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WHEN MULTIPLICATIVE NOISE STYMIES CONTROL

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We consider the stabilization of an unstable discrete-time linear system that is observed over a channel corrupted by continuous multiplicative noise. Our main result shows that if the system growth is large enough, then the system cannot be stabilized. This is done by showing that the probability that the state magnitude remains bounded must go to zero with time. Our proof technique recursively bounds the conditional density of the system state to bound the progress the controller can make. This sidesteps the difficulty encountered in using the standard data-rate theorem style approach; that approach does not work because the mutual information per round between the system state and the observation is potentially unbounded.

It was known that a system with multiplicative observation noise can be stabilized using a simple memoryless linear strategy if the system growth is suitably bounded. The second main result in this paper shows that while memory cannot improve the performance of a linear scheme, a simple nonlinear scheme that uses one-step memory can do better than the best linear scheme.

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MSC2010 subject classifications. 93E03.
Key words and phrases. Control, stability, multiplicative noise.

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LARGE DEVIATIONS FOR FAST TRANSPORT STOCHASTIC RDES WITH APPLICATIONS TO THE EXIT PROBLEM

BY SANDRA CERRAI[1] AND NICHOLAS PASKAL

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We study reaction diffusion equations with a deterministic reaction term as well as two random reaction terms, one that acts on the interior of the domain, and another that acts only on the boundary of the domain. We are interested in the regime where the relative sizes of the diffusion and reaction terms are different. Specifically, we consider the case where the diffusion rate is much larger than the rate of reaction, and the deterministic rate of reaction is much larger than either of the random rate of reactions.

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MSC2010 subject classifications. 60H15, 70K65, 60F10.

Key words and phrases. Stochastic reaction–diffusion equations, averaging principle, large deviations, exit problem, Laplace principle.

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STOCHASTIC APPROXIMATION WITH RANDOM STEP SIZES AND URN MODELS WITH RANDOM REPLACEMENT MATRICES HAVING FINITE MEAN

BY UJAN GANGOPADHYAY AND KRISHANU MAULIK[1]

University of Southern California and Indian Statistical Institute

The stochastic approximation algorithm is a useful technique which has been exploited successfully in probability theory and statistics for a long time. The step sizes used in stochastic approximation are generally taken to be deterministic and same is true for the drift. However, the specific application of urn models with random replacement matrices motivates us to consider stochastic approximation in a setup where both the step sizes and the drift are random, but the sequence is uniformly bounded. The problem becomes interesting when the negligibility conditions on the errors hold only in probability. We first prove a result on stochastic approximation in this setup, which is new in the literature. Then, as an application, we study urn models with random replacement matrices.

In the urn model, the replacement matrices need neither be independent, nor identically distributed. We assume that the replacement matrices are only independent of the color drawn in the same round conditioned on the entire past. We relax the usual second moment assumption on the replacement matrices in the literature and require only first moment to be finite. We require the conditional expectation of the replacement matrix given the past to be close to an irreducible matrix, in an appropriate sense. We do not require any of the matrices to be balanced or nonrandom. We prove convergence of the proportion vector, the composition vector and the count vector in L^1, and hence in probability. It is to be noted that the related differential equation is of Lotka–Volterra type and can be analyzed directly.

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Key words and phrases. Urn model, random replacement matrix, balanced replacement matrix, irreducibility, stochastic approximation, random step size, random drift, uniform integrability, Lotka–Volterra differential equation.

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CRITICAL POINT FOR INFINITE CYCLES IN A RANDOM LOOP MODEL ON TREES

BY ALAN HAMMOND[1] AND MILIND HEGDE

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We study a spatial model of random permutations on trees with a time parameter $T > 0$, a special case of which is the random stirring process. The model on trees was first analysed by Björnberg and Ueltschi [*Ann. Appl. Probab.* **28** (2018) 2063–2082], who established the existence of infinite cycles for T slightly above a putatively identified critical value but left open behaviour at arbitrarily high values of T. We show the existence of infinite cycles for all T greater than a constant, thus classifying behaviour for all values of T and establishing the existence of a sharp phase transition. Numerical studies [*J. Phys. A* **48** Article ID 345002] of the model on $\mathbb{Z}^d$ have shown behaviour with strong similarities to what is proven for trees.

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PARKING ON TRANSITIVE UNIMODULAR GRAPHS

BY MICHAEL DAMRON*,[1], JANKO GRAVNER†,[2], MATTHEW JUNGE‡, HANBAEK LYU§,[3] AND DAVID SIVAKOFF¶,[3]

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Place a car independently with probability p at each site of a graph. Each initially vacant site is a parking spot that can fit one car. Cars simultaneously perform independent random walks. When a car encounters an available parking spot it parks there. Other cars can still drive over the site, but cannot park there. For a large class of transitive and unimodular graphs, we show that the root is almost surely visited infinitely many times when $p \geq 1/2$, and only finitely many times otherwise.

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THE HYDRODYNAMIC LIMIT OF A RANDOMIZED LOAD BALANCING NETWORK

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Randomized load balancing networks arise in a variety of applications, and allow for efficient sharing of resources, while being relatively easy to implement. We consider a network of parallel queues in which incoming jobs with independent and identically distributed service times are assigned to the shortest queue among a subset of d queues chosen uniformly at random, and leave the network on completion of service. Prior work on dynamical properties of this model has focused on the case of exponential service distributions. In this work, we analyze the more realistic case of general service distributions. We first introduce a novel particle representation of the state of the network, and characterize the state dynamics via a countable sequence of interacting stochastic measure-valued evolution equations. Under mild assumptions, we show that the sequence of scaled state processes converges, as the number of servers goes to infinity, to a hydrodynamic limit that is characterized as the unique solution to a countable system of coupled deterministic measure-valued equations. As a simple corollary, we also establish a propagation of chaos result that shows that finite collections of queues are asymptotically independent. The general framework developed here is potentially useful for analyzing a larger class of models arising in diverse fields including biology and materials science.

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A PROBABILISTIC APPROACH TO DIRAC CONCENTRATION IN NONLOCAL MODELS OF ADAPTATION WITH SEVERAL RESOURCES[1]

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This work is devoted to the study of scaling limits in small mutations and large time of the solutions u^ε of two deterministic models of phenotypic adaptation, where the parameter $\varepsilon > 0$ scales the size or frequency of mutations. The second model is the so-called Lotka–Volterra parabolic PDE in $\mathbb{R}^d$ with an arbitrary number of resources and the first one is a version of the second model with finite phenotype space. The solutions of such systems typically concentrate as Dirac masses in the limit $\varepsilon \to 0$. Our main results are, in both cases, the representation of the limits of $\varepsilon \log u^\varepsilon$ as solutions of variational problems and regularity results for these limits. The method mainly relies on Feynman–Kac-type representations of u^ε and Varadhan's lemma. Our probabilistic approach applies to multiresources situations not covered by standard analytical methods and makes the link between variational limit problems and Hamilton–Jacobi equations with irregular Hamiltonians that arise naturally from analytical methods. The finite case presents substantial difficulties since the rate function of the associated large deviation principle (LDP) has noncompact level sets. In that case, we are also able to obtain uniqueness of the solution of the variational problem and of the associated differential problem which can be interpreted as a Hamilton–Jacobi equation in finite state space.

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A VERSION OF ALDOUS' SPECTRAL-GAP CONJECTURE FOR THE ZERO RANGE PROCESS

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We show that the spectral gap of a general zero range process can be controlled in terms of the spectral gap for a single particle. This is in the spirit of Aldous' famous spectral-gap conjecture for the interchange process, now resolved by Caputo et al. Our main inequality decouples the role of the geometry (defined by the jump matrix) from that of the kinetics (specified by the exit rates). Among other consequences, the various spectral gap estimates that were so far only available on the complete graph or the d-dimensional torus now extend effortlessly to arbitrary geometries. As an illustration, we determine the exact order of magnitude of the spectral gap of the rate-one zero-range process on any regular graph and, more generally, for any doubly stochastic jump matrix.

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ITERATIVE MULTILEVEL PARTICLE APPROXIMATION FOR MCKEAN–VLASOV SDES

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The mean field limits of systems of interacting diffusions (also called stochastic interacting particle systems (SIPS)) have been intensively studied since McKean (*Proc. Natl. Acad. Sci. USA* **56** (1966) 1907–1911) as they pave a way to probabilistic representations for many important nonlinear/nonlocal PDEs. The fact that particles are not independent render classical variance reduction techniques not directly applicable, and consequently make simulations of interacting diffusions prohibitive.

In this article, we provide an alternative iterative particle representation, inspired by the fixed-point argument by Sznitman (In *École D'Été de Probabilités de Saint-Flour XIX—1989* (1991) 165–251, Springer). The representation enjoys suitable conditional independence property that is leveraged in our analysis. We establish weak convergence of iterative particle system to the McKean–Vlasov SDEs (McKV–SDEs). One of the immediate advantages of the iterative particle system is that it can be combined with the Multilevel Monte Carlo (MLMC) approach for the simulation of McKV–SDEs. We proved that the MLMC approach reduces the computational complexity of calculating expectations by an order of magnitude. Another perspective on this work is that we analyse the error of nested Multilevel Monte Carlo estimators, which is of independent interest. Furthermore, we work with state dependent functionals, unlike scalar outputs which are common in literature on MLMC. The error analysis is carried out in uniform, and what seems to be new, weighted norms.

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ERGODICITY OF THE ZIGZAG PROCESS[1]

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The zigzag process is a piecewise deterministic Markov process which can be used in a MCMC framework to sample from a given target distribution. We prove the convergence of this process to its target under very weak assumptions, and establish a central limit theorem for empirical averages under stronger assumptions on the decay of the target measure. We use the classical "Meyn–Tweedie" approach (*Markov Chains and Stochastic Stability* (2009) Cambridge Univ. Press; *Adv. in Appl. Probab.* **25** (1993) 487–517). The main difficulty turns out to be the proof that the process can indeed reach all the points in the space, even if we consider the minimal switching rates.

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ON SKOROKHOD EMBEDDINGS AND POISSON EQUATIONS

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The classical Skorokhod embedding problem for a Brownian motion W asks to find a stopping time τ so that W_τ is distributed according to a prescribed probability distribution μ. Many solutions have been proposed during the past 50 years and applications in different fields emerged. This article deals with a generalized Skorokhod embedding problem (SEP): Let X be a Markov process with initial marginal distribution μ_0 and let μ_1 be a probability measure. The task is to find a stopping time τ such that X_τ is distributed according to μ_1. More precisely, we study the question of deciding if a finite mean solution to the SEP can exist for given μ_0, μ_1 and the task of giving a solution which is as explicit as possible.

If μ_0 and μ_1 have positive densities h_0 and h_1 and the generator $\mathcal{A}$ of X has a formal adjoint operator $\mathcal{A}^*$, then we propose necessary and sufficient conditions for the existence of an embedding in terms of the Poisson equation $\mathcal{A}^* H = h_1 - h_0$ and give a fairly explicit construction of the stopping time using the solution of the Poisson equation. For the class of Lévy processes, we carry out the procedure and extend a result of Bertoin and Le Jan to Lévy processes without local times.

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A MCKEAN–VLASOV EQUATION WITH POSITIVE FEEDBACK AND BLOW-UPS

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We study a McKean–Vlasov equation arising from a mean-field model of a particle system with positive feedback. As particles hit a barrier, they cause the other particles to jump in the direction of the barrier and this feedback mechanism leads to the possibility that the system can exhibit contagious blow-ups. Using a fixed-point argument, we construct a differentiable solution up to a first explosion time. Our main contribution is a proof of uniqueness in the class of càdlàg functions, which confirms the validity of related propagation-of-chaos results in the literature. We extend the allowed initial conditions to include densities with any power law decay at the boundary, and connect the exponent of decay with the growth exponent of the solution in small time in a precise way. This takes us asymptotically close to the control on initial conditions required for a global solution theory. A novel minimality result and trapping technique are introduced to prove uniqueness.

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APPROXIMATION OF STOCHASTIC PROCESSES BY NONEXPANSIVE FLOWS AND COMING DOWN FROM INFINITY[1]

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This paper deals with the approximation of semimartingales in finite dimension by dynamical systems. We give trajectorial estimates uniform with respect to the initial condition for a well-chosen distance. This relies on a nonexpansivity property of the flow and allows to consider non-Lipschitz vector fields. The fluctuations of the process are controlled using the martingale technics and stochastic calculus.

Our main motivation is the trajectorial description of stochastic processes starting from large initial values. We state general properties on the coming down from infinity of one-dimensional SDEs, with a focus on stochastically monotone processes. In particular, we recover and complement known results on Λ-coalescent and birth and death processes. Moreover, using Poincaré's compactification techniques for flows close to infinity, we develop this approach in two dimensions for competitive stochastic models. We thus classify the coming down from infinity of Lotka–Volterra diffusions and provide uniform estimates for the scaling limits of competitive birth and death processes.

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MIXING TIME ESTIMATION IN REVERSIBLE MARKOV CHAINS FROM A SINGLE SAMPLE PATH

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The spectral gap $\gamma_\star$ of a finite, ergodic and reversible Markov chain is an important parameter measuring the asymptotic rate of convergence. In applications, the transition matrix $\boldsymbol{P}$ may be unknown, yet one sample of the chain up to a fixed time n may be observed. We consider here the problem of estimating $\gamma_\star$ from this data. Let $\boldsymbol{\pi}$ be the stationary distribution of $\boldsymbol{P}$, and $\pi_\star = \min_x \pi(x)$. We show that if n is at least $\frac{1}{\gamma_\star \pi_\star}$ times a logarithmic correction, then $\gamma_\star$ can be estimated to within a multiplicative factor with high probability. When π is uniform on d states, this nearly matches a lower bound of $\frac{d}{\gamma_\star}$ steps required for precise estimation of $\gamma_\star$. Moreover, we provide the first procedure for computing a fully data-dependent interval, from a single finite-length trajectory of the chain, that traps the mixing time t_{mix} of the chain at a prescribed confidence level. The interval does not require the knowledge of any parameters of the chain. This stands in contrast to previous approaches, which either only provide point estimates, or require a reset mechanism, or additional prior knowledge. The interval is constructed around the relaxation time $t_{\text{relax}} = 1/\gamma_\star$, which is strongly related to the mixing time, and the width of the interval converges to zero roughly at a $1/\sqrt{n}$ rate, where n is the length of the sample path.

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REDUCED-FORM FRAMEWORK UNDER MODEL UNCERTAINTY

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In this paper, we introduce a sublinear conditional expectation with respect to a family of possibly nondominated probability measures on a *progressively enlarged* filtration. In this way, we extend the classic reduced-form setting for credit and insurance markets to the case under model uncertainty, when we consider a family of priors possibly mutually singular to each other. Furthermore, we study the superhedging approach in continuous time for payment streams under model uncertainty, and establish several equivalent versions of dynamic robust superhedging duality. These results close the gap between robust framework for financial market, which is recently studied in an intensive way, and the one for credit and insurance markets, which is limited in the present literature only to some very specific cases.

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A GENERAL CONTINUOUS-STATE NONLINEAR BRANCHING PROCESS

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In this paper, we consider the unique nonnegative solution to the following generalized version of the stochastic differential equation for a continuous-state branching process:

$$X_t = x + \int_0^t \gamma_0(X_s)\,\mathrm{d}s + \int_0^t \int_0^{\gamma_1(X_{s-})} W(\mathrm{d}s, \mathrm{d}u)$$
$$+ \int_0^t \int_0^\infty \int_0^{\gamma_2(X_{s-})} z\tilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u),$$

where $W(\mathrm{d}t, \mathrm{d}u)$ and $\tilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$ denote a Gaussian white noise and an independent compensated spectrally positive Poisson random measure, respectively, and γ_0, γ_1 and γ_2 are functions on $\mathbb{R}_+$ with both γ_1 and γ_2 taking nonnegative values. Intuitively, this process can be identified as a continuous-state branching process with population-size-dependent branching rates and with competition. Using martingale techniques we find rather sharp conditions on extinction, explosion and coming down from infinity behaviors of the process. Some Foster–Lyapunov-type criteria are also developed for such a process. More explicit results are obtained when γ_i, $i = 0, 1, 2$ are power functions.

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EQUILIBRIUM INTERFACES OF BIASED VOTER MODELS[1]

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A one-dimensional interacting particle system is said to exhibit interface tightness if starting in an initial condition describing the interface between two constant configurations of different types, the process modulo translations is positive recurrent. In a biological setting, this describes two populations that do not mix, and it is believed to be a common phenomenon in one-dimensional particle systems. Interface tightness has been proved for voter models satisfying a finite second moment condition on the rates. We extend this to biased voter models. Furthermore, we show that the distribution of the equilibrium interface for the biased voter model converges to that of the voter model when the bias parameter tends to zero. A key ingredient is an identity for the expected number of boundaries in the equilibrium voter model interface, which is of independent interest.

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Key words and phrases. Biased voter model, interface tightness, branching and coalescing random walks.

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PROPAGATION OF CHAOS FOR TOPOLOGICAL INTERACTIONS

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We consider a N-particle model describing an alignment mechanism due to a topological interaction among the agents. We show that the kinetic equation, expected to hold in the mean-field limit $N \to \infty$, as following from the previous analysis in (*J. Stat. Phys.* **163** (2016) 41–60) can be rigorously derived. This means that the statistical independence (propagation of chaos) is indeed recovered in the limit, provided it is assumed at time zero.

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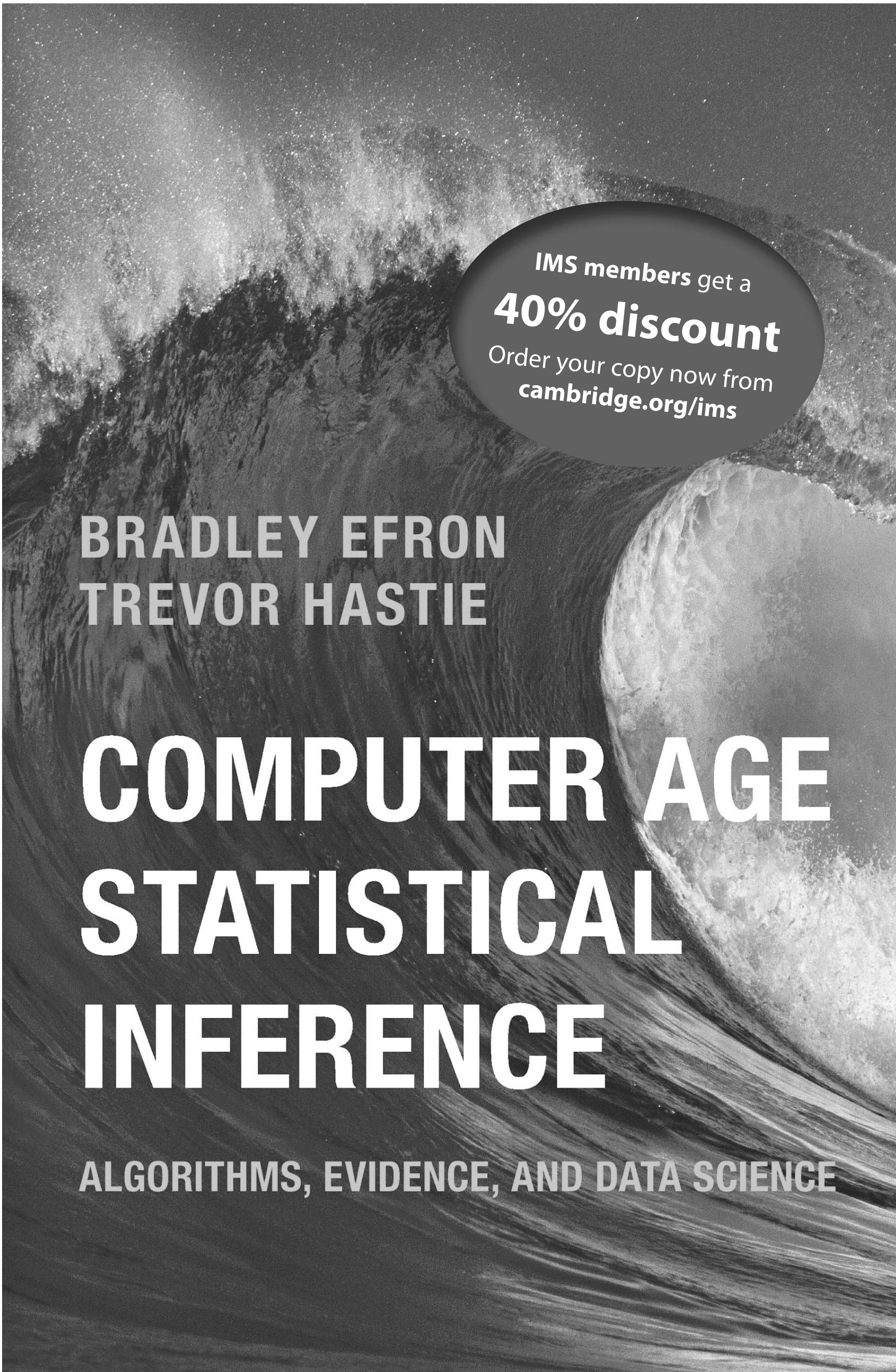
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