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GLOBAL C^1 REGULARITY OF THE VALUE FUNCTION IN OPTIMAL STOPPING PROBLEMS

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We show that if either the process is strong Feller and the boundary point is probabilistically regular for the stopping set, or the process is strong Markov and the boundary point is probabilistically regular for the interior of the stopping set, then the boundary point is Green regular for the stopping set. Combining this implication with the existence of a continuously differentiable flow of the process we show that the value function is continuously differentiable at the optimal stopping boundary whenever the gain function is so. The derived fact holds both in the parabolic and elliptic case of the boundary value problem under the sole hypothesis of probabilistic regularity of the optimal stopping boundary, thus improving upon known analytic results in the PDE literature, and establishing the fact for the first time in the case of integro-differential equations. The method of proof is purely probabilistic and conceptually simple. Examples of application include the first known probabilistic proof of the fact that the time derivative of the value function in the American put problem is continuous across the optimal stopping boundary.

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OPTIMAL REAL-TIME DETECTION OF A DRIFTING BROWNIAN COORDINATE

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Consider the motion of a Brownian particle in three dimensions, whose two spatial coordinates are standard Brownian motions with zero drift, and the remaining (unknown) spatial coordinate is a standard Brownian motion with a (known) nonzero drift. Given that the position of the Brownian particle is being observed in real time, the problem is to detect as soon as possible and with minimal probabilities of the wrong terminal decisions, which spatial coordinate has the nonzero drift. We solve this problem in the Bayesian formulation, under any prior probabilities of the nonzero drift being in any of the three spatial coordinates, when the passage of time is penalised linearly. Finding the exact solution to the problem in three dimensions, including a rigorous treatment of its nonmonotone optimal stopping boundaries, is the main contribution of the present paper. To our knowledge this is the first time that such a problem has been solved in the literature.

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AN INFORMATION-PERCOLATION BOUND FOR SPIN SYNCHRONIZATION ON GENERAL GRAPHS

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This paper considers the problem of reconstructing n independent uniform spins $X_1, \dots, X_n$ living on the vertices of an n-vertex graph G, by observing their interactions on the edges of the graph. This captures instances of models such as (i) broadcasting on trees, (ii) block models, (iii) synchronization on grids, (iv) spiked Wigner models. The paper gives an upper bound on the mutual information between two vertices in terms of a bond percolation estimate. Namely, the information between two vertices' spins is bounded by the probability that these vertices are connected when edges are opened with a probability that "emulates" the edge-information. Both the information and the open-probability are based on the Chi-squared mutual information. The main results allow us to re-derive known results for information-theoretic nonreconstruction in models (i)–(iv), with more direct or improved bounds in some cases, and to obtain new results, such as for a spiked Wigner model on grids. The main result also implies a new subadditivity property for the Chi-squared mutual information for symmetric channels and general graphs, extending the subadditivity property obtained by Evans–Kenyon–Peres–Schulman (*Ann. Appl. Probab.* **10** (2000) 410–433) for trees. Some cases of nonsymmetrical channels are also discussed.

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SPATIAL GROWTH PROCESSES WITH LONG RANGE DISPERSION: MICROSCOPICS, MESOSCOPICS AND DISCREPANCY IN SPREAD RATE

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We consider the speed of propagation of a continuous-time continuous-space branching random walk with the additional restriction that the birth rate at any spatial point cannot exceed 1. The dispersion kernel is taken to have density that decays polynomially as $|x|^{-2\alpha}$, $x \to \infty$. We show that if $\alpha > 2$, then the system spreads at a linear speed, while for $\alpha \in (\frac{1}{2}, 2]$ the spread is faster than linear. We also consider the mesoscopic equation corresponding to the microscopic stochastic system. We show that in contrast to the microscopic process, the solution to the mesoscopic equation spreads exponentially fast for every $\alpha > \frac{1}{2}$.

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ON THE EXIT TIME FROM OPEN SETS OF SOME SEMI-MARKOV PROCESSES

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In this paper we characterize the distribution of the first exit time from an arbitrary open set for a class of semi-Markov processes obtained as time-changed Markov processes. We estimate the asymptotic behaviour of the survival function (for large t) and of the distribution function (for small t) and we provide some conditions for absolute continuity. We have been inspired by a problem of neurophyshiology and our results are particularly usefull in this field, precisely for the so-called Leaky Integrate-and-Fire (LIF) models: the use of semi-Markov processes in these models appear to be realistic under several aspects, for example, it makes the intertimes between spikes a r.v. with infinite expectation, which is a desiderable property. Hence, after the theoretical part, we provide a LIF model based on semi-Markov processes.

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THERMALISATION FOR SMALL RANDOM PERTURBATIONS OF DYNAMICAL SYSTEMS

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We consider an ordinary differential equation with a unique hyperbolic attractor at the origin, to which we add a small random perturbation. It is known that under general conditions, the solution of this stochastic differential equation converges exponentially fast to an equilibrium distribution. We show that the convergence occurs abruptly: in a time window of small size compared to the natural time scale of the process, the distance to equilibrium drops from its maximal possible value to near zero, and only after this time window the convergence is exponentially fast. This is what is known as the cut-off phenomenon in the context of Markov chains of increasing complexity. In addition, we are able to give general conditions to decide whether the distance to equilibrium converges in this time window to a universal function, a fact known as profile cut-off.

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COUPLING AND CONVERGENCE FOR HAMILTONIAN MONTE CARLO

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Based on a new coupling approach, we prove that the transition step of the Hamiltonian Monte Carlo algorithm is contractive w.r.t. a carefully designed Kantorovich (L^1 Wasserstein) distance. The lower bound for the contraction rate is explicit. Global convexity of the potential is not required, and thus multimodal target distributions are included. Explicit quantitative bounds for the number of steps required to approximate the stationary distribution up to a given error ϵ are a direct consequence of contractivity. These bounds show that HMC can overcome diffusive behavior if the duration of the Hamiltonian dynamics is adjusted appropriately.

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THE INVERSE FIRST PASSAGE TIME PROBLEM FOR KILLED BROWNIAN MOTION

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The classical *inverse first passage time problem* asks whether, for a Brownian motion $(B_t)_{t\geq 0}$ and a positive random variable ξ, there exists a barrier $b:\mathbb{R}_+\to\mathbb{R}$ such that $\mathbb{P}\{B_s > b(s), 0\leq s\leq t\} = \mathbb{P}\{\xi > t\}$, for all $t\geq 0$. We study a variant of the inverse first passage time problem for killed Brownian motion. We show that if $\lambda > 0$ is a killing rate parameter and $\mathbb{1}_{(-\infty,0]}$ is the indicator of the set $(-\infty, 0]$ then, under certain compatibility assumptions, there exists a unique continuous function $b:\mathbb{R}_+\to\mathbb{R}$ such that $\mathbb{E}[-\lambda\int_0^t \mathbb{1}_{(-\infty,0]}(B_s - b(s))\,ds] = \mathbb{P}\{\zeta > t\}$ holds for all $t \geq 0$. This is a significant improvement of a result of the first two authors (*Ann. Appl. Probab.* **24** (2014) 1–33).

The main difficulty arises because $\mathbb{1}_{(-\infty,0]}$ is discontinuous. We associate a semilinear parabolic partial differential equation (PDE) coupled with an integral constraint to this version of the inverse first passage time problem. We prove the existence and uniqueness of weak solutions to this constrained PDE system. In addition, we use the recent Feynman–Kac representation results of Glau (*Finance Stoch.* **20** (2016) 1021–1059) to prove that the weak solutions give the correct probabilistic interpretation.

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TRANSPORT-INFORMATION INEQUALITIES FOR MARKOV CHAINS

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This paper is the discrete time counterpart of the previous work in the continuous time case by Guillin, Léonard, the second named author and Yao [*Probab. Theory Related Fields* **144** (2009), 669–695]. We investigate the following transport-information $T_{\mathcal{V}}I$ inequality: $\alpha(T_{\mathcal{V}}(\nu,\mu)) \le I(\nu|P,\mu)$ for all probability measures ν on some metric space $(\mathcal{X},d)$, where μ is an invariant and ergodic probability measure of some given transition kernel $P(x,dy)$, $T_{\mathcal{V}}(\nu,\mu)$ is some transportation cost from ν to μ, $I(\nu|P,\mu)$ is the Donsker–Varadhan information of ν with respect to (P,μ) and $\alpha:[0,\infty)\to[0,\infty]$ is some left continuous increasing function. Using large deviation techniques, we show that $T_{\mathcal{V}}I$ is equivalent to some concentration inequality for the occupation measure of the μ-reversible Markov chain $(X_n)_{n\ge 0}$ with transition probability $P(x,dy)$. Its relationships with the transport-entropy inequalities are discussed. Several easy-to-check sufficient conditions are provided for $T_{\mathcal{V}}I$. We show the usefulness and sharpness of our general results by a number of applications and examples. The main difficulty resides in the fact that the information $I(\nu|P,\mu)$ has no closed expression, contrary to the continuous time or independent and identically distributed case.

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NONEXPONENTIAL SANOV AND SCHILDER THEOREMS ON WIENER SPACE: BSDES, SCHRÖDINGER PROBLEMS AND CONTROL

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We derive new limit theorems for Brownian motion, which can be seen as nonexponential analogues of the large deviation theorems of Sanov and Schilder in their Laplace principle forms. As a first application, we obtain novel scaling limits of backward stochastic differential equations and their related partial differential equations. As a second application, we extend prior results on the small-noise limit of the Schrödinger problem as an optimal transport cost, unifying the control-theoretic and probabilistic approaches initiated respectively by T. Mikami and C. Léonard. Lastly, our results suggest a new scheme for the computation of mean field optimal control problems, distinct from the conventional particle approximation. A key ingredient in our analysis is an extension of the classical variational formula (often attributed to Borell or Boué–Dupuis) for the Laplace transform of Wiener measure.

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THE COALESCENT STRUCTURE OF CONTINUOUS-TIME GALTON–WATSON TREES

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Take a continuous-time Galton–Watson tree. If the system survives until a large time T, then choose k particles uniformly from those alive. What does the ancestral tree drawn out by these k particles look like? Some special cases are known but we give a more complete answer. We concentrate on near-critical cases where the mean number of offspring is $1 + \mu/T$ for some $\mu \in \mathbb{R}$, and show that a scaling limit exists as $T \to \infty$. Viewed backwards in time, the resulting coalescent process is topologically equivalent to Kingman's coalescent, but the times of coalescence have an interesting and highly nontrivial structure. The randomly fluctuating population size, as opposed to constant size populations where the Kingman coalescent more usually arises, have a pronounced effect on both the results and the method of proof required. We give explicit formulas for the distribution of the coalescent times, as well as a construction of the genealogical tree involving a mixture of independent and identically distributed random variables. In general subcritical and supercritical cases it is not possible to give such explicit formulas, but we highlight the special case of birth–death processes.

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ZERO-SUM PATH-DEPENDENT STOCHASTIC DIFFERENTIAL GAMES IN WEAK FORMULATION

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We consider zero-sum stochastic differential games with possibly path-dependent volatility controls. Unlike the previous literature, we allow for weak solutions of the state equation so that the players' controls are automatically of feedback type. In particular, we do not require the controls to be "simple," which has fundamental importance for the possible existence of saddle-points. Under some restrictions, needed for the a priori regularity of the upper and lower value functions of the game, we show that the game value exists when both the appropriate path-dependent Isaacs condition, and the uniqueness of viscosity solutions of the corresponding path-dependent Isaacs-HJB equation hold. We also provide a general verification argument and a characterisation of saddle-points by means of an appropriate notion of second-order backward SDE.

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OPTIMAL CHEEGER CUTS AND BISECTIONS OF RANDOM GEOMETRIC GRAPHS

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Let $d \geq 2$. The Cheeger constant of a graph is the minimum surface-to-volume ratio of all subsets of the vertex set with relative volume at most 1/2. There are several ways to define surface and volume here: the simplest method is to count boundary edges (for the surface) and vertices (for the volume). We show that for a geometric (possibly weighted) graph on n random points in a d-dimensional domain with Lipschitz boundary and with distance parameter decaying more slowly (as a function of n) than the connectivity threshold, the Cheeger constant (under several possible definitions of surface and volume), also known as conductance, suitably rescaled, converges for large n to an analogous Cheeger-type constant of the domain. Previously, García Trillos et al. had shown this for $d \geq 3$ but had required an extra condition on the distance parameter when $d = 2$.

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RANDOM PERMUTATIONS WITHOUT MACROSCOPIC CYCLES

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We consider uniform random permutations of length n conditioned to have no cycle longer than n^β with $0 < \beta < 1$, in the limit of large n. Since in unconstrained uniform random permutations most of the indices are in cycles of macroscopic length, this is a singular conditioning in the limit. Nevertheless, we obtain a fairly complete picture about the cycle number distribution at various lengths. Depending on the scale at which cycle numbers are studied, our results include Poisson convergence, a central limit theorem, a shape theorem and two different functional central limit theorems.

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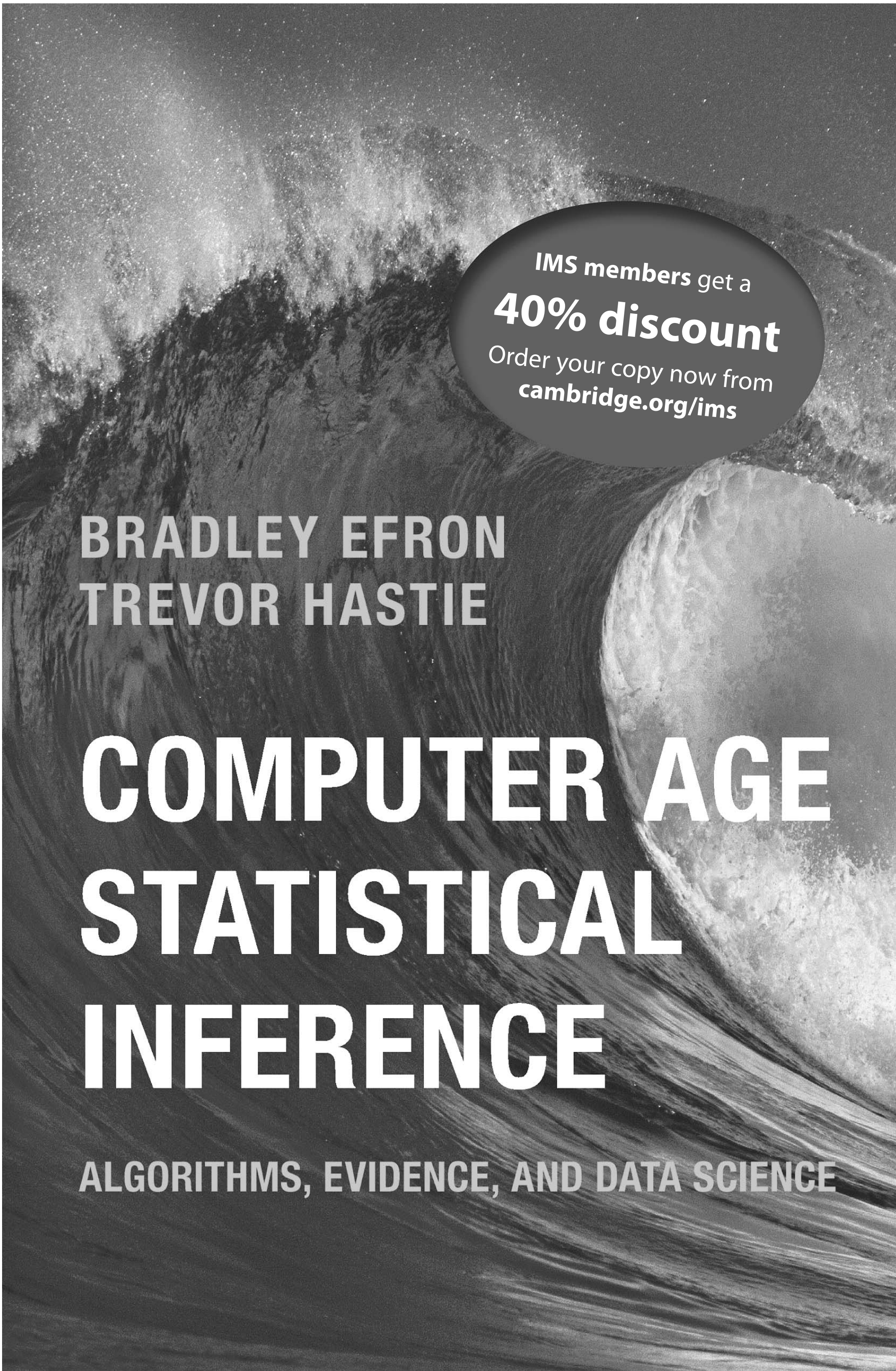
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