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WEAKLY HARMONIC OSCILLATORS PERTURBED BY A CONSERVATIVE NOISE[1]

BY CÉDRIC BERNARDIN, PATRÍCIA GONÇALVES[2] AND MILTON JARA[3]

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We consider a chain of weakly harmonic coupled oscillators perturbed by a conservative noise. We show that by tuning accordingly the coupling constant, energy can diffuse like a Brownian motion or superdiffuse like a maximally $3/2$-stable asymmetric Lévy process. For a critical value of the coupling, the energy diffusion is described by a family of Lévy processes which interpolate between these two processes.

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MSC2010 subject classifications. 60K35, 60G52, 35R11.

Key words and phrases. Fractional diffusion, Lévy process, chains of oscillators, heat conduction, fluctuating hydrodynamics, weakly asymmetric systems.

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DISORDER CHAOS IN SOME DILUTED SPIN GLASS MODELS

BY WEI-KUO CHEN AND DMITRY PANCHENKO

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We prove disorder chaos at zero temperature for three types of diluted models with large connectivity parameter: K-spin antiferromagnetic Ising model for even $K \geq 2$, K-spin spin glass model for even $K \geq 2$, and random K-sat model for all $K \geq 2$. We show that modifying even a small proportion of clauses results in near maximizers of the original and modified Hamiltonians being nearly orthogonal to each other with high probability. We use a standard technique of approximating diluted models by appropriate fully connected models and then apply disorder chaos results in this setting, which include both previously known results as well as new examples motivated by the random K-sat model.

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MSC2010 subject classifications. 60F10, 60G15, 60K35, 82B44.
Key words and phrases. Disorder chaos, p-spin models, diluted spin glasses.

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A LIOUVILLE THEOREM FOR ELLIPTIC SYSTEMS WITH DEGENERATE ERGODIC COEFFICIENTS

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We study the behavior of second-order degenerate elliptic systems in divergence form with random coefficients which are stationary and ergodic. Assuming moment bounds like Chiarini and Deuschel (2014) on the coefficient field a and its inverse, we prove an intrinsic large-scale $C^{1,\alpha}$-regularity estimate for a-harmonic functions and obtain a first-order Liouville theorem for a-harmonic functions.

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MSC2010 subject classifications. Primary 35B53, 35B65, 35J70, 60H25; secondary 60K37.
Key words and phrases. Degenerate elliptic equation, degenerate elliptic system, stochastic homogenization, large-scale regularity, Liouville theorem.

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A CHARACTERIZATION OF PRODUCT-FORM EXCHANGEABLE FEATURE PROBABILITY FUNCTIONS

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We characterize the class of exchangeable feature allocations assigning probability $V_{n,k}\prod_{l=1}^{k} W_{m_l} U_{n-m_l}$ to a feature allocation of n individuals, displaying k features with counts $(m_1, \dots, m_k)$ for these features. Each element of this class is parametrized by a countable matrix V and two sequences U and W of nonnegative weights. Moreover, a consistency condition is imposed to guarantee that the distribution for feature allocations of $(n-1)$ individuals is recovered from that of n individuals, when the last individual is integrated out. We prove that the only members of this class satisfying the consistency condition are mixtures of three-parameter Indian buffet Processes over the mass parameter γ, mixtures of N-dimensional Beta–Bernoulli models over the dimension N, or degenerate limits thereof. Hence, we provide a characterization of these two models as the only consistent exchangeable feature allocations having the required product form, up to randomization of the parameters.

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Key words and phrases. Indian buffet process, exchangeable feature allocations, Gibbs-type partitions.

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STOCHASTIC COAGULATION-FRAGMENTATION PROCESSES WITH A FINITE NUMBER OF PARTICLES AND APPLICATIONS

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Coagulation-fragmentation processes describe the stochastic association and dissociation of particles in clusters. Cluster dynamics with cluster-cluster interactions for a finite number of particles has recently attracted attention especially in stochastic analysis and statistical physics of cellular biology, as novel experimental data are now available, but their interpretation remains challenging. We derive here probability distribution functions for clusters that can either aggregate upon binding to form clusters of arbitrary sizes or a single cluster can dissociate into two sub-clusters. Using combinatorics properties and Markov chain representation, we compute steady-state distributions and moments for the number of particles per cluster in the case where the coagulation and fragmentation rates follow a detailed balance condition. We obtain explicit and asymptotic formulas for the cluster size and the number of clusters in terms of hypergeometric functions. To further characterize clustering, we introduce and discuss two mean times: one is the mean time two particles spend together before they separate and the other is the mean time they spend separated before they meet again for the first time. Finally, we discuss applications of the present stochastic coagulation-fragmentation framework in cell biology.

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MSC2010 subject classifications. Primary 60J20; secondary 05A17.

Key words and phrases. Markov chain, coagulation-fragmentation processes, stochastic processes.

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RIGOROUS RESULTS FOR THE STIGLER–LUCKOCK MODEL FOR THE EVOLUTION OF AN ORDER BOOK[1]

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In 1964, G. J. Stigler introduced a stochastic model for the evolution of an order book on a stock market. This model was independently rediscovered and generalized by H. Luckock in 2003. In his formulation, traders place buy and sell limit orders of unit size according to independent Poisson processes with possibly different intensities. Newly arriving buy (sell) orders are either immediately matched to the best available matching sell (buy) order or stay in the order book until a matching order arrives. Assuming stationarity, Luckock showed that the distribution functions of the best buy and sell order in the order book solve a differential equation, from which he was able to calculate the position of two prices $J^{\mathrm{c}}_{-} < J^{\mathrm{c}}_{+}$ such that buy orders below J^{c}_{-} and sell orders above J^{c}_{+} stay in the order book forever while all other orders are eventually matched. We extend Luckock's model by adding market orders, that is, with a certain rate traders arrive at the market that take the best available buy or sell offer in the order book, if there is one, and do nothing otherwise. We give necessary and sufficient conditions for such an extended model to be positive recurrent and show how these conditions are related to the prices J^{c}_{-} and J^{c}_{+} of Luckock.

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MAX κ-CUT AND THE INHOMOGENEOUS POTTS SPIN GLASS

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We study the asymptotic behavior of the Max κ-cut on a family of sparse, inhomogeneous random graphs. In the large degree limit, the leading term is a variational problem, involving the ground state of a constrained inhomogeneous Potts spin glass. We derive a Parisi-type formula for the free energy of this model, with possible constraints on the proportions, and derive the limiting ground state energy by a suitable zero temperature limit.

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LIMIT DISTRIBUTIONS FOR KPZ GROWTH MODELS WITH SPATIALLY HOMOGENEOUS RANDOM INITIAL CONDITIONS

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For stationary KPZ growth in $1+1$ dimensions, the height fluctuations are governed by the Baik–Rains distribution. Using the totally asymmetric single step growth model, alias TASEP, we investigate height fluctuations for a general class of spatially homogeneous random initial conditions. We prove that for TASEP there is a one-parameter family of limit distributions, labeled by the diffusion coefficient of the initial conditions. The distributions are defined through a variational formula. We use Monte Carlo simulations to obtain their numerical plots. Also discussed is the connection to the six-vertex model at its conical point.

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BORDER AGGREGATION MODEL

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Start with a graph with a subset of vertices called *the border*. A particle released from the origin performs a random walk on the graph until it comes to the immediate neighbourhood of the border, at which point it joins this subset thus increasing the border by one point. Then a new particle is released from the origin and the process repeats until the origin becomes a part of the border itself. We are interested in the total number ξ of particles to be released by this final moment.

We show that this model covers the OK Corral model as well as the erosion model, and obtain distributions and bounds for ξ in cases where the graph is star graph, regular tree and a d-dimensional lattice.

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BACKWARD SDES FOR OPTIMAL CONTROL OF PARTIALLY OBSERVED PATH-DEPENDENT STOCHASTIC SYSTEMS: A CONTROL RANDOMIZATION APPROACH

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We introduce a suitable backward stochastic differential equation (BSDE) to represent the value of an optimal control problem with partial observation for a controlled stochastic equation driven by Brownian motion. Our model is general enough to include cases with latent factors in mathematical finance. By a standard reformulation based on the reference probability method, it also includes the classical model where the observation process is affected by a Brownian motion (even in presence of a correlated noise), a case where a BSDE representation of the value was not available so far. This approach based on BSDEs allows for greater generality beyond the Markovian case, in particular our model may include path-dependence in the coefficients (both with respect to the state and the control), and does not require any nondegeneracy condition on the controlled equation.

We use a randomization method, previously adopted only for cases of full observation, and consisting in a first step, in replacing the control by an exogenous process independent of the driving noise and in formulating an auxiliary ("randomized") control problem where optimization is performed over changes of equivalent probability measures affecting the characteristics of the exogenous process. Our first main result is to prove the equivalence between the original partially observed control problem and the randomized problem. In a second step, we prove that the latter can be associated by duality to a BSDE, which then characterizes the value of the original problem as well.

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A NECESSARY AND SUFFICIENT CONDITION FOR EDGE UNIVERSALITY AT THE LARGEST SINGULAR VALUES OF COVARIANCE MATRICES

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In this paper, we prove a necessary and sufficient condition for the edge universality of sample covariance matrices with general population. We consider sample covariance matrices of the form $\mathcal{Q} = TX(TX)^*$, where X is an $M_2 \times N$ random matrix with $X_{ij} = N^{-1/2}q_{ij}$ such that q_{ij} are *i.i.d.* random variables with zero mean and unit variance, and T is an $M_1 \times M_2$ deterministic matrix such that T^*T is diagonal. We study the asymptotic behavior of the largest eigenvalues of $\mathcal{Q}$ when $M := \min\{M_1, M_2\}$ and N tend to infinity with $\lim_{N\to\infty} N/M = d \in (0,\infty)$. We prove that the Tracy–Widom law holds for the largest eigenvalue of $\mathcal{Q}$ if and only if $\lim_{s\to\infty} s^4\mathbb{P}(|q_{ij}| \geq s) = 0$ under mild assumptions of T. The necessity and sufficiency of this condition for the edge universality was first proved for Wigner matrices by Lee and Yin [*Duke Math. J.* **163** (2014) 117–173].

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TYPICAL DISTANCES IN THE DIRECTED CONFIGURATION MODEL

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We analyze the distribution of the distance between two nodes, sampled uniformly at random, in digraphs generated via the directed configuration model, in the supercritical regime. Under the assumption that the covariance between the in-degree and out-degree is finite, we show that the distance grows logarithmically in the size of the graph. In contrast with the undirected case, this can happen even when the variance of the degrees is infinite. The main tool in the analysis is a new coupling between a breadth-first graph exploration process and a suitable branching process based on the Kantorovich–Rubinstein metric. This coupling holds uniformly for a much larger number of steps in the exploration process than existing ones, and is therefore of independent interest.

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ORDERED AND SIZE-BIASED FREQUENCIES IN GEM AND GIBBS' MODELS FOR SPECIES SAMPLING

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We describe the distribution of frequencies ordered by sample values in a random sample of size n from the two parameter $\mathsf{GEM}(\alpha, \theta)$ random discrete distribution on the positive integers. These frequencies are a (size-α)-biased random permutation of the sample frequencies in either ranked order, or in the order of appearance of values in the sampling process. This generalizes a well-known identity in distribution due to Donnelly and Tavaré [*Adv. in Appl. Probab.* **18** (1986) 1–19] for $\alpha = 0$ to the case $0 \le \alpha < 1$. This description extends to sampling from Gibbs(α) frequencies obtained by suitable conditioning of the $\mathsf{GEM}(\alpha, \theta)$ model, and yields a value-ordered version of the Chinese restaurant construction of $\mathsf{GEM}(\alpha, \theta)$ and Gibbs(α) frequencies in the more usual size-biased order of their appearance. The proofs are based on a general construction of a finite sample $(X_1, \dots, X_n)$ from any random frequencies in size-biased order from the associated exchangeable random partition Π_∞ of $\mathbb{N}$ which they generate.

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LARGE DEVIATIONS THEORY FOR MARKOV JUMP MODELS OF CHEMICAL REACTION NETWORKS

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We prove a sample path Large Deviation Principle (LDP) for a class of jump processes whose rates are not uniformly Lipschitz continuous in phase space. Building on it, we further establish the corresponding Wentzell–Freidlin (W-F) (infinite time horizon) asymptotic theory. These results apply to jump Markov processes that model the dynamics of chemical reaction networks under mass action kinetics, on a microscopic scale. We provide natural sufficient topological conditions for the applicability of our LDP and W-F results. This then justifies the computation of nonequilibrium potential and exponential transition time estimates between different attractors in the large volume limit, for systems that are beyond the reach of standard chemical reaction network theory.

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MULTIPLE-PRIORS OPTIMAL INVESTMENT IN DISCRETE TIME FOR UNBOUNDED UTILITY FUNCTION

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This paper investigates the problem of maximizing expected terminal utility in a discrete-time financial market model with a finite horizon under nondominated model uncertainty. We use a dynamic programming framework together with measurable selection arguments to prove that under mild integrability conditions, an optimal portfolio exists for an unbounded utility function defined on the half-real line.

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COEXISTENCE AND EXTINCTION FOR STOCHASTIC KOLMOGOROV SYSTEMS

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In recent years there has been a growing interest in the study of the dynamics of stochastic populations. A key question in population biology is to understand the conditions under which populations coexist or go extinct. Theoretical and empirical studies have shown that coexistence can be facilitated or negated by both biotic interactions and environmental fluctuations. We study the dynamics of n populations that live in a stochastic environment and which can interact nonlinearly (through competition for resources, predator–prey behavior, etc.). Our models are described by n-dimensional Kolmogorov systems with white noise (stochastic differential equations—SDE). We give sharp conditions under which the populations converge exponentially fast to their unique stationary distribution as well as conditions under which some populations go extinct exponentially fast.

The analysis is done by a careful study of the properties of the invariant measures of the process that are supported on the boundary of the domain. To our knowledge this is one of the first general results describing the asymptotic behavior of stochastic Kolmogorov systems in non-compact domains.

We are able to fully describe the properties of many of the SDE that appear in the literature. In particular, we extend results on two dimensional Lotka-Volterra models, two dimensional predator–prey models, n dimensional simple food chains, and two predator and one prey models. We also show how one can use our methods to classify the dynamics of any two-dimensional stochastic Kolmogorov system satisfying some mild assumptions.

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INITIAL-BOUNDARY VALUE PROBLEM FOR THE HEAT EQUATION—A STOCHASTIC ALGORITHM

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The initial-boundary value problem for the heat equation is solved by using an algorithm based on a random walk on heat balls. Even if it represents a sophisticated generalization of the Walk on Spheres (WOS) algorithm introduced to solve the Dirichlet problem for Laplace's equation, its implementation is rather easy. The construction of this algorithm can be considered as a natural consequence of previous works the authors completed on the hitting time approximation for Bessel processes and Brownian motion [*Ann. Appl. Probab.* **23** (2013) 2259–2289, *Math. Comput. Simulation* **135** (2017) 28–38, *Bernoulli* **23** (2017) 3744–3771]. A similar procedure was introduced previously in the paper [*Random Processes for Classical Equations of Mathematical Physics* (1989) Kluwer Academic].

The definition of the random walk is based on a particular mean value formula for the heat equation. We present here a probabilistic view of this formula.

The aim of the paper is to prove convergence results for this algorithm and to illustrate them by numerical examples. These examples permit to emphasize the efficiency and accuracy of the algorithm.

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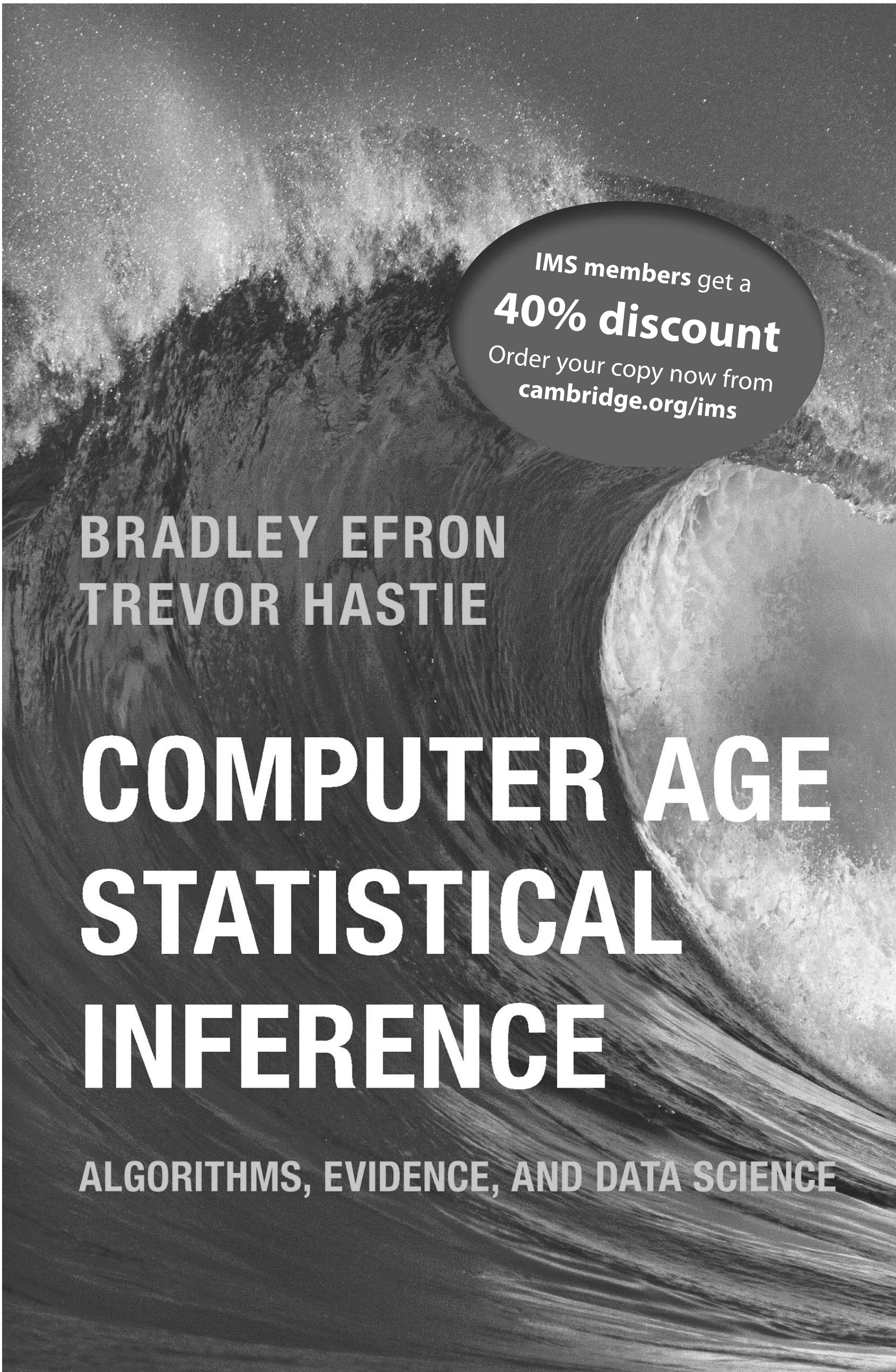
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