birkhäuser, Boston, MA.

[21] JACOD, J. (1979). *Calcul Stochastique et Problèmes de Martingales*. *Lecture Notes in Math.* **714**. Springer, Berlin.

[22] JACOD, J. and SHIRYAEV, A. N. (2003). *Limit Theorems for Stochastic Processes*, 2nd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **288**. Springer, Berlin.

[23] KAC, M. (1956). Foundations of kinetic theory. In *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability*, 1954–1955, *Vol. III* 171–197. Univ. California Press, Berkeley and Los Angeles.

[24] LU, X. and MOUHOT, C. (2012). On measure solutions of the Boltzmann equation, part I: Moment production and stability estimates. *J. Differential Equations* **252** 3305–3363. MR2871802

[25] MCKEAN, H. P. JR. (1967). An exponential formula for solving Boltzmann's equation for a Maxwellian gas. *J. Combin. Theory* **2** 358–382. MR0224348

[26] MISCHLER, S. and MOUHOT, C. (2013). Kac's program in kinetic theory. *Invent. Math.* **193** 1–147.

[27] MISCHLER, S. and WENNBERG, B. (1999). On the spatially homogeneous Boltzmann equation. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **16** 467–501. MR1697562

[28] NANBU, K. (1983). Interrelations between various direct simulation methods for solving the Boltzmann equation. *J. Phys. Soc. Jpn.* **52** 3382–3388.

[29] ROUSSET, M. (2014). A N-uniform quantitative Tanaka's theorem for the conservative Kac's N-particle system with Maxwell molecules. Arxiv preprint. Available at arXiv:1407.1965.

[30] SZNITMAN, A.-S. (1984). Équations de type de Boltzmann, spatialement homogènes. *Z. Wahrsch. Verw. Gebiete* **66** 559–592. MR0753814

[31] TANAKA, H. (1978/79). Probabilistic treatment of the Boltzmann equation of Maxwellian molecules. *Z. Wahrsch. Verw. Gebiete* **46** 67–105.

[32] TOSCANI, G. and VILLANI, C. (1999). Probability metrics and uniqueness of the solution to the Boltzmann equation for a Maxwell gas. *J. Stat. Phys.* **94** 619–637.

[33] VILLANI, C. (1998). On a new class of weak solutions to the spatially homogeneous Boltzmann and Landau equations. *Arch. Ration. Mech. Anal.* **143** 273–307.
[34] VILLANI, C. (2003). *Topics in Optimal Transportation. Graduate Studies in Mathematics* **58**. Amer. Math. Soc., Providence, RI. MR1964483
[35] XU, L. (2016). The multifractal nature of Boltzmann processes. *Stochastic Process. Appl.* **126** 2181–2210.

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A RANDOM MATRIX APPROACH TO NEURAL NETWORKS

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This article studies the Gram random matrix model $G = \frac{1}{T}\Sigma^{\mathsf{T}}\Sigma$, $\Sigma = \sigma(WX)$, classically found in the analysis of random feature maps and random neural networks, where $X = [x_1, \dots, x_T] \in \mathbb{R}^{p \times T}$ is a (data) matrix of bounded norm, $W \in \mathbb{R}^{n \times p}$ is a matrix of independent zero-mean unit variance entries and $\sigma : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous (activation) function—$\sigma(WX)$ being understood entry-wise. By means of a key concentration of measure lemma arising from nonasymptotic random matrix arguments, we prove that, as n, p, T grow large at the same rate, the resolvent $Q = (G + \gamma I_T)^{-1}$, for $\gamma > 0$, has a similar behavior as that met in sample covariance matrix models, involving notably the moment $\Phi = \frac{T}{n}\mathrm{E}[G]$, which provides in passing a deterministic equivalent for the empirical spectral measure of G. Application-wise, this result enables the estimation of the asymptotic performance of single-layer random neural networks. This in turn provides practical insights into the underlying mechanisms into play in random neural networks, entailing several unexpected consequences, as well as a fast practical means to tune the network hyperparameters.

REFERENCES

AKHIEZER, N. I. and GLAZMAN, I. M. (1993). *Theory of Linear Operators in Hilbert Space*. Dover, New York. MR1255973

BAI, Z. D. and SILVERSTEIN, J. W. (1998). No eigenvalues outside the support of the limiting spectral distribution of large-dimensional sample covariance matrices. *Ann. Probab.* **26** 316–345. MR1617051

BAI, Z. D. and SILVERSTEIN, J. W. (2007). On the signal-to-interference-ratio of CDMA systems in wireless communications. *Ann. Appl. Probab.* **17** 81–101. MR2292581

BAI, Z. and SILVERSTEIN, J. W. (2010). *Spectral Analysis of Large Dimensional Random Matrices*, 2nd ed. Springer, New York. MR2567175

BENAYCH-GEORGES, F. and NADAKUDITI, R. R. (2012). The singular values and vectors of low rank perturbations of large rectangular random matrices. *J. Multivariate Anal.* **111** 120–135.

CAMBRIA, E., GASTALDO, P., BISIO, F. and ZUNINO, R. (2015). An ELM-based model for affective analogical reasoning. *Neurocomputing* **149** 443–455.

CHOROMANSKA, A., HENAFF, M., MATHIEU, M., AROUS, G. B. and LECUN, Y. (2015). The loss surfaces of multilayer networks. In *AISTATS*.

COUILLET, R. and BENAYCH-GEORGES, F. (2016). Kernel spectral clustering of large dimensional data. *Electron. J. Stat.* **10** 1393–1454.

COUILLET, R., HOYDIS, J. and DEBBAH, M. (2012). Random beamforming over quasi-static and fading channels: A deterministic equivalent approach. *IEEE Trans. Inform. Theory* **58** 6392–6425. MR2982669

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Key words and phrases. Random matrix theory, random feature maps, neural networks.

COUILLET, R. and KAMMOUN, A. (2016). Random matrix improved subspace clustering. In 2016 *Asilomar Conference on Signals*, *Systems*, *and Computers*.

COUILLET, R., PASCAL, F. and SILVERSTEIN, J. W. (2015). The random matrix regime of Maronna's M-estimator with elliptically distributed samples. *J. Multivariate Anal.* **139** 56–78.

EL KAROUI, N. (2009). Concentration of measure and spectra of random matrices: Applications to correlation matrices, elliptical distributions and beyond. *Ann. Appl. Probab.* **19** 2362–2405. MR2588248

EL KAROUI, N. (2010). The spectrum of kernel random matrices. *Ann. Statist.* **38** 1–50. MR2589315

EL KAROUI, N. (2013). Asymptotic behavior of unregularized and ridge-regularized high-dimensional robust regression estimators: Rigorous results. Preprint. Available at arXiv:1311.2445.

GIRYES, R., SAPIRO, G. and BRONSTEIN, A. M. (2016). Deep neural networks with random Gaussian weights: A universal classification strategy? *IEEE Trans. Signal Process.* **64** 3444–3457. MR3515693

HORNIK, K., STINCHCOMBE, M. and WHITE, H. (1989). Multilayer feedforward networks are universal approximators. *Neural Networks* **2** 359–366.

HUANG, G.-B., ZHU, Q.-Y. and SIEW, C.-K. (2006). Extreme learning machine: Theory and applications. *Neurocomputing* **70** 489–501.

HUANG, G.-B., ZHOU, H., DING, X. and ZHANG, R. (2012). Extreme learning machine for regression and multiclass classification. *Systems*, *Man*, *and Cybernetics*, *Part B*: *Cybernetics*, *IEEE Transactions on* **42** 513–529.

JAEGER, H. and HAAS, H. (2004). Harnessing nonlinearity: Predicting chaotic systems and saving energy in wireless communication. *Science* **304** 78–80.

KAMMOUN, A., KHAROUF, M., HACHEM, W. and NAJIM, J. (2009). A central limit theorem for the sinr at the lmmse estimator output for large-dimensional signals. *IEEE Transactions on Information Theory* **55** 5048–5063.

KRIZHEVSKY, A., SUTSKEVER, I. and HINTON, G. E. (2012). Imagenet classification with deep convolutional neural networks. In *Advances in Neural Information Processing Systems* 1097–1105.

LECUN, Y., CORTES, C. and BURGES, C. (1998). The MNIST database of handwritten digits.

LEDOUX, M. (2005). *The Concentration of Measure Phenomenon* **89**. Amer. Math. Soc., Providence, RI. MR1849347

LIAO, Z. and COUILLET, R. (2017). A large dimensional analysis of least squares support vector machines. *J. Mach. Learn. Res.* To appear. Available at arXiv:1701.02967.

LOUBATON, P. and VALLET, P. (2010). Almost sure localization of the eigenvalues in a Gaussian information plus noise model. Application to the spiked models. *Electron. J. Probab.* **16** 1934–1959.

MAI, X. and COUILLET, R. (2017). The counterintuitive mechanism of graph-based semi-supervised learning in the big data regime. In *IEEE International Conference on Acoustics*, *Speech and Signal Processing* (*ICASSP'*17).

MARČENKO, V. A. and PASTUR, L. A. (1967). Distribution of eigenvalues for some sets of random matrices. *Math. USSR*, *Sb.* **1** 457–483.

PASTUR, L. and ŜERBINA, M. (2011). *Eigenvalue Distribution of Large Random Matrices*. Amer. Math. Soc., Providence, RI. MR2808038

RAHIMI, A. and RECHT, B. (2007). Random features for large-scale kernel machines. In *Advances in Neural Information Processing Systems* 1177–1184.

ROSENBLATT, F. (1958). The perceptron: A probabilistic model for information storage and organization in the brain. *Psychol. Rev.* **65** 386–408.

RUDELSON, M., VERSHYNIN, R. et al. (2013). Hanson–Wright inequality and sub-Gaussian concentration. *Electron. Commun. Probab.* **18** 1–9.

SAXE, A., KOH, P. W., CHEN, Z., BHAND, M., SURESH, B. and NG, A. Y. (2011). On random weights and unsupervised feature learning. In *Proceedings of the* 28*th International Conference on Machine Learning* (*ICML*-11) 1089–1096.

SCHMIDHUBER, J. (2015). Deep learning in neural networks: An overview. *Neural Netw.* **61** 85–117.

SILVERSTEIN, J. W. and BAI, Z. D. (1995). On the empirical distribution of eigenvalues of a class of large dimensional random matrices. *J. Multivariate Anal*. **54** 175–192.

SILVERSTEIN, J. W. and CHOI, S. (1995). Analysis of the limiting spectral distribution of large dimensional random matrices. *J. Multivariate Anal*. **54** 295–309.

TAO, T. (2012). *Topics in Random Matrix Theory* **132**. Amer. Math. Soc., Providence, RI.

TITCHMARSH, E. C. (1939). *The Theory of Functions*. Oxford Univ. Press, New York.

VERSHYNIN, R. (2012). Introduction to the non-asymptotic analysis of random matrices. In *Compressed Sensing*, 210–268, Cambridge Univ. Press, Cambridge.

WILLIAMS, C. K. I. (1998). Computation with infinite neural networks. *Neural Comput*. **10** 1203–1216.

YATES, R. D. (1995). A framework for uplink power control in cellular radio systems. *IEEE Journal on Selected Areas in Communications* **13** 1341–1347.

ZHANG, T., CHENG, X. and SINGER, A. (2014). Marchenko–Pastur Law for Tyler's and Maronna's M-estimators. Available at http://arxiv.org/abs/1401.3424.

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PHASE TRANSITIONS IN THE ONE-DIMENSIONAL COULOMB GAS ENSEMBLES

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We consider the system of particles on a finite interval with pairwise nearest neighbours interaction and external force. This model was introduced by Malyshev [*Probl. Inf. Transm.* **51** (2015) 31–36] to study the flow of charged particles on a rigorous mathematical level. It is a simplified version of a 3-dimensional classical Coulomb gas model. We study Gibbs distribution at finite positive temperature extending recent results on the zero temperature case (ground states). We derive the asymptotics for the mean and for the variances of the distances between the neighbouring charges. We prove that depending on the strength of the external force there are several phase transitions in the local structure of the configuration of the particles in the limit when the number of particles goes to infinity. We identify 5 different phases for any positive temperature.

The proofs rely on a conditional central limit theorem for nonidentical random variables, which has an interest on its own.

REFERENCES

[1] AIZENMAN, M. and MARTIN, P. A. (1980). Structure of Gibbs states of one dimensional Coulomb systems. *Comm. Math. Phys.* **78** 99–116.

[2] AMEUR, Y., HEDENMALM, H. and MAKAROV, N. (2015). Random normal matrices and Ward identities. *Ann. Probab.* **43** 1157–1201.

[3] AMEUR, Y., KANG, N.-G. and MAKAROV, N. Rescaling Ward identities in the random normal matrix model. arXiv:1410.4132.

[4] BAUERSCHMIDT, R., BOURGADE, P., NIKULA, M. and YAU, H.-T. Local density for two-dimensional one-component plasma. arXiv:1510.02074.

[5] BAUERSCHMIDT, R., BOURGADE, P., NIKULA, M. and YAU, H.-T. The two-dimensional Coulomb plasma: Quasi-free approximation and central limit theorem. arXiv:1609.08582.

[6] CUNDEN, F. D., FACCHI, P. and VIVO, P. (2016). A shortcut through the Coulomb gas method for spectral linear statistics on random matrices. *J. Phys. A: Math. Theor.* **49** 35202.

[7] DIACONIS, P. and FREEDMAN, D. A. (1988). Conditional limit theorems for exponential families and finite versions of de Finetti's theorem. *J. Theoret. Probab.* **1** 381–410. MR0958245

[8] EDWARDS, S. F. and LENARD, A. (1962). Exact statistical mechanics of a one-dimensional system with Coulomb forces. II. The method of functional integration. *J. Math. Phys.* **3** 778–792. MR0147214

[9] FEDORYUK, M. V. (1987). *Asimptotika: Integraly i Ryady*. Nauka, Moscow. (Russian) [Asymptotics: Integrals and Series].

[10] FELLER, W. (1971). *An Introduction to Probability Theory and Its Applications, Vol.* 2. Wiley.

MSC2010 subject classifications. 82B21, 82B26, 60F05.
Key words and phrases. Coulomb gas, phase transitions, Gibbs ensemble.

[11] JOHANSSON, K. (1991). Separation of phases at low temperatures in a one-dimensional continuous gas. *Comm. Math. Phys.* **141** 259–278.

[12] JOHANSSON, K. (1995). On separation of phases in one-dimensional gases. *Comm. Math. Phys.* **169** 521–561.

[13] LEBLÉ, T., SERFATY, S. and ZEITOUNI, O. (2017). Large deviations for the two-dimensional two-component plasma. *Comm. Math. Phys.* **350** 301–360.

[14] LENARD, A. (1961). Exact statistical mechanics of a one-dimensional system with Coulomb forces. *J. Math. Phys.* **2** 682.

[15] LENARD, A. (1963). Exact statistical mechanics of a one-dimensional system with Coulomb forces. III. Statistics of the electric field. *J. Math. Phys.* **4** 533.

[16] MALYSHEV, V. A. (2015). Phase transitions in the one-dimensional Coulomb medium. *Probl. Inf. Transm.* **51** 31–36.

[17] MALYSHEV, V. A. and ZAMYATIN, A. A. (2015). One-dimensional Coulomb multiparticle systems. *Adv. Math. Phys.*

[18] PETROV, V. V. (1975). *Sums of Independent Random Variables*. Springer.

[19] SERFATY, S. (2015). *Coulomb Gases and Ginzburg–Landau Vortices*. European Mathematical Society.

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RANDOM CLUSTER DYNAMICS FOR THE ISING MODEL IS RAPIDLY MIXING

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We show that the mixing time of Glauber (single edge update) dynamics for the random cluster model at $q = 2$ on an *arbitrary* n-vertex graph is bounded by a polynomial in n. As a consequence, the Swendsen–Wang algorithm for the ferromagnetic Ising model at any temperature also has a polynomial mixing time bound.

REFERENCES

BLANCA, A. and SINCLAIR, A. (2015). Dynamics for the mean-field random-cluster model. In *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques. LIPIcs. Leibniz Int. Proc. Inform.* **40** 528–543. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern. MR3441983

BLANCA, A. and SINCLAIR, A. (2016). Random-cluster dynamics in $\mathbb{Z}^2$. *Probab. Theory Relat. Fields*. DOI: 10.1007/s00440-016-0725-1.

BOLLOBÁS, B., GRIMMETT, G. and JANSON, S. (1996). The random-cluster model on the complete graph. *Probab. Theory Related Fields* **104** 283–317. MR1376340

BORGS, C., CHAYES, J. T., FRIEZE, A., KIM, J. H., TETALI, P., VIGODA, E. and VU, V. H. (1999). Torpid mixing of some Monte Carlo Markov chain algorithms in statistical physics. In *40th Annual Symposium on Foundations of Computer Science* (*New York*, 1999) 218–229. IEEE Computer Soc., Los Alamitos, CA. MR1917562

CAI, J.-Y., GUO, H. and WILLIAMS, T. (2016). A complete dichotomy rises from the capture of vanishing signatures. *SIAM J. Comput.* **45** 1671–1728. MR3543175

COLLEVECCHIO, A., GARONI, T. M., HYNDMAN, T. and TOKAREV, D. (2016). The worm process for the Ising model is rapidly mixing. *J. Stat. Phys.* **164** 1082–1102. MR3534485

DIACONIS, P. and SALOFF-COSTE, L. (1993). Comparison theorems for reversible Markov chains. *Ann. Appl. Probab.* **3** 696–730. MR1233621

DIACONIS, P. and STROOCK, D. (1991). Geometric bounds for eigenvalues of Markov chains. *Ann. Appl. Probab.* **1** 36–61. MR1097463

DUMINIL-COPIN, H., GAGNEBIN, M., HAREL, M., MANOLESCU, I. and TASSION, V. (2016). Discontinuity of the phase transition for the planar random-cluster and Potts models with $q > 4$. Available at arXiv:abs/1611.09877.

DYER, M., GOLDBERG, L. A., JERRUM, M. and MARTIN, R. (2006). Markov chain comparison. *Probab. Surv.* **3** 89–111. MR2216963

EDWARDS, R. G. and SOKAL, A. D. (1988). Generalization of the Fortuin–Kasteleyn–Swendsen–Wang representation and Monte Carlo algorithm. *Phys. Rev. D* **38** 2009–2012. MR0965465

FORTUIN, C. M. and KASTELEYN, P. W. (1972). On the random-cluster model. I. Introduction and relation to other models. *Physica* **57** 536–564. MR0359655

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GALANIS, A., ŠTEFANKOVIČ, D. and VIGODA, E. (2015). Swendsen–Wang algorithm on the mean-field Potts model. In *Approximation*, *Randomization*, *and Combinatorial Optimization*. *Algorithms and Techniques*. *LIPIcs*. *Leibniz Int*. *Proc*. *Inform*. **40** 815–828. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern. MR3441999

GHEISSARI, R. and LUBETZKY, E. (2016). Mixing times of critical 2D Potts models. Available at arXiv:abs/1607.02182.

GOLDBERG, L. A. and JERRUM, M. (2008). Inapproximability of the Tutte polynomial. *Inform*. *and Comput*. **206** 908–929. MR2433765

GOLDBERG, L. A. and JERRUM, M. (2012). Approximating the partition function of the ferromagnetic Potts model. *J*. *ACM* **59** Art. 25, 31. MR2995824

GOLDBERG, L. A. and JERRUM, M. (2014). The complexity of computing the sign of the Tutte polynomial. *SIAM J*. *Comput*. **43** 1921–1952. MR3291539

GORE, V. K. and JERRUM, M. R. (1999). The Swendsen–Wang process does not always mix rapidly. *J*. *Stat*. *Phys*. **97** 67–86. MR1733467

GRIMMETT, G. (2006). *The Random-Cluster Model*. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **333**. Springer, Berlin. MR2243761

GRIMMETT, G. and JANSON, S. (2009). Random even graphs. *Electron*. *J*. *Combin*. **16** Research Paper, 46, 19. MR2491648

JERRUM, M. and SINCLAIR, A. (1989). Approximating the permanent. *SIAM J*. *Comput*. **18** 1149–1178. MR1025467

JERRUM, M. and SINCLAIR, A. (1993). Polynomial-time approximation algorithms for the Ising model. *SIAM J*. *Comput*. **22** 1087–1116. MR1237164

LONG, Y., NACHMIAS, A., NING, W. and PERES, Y. (2014). A power law of order $1/4$ for critical mean field Swendsen–Wang dynamics. *Mem*. *Am*. *Math*. *Soc*. **232**.

LUBETZKY, E. and SLY, A. (2012). Critical Ising on the square lattice mixes in polynomial time. *Comm*. *Math*. *Phys*. **313** 815–836.

PERES, Y. (2017). Personal communication.

PROKOF'EV, N. and SVISTUNOV, B. (2001). Worm algorithms for classical statistical models. *Phys*. *Rev*. *Lett*. **87** 160601.

RANDALL, D. and WILSON, D. (1999). Sampling spin configurations of an Ising system. In *SODA* 959–960.

SINCLAIR, A. (1992). Improved bounds for mixing rates of Markov chains and multicommodity flow. *Combin*. *Probab*. *Comput*. **1** 351–370.

SINCLAIR, A. and JERRUM, M. (1989). Approximate counting, uniform generation and rapidly mixing Markov chains. *Inform*. *and Comput*. **82** 93–133. MR1003059

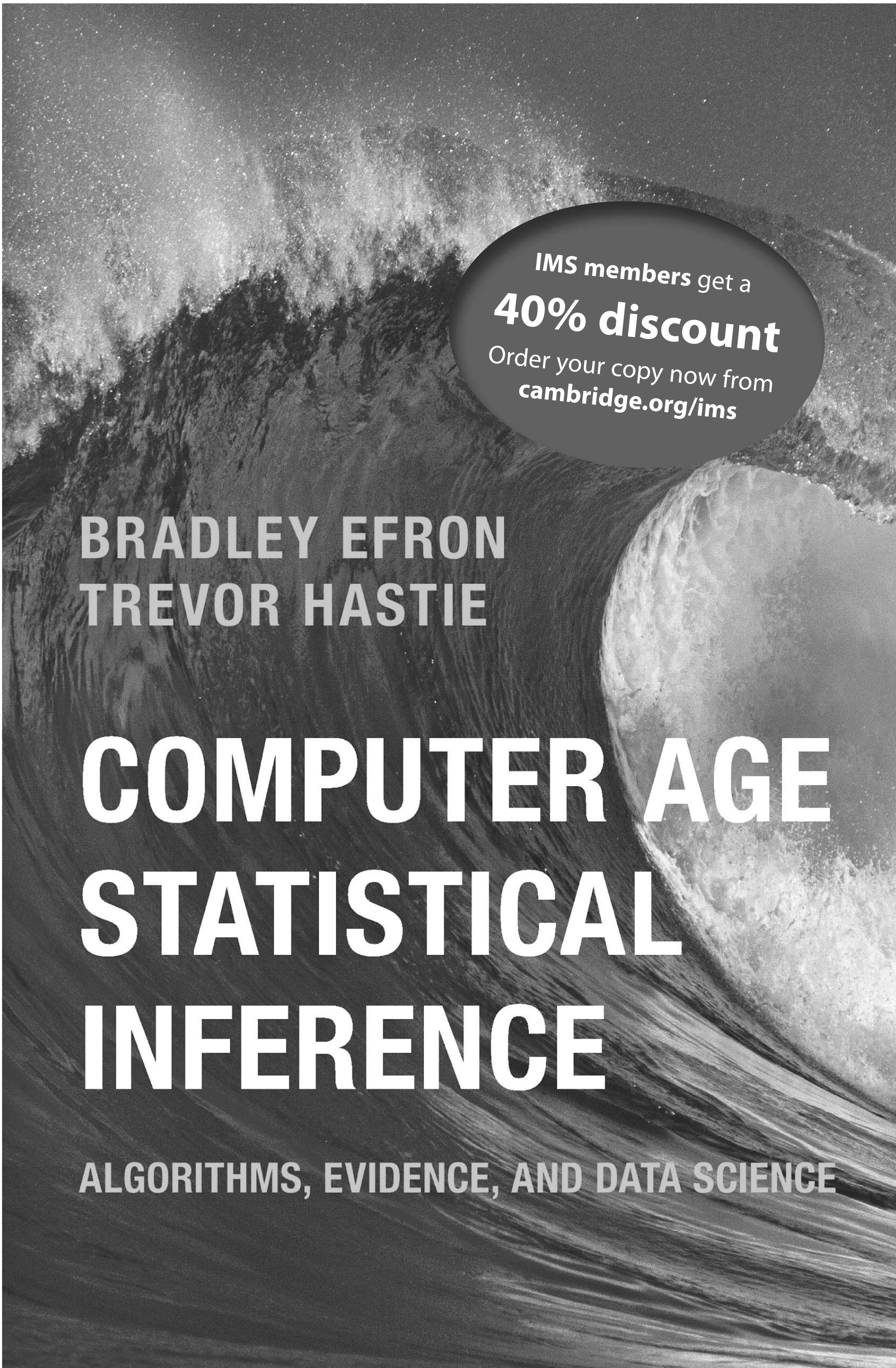
SWENDSEN, R. and WANG, J.-S. (1987). Nonuniversal critical dynamics in Monte Carlo simulations. *Phys*. *Rev*. *Lett*. **58** 86–88.

ULLRICH, M. (2013). Comparison of Swendsen–Wang and heat-bath dynamics. *Random Structures Algorithms* **42** 520–535.

ULLRICH, M. (2014a). Rapid mixing of Swendsen–Wang dynamics in two dimensions. *Dissertationes Math*. (*Rozprawy Mat*.) **502**.

ULLRICH, M. (2014b). Swendsen–Wang is faster than single-bond dynamics. *SIAM J*. *Discrete Math*. **28** 37–48.

VAN DER WAERDEN, B. L. (1941). Die lange Reichweite der regelmäßigen Atomanordnung in Mischkristallen. *Z*. *Phys*. **118** 473–488.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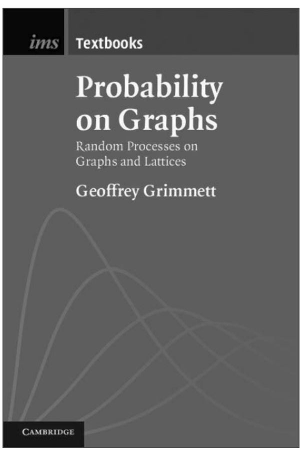

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Probability on Graphs
Random Processes on Graphs and Lattices

Geoffrey Grimmett

This introduction to some of the principal models in the theory of disordered systems leads the reader through the basics, to the very edge of contemporary research, with the minimum of technical fuss. Topics covered include random walk, percolation, self-avoiding walk, interacting particle systems, uniform spanning tree, random graphs, as well as the Ising, Potts, and random-cluster models for ferromagnetism, and the Lorentz model for motion in a random medium. Schramm–Löwner evolutions (SLE) arise in various contexts. The choice of topics is strongly motivated by modern applications and focuses on areas that merit further research. Special features include a simple account of Smirnov's proof of Cardy's formula for critical percolation, and a fairly full account of the theory of influence and sharp-thresholds. Accessible to a wide audience of mathematicians and physicists, this book can be used as a graduate course text. Each chapter ends with a range of exercises.

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