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LOWER BOUNDS FOR TRACE RECONSTRUCTION

BY NINA HOLDEN[1] AND RUSSELL LYONS[2]

[1]*Department of Mathematics, Massachusetts Institute of Technology, ninahold@gmail.com*
[2]*Department of Mathematics, Indiana University, rdlyons@indiana.edu*

In the trace reconstruction problem, an unknown bit string $\mathbf{x} \in \{0,1\}^n$ is sent through a deletion channel where each bit is deleted independently with some probability $q \in (0,1)$, yielding a contracted string $\widetilde{\mathbf{x}}$. How many i.i.d. samples of $\widetilde{\mathbf{x}}$ are needed to reconstruct $\mathbf{x}$ with high probability? We prove that there exist $\mathbf{x}, \mathbf{y} \in \{0,1\}^n$ such that at least $cn^{5/4}/\sqrt{\log n}$ traces are required to distinguish between $\mathbf{x}$ and $\mathbf{y}$ for some absolute constant c, improving the previous lower bound of cn. Furthermore, our result improves the previously known lower bound for reconstruction of random strings from $c\log^2 n$ to $c\log^{9/4} n/\sqrt{\log\log n}$.

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ADAPTIVE EULER–MARUYAMA METHOD FOR SDES WITH NONGLOBALLY LIPSCHITZ DRIFT

BY WEI FANG* AND MICHAEL B. GILES**

Mathematical Institute, University of Oxford, *wei.fang@maths.ox.ac.uk; **mike.giles@maths.ox.ac.uk

This paper proposes an adaptive timestep construction for an Euler–Maruyama approximation of SDEs with nonglobally Lipschitz drift. It is proved that if the timestep is bounded appropriately, then over a finite time interval the numerical approximation is stable, and the expected number of timesteps is finite. Furthermore, the order of strong convergence is the same as usual, that is, order $\frac{1}{2}$ for SDEs with a nonuniform globally Lipschitz volatility, and order 1 for Langevin SDEs with unit volatility and a drift with sufficient smoothness. For a class of ergodic SDEs, we also show that the bound for the moments and the strong error of the numerical solution are uniform in T, which allow us to introduce the adaptive multilevel Monte Carlo method to compute the expectations with respect to the invariant distribution. The analysis is supported by numerical experiments.

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MEAN GEOMETRY FOR 2D RANDOM FIELDS: LEVEL PERIMETER AND LEVEL TOTAL CURVATURE INTEGRALS

BY HERMINE BIERMÉ[1] AND AGNÈS DESOLNEUX[2]

[1]*LMA (UMR CNRS 7348), Université de Poitiers, hermine.bierme@math.univ-poitiers.fr*
[2]*CNRS, CMLA (UMR 8536), Université Paris-Saclay, ENS Cachan, agnes.desolneux@cmla.ens-cachan.fr*

We introduce the level perimeter integral and the total curvature integral associated with a real-valued function f defined on the plane $\mathbb{R}^2$, as integrals allowing to compute the perimeter of the excursion set of f above level t and the total (signed) curvature of its boundary for almost every level t. Thanks to the Gauss–Bonnet theorem, the total curvature is directly related to the Euler characteristic of the excursion set. We show that the level perimeter and the total curvature integrals can be computed in two different frameworks: smooth (at least C^2) functions and piecewise constant functions (also called here elementary functions). Considering 2D random fields (in particular shot noise random fields), we compute their mean perimeter and total curvature integrals, and this provides new "explicit" computations of the mean perimeter and Euler characteristic densities of excursion sets, beyond the Gaussian framework: for piecewise constant shot noise random fields, we give some examples of completely explicit formulas, and for smooth shot noise random fields the provided examples are only partly explicit, since the formulas are given under the form of integrals of some special functions.

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BOUNDS FOR THE ASYMPTOTIC DISTRIBUTION OF THE LIKELIHOOD RATIO

BY ANDREAS ANASTASIOU[1] AND GESINE REINERT[2]

[1]*Department of Mathematics and Statistics, University of Cyprus, anastasiou.andreas@ucy.ac.cy*
[2]*Department of Statistics, University of Oxford, reinert@stats.ox.ac.uk*

In this paper, we give an explicit bound on the distance to chi-square for the likelihood ratio statistic when the data are realisations of independent and identically distributed random elements. To our knowledge, this is the first explicit bound which is available in the literature. The bound depends on the number of samples as well as on the dimension of the parameter space. We illustrate the bound with three examples: samples from an exponential distribution, samples from a normal distribution and logistic regression.

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OPTIMAL POSITION TARGETING VIA DECOUPLING FIELDS

BY STEFAN ANKIRCHNER[1,*], ALEXANDER FROMM[1,**], THOMAS KRUSE[2] AND ALEXANDRE POPIER[3]

[1]*Institute for Mathematics, University of Jena,* [*]*s.ankirchner@uni-jena.de;* [**]*alexander.fromm@uni-jena.de*
[2]*Institute of Mathematics, University of Gießen, thomas.kruse@math.uni-giessen.de*
[3]*Laboratoire Manceau de Mathématiques, Le Mans Université, apopier@univ-lemans.fr*

We consider a variant of the basic problem of the calculus of variations, where the Lagrangian is convex and subject to randomness adapted to a Brownian filtration. We solve the problem by reducing it, via a limiting argument, to an unconstrained control problem that consists in finding an absolutely continuous process minimizing the expected sum of the Lagrangian and the deviation of the terminal state from a given target position. Using the Pontryagin maximum principle, we characterize a solution of the unconstrained control problem in terms of a fully coupled forward–backward stochastic differential equation (FBSDE). We use the method of decoupling fields for proving that the FBSDE has a unique solution. We exploit a monotonicity property of the decoupling field for solving the original constrained problem and characterize its solution in terms of an FBSDE with a free backward part.

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SEMI-IMPLICIT EULER–MARUYAMA APPROXIMATION FOR NONCOLLIDING PARTICLE SYSTEMS

BY HOANG-LONG NGO[1] AND DAI TAGUCHI[2]

[1]*Department of Mathematics and Informatics, Hanoi National University of Education, ngolong@hnue.edu.vn*
[2]*Graduate School of Engineering Science, Osaka University, dai.taguchi.dai@gmail.com*

We introduce a semi-implicit Euler–Maruyama approximation which preserves the noncolliding property for some class of noncolliding particle systems such as Dyson–Brownian motions, Dyson–Ornstein–Uhlenbeck processes and Brownian particle systems with nearest neighbor repulsion, and study its rates of convergence in both L^p-norm and pathwise sense.

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TRADING WITH SMALL NONLINEAR PRICE IMPACT

BY THOMAS CAYÉ[1], MARTIN HERDEGEN[2] AND JOHANNES MUHLE-KARBE[3]

[1]*School of Mathematical Sciences, Dublin City University, thomas.caye@dcu.ie*
[2]*Department of Statistics, University of Warwick, M.Herdegen@warwick.ac.uk*
[3]*Department of Mathematics, Imperial College London, j.muhle-karbe@imperial.ac.uk*

We study portfolio choice with small nonlinear price impact on general market dynamics. Using probabilistic techniques and convex duality, we show that the asymptotic optimum can be described explicitly up to the solution of a nonlinear ODE, which identifies the optimal trading speed and the performance loss due to the trading friction. Previous asymptotic results for proportional and quadratic trading costs are obtained as limiting cases. As an illustration, we discuss how nonlinear trading costs affect the pricing and hedging of derivative securities and active portfolio management.

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OPTIMAL INVESTMENT AND CONSUMPTION WITH LABOR INCOME IN INCOMPLETE MARKETS

BY OLEKSII MOSTOVYI[1] AND MIHAI SÎRBU[2]

[1] *Department of Mathematics, University of Connecticut, oleksii.mostovyi@uconn.edu*
[2] *Department of Mathematics, University of Texas at Austin, sirbu@math.utexas.edu*

We consider the problem of optimal consumption from labor income and investment in a general incomplete semimartingale market. The economic agent cannot borrow against future income, so the total wealth is required to be positive at (all or some) previous times. Under very general conditions, we show that an optimal consumption and investment plan exists and is unique, and provide a dual characterization in terms of an optional strong supermartingale deflator and a decreasing part, which charges only the times when the no-borrowing constraint is binding. The analysis relies on the infinite-dimensional parametrization of the income/liability streams and, therefore, provides the first-order dependence of the optimal investment and consumption plans on future income/liabilities (as well as a pricing rule).

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ORIENTED FIRST PASSAGE PERCOLATION IN THE MEAN FIELD LIMIT, 2. THE EXTREMAL PROCESS

BY NICOLA KISTLER[1,*], ADRIEN SCHERTZER[1,**] AND MARIUS A. SCHMIDT[2]

[1]*Institute for Mathematics, Goethe University Frankfurt am Main,* [*]*kistler@math.uni-frankfurt.de;* [**]*schertzer@math.uni-frankfurt.de*

[2]*Department of Mathematics and Informatics, University of Basel,* *mariusalexander.schmidt@unibas.ch*

This is the second, and last paper in which we address the behavior of oriented first passage percolation on the hypercube in the limit of large dimensions. We prove here that the extremal process converges to a Cox process with exponential intensity. This entails, in particular, that the first passage time converges weakly to a random shift of the Gumbel distribution. The random shift, which has an explicit, universal distribution related to modified Bessel functions of the second kind, is the sole manifestation of correlations ensuing from the geometry of Euclidean space in infinite dimensions. The proof combines the multiscale refinement of the second moment method with a conditional version of the Chen–Stein bounds, and a contraction principle.

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NONLINEAR LARGE DEVIATIONS: BEYOND THE HYPERCUBE

BY JUN YAN

Department of Statistics, Stanford University, junyan65@stanford.edu

By extending (*Adv. Math.* **299** (2016) 396–450), we present a framework to calculate large deviations for nonlinear functions of independent random variables supported on compact sets in Banach spaces. Previous research on nonlinear large deviations has only focused on random variables supported on $\{-1, +1\}^n$, and accordingly we build theory for random variables with general distributions, increasing flexibility in the applications. As examples, we compute the large deviation rate functions for monochromatic subgraph counts in edge-colored complete graphs, and for triangle counts in dense random graphs with continuous edge weights. Moreover, we verify the mean field approximation for a class of vector spin models.

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A BERRY–ESSEEN THEOREM FOR PITMAN'S α-DIVERSITY

BY EMANUELE DOLERA[1] AND STEFANO FAVARO[2]

[1]*Department of Mathematics, University of Pavia*, *emanuele.dolera@unipv.it*
[2]*Department of Economics and Statistics, University of Torino*, *stefano.favaro@unito.it*

This paper contributes to the study of the random number K_n of blocks in the random partition of $\{1,\dots,n\}$ induced by random sampling from the celebrated two parameter Poisson–Dirichlet process. For any $\alpha \in (0,1)$ and $\theta > -\alpha$ Pitman (*Combinatorial Stochastic Processes* (2006) Springer, Berlin) showed that $n^{-\alpha}K_n \xrightarrow{\text{a.s.}} S_{\alpha,\theta}$ as $n \to +\infty$, where the limiting random variable, referred to as Pitman's α-diversity, is distributed according to a polynomially scaled Mittag–Leffler distribution function. Our main result is a Berry–Esseen theorem for Pitman's α-diversity $S_{\alpha,\theta}$, namely we show that

$$\sup_{x\ge 0}\left|\mathsf{P}\left[\frac{K_n}{n^\alpha} \le x\right] - \mathsf{P}[S_{\alpha,\theta} \le x]\right| \le \frac{C(\alpha,\theta)}{n^\alpha}$$

holds for every $n \in \mathbb{N}$ with an explicit constant term $C(\alpha,\theta)$, for $\alpha \in (0,1)$ and $\theta > 0$. The proof relies on three intermediate novel results which are of independent interest: (i) a probabilistic representation of the distribution of K_n in terms of a compound distribution; (ii) a quantitative version of the Laplace's approximation method for integrals; (iii) a refined quantitative bound for Poisson approximation. An application of our Berry–Esseen theorem is presented in the context of Bayesian nonparametric inference for species sampling problems, quantifying the error of a posterior approximation that has been extensively applied to infer the number of unseen species in a population.

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A LOWER BOUND ON THE QUEUEING DELAY IN RESOURCE CONSTRAINED LOAD BALANCING

BY DAVID GAMARNIK[1], JOHN N. TSITSIKLIS[2,*] AND MARTIN ZUBELDIA[2,**]

[1]*Sloan School of Management, Massachusetts Institute of Technology, gamarnik@mit.edu*
[2]*Laboratory for Information and Decision Systems, Massachusetts Institute of Technology, *jnt@mit.edu; **mar.zubeldia@gmail.com*

We consider the following distributed service model: jobs with unit mean, general distribution, and independent processing times arrive as a renewal process of rate λn, with $0 < \lambda < 1$, and are immediately dispatched to one of several queues associated with n identical servers with unit processing rate. We assume that the dispatching decisions are made by a central dispatcher endowed with a finite memory, and with the ability to exchange messages with the servers.

We study the fundamental resource requirements (memory bits and message exchange rate), in order to drive the expected queueing delay in steady-state of a typical job to zero, as n increases. We develop a novel approach to show that, within a certain broad class of "symmetric" policies, every dispatching policy with a message rate of the order of n, and with a memory of the order of $\log n$ bits, results in an expected queueing delay which is bounded away from zero, uniformly as $n \to \infty$. This complements existing results which show that, in the absence of such limitations on the memory or the message rate, there exist policies with vanishing queueing delay (at least with Poisson arrivals and exponential service times).

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THE SOCIAL NETWORK MODEL ON INFINITE GRAPHS

BY JONATHAN HERMON[1], BEN MORRIS[2], CHUAN QIN[3] AND ALLAN SLY[4]

[1]*Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, jonathan.hermon@statslab.cam.ac.uk*

[2]*University of California Davis, morris@math.ucdavis.edu*

[3]*Securities Litigation and Consulting Group, chuanqin@slcg.com*

[4]*Department of Mathematics, Princeton University, asly@math.princeton.edu*

Given an infinite connected regular graph $G = (V, E)$, place at each vertex Poisson(λ) walkers performing independent lazy simple random walks on G simultaneously. When two walkers visit the same vertex at the same time they are declared to be acquainted. We show that when G is vertex-transitive and amenable, for all $\lambda > 0$ a.s. any pair of walkers will eventually have a path of acquaintances between them. In contrast, we show that when G is nonamenable (not necessarily transitive) there is always a phase transition at some $\lambda_c(G) > 0$. We give general bounds on $\lambda_c(G)$ and study the case that G is the d-regular tree in more detail. Finally, we show that in the nonamenable setup, for every λ there exists a finite time $t_\lambda(G)$ such that a.s. there exists an infinite set of walkers having a path of acquaintances between them by time $t_\lambda(G)$.

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VISCOSITY SOLUTIONS TO PARABOLIC MASTER EQUATIONS AND MCKEAN–VLASOV SDES WITH CLOSED-LOOP CONTROLS

BY CONG WU[1] AND JIANFENG ZHANG[2]

[1]*Wells Fargo Securities, wucong085@gmail.com*
[2]*Department of Mathematics, University of Southern California, jianfenz@usc.edu*

The master equation is a type of PDE whose state variable involves the distribution of certain underlying state process. It is a powerful tool for studying the limit behavior of large interacting systems, including mean field games and systemic risk. It also appears naturally in stochastic control problems with partial information and in time inconsistent problems. In this paper we propose a novel notion of viscosity solution for parabolic master equations, arising mainly from control problems, and establish its wellposedness. Our main innovation is to restrict the involved measures to a certain set of semimartingale measures which satisfy the desired compactness. As an important example, we study the HJB master equation associated with the control problems for McKean–Vlasov SDEs. Due to practical considerations, we consider closed-loop controls. It turns out that the regularity of the value function becomes much more involved in this framework than the counterpart in the standard control problems. Finally, we build the whole theory in the path dependent setting, which is often seen in applications. The main result in this part is an extension of Dupire's (2009) functional Itô formula. This Itô formula requires a special structure of the derivatives with respect to the measures, which was originally due to Lions in the state dependent case. We provided an elementary proof for this well known result in the short note (2017), and the same arguments work in the path dependent setting here.

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KINETICALLY CONSTRAINED MODELS WITH RANDOM CONSTRAINTS

BY ASSAF SHAPIRA

LPSM, Paris Diderot University, assafshap@gmail.com

We study two kinetically constrained models in a quenched random environment. The first model is a mixed threshold Fredrickson–Andersen model on $\mathbb{Z}^2$, where the update threshold is either 1 or 2. The second is a mixture of the Fredrickson–Andersen 1-spin facilitated constraint and the North-East constraint in $\mathbb{Z}^2$. We compare three time scales related to these models—the bootstrap percolation time for emptying the origin, the relaxation time of the kinetically constrained model, and the time for emptying the origin of the kinetically constrained model—and understand the effect of the random environment on each of them.

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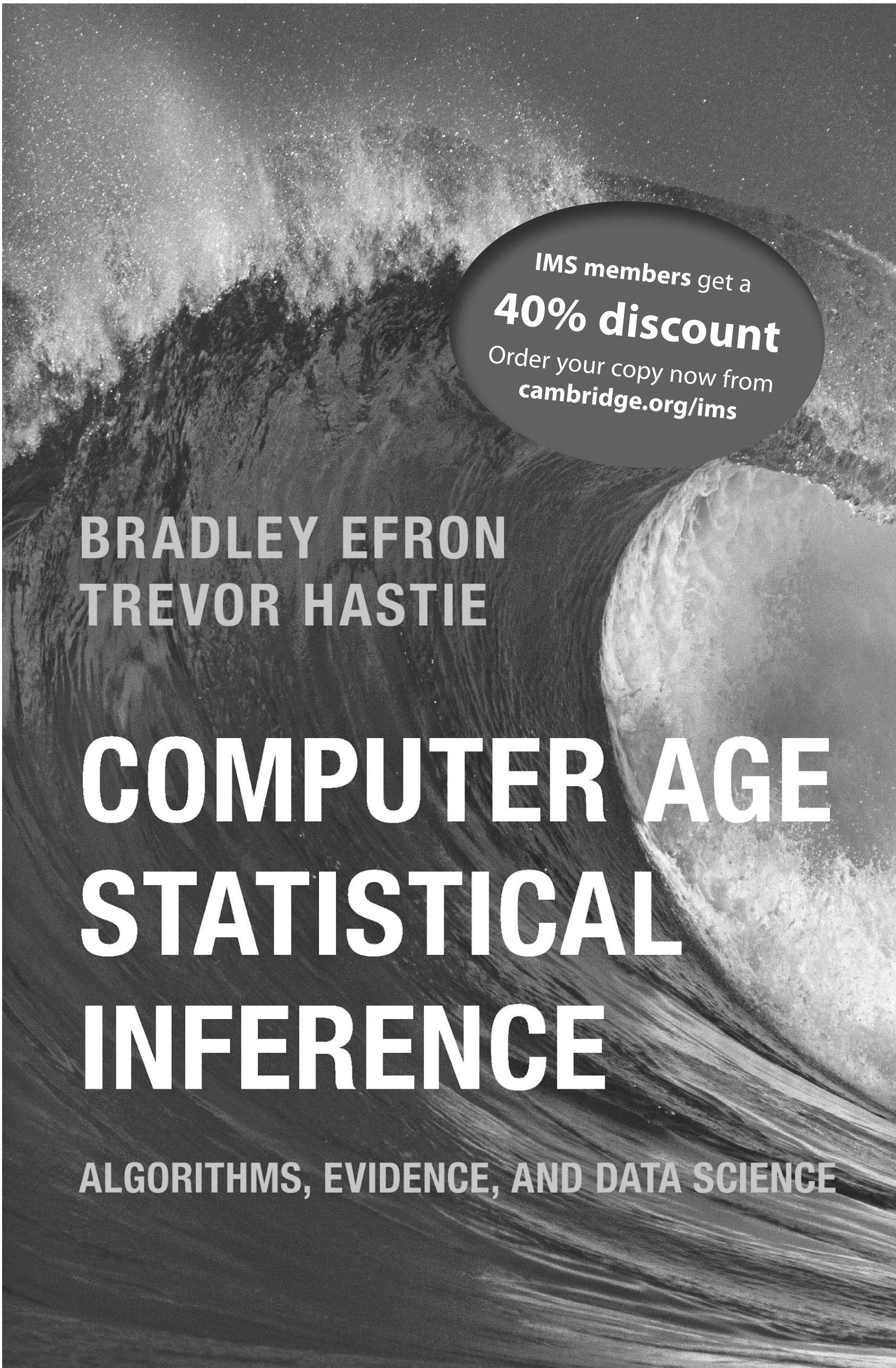
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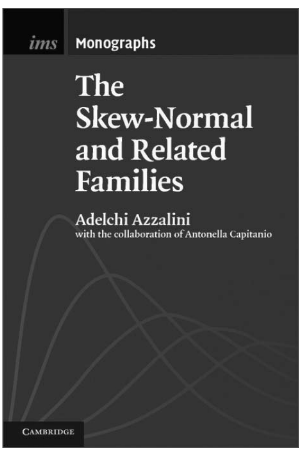

The Skew-Normal and Related Families

Adelchi Azzalini
in collaboration with Antonella Capitanio

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