ISSN 1050-5164 (print)
ISSN 2168-8737 (online)

THE ANNALS *of* APPLIED PROBABILITY

AN OFFICIAL JOURNAL OF THE
INSTITUTE OF MATHEMATICAL STATISTICS

Articles

Vol. 29, No. 5—October 2019

THE ANNALS OF APPLIED PROBABILITY Vol. 29, No. 5, pp. 2613–3269 October 2019

INSTITUTE OF MATHEMATICAL STATISTICS

(Organized September 12, 1935)

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***The Annals of Applied Probability* [ISSN 1050-5164 (print); ISSN 2168-8737 (online)],** Volume 29, Number 5, October 2019. Published bimonthly by the Institute of Mathematical Statistics, 3163 Somerset Drive, Cleveland, Ohio 44122, USA. Periodicals postage paid at Cleveland, Ohio, and at additional mailing offices.

POSTMASTER: Send address changes to *The Annals of Applied Probability*, Institute of Mathematical Statistics, Dues and Subscriptions Office, 9650 Rockville Pike, Suite L 2310, Bethesda, Maryland 20814-3998, USA.

The Annals of Applied Probability
2019, Vol. 29, No. 5, 2613–2653
https://doi.org/10.1214/18-AAP1445

NORMAL CONVERGENCE OF NONLOCALISED GEOMETRIC FUNCTIONALS AND SHOT-NOISE EXCURSIONS

BY RAPHAËL LACHIÈZE-REY

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This article presents a complete second-order theory for a large class of geometric functionals on homogeneous Poisson input. In particular, the results do not require the existence of a radius of stabilisation. Hence they can be applied to geometric functionals of spatial shot-noise fields excursions such as volume, perimeter, or Euler characteristic (the method still applies to stabilising functionals). More generally, it must be checked that a local contribution to the functional is not strongly affected under a perturbation of the input far away. In this case, the exact asymptotic variance is given, as well as the likely optimal speed of convergence in the central limit theorem. This goes through a general mixing-type condition that adapts nicely to both proving asymptotic normality and that variance is of volume order.

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MSC2010 subject classifications. 60D05, 60G60, 60F05.

Key words and phrases. Poisson functionals, shot-noise fields, random excursions, central limit theorem, stabilisation, Berry–Esseen bounds.

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The Annals of Applied Probability
2019, Vol. 29, No. 5, 2654–2699
https://doi.org/10.1214/18-AAP1460

METASTABILITY OF THE CONTACT PROCESS ON FAST EVOLVING SCALE-FREE NETWORKS

BY EMMANUEL JACOB[1], AMITAI LINKER[2] AND PETER MÖRTERS[3]

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We study the contact process in the regime of small infection rates on finite scale-free networks with stationary dynamics based on simultaneous updating of all connections of a vertex. We allow the update rates of individual vertices to increase with the strength of a vertex, leading to a fast evolution of the network. We first develop an approach for inhomogeneous networks with general kernel and then focus on two canonical cases, the factor kernel and the preferential attachment kernel. For these specific networks, we identify and analyse four possible strategies how the infection can survive for a long time. We show that there is fast extinction of the infection when neither of the strategies is successful, otherwise there is slow extinction and the most successful strategy determines the asymptotics of the metastable density as the infection rate goes to zero. We identify the domains in which these strategies dominate in terms of phase diagrams for the exponent describing the decay of the metastable density.

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MSC2010 subject classifications. Primary 05C82; secondary 82C22.

Key words and phrases. Phase transitions, metastable density, evolving network, temporal network, dynamic network, inhomogeneous random graph, preferential attachment network, network dynamics, SIS infection.

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The Annals of Applied Probability
2019, Vol. 29, No. 5, 2700–2743
https://doi.org/10.1214/19-AAP1462

TREE LENGTHS FOR GENERAL Λ-COALESCENTS AND THE ASYMPTOTIC SITE FREQUENCY SPECTRUM AROUND THE BOLTHAUSEN–SZNITMAN COALESCENT[1]

BY CHRISTINA S. DIEHL AND GÖTZ KERSTING

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We study tree lengths in Λ-coalescents without a dust component from a sample of n individuals. For the total length of all branches and the total length of all external branches, we present laws of large numbers in full generality. The other results treat regularly varying coalescents with exponent 1, which cover the Bolthausen–Sznitman coalescent. The theorems contain laws of large numbers for the total length of all internal branches and of internal branches of order a (i.e., branches carrying a individuals out of the sample). These results immediately transform to sampling formulas in the infinite sites model. In particular, we obtain the asymptotic site frequency spectrum of the Bolthausen–Sznitman coalescent. The proofs rely on a new technique to obtain laws of large numbers for certain functionals of decreasing Markov chains.

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MSC2010 subject classifications. Primary 60J75; secondary 60F05, 60J27, 92D25.

Key words and phrases. Λ-coalescent, Bolthausen–Sznitman coalescent, law of large numbers, tree length, infinite sites model, sampling formula, site frequency spectrum, decreasing Markov chain.

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The Annals of Applied Probability
2019, Vol. 29, No. 5, 2744–2781
https://doi.org/10.1214/19-AAP1463

EMPIRICAL OPTIMAL TRANSPORT ON COUNTABLE METRIC SPACES: DISTRIBUTIONAL LIMITS AND STATISTICAL APPLICATIONS[1]

BY CARLA TAMELING, MAX SOMMERFELD AND AXEL MUNK

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We derive distributional limits for empirical transport distances between probability measures supported on countable sets. Our approach is based on sensitivity analysis of optimal values of infinite dimensional mathematical programs and a delta method for nonlinear derivatives. A careful calibration of the norm on the space of probability measures is needed in order to combine differentiability and weak convergence of the underlying empirical process. Based on this, we provide a sufficient and necessary condition for the underlying distribution on the countable metric space for such a distributional limit to hold. We give an explicit form of the limiting distribution for tree spaces.

Finally, we apply our findings to optimal transport based inference in large scale problems. An application to nanoscale microscopy is given.

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MSC2010 subject classifications. Primary 60F05, 60B12, 62E20; secondary 90C08, 90C31, 62G10.

Key words and phrases. Optimal transport, Wasserstein distance, empirical process, limit law, statistical testing.

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The Annals of Applied Probability
2019, Vol. 29, No. 5, 2782–2818
https://doi.org/10.1214/19-AAP1464

EXTINCTION IN LOWER HESSENBERG BRANCHING PROCESSES WITH COUNTABLY MANY TYPES

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We consider a class of branching processes with countably many types which we refer to as *Lower Hessenberg branching processes*. These are multitype Galton–Watson processes with typeset $\mathcal{X} = \{0, 1, 2, \dots\}$, in which individuals of type i may give birth to offspring of type $j \leq i + 1$ only. For this class of processes, we study the set S of fixed points of the progeny generating function. In particular, we highlight the existence of a continuum of fixed points whose minimum is the global extinction probability vector $\boldsymbol{q}$ and whose maximum is the partial extinction probability vector $\tilde{\boldsymbol{q}}$. In the case where $\tilde{\boldsymbol{q}} = \mathbf{1}$, we derive a global extinction criterion which holds under second moment conditions, and when $\tilde{\boldsymbol{q}} < \mathbf{1}$ we develop necessary and sufficient conditions for $\boldsymbol{q} = \tilde{\boldsymbol{q}}$. We also correct a result in the literature on a sequence of finite extinction probability vectors that converge to the infinite vector $\tilde{\boldsymbol{q}}$.

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MSC2010 subject classifications. Primary 60J80; secondary 60J05, 60J22.

Key words and phrases. Infinite-type branching process, extinction probability, extinction criterion, fixed point, varying environment.

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The Annals of Applied Probability
2019, Vol. 29, No. 5, 2819–2848
https://doi.org/10.1214/19-AAP1465

CONTROLLED REFLECTED SDES AND NEUMANN PROBLEM FOR BACKWARD SPDES

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We solve the optimal control problem of a one-dimensional reflected stochastic differential equation, whose coefficients can be path dependent. The value function of this problem is characterized by a backward stochastic partial differential equation (BSPDE) with Neumann boundary conditions. We prove the existence and uniqueness of a sufficiently regular solution for this BSPDE, which is then used to construct the optimal feedback control. In fact, we prove a more general result: the existence and uniqueness of strong solution for the Neumann problem for general nonlinear BSPDEs, which might be of interest even out of the current context.

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MSC2010 subject classifications. 60K35, 93E20, 60H15, 91G80.

Key words and phrases. Optimal control of reflected stochastic differential equations, Neumann problem, backward stochastic partial differential equation.

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The Annals of Applied Probability
2019, Vol. 29, No. 5, 2849–2883
https://doi.org/10.1214/19-AAP1466

DYNAMICS OF OBSERVABLES IN RANK-BASED MODELS AND PERFORMANCE OF FUNCTIONALLY GENERATED PORTFOLIOS

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In the seminal work (*Stochastic Portfolio Theory: Stochastic Modelling and Applied Probability* (2002) Springer), several macroscopic market observables have been introduced, in an attempt to find characteristics capturing the diversity of a financial market. Despite the crucial importance of such observables for investment decisions, a concise mathematical description of their dynamics has been missing. We fill this gap in the setting of rank-based models. The results are then used to study the performance of multiplicatively and additively functionally generated portfolios.

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MSC2010 subject classifications. Primary 60H10, 91G10; secondary 60G15, 60H15.

Key words and phrases. Capital distribution, functionally generated portfolios, Gaussian fluctuations, hitting times, hydrodynamic limits, macroscopic market observables, market diversity, market entropy, porous medium equation, rank-based models, relative return, stochastic partial differential equations, stochastic portfolio theory.

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The Annals of Applied Probability
2019, Vol. 29, No. 5, 2884–2928
https://doi.org/10.1214/19-AAP1467

MEASURING SAMPLE QUALITY WITH DIFFUSIONS[1]

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Stein's method for measuring convergence to a continuous target distribution relies on an operator characterizing the target and *Stein factor* bounds on the solutions of an associated differential equation. While such operators and bounds are readily available for a diversity of univariate targets, few multivariate targets have been analyzed. We introduce a new class of characterizing operators based on Itô diffusions and develop explicit multivariate Stein factor bounds for any target with a fast-coupling Itô diffusion. As example applications, we develop computable and convergence-determining *diffusion Stein discrepancies* for log-concave, heavy-tailed and multimodal targets and use these quality measures to select the hyperparameters of biased Markov chain Monte Carlo (MCMC) samplers, compare random and deterministic quadrature rules and quantify bias-variance tradeoffs in approximate MCMC. Our results establish a near-linear relationship between diffusion Stein discrepancies and Wasserstein distances, improving upon past work even for strongly log-concave targets. The exposed relationship between Stein factors and Markov process coupling may be of independent interest.

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MSC2010 subject classifications. Primary 60J60, 62-04, 62E17, 60E15, 65C60; secondary 62-07, 65C05, 68T05.

Key words and phrases. Multivariate Stein factors, Itô diffusion, Stein's method, Stein discrepancy, sample quality, Wasserstein decay, Markov chain Monte Carlo.

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The Annals of Applied Probability
2019, Vol. 29, No. 5, 2929–2987
https://doi.org/10.1214/19-AAP1470

INTERFERENCE QUEUEING NETWORKS ON GRIDS

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Consider a countably infinite collection of interacting queues, with a queue located at each point of the d-dimensional integer grid, having independent Poisson arrivals, but dependent service rates. The service discipline is of the processor sharing type, with the service rate in each queue slowed down, when the neighboring queues have a larger workload. The interactions are translation invariant in space and is neither of the Jackson Networks type, nor of the mean-field type. Coupling and percolation techniques are first used to show that this dynamics has well-defined trajectories. Coupling from the past techniques are then proposed to build its minimal stationary regime. The rate conservation principle of Palm calculus is then used to identify the stability condition of this system, where the notion of stability is appropriately defined for an infinite dimensional process. We show that the identified condition is also necessary in certain special cases and conjecture it to be true in all cases. Remarkably, the rate conservation principle also provides a closed-form expression for the mean queue size. When the stability condition holds, this minimal solution is the unique translation invariant stationary regime. In addition, there exists a range of small initial conditions for which the dynamics is attracted to the minimal regime. Nevertheless, there exists another range of larger though finite initial conditions for which the dynamics diverges, even though stability criterion holds.

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MSC2010 subject classifications. 60K35, 60J25, 60D05, 90B18, 68M20.

Key words and phrases. Rate conservation principle, mass transport theorem, monotonicity, positive correlation, percolation, stationary distribution, coupling from the past, Loynes' construction, particle systems, interacting queues, queueing theory, stability and instability, information theory, interference field, wireless network.

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The Annals of Applied Probability
2019, Vol. 29, No. 5, 2988–3005
https://doi.org/10.1214/19-AAP1471

APPROXIMATING MIXED HÖLDER FUNCTIONS USING RANDOM SAMPLES

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Suppose $f : [0, 1]^2 \to \mathbb{R}$ is a (c, α)-mixed Hölder function that we sample at l points $X_1, \dots, X_l$ chosen uniformly at random from the unit square. Let the location of these points and the function values $f(X_1), \dots, f(X_l)$ be given. If $l \geq c_1 n \log^2 n$, then we can compute an approximation $\tilde{f}$ such that

$$\|f - \tilde{f}\|_{L^2} = \mathcal{O}(n^{-\alpha} \log^{3/2} n),$$

with probability at least $1 - n^{2-c_1}$, where the implicit constant only depends on the constants $c > 0$ and $c_1 > 0$.

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MSC2010 subject classifications. Primary 26B35; secondary 60G42, 42B35.
Key words and phrases. Hölder condition, sparse grids, randomized Kaczmarz.

The Annals of Applied Probability
2019, Vol. 29, No. 5, 3006–3036
https://doi.org/10.1214/19-AAP1472

LOCAL LAW AND TRACY–WIDOM LIMIT FOR SPARSE SAMPLE COVARIANCE MATRICES

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We consider spectral properties of sparse sample covariance matrices, which includes biadjacency matrices of the bipartite Erdős–Rényi graph model. We prove a local law for the eigenvalue density up to the upper spectral edge. Under a suitable condition on the sparsity, we also prove that the limiting distribution of the rescaled, shifted extremal eigenvalues is given by the GOE Tracy–Widom law with an explicit formula on the deterministic shift of the spectral edge. For the biadjacency matrix of an Erdős–Rényi graph with two vertex sets of comparable sizes M and N, this establishes Tracy–Widom fluctuations of the second largest eigenvalue when the connection probability p is much larger than $N^{-2/3}$ with a deterministic shift of order $(Np)^{-1}$.

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MSC2010 subject classifications. 60B20, 62H10.
Key words and phrases. High-dimensional sample covariance matrices, local law, Tracy–Widom distribution.

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The Annals of Applied Probability
2019, Vol. 29, No. 5, 3037–3061
https://doi.org/10.1214/19-AAP1474

ANOTHER LOOK INTO THE WONG–ZAKAI THEOREM FOR STOCHASTIC HEAT EQUATION

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For the heat equation driven by a smooth, Gaussian random potential:

$$\partial_t u_\varepsilon = \frac{1}{2}\Delta u_\varepsilon + u_\varepsilon(\xi_\varepsilon - c_\varepsilon), \quad t > 0, x \in \mathbb{R},$$

where ξ_ε converges to a spacetime white noise, and c_ε is a diverging constant chosen properly, we prove that u_ε converges in L^n to the solution of the stochastic heat equation for any $n \geq 1$. Our proof is probabilistic, hence provides another perspective of the general result of Hairer and Pardoux (*J. Math. Soc. Japan* **67** (2015) 1551–1604), for the special case of the stochastic heat equation. We also discuss the transition from homogenization to stochasticity.

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MSC2010 subject classifications. Primary 35R60, 60H15; secondary 60H07.

Key words and phrases. Stochastic heat equation, Feynman–Kac formula, Wiener chaos expansion.

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The Annals of Applied Probability
2019, Vol. 29, No. 5, 3062–3127
https://doi.org/10.1214/19-AAP1475

PATHWISE CONVERGENCE OF THE HARD SPHERES KAC PROCESS

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We derive two estimates for the deviation of the N-particle, hard-spheres Kac process from the corresponding Boltzmann equation, measured in expected Wasserstein distance. Particular care is paid to the long-time properties of our estimates, exploiting the stability properties of the limiting Boltzmann equation at the level of realisations of the interacting particle system. As a consequence, we obtain an estimate for the propagation of chaos, uniformly in time and with polynomial rates, as soon as the initial data has a kth moment, $k > 2$. Our approach is similar to Kac's proposal of relating the long-time behaviour of the particle system to that of the limit equation. Along the way, we prove a new estimate for the continuity of the Boltzmann flow measured in Wasserstein distance.

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MSC2010 subject classifications. Primary 60J25, 60K35; secondary 35Q20.

Key words and phrases. Kac process, law of large numbers, Wasserstein distance, Boltzmann equation.

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The Annals of Applied Probability
2019, Vol. 29, No. 5, 3128–3154
https://doi.org/10.1214/19-AAP1476

THE ZEALOT VOTER MODEL[1]

BY RAN HUO AND RICK DURRETT

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Inspired by the spread of discontent as in the 2016 presidential election, we consider a voter model in which 0's are ordinary voters and 1's are zealots. Thinking of a social network, but desiring the simplicity of an infinite object that can have a nontrivial stationary distribution, space is represented by a tree. The dynamics are a variant of the biased voter: if x has degree $d(x)$ then at rate $d(x)p_k$ the individual at x consults $k \geq 1$ neighbors. If at least one neighbor is 1, they adopt state 1, otherwise they become 0. In addition at rate p_0 individuals with opinion 1 change to 0. As in the contact process on trees, we are interested in determining when the zealots survive and when they will survive locally.

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MSC2010 subject classifications. 60K35.
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The Annals of Applied Probability
2019, Vol. 29, No. 5, 3155–3200
https://doi.org/10.1214/19-AAP1477

AFFINE VOLTERRA PROCESSES

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We introduce affine Volterra processes, defined as solutions of certain stochastic convolution equations with affine coefficients. Classical affine diffusions constitute a special case, but affine Volterra processes are neither semimartingales, nor Markov processes in general. We provide explicit exponential-affine representations of the Fourier–Laplace functional in terms of the solution of an associated system of deterministic integral equations of convolution type, extending well-known formulas for classical affine diffusions. For specific state spaces, we prove existence, uniqueness, and invariance properties of solutions of the corresponding stochastic convolution equations. Our arguments avoid infinite-dimensional stochastic analysis as well as stochastic integration with respect to non-semimartingales, relying instead on tools from the theory of finite-dimensional deterministic convolution equations. Our findings generalize and clarify recent results in the literature on rough volatility models in finance.

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MSC2010 subject classifications. Primary 60J20; secondary 60G22, 45D05, 91G20.

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The Annals of Applied Probability
2019, Vol. 29, No. 5, 3201–3229
https://doi.org/10.1214/19-AAP1478

APPROXIMATING STATIONARY DISTRIBUTIONS OF FAST MIXING GLAUBER DYNAMICS, WITH APPLICATIONS TO EXPONENTIAL RANDOM GRAPHS

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We provide a general bound on the Wasserstein distance between two arbitrary distributions of sequences of Bernoulli random variables. The bound is in terms of a mixing quantity for the Glauber dynamics of one of the sequences, and a simple expectation of the other. The result is applied to estimate, with explicit error, expectations of functions of random vectors for some Ising models and exponential random graphs in "high temperature" regimes.

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The Annals of Applied Probability
2019, Vol. 29, No. 5, 3230–3265
https://doi.org/10.1214/19-AAP1479

STEIN'S METHOD FOR STATIONARY DISTRIBUTIONS OF MARKOV CHAINS AND APPLICATION TO ISING MODELS[1]

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We develop a new technique, based on Stein's method, for comparing two stationary distributions of irreducible Markov chains whose update rules are close in a certain sense. We apply this technique to compare Ising models on d-regular expander graphs to the Curie–Weiss model (complete graph) in terms of pairwise correlations and more generally kth order moments. Concretely, we show that d-regular Ramanujan graphs approximate the kth order moments of the Curie–Weiss model to within average error $k/\sqrt{d}$ (averaged over size k subsets), *independent of graph size*. The result applies even in the low-temperature regime; we also derive simpler approximation results for functionals of Ising models that hold only at high temperatures.

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MSC2010 subject classifications. 60C05, 60F05, 60B10.
Key words and phrases. Ising model, Stein's method, graph sparsification, Curie–Weiss.

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The Annals of Applied Probability
2019, Vol. 29, No. 5, 3266–3269
https://doi.org/10.1214/19-AAP1480
Main article: https://doi.org/10.1214/16-AAP1262

CORRECTION NOTE: A STRONG ORDER 1/2 METHOD FOR MULTIDIMENSIONAL SDES WITH DISCONTINUOUS DRIFT

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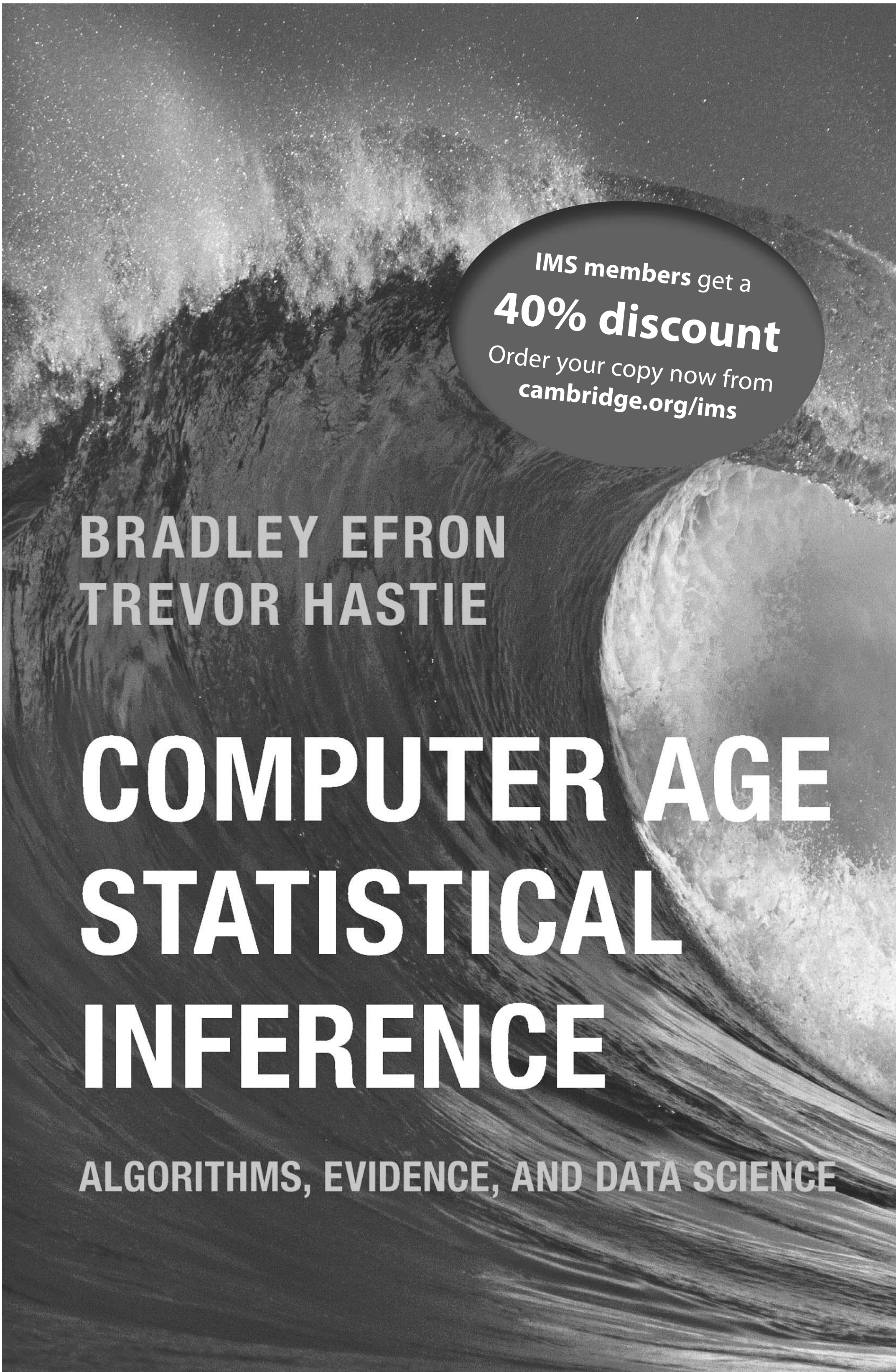
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