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GENERALIZED COUPLINGS AND ERGODIC RATES FOR SPDES AND OTHER MARKOV MODELS

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We establish verifiable general sufficient conditions for exponential or subexponential ergodicity of Markov processes that may lack the strong Feller property. We apply the obtained results to show exponential ergodicity of a variety of nonlinear stochastic partial differential equations with additive forcing, including 2D stochastic Navier–Stokes equations. Our main tool is a new version of the generalized coupling method.

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LOCAL WEAK CONVERGENCE FOR PAGERANK

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PageRank is a well-known algorithm for measuring centrality in networks. It was originally proposed by Google for ranking pages in the World Wide Web. One of the intriguing empirical properties of PageRank is the so-called 'power-law hypothesis': in a scale-free network, the PageRank scores follow a power law with the same exponent as the (in-)degrees. To date, this hypothesis has been confirmed empirically and in several specific random graphs models. In contrast, this paper does not focus on one random graph model but investigates the existence of an asymptotic PageRank distribution, when the graph size goes to infinity, using local weak convergence. This may help to identify general network structures in which the power-law hypothesis holds. We start from the definition of local weak convergence for sequences of (random) undirected graphs, and extend this notion to directed graphs. To this end, we define an exploration process in the directed setting that keeps track of in- and out-degrees of vertices. Then we use this to prove the existence of an asymptotic PageRank distribution. As a result, the limiting distribution of PageRank can be computed directly as a function of the limiting object. We apply our results to the directed configuration model and continuous-time branching processes trees, as well as to preferential attachment models.

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JOIN-THE-SHORTEST QUEUE DIFFUSION LIMIT IN HALFIN–WHITT REGIME: SENSITIVITY ON THE HEAVY-TRAFFIC PARAMETER

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Consider a system of N parallel single-server queues with unit-exponential service time distribution and a single dispatcher where tasks arrive as a Poisson process of rate $\lambda(N)$. When a task arrives, the dispatcher assigns it to one of the servers according to the Join-the-Shortest Queue (JSQ) policy. Eschenfeldt and Gamarnik (*Math. Oper. Res.* **43** (2018) 867–886) identified a novel limiting diffusion process that arises as the weak-limit of the appropriately scaled occupancy measure of the system under the JSQ policy in the Halfin–Whitt regime, where $(N - \lambda(N))/\sqrt{N} \to \beta > 0$ as $N \to \infty$. The analysis of this diffusion goes beyond the state of the art techniques, and even proving its ergodicity is nontrivial, and was left as an open question. Recently, exploiting a generator expansion framework via the Stein's method, Braverman (2018) established its exponential ergodicity, and adapting a regenerative approach, Banerjee and Mukherjee (*Ann. Appl. Probab.* **29** (2018) 1262–1309) analyzed the tail properties of the stationary distribution and path fluctuations of the diffusion.

However, the analysis of the bulk behavior of the stationary distribution, namely, the moments, remained intractable until this work. In this paper, we perform a thorough analysis of the bulk behavior of the stationary distribution of the diffusion process, and discover that it exhibits different qualitative behavior, depending on the value of the heavy-traffic parameter β. Moreover, we obtain precise asymptotic laws of the centered and scaled steady-state distribution, as β tends to 0 and ∞. Of particular interest, we also establish a certain intermittency phenomena in the $\beta \to \infty$ regime and a surprising distributional convergence result in the $\beta \to 0$ regime.

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BOOTSTRAP PERCOLATION ON THE PRODUCT OF THE TWO-DIMENSIONAL LATTICE WITH A HAMMING SQUARE

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Bootstrap percolation on a graph is a deterministic process that iteratively enlarges a set of occupied sites by adjoining points with at least θ occupied neighbors. The initially occupied set is random, given by a uniform product measure with a low density p. Our main focus is on this process on the product graph $\mathbb{Z}^2 \times K_n^2$, where K_n is a complete graph. We investigate how p scales with n so that a typical site is eventually occupied. Under critical scaling, the dynamics with even θ exhibits a sharp phase transition, while odd θ yields a gradual percolation transition. We also establish a gradual transition for bootstrap percolation on $\mathbb{Z}^2 \times K_n$. The community structure of the product graphs connects our process to a heterogeneous bootstrap percolation on $\mathbb{Z}^2$. This natural relation with a generalization of polluted bootstrap percolation is the leading theme in our analysis.

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PROPAGATION OF CHAOS FOR STOCHASTIC SPATIALLY STRUCTURED NEURONAL NETWORKS WITH DELAY DRIVEN BY JUMP DIFFUSIONS

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Spatially structured neural networks driven by jump diffusion noise with monotone coefficients, fully path dependent delay and with a disorder parameter are considered. Well-posedness for the associated McKean–Vlasov equation and a corresponding propagation of chaos result in the infinite population limit are proven. Our existence result for the McKean–Vlasov equation is based on the Euler approximation that is applied to this type of equation for the first time.

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ON AN EPIDEMIC MODEL ON FINITE GRAPHS

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We study a system of random walks, known as the frog model, starting from a profile of independent Poisson(λ) particles per site, with one additional active particle planted at some vertex **o** of a finite connected simple graph $G = (V, E)$. Initially, only the particles occupying **o** are active. Active particles perform $t \in \mathbb{N} \cup \{\infty\}$ steps of the walk they picked before vanishing and activate all inactive particles they hit. This system is often taken as a model for the spread of an epidemic over a population. Let $\mathcal{R}_t$ be the set of vertices which are visited by the process, when active particles vanish after t steps. We study the susceptibility of the process on the underlying graph, defined as the random quantity $\mathcal{S}(G) := \inf\{t : \mathcal{R}_t = V\}$ (essentially, the shortest particles' lifespan required for the entire population to get infected). We consider the cases that the underlying graph is either a regular expander or a d-dimensional torus of side length n (for all $d \geq 1$) $\mathbb{T}_d(n)$ and determine the asymptotic behavior of $\mathcal{S}$ up to a constant factor. In fact, throughout we allow the particle density λ to depend on n and for $d \geq 2$ we determine the asymptotic behavior of $\mathcal{S}(\mathbb{T}_d(n))$ up to smaller order terms for a wide range of $\lambda = \lambda_n$.

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CONVERGENCE TO THE MEAN FIELD GAME LIMIT: A CASE STUDY

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We study the convergence of Nash equilibria in a game of optimal stopping. If the associated mean field game has a unique equilibrium, any sequence of n-player equilibria converges to it as $n \to \infty$. However, both the finite and infinite player versions of the game often admit multiple equilibria. We show that mean field equilibria satisfying a transversality condition are limit points of n-player equilibria, but we also exhibit a remarkable class of mean field equilibria that are not limits, thus questioning their interpretation as "large n" equilibria.

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CENTRAL LIMIT THEOREMS FOR PATTERNS IN MULTISET PERMUTATIONS AND SET PARTITIONS

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We use the recently developed method of weighted dependency graphs to prove central limit theorems for the number of occurrences of any fixed pattern in multiset permutations and in set partitions. This generalizes results for patterns of size 2 in both settings, obtained by Canfield, Janson and Zeilberger and Chern, Diaconis, Kane and Rhoades, respectively.

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LARGE DEVIATION PRINCIPLES FOR FIRST-ORDER SCALAR CONSERVATION LAWS WITH STOCHASTIC FORCING

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In this paper, we established the Freidlin–Wentzell-type large deviation principles for first-order scalar conservation laws perturbed by small multiplicative noise. Due to the lack of the viscous terms in the stochastic equations, the kinetic solution to the Cauchy problem for these first-order conservation laws is studied. Then, based on the well-posedness of the kinetic solutions, we show that the large deviations holds by utilising the weak convergence approach.

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MONTE CARLO WITH DETERMINANTAL POINT PROCESSES

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We show that repulsive random variables can yield Monte Carlo methods with faster convergence rates than the typical $N^{-1/2}$, where N is the number of integrand evaluations. More precisely, we propose stochastic numerical quadratures involving determinantal point processes associated with multivariate orthogonal polynomials, and we obtain root mean square errors that decrease as $N^{-(1+1/d)/2}$, where d is the dimension of the ambient space. First, we prove a central limit theorem (CLT) for the linear statistics of a class of determinantal point processes, when the reference measure is a product measure supported on a hypercube, which satisfies the Nevai-class regularity condition; a result which may be of independent interest. Next, we introduce a Monte Carlo method based on these determinantal point processes, and prove a CLT with explicit limiting variance for the quadrature error, when the reference measure satisfies a stronger regularity condition. As a corollary, by taking a specific reference measure and using a construction similar to importance sampling, we obtain a general Monte Carlo method, which applies to any measure with continuously derivable density. Loosely speaking, our method can be interpreted as a stochastic counterpart to Gaussian quadrature, which at the price of some convergence rate, is easily generalizable to any dimension and has a more explicit error term.

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RANDOM-CLUSTER DYNAMICS IN $\mathbb{Z}^2$: RAPID MIXING WITH GENERAL BOUNDARY CONDITIONS

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The random-cluster model with parameters (p, q) is a random graph model that generalizes bond percolation ($q = 1$) and the Ising and Potts models ($q \geq 2$). We study its Glauber dynamics on $n \times n$ boxes Λ_n of the integer lattice graph $\mathbb{Z}^2$, where the model exhibits a sharp phase transition at $p = p_c(q)$. Unlike traditional spin systems like the Ising and Potts models, the random-cluster model has non-local interactions. Long-range interactions can be imposed as external connections in the boundary of Λ_n, known as *boundary conditions*. For select boundary conditions that do not carry long-range information (namely, wired and free), Blanca and Sinclair proved that when $q > 1$ and $p \neq p_c(q)$, the Glauber dynamics on Λ_n mixes in optimal $O(n^2 \log n)$ time. In this paper, we prove that this mixing time is polynomial in n for every boundary condition that is *realizable* as a configuration on $\mathbb{Z}^2 \setminus \Lambda_n$. We then use this to prove near-optimal $\tilde{O}(n^2)$ mixing time for "typical" boundary conditions. As a complementary result, we construct classes of nonrealizable (nonplanar) boundary conditions inducing slow (stretched-exponential) mixing at $p \ll p_c$.

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THE LARGEST REAL EIGENVALUE IN THE REAL GINIBRE ENSEMBLE AND ITS RELATION TO THE ZAKHAROV–SHABAT SYSTEM

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The real Ginibre ensemble consists of $n \times n$ real matrices $\mathbf{X}$ whose entries are i.i.d. standard normal random variables. In sharp contrast to the complex and quaternion Ginibre ensemble, real eigenvalues in the real Ginibre ensemble attain positive likelihood. In turn, the spectral radius $R_n = \max_{1 \le j \le n} |z_j(\mathbf{X})|$ of the eigenvalues $z_j(\mathbf{X}) \in \mathbb{C}$ of a real Ginibre matrix $\mathbf{X}$ follows a different limiting law (as $n \to \infty$) for $z_j(\mathbf{X}) \in \mathbb{R}$ than for $z_j(\mathbf{X}) \in \mathbb{C} \setminus \mathbb{R}$. Building on previous work by Rider and Sinclair (*Ann. Appl. Probab.* **24** (2014) 1621–1651) and Poplavskyi, Tribe and Zaboronski (*Ann. Appl. Probab.* **27** (2017) 1395–1413), we show that the limiting distribution of $\max_{j:z_j \in \mathbb{R}} z_j(\mathbf{X})$ admits a closed-form expression in terms of a distinguished solution to an inverse scattering problem for the Zakharov–Shabat system. As byproducts of our analysis, we also obtain a new determinantal representation for the limiting distribution of $\max_{j:z_j \in \mathbb{R}} z_j(\mathbf{X})$ and extend recent tail estimates in (*Ann. Appl. Probab.* **27** (2017) 1395–1413) via nonlinear steepest descent techniques.

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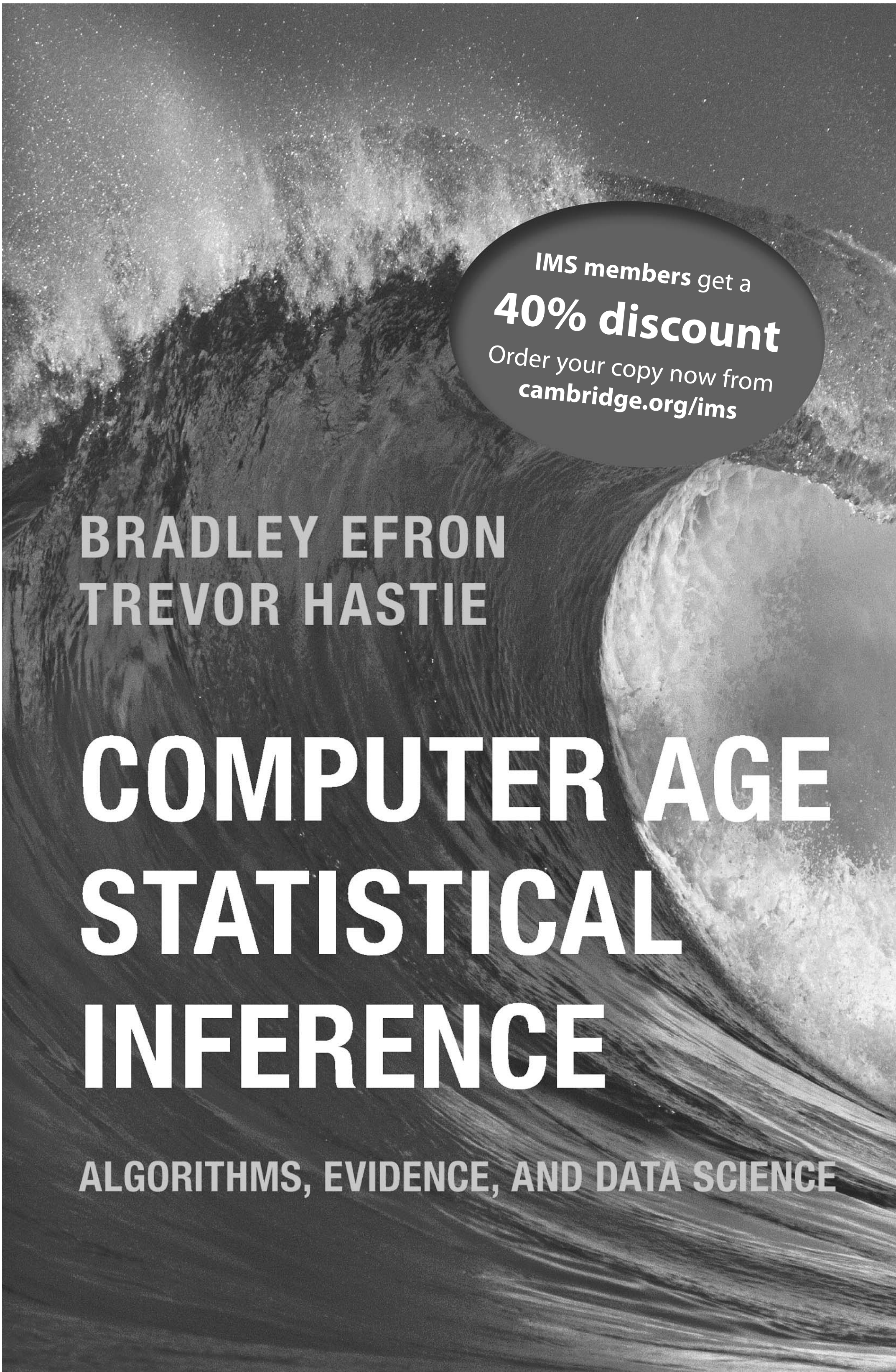
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