ISSN 1050-5164 (print)
ISSN 2168-8737 (online)

THE ANNALS *of* APPLIED PROBABILITY

AN OFFICIAL JOURNAL OF THE
INSTITUTE OF MATHEMATICAL STATISTICS

Articles

Vol. 30, No. 4—August 2020

THE ANNALS OF APPLIED PROBABILITY Vol. 30, No. 4, pp. 1507–2003 August 2020

INSTITUTE OF MATHEMATICAL STATISTICS

(Organized September 12, 1935)

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The Annals of Applied Probability **[ISSN 1050-5164 (print); ISSN 2168-8737 (online)],** Volume 30, Number 4, August 2020. Published bimonthly by the Institute of Mathematical Statistics, 3163 Somerset Drive, Cleveland, Ohio 44122, USA. Periodicals postage paid at Cleveland, Ohio, and at additional mailing offices.

POSTMASTER: Send address changes to *The Annals of Applied Probability*, Institute of Mathematical Statistics, Dues and Subscriptions Office, 9650 Rockville Pike, Suite L 2310, Bethesda, Maryland 20814-3998, USA.

The Annals of Applied Probability
2020, Vol. 30, No. 4, 1507–1533
https://doi.org/10.1214/19-AAP1468

SENSITIVITY ANALYSIS FOR RARE EVENTS BASED ON RÉNYI DIVERGENCE

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Rare events play a key role in many applications and numerous algorithms have been proposed for estimating the probability of a rare event. However, relatively little is known on how to quantify the sensitivity of the rare event's probability with respect to model parameters. In this paper, instead of the direct statistical estimation of rare event sensitivities, we develop novel and general uncertainty quantification and sensitivity bounds which are not tied to specific rare event simulation methods and which apply to families of rare events. Our method is based on a recently derived variational representation for the family of Rényi divergences in terms of risk sensitive functionals associated with the rare events under consideration. Inspired by the derived bounds, we propose new sensitivity indices for rare events and relate them to the moment generating function of the score function. The bounds scale in such a way that we additionally develop sensitivity indices for large deviation rate functions.

REFERENCES

[1] AGARWAL, A., DE MARCO, S., GOBET, E. and LIU, G. (2015). Rare event simulation related to financial risks: Efficient estimation and sensitivity analysis. Preprint.

[2] ARAMPATZIS, G., KATSOULAKIS, M. A. and PANTAZIS, Y. (2015). Accelerated sensitivity analysis in high-dimensional stochastic reaction networks. *PLoS ONE* **10** 1–24.

[3] ASMUSSEN, S. and GLYNN, P. W. (2007). *Stochastic Simulation*: *Algorithms and Analysis*. *Stochastic Modelling and Applied Probability* **57**. Springer, New York. MR2331321

[4] ASMUSSEN, S. and RUBINSTEIN, R. Y. (1999). Sensitivity analysis of insurance risk models via simulation. *Manage*. *Sci*. **45** 1125–1141.

[5] ATAR, R., CHOWDHARY, K. and DUPUIS, P. (2015). Robust bounds on risk-sensitive functionals via Rényi divergence. *SIAM/ASA J. Uncertain*. *Quantificat*. **3** 18–33. MR3299141 https://doi.org/10.1137/130939730

[6] BIRRELL, J., DUPUIS, P., KATSOULAKIS, M. A., REY-BELLET, L. and WANG, J. (2019). Distributional robustness and uncertainty quantification for rare events. arXiv e-prints, arXiv:1911.09580.

[7] BISHOP, C. M. (2006). *Pattern Recognition and Machine Learning*. *Information Science and Statistics*. Springer, New York. MR2247587 https://doi.org/10.1007/978-0-387-45528-0

[8] BOUCHERON, S., LUGOSI, G. and MASSART, P. (2016). *Concentration Inequalities*: *A Nonasymptotic Theory of Independence*. Oxford Univ. Press, Oxford. MR3185193 https://doi.org/10.1093/acprof:oso/9780199535255.001.0001

[9] CAI, W., KALOS, M. H., DE KONING, M. and BULATOV, V. V. (2002). Importance sampling for rare transition events in Markovian processes. *Phys*. *Rev*. *E* **66** 046703.

[10] CHOWDHARY, K. and DUPUIS, P. (2013). Distinguishing and integrating aleatoric and epistemic variation in uncertainty quantification. *ESAIM Math*. *Model*. *Numer. Anal*. **47** 635–662. MR3056403 https://doi.org/10.1051/m2an/2012038

MSC2020 subject classifications. 60F10, 94A17.

Key words and phrases. Rare event, Rényi divergence, variational representation, sensitivity analysis, score function, uncertainty quantification, large deviation, risk sensitive functional.

[11] DEAN, T. and DUPUIS, P. (2011). The design and analysis of a generalized RESTART/DPR algorithm for rare event simulation. *Ann. Oper. Res.* **189** 63–102. MR2833612 https://doi.org/10.1007/s10479-009-0664-7

[12] DEL MORAL, P. (2004). *Feynman–Kac Formulae: Genealogical and Interacting Particle Systems with Applications. Probability and Its Applications (New York)*. Springer, New York. MR2044973 https://doi.org/10.1007/978-1-4684-9393-1

[13] DEL MORAL, P. and GARNIER, J. (2005). Genealogical particle analysis of rare events. *Ann. Appl. Probab.* **15** 2496–2534. MR2187302 https://doi.org/10.1214/105051605000000566

[14] DEL MORAL, P., HU, P. and WU, L. (2012). On the concentration properties of interacting particle processes. *Found. Trends Mach. Learn.* **3** 225–389.

[15] DEMBO, A. and ZEITOUNI, O. (1998). *Large Deviations Techniques and Applications*, 2nd ed. *Applications of Mathematics (New York)* **38**. Springer, New York. MR1619036 https://doi.org/10.1007/978-1-4612-5320-4

[16] DUPUIS, P. and ELLIS, R. S. (1997). *A Weak Convergence Approach to the Theory of Large Deviations. Wiley Series in Probability and Statistics: Probability and Statistics*. Wiley, New York. MR1431744 https://doi.org/10.1002/9781118165904

[17] DUPUIS, P., KATSOULAKIS, M. A., PANTAZIS, Y. and PLECHÁČ, P. (2016). Path-space information bounds for uncertainty quantification and sensitivity analysis of stochastic dynamics. *SIAM/ASA J. Uncertain. Quantificat.* **4** 80–111. MR3455143 https://doi.org/10.1137/15M1025645

[18] DUPUIS, P. and WANG, H. (2007). Subsolutions of an Isaacs equation and efficient schemes for importance sampling. *Math. Oper. Res.* **32** 723–757. MR2348245 https://doi.org/10.1287/moor.1070.0266

[19] EMBRECHTS, P., KLÜPPELBERG, C. and MIKOSCH, T. (1997). *Modelling Extremal Events: For Insurance and Finance. Applications of Mathematics (New York)* **33**. Springer, Berlin. MR1458613 https://doi.org/10.1007/978-3-642-33483-2

[20] FRENKEL, D. and SMIT, B. (2002). *Understanding Molecular Simulation: From Algorithms to Applications*. Academic Press, San Diego.

[21] GARVELS, M. J. J. (2000). *The Splitting Method in Rare Event Simulation*. Thesis (Dr.)–Universiteit Twente (The Netherlands). ProQuest LLC, Ann Arbor, MI. MR2715371

[22] GOLSHANI, L., PASHA, E. and YARI, G. (2009). Some properties of Rényi entropy and Rényi entropy rate. *Inform. Sci.* **179** 2426–2433. MR2554689 https://doi.org/10.1016/j.ins.2009.03.002

[23] GOURGOULIAS, K., KATSOULAKIS, M. A., REY-BELLET, L. and WANG, J. (2020). How biased is your model? Concentration inequalities, information and model bias. *IEEE Trans. Inf. Theory* **66** 3079–3097. https://doi.org/10.1109/TIT.2020.2977067

[24] KATSOULAKIS, M. A., REY-BELLET, L. and WANG, J. (2017). Scalable information inequalities for uncertainty quantification. *J. Comput. Phys.* **336** 513–545. MR3622628 https://doi.org/10.1016/j.jcp.2017.02.020

[25] KAY, S. M. (1993). *Fundamentals of Statistical Signal Processing: Estimation Theory*. Prentice-Hall, Inc., Upper Saddle River, NJ.

[26] L'ECUYER, P., LE GLAND, F., LEZAUD, P. and TUFFIN, B. (2009). Splitting techiques. In *Rare Event Simulation Using Monte Carlo Methods* 39–61. Wiley, Chichester. MR2730761 https://doi.org/10.1002/9780470745403.ch3

[27] LELIÈVRE, T., ROUSSET, M. and STOLTZ, G. (2010). *Free Energy Computations: A Mathematical Perspective*. Imperial College Press, London. MR2681239 https://doi.org/10.1142/9781848162488

[28] LI, J. and XIU, D. (2012). Computation of failure probability subject to epistemic uncertainty. *SIAM J. Sci. Comput.* **34** A2946–A2964. MR3023740 https://doi.org/10.1137/120864155

[29] LIESE, F. and VAJDA, I. (1987). *Convex Statistical Distances. Teubner-Texte zur Mathematik [Teubner Texts in Mathematics]* **95**. BSB B. G. Teubner Verlagsgesellschaft, Leipzig. MR0926905

[30] LIU, J. S. (2001). *Monte Carlo Strategies in Scientific Computing. Springer Series in Statistics*. Springer, New York. MR1842342

[31] PHAM, H. (2007). Some applications and methods of large deviations in finance and insurance. In *Paris–Princeton Lectures on Mathematical Finance* 2004. *Lecture Notes in Math.* **1919** 191–244. Springer, Berlin. MR2384674 https://doi.org/10.1007/978-3-540-73327-0_5

[32] RÉNYI, A. (1961). On measures of entropy and information. In *Proc. 4th Berkeley Sympos. Math. Statist. and Prob., Vol. I* 547–561. Univ. California Press, Berkeley, CA. MR0132570

[33] RUBINO, G. and TUFFIN, B., eds. (2009). *Rare Event Simulation Using Monte Carlo Methods* Wiley, Chichester. MR2742378 https://doi.org/10.1002/9780470745403

[34] RUBINSTEIN, R. Y. and KROESE, D. P. (2017). *Simulation and the Monte Carlo Method*, 3rd ed. *Wiley Series in Probability and Statistics*. Wiley, Hoboken, NJ. MR3617204

[35] SIMON, B. (1993). *The Statistical Mechanics of Lattice Gases. Vol. I. Princeton Series in Physics*. Princeton Univ. Press, Princeton, NJ. MR1239893 https://doi.org/10.1515/9781400863433

[36] Vajda, I. (1990). Distances and discrimination rates for stochastic processes. *Stochastic Process. Appl.* **35** 47–57. MR1062582 https://doi.org/10.1016/0304-4149(90)90121-8

[37] Valsson, O., Tiwary, P. and Parrinello, M. (2016). Enhancing important fluctuations: Rare events and metadynamics from a conceptual viewpoint. *Annu. Rev. Phys. Chem.* **67** 159–184. PMID: 26980304.

[38] van Erven, T. and Harremoës, P. (2014). Rényi divergence and Kullback–Leibler divergence. *IEEE Trans. Inf. Theory* **60** 3797–3820. MR3225930 https://doi.org/10.1109/TIT.2014.2320500

[39] Vazquez-Abad, F. J. and LeQuoc, P. (2001). Sensitivity analysis for ruin probabilities: Canonical risk model. *J. Oper. Res. Soc.* **52** 71–81.

[40] Villen-Altamirano, M. and Villen-Altamirano, J. (1991). Restart: A method for accelerating rare event simulations. In *Proc. of the* 13*th International Teletraffic Congress*, *Queueing*, *Performance and Control in ATM* 71–76. Elsevier, Amsterdam.

The Annals of Applied Probability
2020, Vol. 30, No. 4, 1534–1581
https://doi.org/10.1214/19-AAP1535

NONASYMPTOTIC BOUNDS FOR SAMPLING ALGORITHMS WITHOUT LOG-CONCAVITY

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Discrete time analogues of ergodic stochastic differential equations (SDEs) are one of the most popular and flexible tools for sampling high-dimensional probability measures. Non-asymptotic analysis in the L^2 Wasserstein distance of sampling algorithms based on Euler discretisations of SDEs has been recently developed by several authors for log-concave probability distributions. In this work we replace the log-concavity assumption with a log-concavity at infinity condition. We provide novel L^2 convergence rates for Euler schemes, expressed explicitly in terms of problem parameters. From there we derive nonasymptotic bounds on the distance between the laws induced by Euler schemes and the invariant laws of SDEs, both for schemes with standard and with randomised (inaccurate) drifts. We also obtain bounds for the hierarchy of discretisation, which enables us to deploy a multi-level Monte Carlo estimator. Our proof relies on a novel construction of a coupling for the Markov chains that can be used to control both the L^1 and L^2 Wasserstein distances simultaneously. Finally, we provide a weak convergence analysis that covers both the standard and the randomised (inaccurate) drift case. In particular, we reveal that the variance of the randomised drift does not influence the rate of weak convergence of the Euler scheme to the SDE.

REFERENCES

[1] ARJOVSKY, M., CHINTALA, S. and BOTTOU, L. (2017). Wasserstein generative adversarial networks. In *Proceedings of the* 34*th International Conference on Machine Learning* (*International Convention Centre*, *Sydney*, *Australia*, 06–11 *August* 2017) (D. D. Precup and Y. W. Teh, eds.). *Proc*. *Mach*. *Learn*. *Res*. (*PMLR*) **70** 214–223.

[2] BAKER, J., FEARNHEAD, P., FOX, E. B. and NEMETH, C. (2019). Control variates for stochastic gradient MCMC. *Stat*. *Comput*. **29** 599–615. MR3969063 https://doi.org/10.1007/s11222-018-9826-2

[3] CHATTERJI, N., FLAMMARION, N., MA, Y., BARTLETT, P. and JORDAN, M. (2018). On the theory of variance reduction for stochastic gradient Monte Carlo. In *Proceedings of the* 35*th International Conference on Machine Learning* (J. J. Dy and A. A. Krause, eds.). *Proc*. *Mach*. *Learn*. *Res*. (*PMLR*) **80** 764–773.

[4] CHENG, X. and BARTLETT, P. (2018). Convergence of Langevin MCMC in KL-divergence. In *Algorithmic Learning Theory* 2018 (F. Janoos, M. Mohri and K. Sridharan, eds.). *Proc*. *Mach*. *Learn*. *Res*. (*PMLR*) **83** 26. MR3857306

[5] CHENG, X., CHATTERJI, N. S., ABBASI-YADKORI, Y., BARTLETT, P. L. and JORDAN, M. I. (2018). Sharp convergence rates for Langevin dynamics in the nonconvex setting. Preprint. Available at arXiv:1805.01648.

[6] DALALYAN, A. S. (2017). Theoretical guarantees for approximate sampling from smooth and log-concave densities. *J*. *R*. *Stat*. *Soc*. *Ser*. *B*. *Stat*. *Methodol*. **79** 651–676. MR3641401 https://doi.org/10.1111/rssb.12183

[7] DALALYAN, A. S. and KARAGULYAN, A. (2019). User-friendly guarantees for the Langevin Monte Carlo with inaccurate gradient. *Stochastic Process*. *Appl*. **129** 5278–5311. MR4025705 https://doi.org/10.1016/j.spa.2019.02.016

MSC2020 subject classifications. Primary 65C05, 65C30, 65C40; secondary 60J22, 62H12, 60H10.

Key words and phrases. Euler schemes, Langevin Monte Carlo, Markov chain Monte Carlo, sampling, multi-level Monte Carlo, coupling.

[8] DJELLOUT, H., GUILLIN, A. and WU, L. (2004). Transportation cost-information inequalities and applications to random dynamical systems and diffusions. *Ann. Probab.* **32** 2702–2732. MR2078555 https://doi.org/10.1214/009117904000000531

[9] DURMUS, A. and MOULINES, É. (2017). Nonasymptotic convergence analysis for the unadjusted Langevin algorithm. *Ann. Appl. Probab.* **27** 1551–1587. MR3678479 https://doi.org/10.1214/16-AAP1238

[10] DURMUS, A. and MOULINES, É. (2019). High-dimensional Bayesian inference via the unadjusted Langevin algorithm. *Bernoulli* **25** 2854–2882. MR4003567 https://doi.org/10.3150/18-BEJ1073

[11] DURMUS, A., ROBERTS, G. O., VILMART, G. and ZYGALAKIS, K. C. (2017). Fast Langevin based algorithm for MCMC in high dimensions. *Ann. Appl. Probab.* **27** 2195–2237. MR3693524 https://doi.org/10.1214/16-AAP1257

[12] EBERLE, A. (2011). Reflection coupling and Wasserstein contractivity without convexity. *C. R. Math. Acad. Sci. Paris* **349** 1101–1104. MR2843007 https://doi.org/10.1016/j.crma.2011.09.003

[13] EBERLE, A. (2016). Reflection couplings and contraction rates for diffusions. *Probab. Theory Related Fields* **166** 851–886. MR3568041 https://doi.org/10.1007/s00440-015-0673-1

[14] EBERLE, A., GUILLIN, A. and ZIMMER, R. (2019). Quantitative Harris-type theorems for diffusions and McKean–Vlasov processes. *Trans. Amer. Math. Soc.* **371** 7135–7173. MR3939573 https://doi.org/10.1090/tran/7576

[15] EBERLE, A. and MAJKA, M. B. (2019). Quantitative contraction rates for Markov chains on general state spaces. *Electron. J. Probab.* **24** Paper No. 26, 36. MR3933205 https://doi.org/10.1214/19-EJP287

[16] FANG, W. and GILES, M. B. (2019). Multilevel Monte Carlo method for ergodic SDEs without contractivity. *J. Math. Anal. Appl.* **476** 149–176. MR3944422 https://doi.org/10.1016/j.jmaa.2018.12.032

[17] GILES, M. B. (2008). Multilevel Monte Carlo path simulation. *Oper. Res.* **56** 607–617. MR2436856 https://doi.org/10.1287/opre.1070.0496

[18] GILES, M. B. (2015). Multilevel Monte Carlo methods. *Acta Numer.* **24** 259–328. MR3349310 https://doi.org/10.1017/S096249291500001X

[19] GILES, M. B., MAJKA, M. B., SZPRUCH, L., VOLLMER, S. J. and ZYGALAKIS, K. C. (2019). Multi-level Monte Carlo methods for the approximation of invariant measures of stochastic differential equations. *Stat. Comput.*

[20] GORHAM, J., DUNCAN, A. B., VOLLMER, S. J. and MACKEY, L. (2019). Measuring sample quality with diffusions. *Ann. Appl. Probab.* **29** 2884–2928. MR4019878 https://doi.org/10.1214/19-AAP1467

[21] HEINRICH, S. (2001). Multilevel Monte Carlo methods. In *Large-Scale Scientific Computing* 58–67. Springer, Berlin.

[22] HUGGINS, J. H. and ZOU, J. (2016). Quantifying the accuracy of approximate diffusions and Markov chains. In *Proceedings of the* 20*th International Conference on Artificial Intelligence and Statistics* (A. A. Singh and J. J. Zhu, eds.). *Proc. Mach. Learn. Res.* (*PMLR*) **54** 382–391.

[23] JENTZEN, A. and NEUENKIRCH, A. (2009). A random Euler scheme for Carathéodory differential equations. *J. Comput. Appl. Math.* **224** 346–359. MR2474237 https://doi.org/10.1016/j.cam.2008.05.060

[24] JOULIN, A. and OLLIVIER, Y. (2010). Curvature, concentration and error estimates for Markov chain Monte Carlo. *Ann. Probab.* **38** 2418–2442. MR2683634 https://doi.org/10.1214/10-AOP541

[25] KEBAIER, A. (2005). Statistical Romberg extrapolation: A new variance reduction method and applications to option pricing. *Ann. Appl. Probab.* **15** 2681–2705. MR2187308 https://doi.org/10.1214/105051605000000511

[26] KHASMINSKII, R. (2012). *Stochastic Stability of Differential Equations*, 2nd ed. *Stochastic Modelling and Applied Probability* **66**. Springer, Heidelberg. MR2894052 https://doi.org/10.1007/978-3-642-23280-0

[27] KRUSE, R. and WU, Y. (2017). Error analysis of randomized Runge–Kutta methods for differential equations with time-irregular coefficients. *Comput. Methods Appl. Math.* **17** 479–498. MR3667085 https://doi.org/10.1515/cmam-2016-0048

[28] KRYLOV, N. V. (1999). On Kolmogorov's equations for finite-dimensional diffusions. In *Stochastic PDE's and Kolmogorov Equations in Infinite Dimensions* (*Cetraro*, 1998). *Lecture Notes in Math.* **1715** 1–63. Springer, Berlin. MR1731794 https://doi.org/10.1007/BFb0092417

[29] LAMBERTON, D. and PAGÈS, G. (2002). Recursive computation of the invariant distribution of a diffusion. *Bernoulli* **8** 367–405. MR1913112 https://doi.org/10.1142/S0219493703000838

[30] LINDVALL, T. (1992). *Lectures on the Coupling Method. Wiley Series in Probability and Mathematical Statistics*: *Probability and Mathematical Statistics*. Wiley, New York. MR1180522

[31] LUO, D. and WANG, J. (2016). Exponential convergence in L^p-Wasserstein distance for diffusion processes without uniformly dissipative drift. *Math. Nachr.* **289** 1909–1926. MR3563910 https://doi.org/10.1002/mana.201500351

[32] MCCANN, R. J. (1999). Exact solutions to the transportation problem on the line. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **455** 1341–1380. MR1701760 https://doi.org/10.1098/rspa.1999.0364

[33] MILSTEIN, G. N. and TRETYAKOV, M. V. (2004). *Stochastic Numerics for Mathematical Physics*. *Scientific Computation*. Springer, Berlin. MR2069903 https://doi.org/10.1007/978-3-662-10063-9

[34] NAGAPETYAN, T., DUNCAN, A. B., HASENCLEVER, L., VOLLMER, S. J., SZPRUCH, L. and ZYGALAKIS, K. (2017). The true cost of stochastic gradient Langevin dynamics. Preprint. Available at arXiv:1706.02692.

[35] NÜSKEN, N. and PAVLIOTIS, G. A. (2019). Constructing sampling schemes via coupling: Markov semigroups and optimal transport. *SIAM/ASA J. Uncertain. Quantificat.* **7** 324–382. MR3928355 https://doi.org/10.1137/18M119896X

[36] PAGÈS, G. and PANLOUP, F. (2012). Ergodic approximation of the distribution of a stationary diffusion: Rate of convergence. *Ann. Appl. Probab.* **22** 1059–1100. MR2977986 https://doi.org/10.1214/11-AAP779

[37] PANLOUP, F. (2008). Recursive computation of the invariant measure of a stochastic differential equation driven by a Lévy process. *Ann. Appl. Probab.* **18** 379–426. MR2398761 https://doi.org/10.1214/105051607000000285

[38] PILLAI, N. S. and SMITH, A. (2014). Ergodicity of approximate MCMC chains with applications to large data sets. Preprint. Available at arXiv:1405.0182.

[39] PRZYBYŁOWICZ, P. and MORKISZ, P. (2014). Strong approximation of solutions of stochastic differential equations with time-irregular coefficients via randomized Euler algorithm. *Appl. Numer. Math.* **78** 80–94. MR3158733 https://doi.org/10.1016/j.apnum.2013.12.003

[40] QIN, Q. and HOBERT, J. P. (2019). Geometric convergence bounds for Markov chains in Wasserstein distance based on generalized drift and contraction conditions. Available at arXiv:1902.02964.

[41] RAGINSKY, M., RAKHLIN, A. and TELGARSKY, M. (2017). Non-convex learning via stochastic gradient Langevin dynamics: A nonasymptotic analysis. In *Proceedings of the* 2017 *Conference on Learning Theory* (S. S. Kale and O. O. Shamir, eds.). *Proc. Mach. Learn. Res.* (*PMLR*) **65** 1674–1703.

[42] RICE, J. A. (2007). *Mathematical Statistics and Data Analysis*, 3rd ed. Thomson Brooks/Cole, Belmont.

[43] RUDOLF, D. and SCHWEIZER, N. (2018). Perturbation theory for Markov chains via Wasserstein distance. *Bernoulli* **24** 2610–2639. MR3779696 https://doi.org/10.3150/17-BEJ938

[44] SHAMIR, O. (2016). Without-replacement sampling for stochastic gradient methods. In *Advances in Neural Information Processing Systems* 29 (D. D. Lee, M. Sugiyama, U. V. Luxburg, I. Guyon and R. Garnett, eds.) 46–54. Curran Associates, Red Hook.

[45] SZPRUCH, Ł. and ZHĀNG, X. (2018). V-integrability, asymptotic stability and comparison property of explicit numerical schemes for non-linear SDEs. *Math. Comp.* **87** 755–783. MR3739216 https://doi.org/10.1090/mcom/3219

[46] TALAGRAND, M. (1996). Transportation cost for Gaussian and other product measures. *Geom. Funct. Anal.* **6** 587–600. MR1392331 https://doi.org/10.1007/BF02249265

[47] TALAY, D. (1990). Second-order discretization schemes of stochastic differential systems for the computation of the invariant law. *Stochastics* **29** 13–36.

[48] TEH, Y. W., THIERY, A. H. and VOLLMER, S. J. (2016). Consistency and fluctuations for stochastic gradient Langevin dynamics. *J. Mach. Learn. Res.* **17** Paper No. 7, 33. MR3482927

[49] THORISSON, H. (2000). *Coupling*, *Stationarity*, *and Regeneration*. *Probability and Its Applications* (*New York*). Springer, New York. MR1741181 https://doi.org/10.1007/978-1-4612-1236-2

[50] VILLANI, C. (2009). *Optimal Transport*: *Old and New*. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **338**. Springer, Berlin. MR2459454 https://doi.org/10.1007/978-3-540-71050-9

[51] VOLLMER, S. J., ZYGALAKIS, K. C. and TEH, Y. W. (2016). Exploration of the (non-)asymptotic bias and variance of stochastic gradient Langevin dynamics. *J. Mach. Learn. Res.* **17** Paper No. 159, 45. MR3555050

[52] WELLING, M. and TEH, Y. W. (2011). Bayesian learning via stochastic gradient Langevin dynamics. In *Proceedings of the* 28*th International Conference on Machine Learning* (*ICML*-11) 681–688.

[53] XIFARA, T., SHERLOCK, C., LIVINGSTONE, S., BYRNE, S. and GIROLAMI, M. (2014). Langevin diffusions and the Metropolis-adjusted Langevin algorithm. *Statist. Probab. Lett.* **91** 14–19. MR3208109 https://doi.org/10.1016/j.spl.2014.04.002

The Annals of Applied Probability
2020, Vol. 30, No. 4, 1582–1613
https://doi.org/10.1214/19-AAP1536

HYDRODYNAMIC LIMIT AND PROPAGATION OF CHAOS FOR BROWNIAN PARTICLES REFLECTING FROM A NEWTONIAN BARRIER

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In 2001, Frank Knight constructed a stochastic process modeling the one-dimensional interaction of two particles, one being Newtonian in the sense that it obeys Newton's laws of motion, and the other particle being Brownian. We construct a multi-particle analog, using Skorohod map estimates in proving a propagation of chaos, and characterizing the hydrodynamic limit as the solution to a PDE with free boundary condition. This PDE resembles the Stefan problem but has a Neumann type boundary condition. Stochastic methods are used to show existence and uniqueness for this free boundary problem.

REFERENCES

[1] BARLOW, M. T. and YOR, M. (1981). (Semi-) martingale inequalities and local times. *Z. Wahrsch. Verw. Gebiete* **55** 237–254. MR0608019 https://doi.org/10.1007/BF00532117

[2] BASS, R. F., BURDZY, K., CHEN, Z.-Q. and HAIRER, M. (2010). Stationary distributions for diffusions with inert drift. *Probab. Theory Related Fields* **146** 1–47. MR2550357 https://doi.org/10.1007/s00440-008-0182-6

[3] BURDZY, K., CHEN, Z.-Q. and SYLVESTER, J. (2004). The heat equation and reflected Brownian motion in time-dependent domains. *Ann. Probab.* **32** 775–804. MR2039943 https://doi.org/10.1214/aop/1079021464

[4] CABEZAS, M., DEMBO, A., SARANTSEV, A. and SIDORAVICIUS, V. (2019). Brownian particles with rank-dependent drifts: Out-of-equilibrium behavior. *Comm. Pure Appl. Math.* **72** 1424–1458. MR3957396 https://doi.org/10.1002/cpa.21825

[5] CHAYES, L. and SWINDLE, G. (1996). Hydrodynamic limits for one-dimensional particle systems with moving boundaries. *Ann. Probab.* **24** 559–598. MR1404521 https://doi.org/10.1214/aop/1039639355

[6] CHEN, Z.-Q. and FAN, W.-T. (2017). Systems of interacting diffusions with partial annihilation through membranes. *Ann. Probab.* **45** 100–146. MR3601647 https://doi.org/10.1214/15-AOP1047

[7] DE MASI, A., GALVES, A., LÖCHERBACH, E. and PRESUTTI, E. (2015). Hydrodynamic limit for interacting neurons. *J. Stat. Phys.* **158** 866–902. MR3311484 https://doi.org/10.1007/s10955-014-1145-1

[8] FASANO, A. and PRIMICERIO, M. (1977). General free-boundary problems for the heat equation. I. *J. Math. Anal. Appl.* **57** 694–723. MR0487016 https://doi.org/10.1016/0022-247X(77)90256-6

[9] FASANO, A. and PRIMICERIO, M. (1977). General free-boundary problems for the heat equation. II. *J. Math. Anal. Appl.* **58** 202–231. MR0487017 https://doi.org/10.1016/0022-247X(77)90239-6

[10] FASANO, A. and PRIMICERIO, M. (1977). General free-boundary problems for the heat equation. III. *J. Math. Anal. Appl.* **59** 1–14. MR0487018 https://doi.org/10.1016/0022-247X(77)90088-9

[11] FISCHER, M. and NAPPO, G. (2010). On the moments of the modulus of continuity of Itô processes. *Stoch. Anal. Appl.* **28** 103–122. MR2597982 https://doi.org/10.1080/07362990903415825

[12] FOLLAND, G. B. (1999). *Real Analysis*: *Modern Techniques and Their Applications*, 2nd ed. *Pure and Applied Mathematics* (*New York*). Wiley, New York. MR1681462

[13] FOURNIER, N. and GUILLIN, A. (2015). On the rate of convergence in Wasserstein distance of the empirical measure. *Probab. Theory Related Fields* **162** 707–738. MR3383341 https://doi.org/10.1007/s00440-014-0583-7

[14] FRIEDMAN, A. (1964). *Partial Differential Equations of Parabolic Type*. Prentice-Hall, Englewood Cliffs, NJ. MR0181836

MSC2020 subject classifications. Primary 60, 60K35; secondary 60J55, 82.

Key words and phrases. Propagation of chaos, Brownian motion, reflected diffusions, hydrodynamic limit, Skorohod maps, free boundary problems.

[15] GÄRTNER, J. (1988). On the McKean–Vlasov limit for interacting diffusions. *Math. Nachr.* **137** 197–248. MR0968996 https://doi.org/10.1002/mana.19881370116

[16] GOLSE, F. (2005). Hydrodynamic limits. In *European Congress of Mathematics* 699–717. Eur. Math. Soc., Zürich. MR2185776

[17] GREENWOOD, P. E. and WARD, L. M. (2016). *Stochastic Neuron Models*. *Mathematical Biosciences Institute Lecture Series*. *Stochastics in Biological Systems* **1**. Springer, Cham. MR3445967 https://doi.org/10.1007/978-3-319-26911-5

[18] KAC, M. (1956). Foundations of kinetic theory. In *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability*, 1954–1955, *Vol. III* 171–197. Univ. California Press, Berkeley, CA. MR0084985

[19] KARATZAS, I., PAL, S. and SHKOLNIKOV, M. (2016). Systems of Brownian particles with asymmetric collisions. *Ann. Inst. Henri Poincaré Probab. Stat.* **52** 323–354. MR3449305 https://doi.org/10.1214/14-AIHP646

[20] KARATZAS, I. and SHREVE, S. E. (1991). *Brownian Motion and Stochastic Calculus*, 2nd ed. *Graduate Texts in Mathematics* **113**. Springer, New York. MR1121940 https://doi.org/10.1007/978-1-4612-0949-2

[21] KELLER-RESSEL, M. and MÜLLER, M. S. (2016). A Stefan-type stochastic moving boundary problem. *Stoch. Partial Differ. Equ. Anal. Comput.* **4** 746–790. MR3554430 https://doi.org/10.1007/s40072-016-0076-z

[22] KIM, K., ZHENG, Z. and SOWERS, R. B. (2012). A stochastic Stefan problem. *J. Theoret. Probab.* **25** 1040–1080. MR2993014 https://doi.org/10.1007/s10959-011-0392-1

[23] KIPNIS, C. and LANDIM, C. (1999). *Scaling Limits of Interacting Particle Systems*. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **320**. Springer, Berlin. MR1707314 https://doi.org/10.1007/978-3-662-03752-2

[24] KNIGHT, F. B. (2001). On the path of an inert object impinged on one side by a Brownian particle. *Probab. Theory Related Fields* **121** 577–598. MR1872429 https://doi.org/10.1007/s004400100160

[25] LANG, S. (1968). *Analysis I*. Addison-Wesley, Reading, MA.

[26] MCKEAN, H. P. JR. (1967). Propagation of chaos for a class of non-linear parabolic equations. In *Stochastic Differential Equations* (*Lecture Series in Differential Equations*, *Session* 7, *Catholic Univ.*, 1967) 41–57. Air Force Office Sci. Res., Arlington, VA. MR0233437

[27] MÉLÉARD, S. (1996). Asymptotic behaviour of some interacting particle systems; McKean–Vlasov and Boltzmann models. In *Probabilistic Models for Nonlinear Partial Differential Equations* (*Montecatini Terme*, 1995). *Lecture Notes in Math.* **1627** 42–95. Springer, Berlin. MR1431299 https://doi.org/10.1007/BFb0093177

[28] NADTOCHIY, S. and SHKOLNIKOV, M. (2019). Particle systems with singular interaction through hitting times: Application in systemic risk modeling. *Ann. Appl. Probab.* **29** 89–129. MR3910001 https://doi.org/10.1214/18-AAP1403

[29] OELSCHLÄGER, K. (1984). A martingale approach to the law of large numbers for weakly interacting stochastic processes. *Ann. Probab.* **12** 458–479. MR0735849

[30] PAKDAMAN, K., THIEULLEN, M. and WAINRIB, G. (2010). Fluid limit theorems for stochastic hybrid systems with application to neuron models. *Adv. in Appl. Probab.* **42** 761–794. MR2779558 https://doi.org/10.1239/aap/1282924062

[31] SARANTSEV, A. (2015). Triple and simultaneous collisions of competing Brownian particles. *Electron. J. Probab.* **20** Art. ID 29. MR3325099 https://doi.org/10.1214/EJP.v20-3279

[32] SARANTSEV, A. (2017). Infinite systems of competing Brownian particles. *Ann. Inst. Henri Poincaré Probab. Stat.* **53** 2279–2315. MR3729655 https://doi.org/10.1214/16-AIHP791

[33] SKOROHOD, A. V. (1961). Stochastic equations for diffusion processes with a boundary. *Teor. Veroyatn. Primen.* **6** 287–298. MR0145598

[34] SKOROHOD, A. V. (1988). *Stochastic Equations for Complex Systems*. *Mathematics and Its Applications* (*Soviet Series*) **13**. D. Reidel, Dordrecht. MR0924158

[35] SZNITMAN, A.-S. (1984). Équations de type de Boltzmann, spatialement homogènes. *Z. Wahrsch. Verw. Gebiete* **66** 559–592. MR0753814 https://doi.org/10.1007/BF00531891

[36] SZNITMAN, A.-S. (1984). Nonlinear reflecting diffusion process, and the propagation of chaos and fluctuations associated. *J. Funct. Anal.* **56** 311–336. MR0743844 https://doi.org/10.1016/0022-1236(84)90080-6

[37] TANAKA, H. (1984). Limit theorems for certain diffusion processes with interaction. In *Stochastic Analysis* (*Katata/Kyoto*, 1982). *North-Holland Math. Library* **32** 469–488. North-Holland, Amsterdam. MR0780770 https://doi.org/10.1016/S0924-6509(08)70405-7

[38] VARADHAN, S. R. S. (1993). Entropy methods in hydrodynamic scaling. In *Nonequilibrium Problems in Many-Particle Systems* (*Montecatini*, 1992). *Lecture Notes in Math*. **1551** 112–145. Springer, Berlin. MR1296260 https://doi.org/10.1007/BFb0090931

[39] VILLANI, C. (2003). *Topics in Optimal Transportation*. *Graduate Studies in Mathematics* **58**. Amer. Math. Soc., Providence, RI. MR1964483 https://doi.org/10.1090/gsm/058

[40] WHITE, D. (2007). Processes with inert drift. *Electron*. *J*. *Probab*. **12** 1509–1546. MR2365876 https://doi.org/10.1214/EJP.v12-465

The Annals of Applied Probability
2020, Vol. 30, No. 4, 1614–1641
https://doi.org/10.1214/19-AAP1537

RANDOM WALK ON RANDOM WALKS: LOW DENSITIES

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We consider a random walker in a dynamic random environment given by a system of independent discrete-time simple symmetric random walks. We obtain ballisticity results under two types of perturbations: low particle density, and strong local drift on particles. Surprisingly, the random walker may behave very differently depending on whether the underlying environment particles perform lazy or nonlazy random walks, which is related to a notion of permeability of the system. We also provide a strong law of large numbers, a functional central limit theorem and large deviation bounds under an ellipticity condition.

REFERENCES

[1] AVENA, L. (2010). Random walks in dynamic random environments. Ph.D. thesis, Mathematical Institute, Faculty of Science, Leiden Univ.

[2] AVENA, L., BLONDEL, O. and FAGGIONATO, A. (2016). A class of random walks in reversible dynamic environments: Antisymmetry and applications to the East model. *J. Stat. Phys.* **165** 1–23. MR3547832 https://doi.org/10.1007/s10955-016-1596-7

[3] AVENA, L., BLONDEL, O. and FAGGIONATO, A. (2016). L^2-perturbed Markov processes and applications to random walks in dynamic random environments. ArXiv E-prints.

[4] AVENA, L., DEN HOLLANDER, F. and REDIG, F. (2010). Large deviation principle for one-dimensional random walk in dynamic random environment: Attractive spin-flips and simple symmetric exclusion. *Markov Process. Related Fields* **16** 139–168. MR2664339

[5] AVENA, L., DEN HOLLANDER, F. and REDIG, F. (2011). Law of large numbers for a class of random walks in dynamic random environments. *Electron. J. Probab.* **16** 587–617. MR2786643 https://doi.org/10.1214/EJP.v16-866

[6] AVENA, L., DOS SANTOS, R. S. and VÖLLERING, F. (2013). Transient random walk in symmetric exclusion: Limit theorems and an Einstein relation. *ALEA Lat. Am. J. Probab. Math. Stat.* **10** 693–709. MR3108811

[7] AVENA, L., FRANCO, T., JARA, M. and VÖLLERING, F. (2015). Symmetric exclusion as a random environment: Hydrodynamic limits. *Ann. Inst. Henri Poincaré Probab. Stat.* **51** 901–916. MR3365966 https://doi.org/10.1214/14-AIHP607

[8] AVENA, L., JARA, M. and VÖLLERING, F. (2018). Explicit LDP for a slowed RW driven by a symmetric exclusion process. *Probab. Theory Related Fields* **171** 865–915. MR3827224 https://doi.org/10.1007/s00440-017-0797-6

[9] BETHUELSEN, S. A. (2018). The contact process as seen from a random walk. *ALEA Lat. Am. J. Probab. Math. Stat.* **15** 571–585. MR3800486

[10] BETHUELSEN, S. A. and HEYDENREICH, M. (2017). Law of large numbers for random walks on attractive spin-flip dynamics. *Stochastic Process. Appl.* **127** 2346–2372. MR3652417 https://doi.org/10.1016/j.spa.2016.09.016

[11] BETHUELSEN, S. A. and VÖLLERING, F. (2016). Absolute continuity and weak uniform mixing of random walk in dynamic random environment. *Electron. J. Probab.* **21** 71. MR3580037 https://doi.org/10.1214/16-EJP10

MSC2020 subject classifications. Primary 60F15, 60K35, 60K37; secondary 82B41, 82C22, 82C44.
Key words and phrases. Random walk, dynamic random environment, strong law of large numbers, functional central limit theorem, large deviation bound, renormalisation, regeneration times.

[12] Blondel, O., Hilario, M. R., dos Santos, R. S., Sidoravicius, V. and Teixeira, A. (2019). Random walk on random walks: Higher dimensions. *Electron. J. Probab.* **24** 80. MR4003133 https://doi.org/doi:10.1214/19-EJP337

[13] Boldrighini, C., Ignatyuk, I. A., Malyshev, V. A. and Pellegrinotti, A. (1992). Random walk in dynamic environment with mutual influence. *Stochastic Process. Appl.* **41** 157–177. MR1162723 https://doi.org/10.1016/0304-4149(92)90151-F

[14] Campos, D., Drewitz, A., Ramírez, A. F., Rassoul-Agha, F. and Seppäläinen, T. (2013). Level 1 quenched large deviation principle for random walk in dynamic random environment. *Bull. Inst. Math. Acad. Sin.* (*N.S.*) **8** 1–29. MR3097414

[15] Comets, F. and Zeitouni, O. (2004). A law of large numbers for random walks in random mixing environments. *Ann. Probab.* **32** 880–914. MR2039946 https://doi.org/10.1214/aop/1079021467

[16] den Hollander, F., dos Santos, R. and Sidoravicius, V. (2013). Law of large numbers for non-elliptic random walks in dynamic random environments. *Stochastic Process. Appl.* **123** 156–190. MR2988114 https://doi.org/10.1016/j.spa.2012.09.002

[17] den Hollander, F. and dos Santos, R. S. (2014). Scaling of a random walk on a supercritical contact process. *Ann. Inst. Henri Poincaré Probab. Stat.* **50** 1276–1300. MR3269994 https://doi.org/10.1214/13-AIHP561

[18] den Hollander, F., Kesten, H. and Sidoravicius, V. (2014). Random walk in a high density dynamic random environment. *Indag. Math.* (*N.S.*) **25** 785–799. MR3217035 https://doi.org/10.1016/j.indag.2014.04.010

[19] dos Santos, R. S. (2012). Some case studies of random walks in dynamic random environments. Ph.D. thesis, Mathematical Institute, Faculty of Science, Leiden Univ.

[20] dos Santos, R. S. (2014). Non-trivial linear bounds for a random walk driven by a simple symmetric exclusion process. *Electron. J. Probab.* **19** 49. MR3217337 https://doi.org/10.1214/ejp.v19-3159

[21] Hilário, M. R., den Hollander, F., dos Santos, R. S., Sidoravicius, V. and Teixeira, A. (2015). Random walk on random walks. *Electron. J. Probab.* **20** 95. MR3399831 https://doi.org/10.1214/EJP.v20-4437

[22] Huveneers, F. and Simenhaus, F. (2015). Random walk driven by the simple exclusion process. *Electron. J. Probab.* **20** 105. MR3407222 https://doi.org/10.1214/EJP.v20-3906

[23] Kesten, H., Kozlov, M. V. and Spitzer, F. (1975). A limit law for random walk in a random environment. *Compos. Math.* **30** 145–168. MR0380998

[24] Madras, N. (1986). A process in a randomly fluctuating environment. *Ann. Probab.* **14** 119–135. MR0815962

[25] Mountford, T. and Vares, M. E. (2015). Random walks generated by equilibrium contact processes. *Electron. J. Probab.* **20** 3. MR3311216 https://doi.org/10.1214/EJP.v20-3439

[26] Orenshtein, T. and dos Santos, R. S. (2016). Zero-one law for directional transience of one-dimensional random walks in dynamic random environments. *Electron. Commun. Probab.* **21** 15. MR3485384 https://doi.org/10.1214/16-ECP4426

[27] Redig, F. and Völlering, F. (2013). Random walks in dynamic random environments: A transference principle. *Ann. Probab.* **41** 3157–3180. MR3127878 https://doi.org/10.1214/12-AOP819

[28] Sinaĭ, Y. G. (1982). The limit behavior of a one-dimensional random walk in a random environment. *Teor. Veroyatn. Primen.* **27** 247–258. MR0657919

[29] Solomon, F. (1975). Random walks in a random environment. *Ann. Probab.* **3** 1–31. MR0362503 https://doi.org/10.1214/aop/1176996444

[30] Sznitman, A.-S. (2002). An effective criterion for ballistic behavior of random walks in random environment. *Probab. Theory Related Fields* **122** 509–544. MR1902189 https://doi.org/10.1007/s004400100177

[31] Sznitman, A.-S. (2004). Topics in random walks in random environment. In *School and Conference on Probability Theory. ICTP Lect. Notes*, *XVII* 203–266. Abdus Salam Int. Cent. Theoret. Phys., Trieste. MR2198849

The Annals of Applied Probability
2020, Vol. 30, No. 4, 1642–1668
https://doi.org/10.1214/19-AAP1539

HIGH-DIMENSIONAL LIMITS OF EIGENVALUE DISTRIBUTIONS FOR GENERAL WISHART PROCESS

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In this article, we obtain an equation for the high-dimensional limit measure of eigenvalues of generalized Wishart processes, and the results are extended to random particle systems that generalize SDEs of eigenvalues. We also introduce a new set of conditions on the coefficient matrices for the existence and uniqueness of a strong solution for the SDEs of eigenvalues. The equation of the limit measure is further discussed assuming self-similarity on the eigenvalues.

REFERENCES

ANDERSON, G. W., GUIONNET, A. and ZEITOUNI, O. (2010). *An Introduction to Random Matrices*. *Cambridge Studies in Advanced Mathematics* **118**. Cambridge Univ. Press, Cambridge. MR2760897

BAI, Z. and SILVERSTEIN, J. W. (2010). *Spectral Analysis of Large Dimensional Random Matrices*, 2nd ed. *Springer Series in Statistics*. Springer, New York. MR2567175 https://doi.org/10.1007/978-1-4419-0661-8

BRU, M.-F. (1989). Diffusions of perturbed principal component analysis. *J. Multivariate Anal.* **29** 127–136. MR0991060 https://doi.org/10.1016/0047-259X(89)90080-8

CABANAL-DUVILLARD, T. and GUIONNET, A. (2001). Large deviations upper bounds for the laws of matrix-valued processes and non-communicative entropies. *Ann. Probab.* **29** 1205–1261. MR1872742 https://doi.org/10.1214/aop/1015345602

CÉPA, E. and LÉPINGLE, D. (1997). Diffusing particles with electrostatic repulsion. *Probab. Theory Related Fields* **107** 429–449. MR1440140 https://doi.org/10.1007/s004400050092

CHAN, T. (1992). The Wigner semi-circle law and eigenvalues of matrix-valued diffusions. *Probab. Theory Related Fields* **93** 249–272. MR1176727 https://doi.org/10.1007/BF01195231

DA FONSECA, J., GRASSELLI, M. and IELPO, F. (2014). Estimating the Wishart affine stochastic correlation model using the empirical characteristic function. *Stud. Nonlinear Dyn. Econom.* **18** 253–289. MR3258801 https://doi.org/10.1515/snde-2012-0009

DA FONSECA, J., GRASSELLI, M. and TEBALDI, C. (2008). Option pricing when correlations are stochastic: An analytical framework. *Rev. Deriv. Res.* **10** 151–180.

DEMNI, N. (2007). The Laguerre process and generalized Hartman–Watson law. *Bernoulli* **13** 556–580. MR2331264 https://doi.org/10.3150/07-BEJ6048

DYSON, F. J. (1962). A Brownian-motion model for the eigenvalues of a random matrix. *J. Math. Phys.* **3** 1191–1198. MR0148397 https://doi.org/10.1063/1.1703862

GNOATTO, A. (2012). The Wishart short rate model. *Int. J. Theor. Appl. Finance* **15** 1250056, 24. MR3011743 https://doi.org/10.1142/S0219024912500562

GNOATTO, A. and GRASSELLI, M. (2014). An affine multicurrency model with stochastic volatility and stochastic interest rates. *SIAM J. Financial Math.* **5** 493–531. MR3252805 https://doi.org/10.1137/130922902

GÖING-JAESCHKE, A. and YOR, M. (2003). A survey and some generalizations of Bessel processes. *Bernoulli* **9** 313–349. MR1997032 https://doi.org/10.3150/bj/1068128980

GOURIEROUX, C. (2006). Continuous time Wishart process for stochastic risk. *Econometric Rev.* **25** 177–217. MR2256286 https://doi.org/10.1080/07474930600713234

GOURIEROUX, C. and SUFANA, R. (2010). Derivative pricing with Wishart multivariate stochastic volatility. *J. Bus. Econom. Statist.* **28** 438–451. MR2723611 https://doi.org/10.1198/jbes.2009.08105

MSC2020 subject classifications. 60H15, 60F05.

Key words and phrases. Dyson Brownian motion, eigenvalue distribution, generalized Wishart process, high-dimensional limit, squared Bessel particle system, Wishart process.

GRACZYK, P. and MAŁECKI, J. (2013). Multidimensional Yamada–Watanabe theorem and its applications to particle systems. *J. Math. Phys.* **54** 021503, 15. MR3076363 https://doi.org/10.1063/1.4790507

GRACZYK, P. and MAŁECKI, J. (2014). Strong solutions of non-colliding particle systems. *Electron. J. Probab.* **19** no. 119, 21. MR3296535 https://doi.org/10.1214/EJP.v19-3842

KATORI, M. (2016). *Bessel Processes, Schramm–Loewner Evolution, and the Dyson Model. SpringerBriefs in Mathematical Physics* **11**. Springer, Tokyo. MR3445783

KONDOR, R. and JEBARA, T. (2007). Gaussian and Wishart hyperkernels. In *Advances in Neural Information Processing Systems* 729–736.

KÖNIG, W. and O'CONNELL, N. (2001). Eigenvalues of the Laguerre process as non-colliding squared Bessel processes. *Electron. Commun. Probab.* **6** 107–114. MR1871699 https://doi.org/10.1214/ECP.v6-1040

KRYLOV, N. V. and RÖCKNER, M. (2005). Strong solutions of stochastic equations with singular time dependent drift. *Probab. Theory Related Fields* **131** 154–196. MR2117951 https://doi.org/10.1007/s00440-004-0361-z

LI, W.-J., ZHANG, Z. and YEUNG, D.-Y. (2009). Latent Wishart processes for relational kernel learning. *J. Mach. Learn. Res.* **5** 336–343.

LI, J., ZHAO, B., DENG, C. and XU, R. (2016). Time varying metric learning for visual tracking. *Pattern Recogn. Lett.* **80** 157–164.

ROGERS, L. C. G. and SHI, Z. (1993). Interacting Brownian particles and the Wigner law. *Probab. Theory Related Fields* **95** 555–570. MR1217451 https://doi.org/10.1007/BF01196734

SCHÖLKOPF, B. and SMOLA, A. J. (2002). *Learning with Kernels*. MIT Press, Cambridge.

STEIN, E. M. and SHAKARCHI, R. (2011). *Functional Analysis: Introduction to Further Topics in Analysis. Princeton Lectures in Analysis* **4**. Princeton Univ. Press, Princeton, NJ. MR2827930

WU, S.-J., GHOSH, S. K., KU, Y.-C. and BLOOMFIELD, P. (2018). Dynamic correlation multivariate stochastic volatility with latent factors. *Stat. Neerl.* **72** 48–69. MR3746787 https://doi.org/10.1111/stan.12115

ZAMBOTTI, L. (2017). Bessel processes. In *Random Obstacle Problems. Lecture Notes in Math.* **2181** 31–50. Springer, Cham. MR3616274 https://doi.org/10.1007/978-3-319-52096-4

ZHANG, Z., KWOK, J. and YEUNG, D.-Y. (2006). Model-based transductive learning of the kernel matrix. *Mach. Learn.* **63** 69–101.

The Annals of Applied Probability
2020, Vol. 30, No. 4, 1669–1692
https://doi.org/10.1214/19-AAP1540

CONDITIONAL OPTIMAL STOPPING: A TIME-INCONSISTENT OPTIMIZATION

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Inspired by recent work of P.-L. Lions on conditional optimal control, we introduce a problem of optimal stopping under bounded rationality: the objective is the expected payoff at the time of stopping, conditioned on another event. For instance, an agent may care only about states where she is still alive at the time of stopping, or a company may condition on not being bankrupt. We observe that conditional optimization is time-inconsistent due to the dynamic change of the conditioning probability and develop an equilibrium approach in the spirit of R. H. Strotz' work for sophisticated agents in discrete time. Equilibria are found to be essentially unique in the case of a finite time horizon whereas an infinite horizon gives rise to nonuniqueness and other interesting phenomena. We also introduce a theory which generalizes the classical Snell envelope approach for optimal stopping by considering a pair of processes with Snell-type properties.

REFERENCES

[1] BARBERIS, N. (2012). A model of casino gambling. *Manage. Sci.* **58** 35–51.

[2] BASAK, S. and CHABAKAURI, G. (2010). Dynamic mean-variance asset allocation. *Rev. Financ. Stud.* **23** 2970–3016.

[3] BJÖRK, T., KHAPKO, M. and MURGOCI, A. (2017). On time-inconsistent stochastic control in continuous time. *Finance Stoch.* **21** 331–360. MR3626618 https://doi.org/10.1007/s00780-017-0327-5

[4] BJÖRK, T. and MURGOCI, A. (2014). A theory of Markovian time-inconsistent stochastic control in discrete time. *Finance Stoch.* **18** 545–592. MR3232016 https://doi.org/10.1007/s00780-014-0234-y

[5] BJÖRK, T., MURGOCI, A. and ZHOU, X. Y. (2014). Mean-variance portfolio optimization with state-dependent risk aversion. *Math. Finance* **24** 1–24. MR3157686 https://doi.org/10.1111/j.1467-9965.2011.00515.x

[6] CHRISTENSEN, S. and LINDENSJÖ, K. (2018). On finding equilibrium stopping times for time-inconsistent Markovian problems. *SIAM J. Control Optim.* **56** 4228–4255. MR3880244 https://doi.org/10.1137/17M1153029

[7] CHRISTENSEN, S. and LINDENSJÖ, K. (2018). On time-inconsistent stopping problems and mixed strategy stopping times. *Stochastic Process. Appl.* To appear.

[8] CZICHOWSKY, C. (2013). Time-consistent mean-variance portfolio selection in discrete and continuous time. *Finance Stoch.* **17** 227–271. MR3038591 https://doi.org/10.1007/s00780-012-0189-9

[9] EBERT, S. and STRACK, P. (2018). Never, ever getting started: On prospect theory without commitment. Preprint SSRN:2765550.

[10] EKELAND, I. and LAZRAK, A. (2006). Being serious about non-commitment: Subgame perfect equilibrium in continuous time. Preprint. Available at arXiv:0604264v1.

[11] EKELAND, I. and LAZRAK, A. (2010). The golden rule when preferences are time inconsistent. *Math. Financ. Econ.* **4** 29–55. MR2746576 https://doi.org/10.1007/s11579-010-0034-x

[12] EKELAND, I., MBODJI, O. and PIRVU, T. A. (2012). Time-consistent portfolio management. *SIAM J. Financial Math.* **3** 1–32. MR2968026 https://doi.org/10.1137/100810034

[13] EKELAND, I. and PIRVU, T. A. (2008). Investment and consumption without commitment. *Math. Financ. Econ.* **2** 57–86. MR2461340 https://doi.org/10.1007/s11579-008-0014-6

[14] FREDERICK, S., LOEWENSTEIN, G. and O'DONOGHUE, T. (2002). Time discounting and time preference: A critical review. *J. Econ. Lit.* **40** 351–401.

MSC2020 subject classifications. 60G40, 93E20, 91A13, 91A15.
Key words and phrases. Conditional optimal stopping, time-inconsistency, equilibrium.

[15] HE, X. D. and ZHOU, X. Y. (2011). Portfolio choice under cumulative prospect theory: An analytical treatment. *Manage*. *Sci*. **57** 315–331.

[16] HU, Y., JIN, H. and ZHOU, X. Y. (2012). Time-inconsistent stochastic linear-quadratic control. *SIAM J*. *Control Optim*. **50** 1548–1572. MR2968066 https://doi.org/10.1137/110853960

[17] HU, Y., JIN, H. and ZHOU, X. Y. (2017). Time-inconsistent stochastic linear-quadratic control: Characterization and uniqueness of equilibrium. *SIAM J*. *Control Optim*. **55** 1261–1279. MR3639569 https://doi.org/10.1137/15M1019040

[18] HUANG, Y.-J. and ZHOU, Z. (2018). Strong and weak equilibria for time-inconsistent stochastic control in continuous time. Preprint. Available at arXiv:1809.09243v1.

[19] HUANG, Y.-J. and NGUYEN-HUU, A. (2018). Time-consistent stopping under decreasing impatience. *Finance Stoch*. **22** 69–95. MR3738666 https://doi.org/10.1007/s00780-017-0350-6

[20] HUANG, Y.-J., NGUYEN-HUU, A. and ZHOU, X. Y. (2020). General stopping behaviors of naïve and noncommitted sophisticated agents, with application to probability distortion. *Math*. *Finance*. **30** 310–340.

[21] HUANG, Y.-J. and ZHOU, Z. (2017). Optimal equilibria for time-inconsistent stopping problems in continuous time. *Math*. *Finance*. To appear.

[22] HUANG, Y.-J. and ZHOU, Z. (2019). The optimal equilibrium for time-inconsistent stopping problems—The discrete-time case. *SIAM J*. *Control Optim*. **57** 590–609. MR3911711 https://doi.org/10.1137/17M1139187

[23] JIN, H. and ZHOU, X. Y. (2008). Behavioral portfolio selection in continuous time. *Math*. *Finance* **18** 385–426. MR2427728 https://doi.org/10.1111/j.1467-9965.2008.00339.x

[24] KARNAM, C., MA, J. and ZHANG, J. (2017). Dynamic approaches for some time-inconsistent optimization problems. *Ann*. *Appl*. *Probab*. **27** 3435–3477. MR3737929 https://doi.org/10.1214/17-AAP1284

[25] LIONS, P.-L. (2017). HJB equations and extensions of classical stochastic control theory. *Lectures at Collège de France*. Video available at https://www.college-de-france.fr/site/en-pierre-louis-lions/course-2016-10-21-09h00.htm. Transcript by Charafeddine Mouzouni available at https://hal.archives-ouvertes.fr/cel-01568969.

[26] PELEG, B. and YAARI, M. E. (1973). On the existence of a consistent course of action when tastes are changing. *Rev*. *Econ*. *Stud*. **40** 391–401.

[27] POLLAK, R. A. (1968). Consistent planning. *Rev*. *Econ*. *Stud*. **35** 201–208.

[28] SAMUELSON, P. A. (1958). An exact consumption-loan model of interest with or without the social contrivance of money. *Ann*. *Math*. *Stat*. **66** 467–482.

[29] STROTZ, R. H. (1955). Myopia and inconsistency in dynamic utility maximization. *Rev*. *Econ*. *Stud*. **23** 165–180.

[30] TAN, K. S., WEI, W. and ZHOU, X. Y. (2018). Failure of smooth pasting principle and nonexistence of equilibrium stopping rules under time-inconsistency. Preprint. Available at arXiv:1807.01785v1.

[31] YONG, J. (2012). Time-inconsistent optimal control problems and the equilibrium HJB equation. *Math*. *Control Relat*. *Fields* **2** 271–329. MR2991570 https://doi.org/10.3934/mcrf.2012.2.271

The Annals of Applied Probability
2020, Vol. 30, No. 4, 1693–1761
https://doi.org/10.1214/19-AAP1541

ON THE CONVERGENCE OF CLOSED-LOOP NASH EQUILIBRIA TO THE MEAN FIELD GAME LIMIT

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This paper continues the study of the mean field game (MFG) convergence problem: In what sense do the Nash equilibria of n-player stochastic differential games converge to the mean field game as $n \to \infty$? Previous work on this problem took two forms. First, when the n-player equilibria are open-loop, compactness arguments permit a characterization of all limit points of n-player equilibria as weak MFG equilibria, which contain additional randomness compared to the standard (strong) equilibrium concept. On the other hand, when the n-player equilibria are closed-loop, the convergence to the MFG equilibrium is known only when the MFG equilibrium is unique and the associated "master equation" is solvable and sufficiently smooth. This paper adapts the compactness arguments to the closed-loop case, proving a convergence theorem that holds even when the MFG equilibrium is nonunique. Every limit point of n-player equilibria is shown to be the same kind of weak MFG equilibrium as in the open-loop case. Some partial results and examples are discussed for the converse question, regarding which of the weak MFG equilibria can arise as the limit of n-player (approximate) equilibria.

REFERENCES

[1] BAFICO, R. and BALDI, P. (1981/82). Small random perturbations of Peano phenomena. *Stochastics* **6** 279–292. MR0665404 https://doi.org/10.1080/17442508208833208

[2] BAYRAKTAR, E. and COHEN, A. (2018). Analysis of a finite state many player game using its master equation. *SIAM J. Control Optim.* **56** 3538–3568. MR3860894 https://doi.org/10.1137/17M113887X

[3] BEN AROUS, G. and ZEITOUNI, O. (1999). Increasing propagation of chaos for mean field models. *Ann. Inst. Henri Poincaré Probab. Stat.* **35** 85–102. MR1669916 https://doi.org/10.1016/S0246-0203(99)80006-5

[4] BERGIN, J. and BERNHARDT, D. (1992). Anonymous sequential games with aggregate uncertainty. *J. Math. Econom.* **21** 543–562. MR1195676 https://doi.org/10.1016/0304-4068(92)90026-4

[5] BERTSEKAS, D. and SHREVE, S. (1996). *Stochastic Optimal Control: The Discrete Time Case*. Athena Scientific.

[6] BORKAR, V. S. and GHOSH, M. K. (1992). Stochastic differential games: Occupation measure based approach. *J. Optim. Theory Appl.* **73** 359–385. MR1163266 https://doi.org/10.1007/BF00940187

[7] BRÉMAUD, P. and YOR, M. (1978). Changes of filtrations and of probability measures. *Z. Wahrsch. Verw. Gebiete* **45** 269–295. MR0511775 https://doi.org/10.1007/BF00537538

[8] BRUNICK, G. and SHREVE, S. (2013). Mimicking an Itô process by a solution of a stochastic differential equation. *Ann. Appl. Probab.* **23** 1584–1628. MR3098443 https://doi.org/10.1214/12-aap881

[9] CARDALIAGUET, P. ., CIRANT, M. and PORRETTA, A. (2018). Remarks on Nash equilibria in mean field game models with a major player. arXiv preprint arXiv:1811.02811.

[10] CARDALIAGUET, P. (2017). The convergence problem in mean field games with local coupling. *Appl. Math. Optim.* **76** 177–215. MR3679342 https://doi.org/10.1007/s00245-017-9434-0

[11] CARDALIAGUET, P., DELARUE, F., LASRY, J.-M. and LIONS, P.-L. (2019). *The Master Equation and the Convergence Problem in Mean Field Games*. *Annals of Mathematics Studies* **201**. Princeton Univ. Press, Princeton, NJ. MR3967062

[12] CARDALIAGUET, P. and RAINER, C. (2009). Stochastic differential games with asymmetric information. *Appl. Math. Optim.* **59** 1–36. MR2465706 https://doi.org/10.1007/s00245-008-9042-0

MSC2020 subject classifications. 60H30, 93E20, 91A06.

Key words and phrases. Mean field games, stochastic differential games, closed-loop controls, McKean–Vlasov equations, relaxed controls.

[13] CARMONA, R. (2016). *Lectures on BSDEs, Stochastic Control, and Stochastic Differential Games with Financial Applications*. *Financial Mathematics* **1**. SIAM, Philadelphia, PA. MR3629171

[14] CARMONA, R., CERENZIA, M. and PALMER, A. Z. (2018). The Dyson game. arXiv preprint arXiv:1808.02464.

[15] CARMONA, R. and DELARUE, F. (2013). Probabilistic analysis of mean-field games. *SIAM J. Control Optim*. **51** 2705–2734. MR3072222 https://doi.org/10.1137/120883499

[16] CARMONA, R. and DELARUE, F. (2017). *Probabilistic Theory of Mean Field Games*: *Vol. I*, *Mean Field FBSDEs*, *Control*, *and Games*. *Stochastic Analysis and Applications*. Springer Verlag.

[17] CARMONA, R. and DELARUE, F. (2018). *Probabilistic Theory of Mean Field Games with Applications. II*: *Mean Field Games with Common Noise and Master Equations*. *Probability Theory and Stochastic Modelling* **84**. Springer, Cham. MR3753660

[18] CARMONA, R., DELARUE, F. and LACKER, D. (2016). Mean field games with common noise. *Ann. Probab.* **44** 3740–3803. MR3572323 https://doi.org/10.1214/15-AOP1060

[19] CARMONA, R., FOUQUE, J.-P. and SUN, L.-H. (2015). Mean field games and systemic risk. *Commun. Math. Sci*. **13** 911–933. MR3325083 https://doi.org/10.4310/CMS.2015.v13.n4.a4

[20] CECCHIN, A., DAI PRA, P., FISCHER, M. and PELINO, G. (2019). On the convergence problem in mean field games: A two state model without uniqueness. *SIAM J. Control Optim*. **57** 2443–2466. MR3981375 https://doi.org/10.1137/18M1222454

[21] CECCHIN, A. and PELINO, G. (2019). Convergence, fluctuations and large deviations for finite state mean field games via the master equation. *Stochastic Process. Appl*. **129** 4510–4555. MR4013871 https://doi.org/10.1016/j.spa.2018.12.002

[22] CHASSAGNEUX, J.-F., CRISAN, D. and DELARUE, F. (2014). A probabilistic approach to classical solutions of the master equation for large population equilibria. arXiv preprint arXiv:1411.3009.

[23] DELARUE, F. (2017). Mean field games: A toy model on an Erdös–Renyi graph. In *Journées MAS* 2016 *de la SMAI—Phénomènes Complexes et Hétérogènes*. *ESAIM Proc. Surveys* **60** 1–26. EDP Sci., Les Ulis. MR3772471

[24] DELARUE, F. and FOGUEN TCHUENDOM, R. (2020). Selection of equilibria in a linear quadratic mean-field game. *Stochastic Process. Appl*. **130** 1000–1040. MR4046528 https://doi.org/10.1016/j.spa.2019.04.005

[25] DELARUE, F., LACKER, D. and RAMANAN, K. (2020). From the master equation to mean field game limit theory: Large deviations and concentration of measure. *Ann. Probab.* **48** 211–263. MR4079435 http://dx.doi.org/10.1214/19-AOP1359

[26] DELARUE, F., LACKER, D. and RAMANAN, K. (2019). From the master equation to mean field game limit theory: A central limit theorem. *Electron. J. Probab.* **24** Paper No. 51, 54. MR3954791 https://doi.org/10.1214/19-EJP298

[27] EL KAROUI, N. and MÉLÉARD, S. (1990). Martingale measures and stochastic calculus. *Probab. Theory Related Fields* **84** 83–101. MR1027822 https://doi.org/10.1007/BF01288560

[28] EL KAROUI, N., NGUYEN, D. and JEANBLANC-PICQUÉ, M. (1987). Compactification methods in the control of degenerate diffusions: Existence of an optimal control. *Stochastics* **20** 169–219. MR0878312 https://doi.org/10.1080/17442508708833443

[29] EL KAROUI, N., NGUYEN, D. H. and JEANBLANC-PICQUÉ, M. (1988). Existence of an optimal Markovian filter for the control under partial observations. *SIAM J. Control Optim*. **26** 1025–1061. MR0957652 https://doi.org/10.1137/0326057

[30] FELEQI, E. (2013). The derivation of ergodic mean field game equations for several populations of players. *Dyn. Games Appl*. **3** 523–536. MR3127148 https://doi.org/10.1007/s13235-013-0088-5

[31] FIGALLI, A. (2008). Existence and uniqueness of martingale solutions for SDEs with rough or degenerate coefficients. *J. Funct. Anal*. **254** 109–153. MR2375067 https://doi.org/10.1016/j.jfa.2007.09.020

[32] FILIPPOV, A. F. (1962). On certain questions in the theory of optimal control. *J. SIAM Control Ser. A* **1** 76–84. MR0149985

[33] FISCHER, M. (2017). On the connection between symmetric N-player games and mean field games. *Ann. Appl. Probab.* **27** 757–810. MR3655853 https://doi.org/10.1214/16-AAP1215

[34] FLEMING, W. H. (1977). Generalized solutions in optimal stochastic control. In *Differential Games and Control Theory*, *II* (*Proc*. 2*nd Conf*., *Univ. Rhode Island*, *Kingston*, *R.I.*, 1976) *Lecture Notes in Pure and Appl. Math*. **30** 147–165. MR0688675

[35] FLEMING, W. H. and NISIO, M. (1984). On stochastic relaxed control for partially observed diffusions. *Nagoya Math. J.* **93** 71–108. MR0738919 https://doi.org/10.1017/S0027763000020742

[36] FRIEDMAN, A. (1972). Stochastic differential games. *J. Differential Equations* **11** 79–108. MR0292528 https://doi.org/10.1016/0022-0396(72)90082-4

[37] FUDENBERG, D. and LEVINE, D. (1986). Limit games and limit equilibria. *J. Econom. Theory* **38** 261–279. MR0841698 https://doi.org/10.1016/0022-0531(86)90118-3

[38] FUDENBERG, D. and LEVINE, D. (2009). Open-loop and closed-loop equilibria in dynamic games with many players. In *A Long-Run Collaboration on Long-Run Games* 41–58. World Scientific.
[39] GANGBO, W. and SWIECH, A. (2015). Existence of a solution to an equation arising from the theory of mean field games. *J. Differential Equations* **259** 6573–6643. MR3397332 https://doi.org/10.1016/j.jde.2015.08.001
[40] GÄRTNER, J. (1988). On the McKean–Vlasov limit for interacting diffusions. *Math. Nachr.* **137** 197–248. MR0968996 https://doi.org/10.1002/mana.19881370116
[41] GYÖNGY, I. (1986). Mimicking the one-dimensional marginal distributions of processes having an Itô differential. *Probab. Theory Related Fields* **71** 501–516. MR0833267 https://doi.org/10.1007/BF00699039
[42] HAMADENE, S. and LEPELTIER, J.-P. (1995). Zero-sum stochastic differential games and backward equations. *Systems Control Lett.* **24** 259–263. MR1321134 https://doi.org/10.1016/0167-6911(94)00011-J
[43] HAUSSMANN, U. G. and LEPELTIER, J.-P. (1990). On the existence of optimal controls. *SIAM J. Control Optim.* **28** 851–902. MR1051628 https://doi.org/10.1137/0328049
[44] HUANG, M., CAINES, P. E. and MALHAMÉ, R. P. (2007). Large-population cost-coupled LQG problems with nonuniform agents: Individual-mass behavior and decentralized ϵ-Nash equilibria. *IEEE Trans. Automat. Control* **52** 1560–1571. MR2352434 https://doi.org/10.1109/TAC.2007.904450
[45] HUANG, M., MALHAMÉ, R. P. and CAINES, P. E. (2006). Large population stochastic dynamic games: Closed-loop McKean–Vlasov systems and the Nash certainty equivalence principle. *Commun. Inf. Syst.* **6** 221–251. MR2346927
[46] JABIN, P.-E. and WANG, Z. (2016). Mean field limit and propagation of chaos for Vlasov systems with bounded forces. *J. Funct. Anal.* **271** 3588–3627. MR3558251 https://doi.org/10.1016/j.jfa.2016.09.014
[47] JACOD, J. and MÉMIN, J. (1981). Weak and strong solutions of stochastic differential equations: Existence and stability. In *Stochastic Integrals* (*Proc. Sympos., Univ. Durham, Durham*, 1980). *Lecture Notes in Math.* **851** 169–212. Springer, Berlin. MR0620991
[48] KALLENBERG, O. (2002). *Foundations of Modern Probability*, 2nd ed. *Probability and Its Applications* (*New York*). Springer, New York. MR1876169 https://doi.org/10.1007/978-1-4757-4015-8
[49] KRYLOV, N. V. and RÖCKNER, M. (2005). Strong solutions of stochastic equations with singular time dependent drift. *Probab. Theory Related Fields* **131** 154–196. MR2117951 https://doi.org/10.1007/s00440-004-0361-z
[50] LACKER, D. (2015). Mean field games via controlled martingale problems: Existence of Markovian equilibria. *Stochastic Process. Appl.* **125** 2856–2894. MR3332857 https://doi.org/10.1016/j.spa.2015.02.006
[51] LACKER, D. (2016). A general characterization of the mean field limit for stochastic differential games. *Probab. Theory Related Fields* **165** 581–648. MR3520014 https://doi.org/10.1007/s00440-015-0641-9
[52] LACKER, D. (2018). On a strong form of propagation of chaos for McKean–Vlasov equations. *Electron. Commun. Probab.* **23** Paper No. 45, 11. MR3841406 https://doi.org/10.1214/18-ECP150
[53] LASRY, J.-M. and LIONS, P.-L. (2006). Jeux à champ moyen. I. Le cas stationnaire. *C. R. Math. Acad. Sci. Paris* **343** 619–625. MR2269875 https://doi.org/10.1016/j.crma.2006.09.019
[54] LASRY, J.-M. and LIONS, P.-L. (2006). Jeux à champ moyen. II. Horizon fini et contrôle optimal. *C. R. Math. Acad. Sci. Paris* **343** 679–684. MR2271747 https://doi.org/10.1016/j.crma.2006.09.018
[55] LASRY, J.-M. and LIONS, P.-L. (2007). Mean field games. *Jpn. J. Math.* **2** 229–260. MR2295621 https://doi.org/10.1007/s11537-007-0657-8
[56] NEUFELD, A. and NUTZ, M. (2014). Measurability of semimartingale characteristics with respect to the probability law. *Stochastic Process. Appl.* **124** 3819–3845. MR3249357 https://doi.org/10.1016/j.spa.2014.07.006
[57] NUTZ, M., SAN MARTIN, J. and TAN, X. (2018). Convergence to the mean field game limit: A case study. arXiv preprint arXiv:1806.00817.
[58] OELSCHLÄGER, K. (1984). A martingale approach to the law of large numbers for weakly interacting stochastic processes. *Ann. Probab.* **12** 458–479. MR0735849
[59] PROTTER, P. E. (2005). *Stochastic Integration and Differential Equations*, 2nd ed. *Stochastic Modelling and Applied Probability* **21**. Springer, Berlin. Version 2.1, Corrected third printing. MR2273672 https://doi.org/10.1007/978-3-662-10061-5
[60] ROXIN, E. (1962). The existence of optimal controls. *Michigan Math. J.* **9** 109–119. MR0136844
[61] ROYDEN, H. L. (1968). *Real Analysis*, 2nd ed. Macmillan. MR0151555
[62] STROOCK, D. W. and VARADHAN, S. R. S. (2006). *Multidimensional Diffusion Processes*. *Classics in Mathematics*. Springer, Berlin. Reprint of the 1997 edition. MR2190038
[63] SZNITMAN, A.-S. (1991). Topics in propagation of chaos. In *École D'Été de Probabilités de Saint-Flour XIX*—1989. *Lecture Notes in Math.* **1464** 165–251. Springer, Berlin. MR1108185 https://doi.org/10.1007/BFb0085169
[64] TREVISAN, D. (2013). Zero noise limits using local times. *Electron. Commun. Probab.* **18** no. 31, 7. MR3056068 https://doi.org/10.1214/ECP.v18-2587

[65] TREVISAN, D. (2016). Well-posedness of multidimensional diffusion processes with weakly differentiable coefficients. *Electron. J. Probab.* **21** Paper No. 22, 41. MR3485364 https://doi.org/10.1214/16-EJP4453

[66] VERETENNIKOV, A. Y. (1981). On strong solutions and explicit formulas for solutions of stochastic integral equations. *Math. USSR, Sb.* **39** 387.

[67] VILLANI, C. (2003). *Topics in Optimal Transportation*. *Graduate Studies in Mathematics* **58**. Amer. Math. Soc., Providence, RI. MR1964483 https://doi.org/10.1090/gsm/058

[68] WALSH, J. B. (1986). An introduction to stochastic partial differential equations. In *École D'été de Probabilités de Saint-Flour, XIV*—1984. *Lecture Notes in Math.* **1180** 265–439. Springer, Berlin. MR0876085 https://doi.org/10.1007/BFb0074920

The Annals of Applied Probability
2020, Vol. 30, No. 4, 1762–1783
https://doi.org/10.1214/19-AAP1542

ON BAYESIAN CONSISTENCY FOR FLOWS OBSERVED THROUGH A PASSIVE SCALAR

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We consider the statistical inverse problem of estimating a background fluid flow field $\mathbf{v}$ from the partial, noisy observations of the concentration θ of a substance passively advected by the fluid, so that θ is governed by the partial differential equation

$$\frac{\partial}{\partial t}\theta(t,\mathbf{x}) = -\mathbf{v}(\mathbf{x})\cdot\nabla\theta(t,\mathbf{x}) + \kappa\Delta\theta(t,\mathbf{x}), \quad \theta(0,\mathbf{x}) = \theta_0(\mathbf{x})$$

for $t \in [0,T]$, $T > 0$ and $\mathbf{x} \in \mathbb{T}^2 = [0,1]^2$. The initial condition θ_0 and diffusion coefficient κ are assumed to be known and the data consist of point observations of the scalar field θ corrupted by additive, i.i.d. Gaussian noise. We adopt a Bayesian approach to this estimation problem and establish that the inference is consistent, that is, that the posterior measure identifies the true background flow as the number of scalar observations grows large. Since the inverse map is ill-defined for some classes of problems even for perfect, infinite measurements of θ, multiple experiments (initial conditions) are required to resolve the true fluid flow. Under this assumption, suitable conditions on the observation points, and given support and tail conditions on the prior measure, we show that the posterior measure converges to a Dirac measure centered on the true flow as the number of observations goes to infinity.

REFERENCES

ABRAHAM, K. (2019). Nonparametric Bayesian posterior contraction rates for scalar diffusions with high-frequency data. *Bernoulli* **25** 2696–2728. MR4003562 https://doi.org/10.3150/18-BEJ1067

BILLINGSLEY, P. (2013). *Convergence of Probability Measures*. Wiley, New York. MR0233396

BORGGAARD, J., GLATT-HOLTZ, N. and KROMETIS, J. (2018). A Bayesian approach to estimating background flows from a passive scalar. Preprint. Available at arXiv:1808.01084.

DASHTI, M. and STUART, A. M. (2017). The Bayesian approach to inverse problems. In *Handbook of Uncertainty Quantification* 311–428. Springer, Cham. MR3839555

DIACONIS, P. and FREEDMAN, D. (1986). On the consistency of Bayes estimates. *Ann. Statist.* **14** 1–67. MR0829555 https://doi.org/10.1214/aos/1176349830

DOOB, J. L. (1949). Application of the theory of martingales. In *Le Calcul des Probabilités et Ses Applications. Colloques Internationaux du Centre National de la Recherche Scientifique* **13** 23–27. Centre National de la Recherche Scientifique, Paris. MR0033460

EVANS, L. C. (2010). *Partial Differential Equations*, 2nd ed. *Graduate Studies in Mathematics* **19**. Amer. Math. Soc., Providence, RI. MR2597943 https://doi.org/10.1090/gsm/019

FREEDMAN, D. A. (1963). On the asymptotic behavior of Bayes' estimates in the discrete case. *Ann. Math. Stat.* **34** 1386–1403. MR0158483 https://doi.org/10.1214/aoms/1177703871

GHOSAL, S. and VAN DER VAART, A. (2017). *Fundamentals of Nonparametric Bayesian Inference*. *Cambridge Series in Statistical and Probabilistic Mathematics* **44**. Cambridge Univ. Press, Cambridge. MR3587782 https://doi.org/10.1017/9781139029834

GRAFAKOS, L. and OH, S. (2014). The Kato–Ponce inequality. *Comm. Partial Differential Equations* **39** 1128–1157. MR3200091 https://doi.org/10.1080/03605302.2013.822885

GUGUSHVILI, S. and SPREIJ, P. (2014). Nonparametric Bayesian drift estimation for multidimensional stochastic differential equations. *Lith. Math. J.* **54** 127–141. MR3212631 https://doi.org/10.1007/s10986-014-9232-1

MSC2020 subject classifications. Primary 62G20; secondary 65N21, 76R05.
Key words and phrases. Bayesian statistical inversion, Bayesian consistency, passive scalars.

KAIPIO, J. and SOMERSALO, E. (2005). *Statistical and Computational Inverse Problems*. *Applied Mathematical Sciences* **160**. Springer, New York. MR2102218

KARCH, G. K., SADLO, F., WEISKOPF, D., HANSEN, C. D., LI, G. S. and ERTL, T. (2012). Dye-based flow visualization. *Comput*. *Sci*. *Eng*. **14** 80–86. https://doi.org/10.1109/MCSE.2012.118

KELLAY, H. and GOLDBURG, W. I. (2002). Two-dimensional turbulence: A review of some recent experiments. *Rep*. *Progr*. *Phys*. **65** 845–894.

KOSKELA, J., SPANÒ, D. and JENKINS, P. A. (2019). Consistency of Bayesian nonparametric inference for discretely observed jump diffusions. *Bernoulli* **25** 2183–2205. MR3961245 https://doi.org/10.3150/18-BEJ1050

KROMETIS, J. (2018). A Bayesian approach to estimating background flows from a passive scalar. Ph.D. thesis, Virginia Polytechnic Institute and State Univ.

LAPLACE, P.-S. (1810). Mémoire sur les approximations des formules qui sont fonctions de très grands nombres et sur leur applications aux probabilités. *Mém*. *Acad*. *Sci*. *Paris*.

LE CAM, L. (1953). On some asymptotic properties of maximum likelihood estimates and related Bayes' estimates. *Univ*. *Calif*. *Publ*. *Statist*. **1** 277–329. MR0054913

LE CAM, L. and YANG, G. L. (2000). *Asymptotics in Statistics*: *Some Basic Concepts*, 2nd ed. *Springer Series in Statistics*. Springer, New York. MR1784901 https://doi.org/10.1007/978-1-4612-1166-2

LIEBERMAN, G. M. (1996). *Second Order Parabolic Differential Equations*. World Scientific, River Edge, NJ. MR1465184 https://doi.org/10.1142/3302

MUSCALU, C. and SCHLAG, W. (2013). *Classical and Multilinear Harmonic Analysis*, *Vol*. *II*. *Cambridge Studies in Advanced Mathematics* **138**. Cambridge Univ. Press, Cambridge. MR3052499

NEWEY, W. K. and MCFADDEN, D. (1994). Large sample estimation and hypothesis testing. In *Handbook of Econometrics*, *Vol*. *IV*. *Handbooks in Econom*. **2** 2111–2245. North-Holland, Amsterdam. MR1315971

NICKL, R. (2013). Statistical theory. Lecture notes.

NICKL, R. (2017). Bernstein–von Mises theorems for statistical inverse problems I: Schrödinger equation. Preprint. Available at arXiv:1707.01764.

NICKL, R. and SÖHL, J. (2017). Nonparametric Bayesian posterior contraction rates for discretely observed scalar diffusions. *Ann*. *Statist*. **45** 1664–1693. MR3670192 https://doi.org/10.1214/16-AOS1504

ØKSENDAL, B. (2013). *Stochastic Differential Equations*: *An Introduction with Applications*, 6th ed. *Universitext*. Springer, Berlin. MR2001996 https://doi.org/10.1007/978-3-642-14394-6

PAPASPILIOPOULOS, O., POKERN, Y., ROBERTS, G. O. and STUART, A. M. (2012). Nonparametric estimation of diffusions: A differential equations approach. *Biometrika* **99** 511–531. MR2966767 https://doi.org/10.1093/biomet/ass034

ROBINSON, J. C. (2001). *Infinite-Dimensional Dynamical Systems*: *An Introduction to Dissipative Parabolic PDEs and the Theory of Global Attractors*. *Cambridge Texts in Applied Mathematics*. Cambridge Univ. Press, Cambridge. MR1881888 https://doi.org/10.1007/978-94-010-0732-0

RUDIN, W. (1964). *Principles of Mathematical Analysis*. 2nd ed. McGraw-Hill, New York. MR0166310

SCHWARTZ, L. (1965). On Bayes procedures. *Z*. *Wahrsch*. *Verw*. *Gebiete* **4** 10–26. MR0184378 https://doi.org/10.1007/BF00535479

SMITS, A. J. (2012). *Flow Visualization*: *Techniques and Examples*. World Scientific, Singapore.

STUART, A. M. (2010). Inverse problems: A Bayesian perspective. *Acta Numer*. **19** 451–559. MR2652785 https://doi.org/10.1017/S0962492910000061

TEMAM, R. (1995). *Navier–Stokes Equations and Nonlinear Functional Analysis*, 2nd ed. *CBMS–NSF Regional Conference Series in Applied Mathematics* **66**. SIAM, Philadelphia, PA. MR1318914 https://doi.org/10.1137/1.9781611970050

VAN DER MEULEN, F. and VAN ZANTEN, H. (2013). Consistent nonparametric Bayesian inference for discretely observed scalar diffusions. *Bernoulli* **19** 44–63. MR3019485 https://doi.org/10.3150/11-BEJ385

VOLLMER, S. J. (2013). Posterior consistency for Bayesian inverse problems through stability and regression results. *Inverse Probl*. **29** Art. ID 125011. MR3141858 https://doi.org/10.1088/0266-5611/29/12/125011

WASSERMAN, L. (1998). Asymptotic properties of nonparametric Bayesian procedures. In *Practical Nonparametric and Semiparametric Bayesian Statistics*. *Lect*. *Notes Stat*. **133** 293–304. Springer, New York. MR1630088 https://doi.org/10.1007/978-1-4612-1732-9_16

WOLFGANG, M. (1987). *Flow Visualization*. Academic Press, London.

The Annals of Applied Probability
2020, Vol. 30, No. 4, 1784–1823
https://doi.org/10.1214/19-AAP1543

APPROXIMATION SCHEMES FOR VISCOSITY SOLUTIONS OF FULLY NONLINEAR STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

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The aim of this paper is to develop a general method for constructing approximation schemes for viscosity solutions of fully nonlinear pathwise stochastic partial differential equations, and for proving their convergence. Our results apply to approximations such as explicit finite difference schemes and Trotter–Kato type mixing formulas. The irregular time dependence disrupts the usual methods from the classical viscosity theory for creating schemes that are both monotone and convergent, an obstacle that cannot be overcome by incorporating higher order correction terms, as is done for numerical approximations of stochastic or rough ordinary differential equations. The novelty here is to regularize those driving paths with nontrivial quadratic variation in order to guarantee both monotonicity and convergence.

We present qualitative and quantitative results, the former covering a wide variety of schemes for second-order equations. An error estimate is established in the Hamilton–Jacobi case, its merit being that it depends on the path only through the modulus of continuity, and not on the derivatives or total variation. As a result, it is possible to choose a regularization of the path so as to obtain efficient rates of convergence. This is demonstrated in the specific setting of equations with multiplicative white noise in time, in which case the convergence holds with probability one. We also present an example using scaled random walks that exhibits convergence in distribution.

REFERENCES

[1] BARLES, G. and JAKOBSEN, E. R. (2002). On the convergence rate of approximation schemes for Hamilton–Jacobi–Bellman equations. *M2AN Math. Model. Numer. Anal.* **36** 33–54. MR1916291 https://doi.org/10.1051/m2an:2002002

[2] BARLES, G. and JAKOBSEN, E. R. (2005). Error bounds for monotone approximation schemes for Hamilton–Jacobi–Bellman equations. *SIAM J. Numer. Anal.* **43** 540–558. MR2177879 https://doi.org/10.1137/S003614290343815X

[3] BARLES, G. and JAKOBSEN, E. R. (2007). Error bounds for monotone approximation schemes for parabolic Hamilton–Jacobi–Bellman equations. *Math. Comp.* **76** 1861–1893. MR2336272 https://doi.org/10.1090/S0025-5718-07-02000-5

[4] BARLES, G. and SOUGANIDIS, P. E. (1991). Convergence of approximation schemes for fully nonlinear second order equations. *Asymptot. Anal.* **4** 271–283. MR1115933

[5] BUCKDAHN, R. and MA, J. (2001). Stochastic viscosity solutions for nonlinear stochastic partial differential equations. I. *Stochastic Process. Appl.* **93** 181–204. MR1828772 https://doi.org/10.1016/S0304-4149(00)00093-4

[6] BUCKDAHN, R. and MA, J. (2001). Stochastic viscosity solutions for nonlinear stochastic partial differential equations. II. *Stochastic Process. Appl.* **93** 205–228. MR1831830 https://doi.org/10.1016/S0304-4149(00)00092-2

[7] CAFFARELLI, L. A. and SOUGANIDIS, P. E. (2008). A rate of convergence for monotone finite difference approximations to fully nonlinear, uniformly elliptic PDEs. *Comm. Pure Appl. Math.* **61** 1–17. MR2361302 https://doi.org/10.1002/cpa.20208

[8] CARUANA, M., FRIZ, P. K. and OBERHAUSER, H. (2011). A (rough) pathwise approach to a class of nonlinear stochastic partial differential equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **28** 27–46. MR2765508 https://doi.org/10.1016/j.anihpc.2010.11.002

MSC2020 subject classifications. 65M12, 65M06, 60H15, 35R60, 35K55, 35D40.

Key words and phrases. Stochastic viscosity solutions, finite difference schemes, monotone schemes, splitting formulae, error estimates.

[9] COURANT, R., FRIEDRICHS, K. and LEWY, H. (1967). On the partial difference equations of mathematical physics. *IBM J. Res. Develop.* **11** 215–234. MR0213764 https://doi.org/10.1147/rd.112.0215

[10] CRANDALL, M. G., ISHII, H. and LIONS, P.-L. (1992). User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc.* (*N.S.*) **27** 1–67. MR1118699 https://doi.org/10.1090/S0273-0979-1992-00266-5

[11] CRANDALL, M. G. and LIONS, P.-L. (1984). Two approximations of solutions of Hamilton–Jacobi equations. *Math. Comp.* **43** 1–19. MR0744921 https://doi.org/10.2307/2007396

[12] FRIZ, P. K., GASSIAT, P., LIONS, P.-L. and SOUGANIDIS, P. E. (2017). Eikonal equations and pathwise solutions to fully non-linear SPDEs. *Stoch. Partial Differ. Equ. Anal. Comput.* **5** 256–277. MR3640072 https://doi.org/10.1007/s40072-016-0087-9

[13] GASSIAT, P. and GESS, B. (2019). Regularization by noise for stochastic Hamilton–Jacobi equations. *Probab. Theory Related Fields* **173** 1063–1098. MR3936151 https://doi.org/10.1007/s00440-018-0848-7

[14] GUBINELLI, M., TINDEL, S. and TORRECILLA, I. Controlled viscosity solutions of fully nonlinear rough PDEs. Available at arXiv:1403.2832 [math.PR].

[15] HOEL, H., KARLSEN, K. H., RISEBRO, N. H. and STORRØSTEN, E. B. (2019). Numerical methods for conservation laws with rough flux. *Stoch PDE: Anal Comp.* https://doi.org/10.1007/s40072-019-00145-7

[16] ISHII, H. (1985). Hamilton–Jacobi equations with discontinuous Hamiltonians on arbitrary open sets. *Bull. Fac. Sci. Engrg. Chuo Univ.* **28** 33–77. MR0845397

[17] JAKOBSEN, E. R. (2004). On error bounds for approximation schemes for non-convex degenerate elliptic equations. *BIT* **44** 269–285. MR2093506 https://doi.org/10.1023/B:BITN.0000039390.33444.f2

[18] JAKOBSEN, E. R. (2006). On error bounds for monotone approximation schemes for multi-dimensional Isaacs equations. *Asymptot. Anal.* **49** 249–273. MR2270893

[19] KRYLOV, N. V. (2015). On the rate of convergence of finite-difference approximations for elliptic Isaacs equations in smooth domains. *Comm. Partial Differential Equations* **40** 1393–1407. MR3355497 https://doi.org/10.1080/03605302.2015.1029074

[20] KUO, H.-J. and TRUDINGER, N. S. (1996). Positive difference operators on general meshes. *Duke Math. J.* **83** 415–433. MR1390653 https://doi.org/10.1215/S0012-7094-96-08314-3

[21] KUO, H. J. and TRUDINGER, N. S. (1992). Discrete methods for fully nonlinear elliptic equations. *SIAM J. Numer. Anal.* **29** 123–135. MR1149088 https://doi.org/10.1137/0729008

[22] LIONS, P.-L. and PERTHAME, B. (1987). Remarks on Hamilton–Jacobi equations with measurable time-dependent Hamiltonians. *Nonlinear Anal.* **11** 613–621. MR0886652 https://doi.org/10.1016/0362-546X(87)90076-9

[23] LIONS, P.-L. and SOUGANIDIS, P. E. Forthcoming book on fully nonlinear stochastic partial differential equations.

[24] LIONS, P.-L. and SOUGANIDIS, P. E. (1998). Fully nonlinear stochastic partial differential equations. *C. R. Acad. Sci. Paris Sér. I Math.* **326** 1085–1092. MR1647162 https://doi.org/10.1016/S0764-4442(98)80067-0

[25] LIONS, P.-L. and SOUGANIDIS, P. E. (1998). Fully nonlinear stochastic partial differential equations: Nonsmooth equations and applications. *C. R. Acad. Sci. Paris Sér. I Math.* **327** 735–741. MR1659958 https://doi.org/10.1016/S0764-4442(98)80161-4

[26] LIONS, P.-L. and SOUGANIDIS, P. E. (2000). Uniqueness of weak solutions of fully nonlinear stochastic partial differential equations. *C. R. Acad. Sci. Paris Sér. I Math.* **331** 783–790. MR1807189 https://doi.org/10.1016/S0764-4442(00)01597-4

[27] LIONS, P.-L. and SOUGANIDIS, P. E. (2000). Fully nonlinear stochastic pde with semilinear stochastic dependence. *C. R. Acad. Sci. Paris Sér. I Math.* **331** 617–624. MR1799099 https://doi.org/10.1016/S0764-4442(00)00583-8

[28] NUNZIANTE, D. (1990). Uniqueness of viscosity solutions of fully nonlinear second order parabolic equations with discontinuous time-dependence. *Differential Integral Equations* **3** 77–91. MR1014727

[29] SEEGER, B. (2018). Homogenization of pathwise Hamilton–Jacobi equations. *J. Math. Pures Appl.* (9) **110** 1–31. MR3744918 https://doi.org/10.1016/j.matpur.2017.07.012

[30] SEEGER, B. (2018). Perron's method for pathwise viscosity solutions. *Comm. Partial Differential Equations* **43** 998–1018. MR3909032 https://doi.org/10.1080/03605302.2018.1488262

[31] SOUGANIDIS, P. E. Fully nonlinear first- and second-order stochastic partial differential equations. Preprint. Available at arXiv:1809.01748 [math.AP].

[32] SOUGANIDIS, P. E. (1985). Approximation schemes for viscosity solutions of Hamilton–Jacobi equations. *J. Differential Equations* **59** 1–43. MR0803085 https://doi.org/10.1016/0022-0396(85)90136-6

[33] Souganidis, P. E. (1985). Max-min representations and product formulas for the viscosity solutions of Hamilton–Jacobi equations with applications to differential games. *Nonlinear Anal*. **9** 217–257. MR0784388 https://doi.org/10.1016/0362-546X(85)90062-8

[34] Souganidis, P. E. and Yip, N. K. (2004). Uniqueness of motion by mean curvature perturbed by stochastic noise. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **21** 1–23. MR2037245 https://doi.org/10.1016/S0294-1449(03)00029-5

[35] Turanova, O. (2015). Error estimates for approximations of nonhomogeneous nonlinear uniformly elliptic equations. *Calc. Var. Partial Differential Equations* **54** 2939–2983. MR3412399 https://doi.org/10.1007/s00526-015-0890-6

The Annals of Applied Probability
2020, Vol. 30, No. 4, 1824–1846
https://doi.org/10.1214/19-AAP1544

A THRESHOLD FOR CUTOFF IN TWO-COMMUNITY RANDOM GRAPHS

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In this paper, we are interested in the impact of communities on the mixing behavior of the nonbacktracking random walk. We consider sequences of sparse random graphs of size N generated according to a variant of the classical configuration model which incorporates a two-community structure. The strength of the bottleneck is measured by a parameter α which roughly corresponds to the fraction of edges that go from one community to the other. We show that if $\alpha \gg \frac{1}{\log N}$, then the nonbacktracking random walk exhibits cutoff at the same time as in the one-community case, but with a larger cutoff window, and that the distance profile inside this window converges to the Gaussian tail function. On the other hand, if $\alpha \ll \frac{1}{\log N}$ or $\alpha \asymp \frac{1}{\log N}$, then the mixing time is of order $1/\alpha$ and there is no cutoff.

REFERENCES

[1] ABBE, E. (2017). Community detection and stochastic block models: Recent developments. *J. Mach. Learn. Res.* **18** Paper No. 177, 86. MR3827065

[2] ALDOUS, D. (1983). Random walks on finite groups and rapidly mixing Markov chains. In *Seminar on Probability, XVII. Lecture Notes in Math.* **986** 243–297. Springer, Berlin. MR0770418 https://doi.org/10.1007/BFb0068322

[3] AVENA, L., GÜLDAŞ, H., HOFSTAD, R. V. D. and DEN HOLLANDER, F. (2019). Random walks on dynamic configuration models: A trichotomy. *Stochastic Process. Appl.* **129** 3360–3375. MR3985565 https://doi.org/10.1016/j.spa.2018.09.010

[4] AVENA, L., GÜLDAŞ, H., VAN DER HOFSTAD, R. and DEN HOLLANDER, F. (2018). Mixing times of random walks on dynamic configuration models. *Ann. Appl. Probab.* **28** 1977–2002. MR3843821 https://doi.org/10.1214/17-AAP1289

[5] BASU, R., HERMON, J. and PERES, Y. (2017). Characterization of cutoff for reversible Markov chains. *Ann. Probab.* **45** 1448–1487. MR3650406 https://doi.org/10.1214/16-AOP1090

[6] BEN-HAMOU, A. and SALEZ, J. (2017). Cutoff for nonbacktracking random walks on sparse random graphs. *Ann. Probab.* **45** 1752–1770. MR3650414 https://doi.org/10.1214/16-AOP1100

[7] BENJAMINI, I., KOZMA, G. and WORMALD, N. (2014). The mixing time of the giant component of a random graph. *Random Structures Algorithms* **45** 383–407. MR3252922 https://doi.org/10.1002/rsa.20539

[8] BERESTYCKI, N., LUBETZKY, E., PERES, Y. and SLY, A. (2018). Random walks on the random graph. *Ann. Probab.* **46** 456–490. MR3758735 https://doi.org/10.1214/17-AOP1189

[9] BORCEA, J., BRÄNDÉN, P. and LIGGETT, T. M. (2009). Negative dependence and the geometry of polynomials. *J. Amer. Math. Soc.* **22** 521–567. MR2476782 https://doi.org/10.1090/S0894-0347-08-00618-8

[10] BORDENAVE, C., CAPUTO, P. and SALEZ, J. (2018). Random walk on sparse random digraphs. *Probab. Theory Related Fields* **170** 933–960. MR3773804 https://doi.org/10.1007/s00440-017-0796-7

[11] CHATTERJEE, S. (2007). Stein's method for concentration inequalities. *Probab. Theory Related Fields* **138** 305–321. MR2288072 https://doi.org/10.1007/s00440-006-0029-y

[12] DIACONIS, P. and SHAHSHAHANI, M. (1981). Generating a random permutation with random transpositions. *Z. Wahrsch. Verw. Gebiete* **57** 159–179. MR0626813 https://doi.org/10.1007/BF00535487

[13] GIRVAN, M. and NEWMAN, M. E. J. (2002). Community structure in social and biological networks. *Proc. Natl. Acad. Sci. USA* **99** 7821–7826. MR1908073 https://doi.org/10.1073/pnas.122653799

[14] GRIFFITHS, S., KANG, R. J., OLIVEIRA, R. I. and PATEL, V. (2014). Tight inequalities among set hitting times in Markov chains. *Proc. Amer. Math. Soc.* **142** 3285–3298. MR3223383 https://doi.org/10.1090/S0002-9939-2014-12045-4

MSC2020 subject classifications. Primary 60J10; secondary 05C80, 05C81.
Key words and phrases. Nonbacktracking random walk, random graphs, mixing times, cutoff, bottleneck.

[15] HERMON, J. (2018). A technical report on hitting times, mixing and cutoff. *ALEA Lat. Am. J. Probab. Math. Stat.* **15** 101–120. MR3765366 https://doi.org/10.30757/alea.v15-05

[16] HOLLAND, P. W., LASKEY, K. B. and LEINHARDT, S. (1983). Stochastic blockmodels: First steps. *Soc. Netw.* **5** 109–137. MR0718088 https://doi.org/10.1016/0378-8733(83)90021-7

[17] LABBÉ, C. and LACOIN, H. (2019). Cutoff phenomenon for the asymmetric simple exclusion process and the biased card shuffling. *Ann. Probab.* **47** 1541–1586. MR3945753 https://doi.org/10.1214/18-AOP1290

[18] LEVIN, D. A. and PERES, Y. (2017). *Markov Chains and Mixing Times*. Amer. Math. Soc., Providence, RI. Second edition of [MR2466937], With contributions by Elizabeth L. Wilmer, With a chapter on "Coupling from the past" by James G. Propp and David B. Wilson. MR3726904

[19] LEZAUD, P. (1998). Quantitative analysis of Markov chains using perturbation of their kernel. Theses, Univ. Toulouse 3 Paul Sabatier. https://hal-enac.archives-ouvertes.fr/tel-01084797.

[20] LUBETZKY, E. and SLY, A. (2010). Cutoff phenomena for random walks on random regular graphs. *Duke Math. J.* **153** 475–510. MR2667423 https://doi.org/10.1215/00127094-2010-029

[21] LUBETZKY, E. and SLY, A. (2011). Explicit expanders with cutoff phenomena. *Electron. J. Probab.* **16** 419–435. MR2774096 https://doi.org/10.1214/EJP.v16-869

[22] LUBETZKY, E. and SLY, A. (2017). Universality of cutoff for the Ising model. *Ann. Probab.* **45** 3664–3696. MR3729612 https://doi.org/10.1214/16-AOP1146

[23] MANN, B. W. (1996). *Berry–Esseen Central Limit Theorems for Markov Chains*. ProQuest LLC, Ann Arbor, MI. Thesis (Ph.D.), Harvard Univ. MR2694356

[24] OLIVEIRA, R. I. (2012). Mixing and hitting times for finite Markov chains. *Electron. J. Probab.* **17** no. 70, 12. MR2968677 https://doi.org/10.1214/EJP.v17-2274

[25] PEMANTLE, R. and PERES, Y. (2014). Concentration of Lipschitz functionals of determinantal and other strong Rayleigh measures. *Combin. Probab. Comput.* **23** 140–160. MR3197973 https://doi.org/10.1017/S0963548313000345

[26] PERES, Y. and SOUSI, P. (2015). Mixing times are hitting times of large sets. *J. Theoret. Probab.* **28** 488–519. MR3370663 https://doi.org/10.1007/s10959-013-0497-9

[27] POIRÉE, T. (2018). Temps de mélange sur des graphes à communautés. Master's thesis, supervised by A. Ben-Hamou and A. Guyader, Sorbonne Univ.

[28] ROSS, N. (2011). Fundamentals of Stein's method. *Probab. Surv.* **8** 210–293. MR2861132 https://doi.org/10.1214/11-PS182

[29] SAH, P., SINGH, L. O., CLAUSET, A. and BANSAL, S. (2014). Exploring community structure in biological networks with random graphs. *BMC Bioinform.* **15** 220. https://doi.org/10.1186/1471-2105-15-220

[30] STEGEHUIS, C., VAN DER HOFSTAD, R. and VAN LEEUWAARDEN, J. S. H. (2016). Epidemic spreading on complex networks with community structures. *Sci. Rep.* **6** 29748. https://doi.org/10.1038/srep29748

[31] VAN DER HOFSTAD, R., VAN LEEUWAARDEN, J. S. and STEGEHUIS, C. (2015). Hierarchical configuration model. arXiv preprint arXiv:1512.08397.

The Annals of Applied Probability
2020, Vol. 30, No. 4, 1847–1883
https://doi.org/10.1214/19-AAP1545

MIXING TIME AND CUTOFF FOR THE WEAKLY ASYMMETRIC SIMPLE EXCLUSION PROCESS

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We consider the simple exclusion process with k particles on a segment of length N performing random walks with transition $p > 1/2$ to the right and $q = 1 - p$ to the left. We focus on the case where the asymmetry in the jump rates $b = p - q > 0$ vanishes in the limit when N and k tend to infinity, and obtain sharp asymptotics for the mixing times of this sequence of Markov chains in the two cases where the asymmetry is either much larger or much smaller than $(\log k)/N$. We show that in the former case ($b \gg (\log k)/N$), the mixing time corresponds to the time needed to reach macroscopic equilibrium, like for the strongly asymmetric (i.e., constant b) case studied in (*Ann. Probab.* **47** (2019) 1541–1586), while the latter case ($b \ll (\log k)/N$) macroscopic equilibrium is not sufficient for mixing and one must wait till local fluctuations equilibrate, similarly to what happens in the symmetric case worked out in (*Ann. Probab.* **44** (2016) 1426–1487). In both cases, convergence to equilibrium is abrupt: we have a cutoff phenomenon for the total-variation distance. We present a conjecture for the remaining regime when the asymmetry is of order $(\log k)/N$.

REFERENCES

[1] ALDOUS, D. and DIACONIS, P. (1986). Shuffling cards and stopping times. *Amer. Math. Monthly* **93** 333–348. MR0841111 https://doi.org/10.2307/2323590

[2] BENJAMINI, I., BERGER, N., HOFFMAN, C. and MOSSEL, E. (2005). Mixing times of the biased card shuffling and the asymmetric exclusion process. *Trans. Amer. Math. Soc.* **357** 3013–3029. MR2135733 https://doi.org/10.1090/S0002-9947-05-03610-X

[3] BERTINI, L. and GIACOMIN, G. (1997). Stochastic Burgers and KPZ equations from particle systems. *Comm. Math. Phys.* **183** 571–607. MR1462228 https://doi.org/10.1007/s002200050044

[4] DE MASI, A., PRESUTTI, E. and SCACCIATELLI, E. (1989). The weakly asymmetric simple exclusion process. *Ann. Inst. Henri Poincaré Probab. Stat.* **25** 1–38. MR0995290

[5] DIACONIS, P. and SHAHSHAHANI, M. (1981). Generating a random permutation with random transpositions. *Z. Wahrsch. Verw. Gebiete* **57** 159–179. MR0626813 https://doi.org/10.1007/BF00535487

[6] DITTRICH, P. and GÄRTNER, J. (1991). A central limit theorem for the weakly asymmetric simple exclusion process. *Math. Nachr.* **151** 75–93. MR1121199 https://doi.org/10.1002/mana.19911510107

[7] GÄRTNER, J. (1988). Convergence towards Burgers' equation and propagation of chaos for weakly asymmetric exclusion processes. *Stochastic Process. Appl.* **27** 233–260. MR0931030 https://doi.org/10.1016/0304-4149(87)90040-8

[8] KIPNIS, C. and LANDIM, C. (1999). *Scaling Limits of Interacting Particle Systems. Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **320**. Springer, Berlin. MR1707314 https://doi.org/10.1007/978-3-662-03752-2

[9] KIPNIS, C., OLLA, S. and VARADHAN, S. R. S. (1989). Hydrodynamics and large deviation for simple exclusion processes. *Comm. Pure Appl. Math.* **42** 115–137. MR0978701 https://doi.org/10.1002/cpa.3160420202

[10] LABBÉ, C. (2017). Weakly asymmetric bridges and the KPZ equation. *Comm. Math. Phys.* **353** 1261–1298. MR3652491 https://doi.org/10.1007/s00220-017-2875-0

[11] LABBÉ, C. and LACOIN, H. (2019). Cutoff phenomenon for the asymmetric simple exclusion process and the biased card shuffling. *Ann. Probab.* **47** 1541–1586. MR3945753 https://doi.org/10.1214/18-AOP1290

MSC2020 subject classifications. Primary 60J27; secondary 37A25, 82C22.
Key words and phrases. Exclusion process, WASEP, mixing time, cutoff.

[12] LACOIN, H. (2016). The cutoff profile for the simple exclusion process on the circle. *Ann. Probab.* **44** 3399–3430. MR3551201 https://doi.org/10.1214/15-AOP1053

[13] LACOIN, H. (2016). Mixing time and cutoff for the adjacent transposition shuffle and the simple exclusion. *Ann. Probab.* **44** 1426–1487. MR3474475 https://doi.org/10.1214/15-AOP1004

[14] LEVIN, D. A. and PERES, Y. (2016). Mixing of the exclusion process with small bias. *J. Stat. Phys.* **165** 1036–1050. MR3575636 https://doi.org/10.1007/s10955-016-1664-z

[15] LEVIN, D. A. and PERES, Y. (2017). *Markov Chains and Mixing Times*. Amer. Math. Soc., Providence, RI. Second edition of [MR2466937], With contributions by Elizabeth L. Wilmer, With a chapter on "Coupling from the past" by James G. Propp and David B. Wilson. MR3726904

[16] LIGGETT, T. M. (2005). *Interacting Particle Systems*. *Classics in Mathematics*. Springer, Berlin. Reprint of the 1985 original. MR2108619 https://doi.org/10.1007/b138374

[17] MORRIS, B. (2006). The mixing time for simple exclusion. *Ann. Appl. Probab.* **16** 615–635. MR2244427 https://doi.org/10.1214/105051605000000728

[18] REZAKHANLOU, F. (1991). Hydrodynamic limit for attractive particle systems on $\mathbf{Z}^d$. *Comm. Math. Phys.* **140** 417–448. MR1130693

[19] WILSON, D. B. (2004). Mixing times of Lozenge tiling and card shuffling Markov chains. *Ann. Appl. Probab.* **14** 274–325. MR2023023 https://doi.org/10.1214/aoap/1075828054

The Annals of Applied Probability
2020, Vol. 30, No. 4, 1884–1909
https://doi.org/10.1214/19-AAP1546

PARTICLES SYSTEMS AND NUMERICAL SCHEMES FOR MEAN REFLECTED STOCHASTIC DIFFERENTIAL EQUATIONS

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This paper is devoted to the study of reflected Stochastic Differential Equations when the constraint is not on the paths of the solution but acts on its law. These reflected equations have been introduced recently in a backward form by Briand, Elie and Hu (*Ann. Appl. Probab.* **28** (2018) 482–510) in the context of risk measures. We here focus on the forward version of such reflected equations. Our main objective is to provide an approximation of the solutions with the help of interacting particles systems. This approximation allows to design a numerical scheme for this kind of equations.

REFERENCES

[1] ARTZNER, P., DELBAEN, F., EBER, J.-M. and HEATH, D. (1999). Coherent measures of risk. *Math. Finance* **9** 203–228. MR1850791 https://doi.org/10.1111/1467-9965.00068

[2] BAUDOIN, F. (2002). Conditioned stochastic differential equations: Theory, examples and application to finance. *Stochastic Process. Appl.* **100** 109–145. MR1919610 https://doi.org/10.1016/S0304-4149(02)00109-6

[3] BRIAND, P., ELIE, R. and HU, Y. (2018). BSDEs with mean reflection. *Ann. Appl. Probab.* **28** 482–510. MR3770882 https://doi.org/10.1214/17-AAP1310

[4] CATTIAUX, P. and LÉONARD, C. (1995). Large deviations and Nelson processes. *Forum Math.* **7** 95–115. MR1307957 https://doi.org/10.1515/form.1995.7.95

[5] FÖLLMER, H. and SCHIED, A. (2002). Convex measures of risk and trading constraints. *Finance Stoch.* **6** 429–447. MR1932379 https://doi.org/10.1007/s007800200072

[6] FOURNIER, N. and GUILLIN, A. (2015). On the rate of convergence in Wasserstein distance of the empirical measure. *Probab. Theory Related Fields* **162** 707–738. MR3383341 https://doi.org/10.1007/s00440-014-0583-7

[7] JABIR, J.-F. (2017). Diffusion processes with weak constraint through penalization approximation. Available at arXiv:1704.01505v2.

[8] LIONS, P.-L. and SZNITMAN, A.-S. (1984). Stochastic differential equations with reflecting boundary conditions. *Comm. Pure Appl. Math.* **37** 511–537. MR0745330 https://doi.org/10.1002/cpa.3160370408

[9] MAURY, B., ROUDNEFF-CHUPIN, A. and SANTAMBROGIO, F. (2010). A macroscopic crowd motion model of gradient flow type. *Math. Models Methods Appl. Sci.* **20** 1787–1821. MR2735914 https://doi.org/10.1142/S0218202510004799

[10] MIKAMI, T. and THIEULLEN, M. (2008). Optimal transportation problem by stochastic optimal control. *SIAM J. Control Optim.* **47** 1127–1139. MR2407010 https://doi.org/10.1137/050631264

[11] RACHEV, S. T. and RÜSCHENDORF, L. (1998). *Mass Transportation Problems, Vol. II: Applications. Probability and Its Applications (New York)*. Springer, New York. MR1619171

[12] SZNITMAN, A.-S. (1991). Topics in propagation of chaos. In *École D'Été de Probabilités de Saint-Flour XIX—1989. Lecture Notes in Math.* **1464** 165–251. Springer, Berlin. MR1108185 https://doi.org/10.1007/BFb0085169

[13] TAN, X. and TOUZI, N. (2013). Optimal transportation under controlled stochastic dynamics. *Ann. Probab.* **41** 3201–3240. MR3127880 https://doi.org/10.1214/12-AOP797

[14] TANAKA, H. (1979). Stochastic differential equations with reflecting boundary condition in convex regions. *Hiroshima Math. J.* **9** 163–177. MR0529332

MSC2020 subject classifications. 60H10.
Key words and phrases. McKean–Vlasov SDE, mean-reflected, particle system.

The Annals of Applied Probability
2020, Vol. 30, No. 4, 1910–1933
https://doi.org/10.1214/19-AAP1547

STATISTICAL THRESHOLDS FOR TENSOR PCA

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We study the statistical limits of testing and estimation for a rank one deformation of a Gaussian random tensor. We compute the sharp thresholds for hypothesis testing and estimation by maximum likelihood and show that they are the same. Furthermore, we find that the maximum likelihood estimator achieves the maximal correlation with the planted vector among measurable estimators above the estimation threshold. In this setting, the maximum likelihood estimator exhibits a discontinuous BBP-type transition: below the critical threshold the estimator is orthogonal to the planted vector, but above the critical threshold, it achieves positive correlation which is uniformly bounded away from zero.

REFERENCES

[1] ADLER, R. J. and TAYLOR, J. E. (2007). *Random Fields and Geometry*. *Springer Monographs in Mathematics*. Springer, New York. MR2319516

[2] ANANDKUMAR, A., GE, R., HSU, D., KAKADE, S. M. and TELGARSKY, M. (2014). Tensor decompositions for learning latent variable models. *J. Mach. Learn. Res.* **15** 2773–2832. MR3270750

[3] BAIK, J., BEN AROUS, G. and PÉCHÉ, S. (2005). Phase transition of the largest eigenvalue for nonnull complex sample covariance matrices. *Ann. Probab.* **33** 1643–1697. MR2165575 https://doi.org/10.1214/009117905000000233

[4] BAIK, J. and SILVERSTEIN, J. W. (2006). Eigenvalues of large sample covariance matrices of spiked population models. *J. Multivariate Anal.* **97** 1382–1408. MR2279680 https://doi.org/10.1016/j.jmva.2005.08.003

[5] BANKS, J., MOORE, C., VERSHYNIN, R., VERZELEN, N. and XU, J. (2018). Information-theoretic bounds and phase transitions in clustering, sparse PCA, and submatrix localization. *IEEE Trans. Inform. Theory* **64** 4872–4994. MR3819345 https://doi.org/10.1109/tit.2018.2810020

[6] BARBIER, J., DIA, M., MACRIS, N., KRZAKALA, F., LESIEUR, T. and ZDEBOROVÁ, L. (2016). Mutual information for symmetric rank-one matrix estimation: A proof of the replica formula. *Adv. Neural Inf. Process. Syst.* 424–432.

[7] BARBIER, J. and MACRIS, N. (2017). The stochastic interpolation method: A simple scheme to prove replica formulas in bayesian inference. ArXiv preprint. Available at arXiv:1705.02780.

[8] BEN AROUS, G., GHEISSARI, R. and JAGANNATH, A. (2020). Algorithmic thresholds for tensor PCA. *Ann. Probab.* **48** 2052–2087. MR4124533 https://doi.org/10.1214/19-AOP1415

[9] BEN AROUS, G. and JAGANNATH, A. (2018). Spectral gap estimates in mean field spin glasses. *Comm. Math. Phys.* **361** 1–52. MR3825934 https://doi.org/10.1007/s00220-018-3152-6

[10] BEN AROUS, G., MEI, S., MONTANARI, A. and NICA, M. (2019). The landscape of the spiked tensor model. *Comm. Pure Appl. Math.* **72** 2282–2330. MR4011861 https://doi.org/10.1002/cpa.21861

[11] BENAYCH-GEORGES, F., GUIONNET, A. and MAIDA, M. (2012). Large deviations of the extreme eigenvalues of random deformations of matrices. *Probab. Theory Related Fields* **154** 703–751. MR3000560 https://doi.org/10.1007/s00440-011-0382-3

[12] BENAYCH-GEORGES, F. and NADAKUDITI, R. R. (2011). The eigenvalues and eigenvectors of finite, low rank perturbations of large random matrices. *Adv. Math.* **227** 494–521. MR2782201 https://doi.org/10.1016/j.aim.2011.02.007

MSC2020 subject classifications. 62F05, 62F10, 82D30, 82B26.
Key words and phrases. Tensor principal component analysis, principal component analysis, BBP transition, spherical spin glasses, spiked tensor, spiked matrix.

[13] BLOEMENDAL, A., KNOWLES, A., YAU, H.-T. and YIN, J. (2016). On the principal components of sample covariance matrices. *Probab. Theory Related Fields* **164** 459–552. MR3449395 https://doi.org/10.1007/s00440-015-0616-x

[14] BLOEMENDAL, A. and VIRÁG, B. (2013). Limits of spiked random matrices I. *Probab. Theory Related Fields* **156** 795–825. MR3078286 https://doi.org/10.1007/s00440-012-0443-2

[15] BOUCHERON, S., LUGOSI, G. and MASSART, P. (2013). *Concentration Inequalities*: *A Nonasymptotic Theory of Independence*. Oxford Univ. Press, Oxford. MR3185193 https://doi.org/10.1093/acprof:oso/9780199535255.001.0001

[16] CAPITAINE, M., DONATI-MARTIN, C. and FÉRAL, D. (2009). The largest eigenvalues of finite rank deformation of large Wigner matrices: Convergence and nonuniversality of the fluctuations. *Ann. Probab.* **37** 1–47. MR2489158 https://doi.org/10.1214/08-AOP394

[17] CHEN, W.-K. (2013). The Aizenman–Sims–Starr scheme and Parisi formula for mixed p-spin spherical models. *Electron. J. Probab.* **18** 94. MR3126577 https://doi.org/10.1214/EJP.v18-2580

[18] CHEN, W.-K. (2019). Phase transition in the spiked random tensor with Rademacher prior. *Ann. Statist.* **47** 2734–2756. MR3988771 https://doi.org/10.1214/18-AOS1763

[19] CHEN, W.-K., HANDSCHY, M. and LERMAN, G. (2018). Phase transition in random tensors with multiple spikes. ArXiv preprint. Available at arXiv:1809.06790.

[20] CHEN, W.-K. and SEN, A. (2017). Parisi formula, disorder chaos and fluctuation for the ground state energy in the spherical mixed p-spin models. *Comm. Math. Phys.* **350** 129–173. MR3606472 https://doi.org/10.1007/s00220-016-2808-3

[21] CRISANTI, A. and SOMMERS, H.-J. (1992). The sphericalp-spin interaction spin glass model: The statics. *Z. Phys. B, Condens. Matter* **87** 341–354.

[22] DESHPANDE, Y., ABBE, E. and MONTANARI, A. (2017). Asymptotic mutual information for the balanced binary stochastic block model. *Inf. Inference* **6** 125–170. MR3671474 https://doi.org/10.1093/imaiai/iaw017

[23] DUCHENNE, O., BACH, F., KWEON, I.-S. and PONCE, J. (2011). A tensor-based algorithm for high-order graph matching. *IEEE Trans. Pattern Anal. Mach. Intell.* **33** 2383–2395.

[24] EDWARDS, S. F. and JONES, R. C. (1976). The eigenvalue spectrum of a large symmetric random matrix. *J. Phys. A*: *Math. Gen.* **9** 1595.

[25] EL ALAOUI, A., KRZAKALA, F. and JORDAN, M. I. (2017). Finite size corrections and likelihood ratio fluctuations in the spiked Wigner model. ArXiv preprint. Available at arXiv:1710.02903.

[26] FÉRAL, D. and PÉCHÉ, S. (2007). The largest eigenvalue of rank one deformation of large Wigner matrices. *Comm. Math. Phys.* **272** 185–228. MR2291807 https://doi.org/10.1007/s00220-007-0209-3

[27] GILLIN, P., NISHIMORI, H. and SHERRINGTON, D. (2001). Multispin Ising spin glasses with ferromagnetic interactions. *J. Phys. A* **34** 2949–2964. MR1832363 https://doi.org/10.1088/0305-4470/34/14/303

[28] GILLIN, P. and SHERRINGTON, D. (2000). $p > 2$ spin glasses with first-order ferromagnetic transitions. *J. Phys. A* **33** 3081–3091. MR1766894 https://doi.org/10.1088/0305-4470/33/16/302

[29] GUO, D., SHAMAI, S. and VERDÚ, S. (2005). Mutual information and minimum mean-square error in Gaussian channels. *IEEE Trans. Inform. Theory* **51** 1261–1282. MR2241490 https://doi.org/10.1109/TIT.2005.844072

[30] HILLAR, C. J. and LIM, L.-H. (2013). Most tensor problems are NP-hard. *J. ACM* **60** 45. MR3144915 https://doi.org/10.1145/2512329

[31] HOPKINS, S. B., SHI, J., SCHRAMM, T. and STEURER, D. (2016). Fast spectral algorithms from sum-of-squares proofs: Tensor decomposition and planted sparse vectors. In *STOC'*16—*Proceedings of the* 48*th Annual ACM SIGACT Symposium on Theory of Computing* 178–191. ACM, New York. MR3536564

[32] HOPKINS, S. B., SHI, J. and STEURER, D. (2015). Tensor principal component analysis via sum-of-square proofs. In *Conference on Learning Theory* 956–1006.

[33] JAGANNATH, A. and TOBASCO, I. (2017). Low temperature asymptotics of spherical mean field spin glasses. *Comm. Math. Phys.* **352** 979–1017. MR3631397 https://doi.org/10.1007/s00220-017-2864-3

[34] JOHNSTONE, I. M. (2001). On the distribution of the largest eigenvalue in principal components analysis. *Ann. Statist.* **29** 295–327. MR1863961 https://doi.org/10.1214/aos/1009210544

[35] KIM, C., BANDEIRA, A. S. and GOEMANS, M. X. (2017). Community detection in hypergraphs, spiked tensor models, and sum-of-squares. In *Sampling Theory and Applications* (*SampTA*), 2017 *International Conference on* 124–128. IEEE.

[36] KIM, J. and POLLARD, D. (1990). Cube root asymptotics. *Ann. Statist.* **18** 191–219. MR1041391 https://doi.org/10.1214/aos/1176347498

[37] KO, J. (2018). Free energy of multiple systems of spherical spin glasses with constrained overlaps. ArXiv preprint. Available at arXiv:1806.09772.

[38] KORADA, S. B. and MACRIS, N. (2009). Exact solution of the gauge symmetric p-spin glass model on a complete graph. *J. Stat. Phys.* **136** 205–230. MR2525244 https://doi.org/10.1007/s10955-009-9781-6

[39] LELARGE, M. and MIOLANE, L. (2019). Fundamental limits of symmetric low-rank matrix estimation. *Probab. Theory Related Fields* **173** 859–929. MR3936148 https://doi.org/10.1007/s00440-018-0845-x

[40] LESIEUR, T., KRZAKALA, F. and ZDEBOROVÁ, L. (2017). Constrained low-rank matrix estimation: Phase transitions, approximate message passing and applications. *J. Stat. Mech. Theory Exp.* **2017** 073403. MR3683819 https://doi.org/10.1088/1742-5468/aa7284

[41] LESIEUR, T., MIOLANE, L., LELARGE, M., KRZAKALA, F. and ZDEBOROVÁ, L. (2017). Statistical and computational phase transitions in spiked tensor estimation. In *IEEE International Symposium on Information Theory* (*ISIT*) 511–515.

[42] LI, N. and LI, B. (2010). Tensor completion for on-board compression of hyperspectral images. In *Image Processing* (*ICIP*), 2010 17*th IEEE International Conference on* 517–520. IEEE.

[43] MAÏDA, M. (2007). Large deviations for the largest eigenvalue of rank one deformations of Gaussian ensembles. *Electron. J. Probab.* **12** 1131–1150. MR2336602 https://doi.org/10.1214/EJP.v12-438

[44] MÉZARD, M., PARISI, G. and VIRASORO, M. A. (1987). *Spin Glass Theory and Beyond. World Scientific Lecture Notes in Physics* **9**. World Scientific Co., Inc., Teaneck, NJ. MR1026102

[45] MILGROM, P. and SEGAL, I. (2002). Envelope theorems for arbitrary choice sets. *Econometrica* **70** 583–601. MR1913824 https://doi.org/10.1111/1468-0262.00296

[46] MIOLANE, L. (2018). Phase transitions in spiked matrix estimation: Information-theoretic analysis. ArXiv preprint. Available at arXiv:1806.04343.

[47] MONTANARI, A., REICHMAN, D. and ZEITOUNI, O. (2015). On the limitation of spectral methods: From the Gaussian hidden clique problem to rank-one perturbations of Gaussian tensors. *Adv. Neural Inf. Process. Syst.* 217–225.

[48] PANCHENKO, D. (2009). Cavity method in the spherical SK model. *Ann. Inst. Henri Poincaré Probab. Stat.* **45** 1020–1047. MR2572162 https://doi.org/10.1214/08-AIHP193

[49] PANCHENKO, D. (2013). *The Sherrington–Kirkpatrick Model*. Springer.

[50] PANCHENKO, D. and TALAGRAND, M. (2007). On the overlap in the multiple spherical SK models. *Ann. Probab.* **35** 2321–2355. MR2353390 https://doi.org/10.1214/009117907000000015

[51] PÉCHÉ, S. (2006). The largest eigenvalue of small rank perturbations of Hermitian random matrices. *Probab. Theory Related Fields* **134** 127–173. MR2221787 https://doi.org/10.1007/s00440-005-0466-z

[52] PERRY, A., WEIN, A. S. and BANDEIRA, A. S. (2016). Statistical limits of spiked tensor models. ArXiv preprint. Available at arXiv:1612.07728.

[53] PERRY, A., WEIN, A. S., BANDEIRA, A. S. and MOITRA, A. (2016). Optimality and sub-optimality of PCA for spiked random matrices and synchronization. ArXiv preprint. Available at arXiv:1609.05573.

[54] RICHARD, E. and MONTANARI, A. (2014). A statistical model for tensor PCA. *Adv. Neural Inf. Process. Syst.* 2897–2905.

[55] ROS, V., BEN AROUS, G., BIROLI, G. and CAMMAROTA, C. (2018). Complex energy landscapes in spiked-tensor and simple glassy models: Ruggedness, arrangements of local minima and phase transitions. ArXiv preprint. Available at arXiv:1804.02686.

[56] TALAGRAND, M. (2006). Free energy of the spherical mean field model. *Probab. Theory Related Fields* **134** 339–382. MR2226885 https://doi.org/10.1007/s00440-005-0433-8

[57] TALAGRAND, M. (2011). *Mean Field Models for Spin Glasses. Volume I: Basic Examples. Ergebnisse der Mathematik und Ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics* [*Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics*] **54**. Springer, Berlin. MR2731561 https://doi.org/10.1007/978-3-642-15202-3

The Annals of Applied Probability
2020, Vol. 30, No. 4, 1934–1970
https://doi.org/10.1214/19-AAP1548

EXACT FORMULAS FOR TWO INTERACTING PARTICLES AND APPLICATIONS IN PARTICLE SYSTEMS WITH DUALITY

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We consider two particles performing continuous-time nearest neighbor random walk on $\mathbb{Z}$ and interacting with each other when they are at neighboring positions. The interaction is either repulsive (partial exclusion process) or attractive (inclusion process). We provide an *exact formula* for the Laplace–Fourier transform of the transition probabilities of the two-particle dynamics. From this we derive a general *scaling limit* result, which shows that the possible scaling limits are coalescing Brownian motions, reflected Brownian motions and sticky Brownian motions.

In particle systems with duality, the solution of the dynamics of two dual particles provides relevant information. We apply the exact formula to the the symmetric inclusion process, that is self-dual, in the *condensation regime*. We thus obtain two results. First, by computing the time-dependent covariance of the particle occupation number at two lattice sites we characterise the time-dependent coarsening in infinite volume when the process is started from a homogeneous product measure. Second, we identify the limiting variance of the density field in the diffusive scaling limit, relating it to the local time of sticky Brownian motion.

REFERENCES

[1] AMIR, M. (1991). Sticky Brownian motion as the strong limit of a sequence of random walks. *Stochastic Process*. *Appl*. **39** 221–237. MR1136247 https://doi.org/10.1016/0304-4149(91)90080-V

[2] ARMENDÁRIZ, I., GROSSKINSKY, S. and LOULAKIS, M. (2013). Zero-range condensation at criticality. *Stochastic Process*. *Appl*. **123** 3466–3496. MR3071386 https://doi.org/10.1016/j.spa.2013.04.021

[3] ARMENDÁRIZ, I. and LOULAKIS, M. (2009). Thermodynamic limit for the invariant measures in supercritical zero range processes. *Probab. Theory Related Fields* **145** 175–188. MR2520125 https://doi.org/10.1007/s00440-008-0165-7

[4] BELTRÁN, J. and LANDIM, C. (2012). Metastability of reversible condensed zero range processes on a finite set. *Probab. Theory Related Fields* **152** 781–807. MR2892962 https://doi.org/10.1007/s00440-010-0337-0

[5] BIANCHI, A., DOMMERS, S. and GIARDINÀ, C. (2017). Metastability in the reversible inclusion process. *Electron*. *J*. *Probab*. **22** Paper No. 70, 34. MR3698739 https://doi.org/10.1214/17-EJP98

[6] BOBROWSKI, A. (2016). *Convergence of One-Parameter Operator Semigroups*: *In Models of Mathematical Biology and Elsewhere*. *New Mathematical Monographs* **30**. Cambridge Univ. Press, Cambridge. MR3526064 https://doi.org/10.1017/CBO9781316480663

[7] BORODIN, A., CORWIN, I. and SASAMOTO, T. (2014). From duality to determinants for q-TASEP and ASEP. *Ann*. *Probab*. **42** 2314–2382. MR3265169 https://doi.org/10.1214/13-AOP868

[8] BORODIN, A. and PETROV, L. (2014). Integrable probability: From representation theory to Macdonald processes. *Probab*. *Surv*. **11** 1–58. MR3178541 https://doi.org/10.1214/13-PS225

[9] CARINCI, G., GIARDINÀ, C., REDIG, F. and SASAMOTO, T. (2016). A generalized asymmetric exclusion process with $\mathscr{U}_q(\mathfrak{sl}_2)$ stochastic duality. *Probab. Theory Related Fields* **166** 887–933. MR3568042 https://doi.org/10.1007/s00440-015-0674-0

[10] CARINCI, G., GIARDINÀ, C., REDIG, F. and SASAMOTO, T. (2016). Asymmetric stochastic transport models with $\mathscr{U}_q(\mathfrak{su}(1,1))$ symmetry. *J*. *Stat*. *Phys*. **163** 239–279. MR3478310 https://doi.org/10.1007/s10955-016-1473-4

MSC2020 subject classifications. 82C22, 60K35.
Key words and phrases. Interacting particle systems, duality, inclusion process, condensation.

[11] CARINCI, G., JARA, M. and REDIG, F. (2019). KPZ scaling for the weakly asymmetric inclusion process. Preprint.

[12] DE MASI, A. and PRESUTTI, E. (1991). *Mathematical Methods for Hydrodynamic Limits*. *Lecture Notes in Math*. **1501**. Springer, Berlin. MR1175626 https://doi.org/10.1007/BFb0086457

[13] GIARDINÀ, C., KURCHAN, J., REDIG, F. and VAFAYI, K. (2009). Duality and hidden symmetries in interacting particle systems. *J. Stat. Phys.* **135** 25–55. MR2505724 https://doi.org/10.1007/s10955-009-9716-2

[14] GIARDINÀ, C., REDIG, F. and VAFAYI, K. (2010). Correlation inequalities for interacting particle systems with duality. *J. Stat. Phys.* **141** 242–263. MR2726642 https://doi.org/10.1007/s10955-010-0055-0

[15] GODRÈCHE, C. and LUCK, J. M. (2005). Dynamics of the condensate in zero-range processes. *J. Phys. A* **38** 7215–7237. MR2165701 https://doi.org/10.1088/0305-4470/38/33/002

[16] GONÇALVES, P., JARA, M. and SIMON, M. (2017). Second order Boltzmann–Gibbs principle for polynomial functions and applications. *J. Stat. Phys.* **166** 90–113. MR3592852 https://doi.org/10.1007/s10955-016-1686-6

[17] GRIMMETT, G. R. and STIRZAKER, D. R. (2001). *Probability and Random Processes*, 3rd ed. Oxford Univ. Press, New York. MR2059709

[18] GROSSKINSKY, S., REDIG, F. and VAFAYI, K. (2011). Condensation in the inclusion process and related models. *J. Stat. Phys.* **142** 952–974. MR2781716 https://doi.org/10.1007/s10955-011-0151-9

[19] GROSSKINSKY, S., REDIG, F. and VAFAYI, K. (2013). Dynamics of condensation in the symmetric inclusion process. *Electron. J. Probab.* **18** no. 66, 23. MR3078025 https://doi.org/10.1214/EJP.v18-2720

[20] GROSSKINSKY, S., SCHÜTZ, G. M. and SPOHN, H. (2003). Condensation in the zero range process: Stationary and dynamical properties. *J. Stat. Phys.* **113** 389–410. MR2013129 https://doi.org/10.1023/A:1026008532442

[21] HARRISON, J. M. and LEMOINE, A. J. (1981). Sticky Brownian motion as the limit of storage processes. *J. Appl. Probab.* **18** 216–226. MR0598937 https://doi.org/10.2307/3213181

[22] HOWITT, C. J. (2007). Stochastic flows and sticky Brownian motion. PhD thesis, Univ. Warwick institutional repository. Available at http://go.warwick.ac.uk/wrap.

[23] KARATZAS, I. and SHREVE, S. E. (2012). *Brownian Motion and Stochastic Calculus*, 2nd ed. *Graduate Texts in Mathematics* **113**. Springer, New York. MR1121940 https://doi.org/10.1007/978-1-4612-0949-2

[24] KIPNIS, C. and LANDIM, C. (1999). *Scaling Limits of Interacting Particle Systems*. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **320**. Springer, Berlin. MR1707314 https://doi.org/10.1007/978-3-662-03752-2

[25] KIPNIS, C., MARCHIORO, C. and PRESUTTI, E. (1982). Heat flow in an exactly solvable model. *J. Stat. Phys.* **27** 65–74. MR0656869 https://doi.org/10.1007/BF01011740

[26] KOSTRYKIN, V., POTTHOFF, J. and SCHRADER, R. (2010). Brownian motions on metric graphs: Feller Brownian motions on intervals revisited. Preprint. Available at arXiv:1008.3761v2.

[27] LIGGETT, T. M. (1985). *Interacting Particle Systems*. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **276**. Springer, New York. MR0776231 https://doi.org/10.1007/978-1-4613-8542-4

[28] POPKOV, V. and SCHÜTZ, G. M. (2002). Integrable Markov processes and quantum spin chains. *Mat. Fiz. Anal. Geom.* **9** 401–411. MR1949795

[29] RÁCZ, M. Z. and SHKOLNIKOV, M. (2015). Multidimensional sticky Brownian motions as limits of exclusion processes. *Ann. Appl. Probab.* **25** 1155–1188. MR3325271 https://doi.org/10.1214/14-AAP1019

[30] SCHÜTZ, G. and SANDOW, S. (1994). Non-Abelian symmetries of stochastic processes: Derivation of correlation functions for random-vertex models and disordered-interacting-particle systems. *Phys. Rev. E* (3) **49** 2726.

[31] SCHÜTZ, G. M. (1997). Exact solution of the master equation for the asymmetric exclusion process. *J. Stat. Phys.* **88** 427–445. MR1468391 https://doi.org/10.1007/BF02508478

[32] SCHÜTZ, G. M. (2001). Exactly solvable models for many-body systems far from equilibrium. In *Phase Transitions and Critical Phenomena*, *Vol*. 19 1–251. Academic Press, San Diego, CA. MR2033539 https://doi.org/10.1016/S1062-7901(01)80015-X

[33] SPOHN, H. (2012). Stochastic integrability and the KPZ equation, published in IAMP News Bulletin. Preprint. Available at arXiv:1204.2657.

[34] TRACY, C. A. and WIDOM, H. (2008). Integral formulas for the asymmetric simple exclusion process. *Comm. Math. Phys.* **279** 815–844. MR2386729 https://doi.org/10.1007/s00220-008-0443-3

The Annals of Applied Probability
2020, Vol. 30, No. 4, 1971–2003
https://doi.org/10.1214/19-AAP1549

EDGEWORTH EXPANSION FOR EULER APPROXIMATION OF CONTINUOUS DIFFUSION PROCESSES

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In this paper we present the Edgeworth expansion for the Euler approximation scheme of a continuous diffusion process driven by a Brownian motion. Our methodology is based upon a recent work (*Stochastic Process. Appl.* **123** (2013) 887–933), which establishes Edgeworth expansions associated with asymptotic mixed normality using elements of Malliavin calculus. Potential applications of our theoretical results include higher order expansions for weak and strong approximation errors associated to the Euler scheme, and for studentized version of the error process.

REFERENCES

[1] ALDOUS, D. J. and EAGLESON, G. K. (1978). On mixing and stability of limit theorems. *Ann. Probab.* **6** 325–331. MR0517416 https://doi.org/10.1214/aop/1176995577

[2] ARNOLD, S. (2006). Approximation schemes for SDEs with discontinuous coefficients. Ph.D. thesis, ETH Zürich.

[3] BALLY, V. and TALAY, D. (1996). The law of the Euler scheme for stochastic differential equations. I. Convergence rate of the distribution function. *Probab. Theory Related Fields* **104** 43–60. MR1367666 https://doi.org/10.1007/BF01303802

[4] BALLY, V. and TALAY, D. (1996). The law of the Euler scheme for stochastic differential equations. II. Convergence rate of the density. *Monte Carlo Methods Appl.* **2** 93–128. MR1401964 https://doi.org/10.1515/mcma.1996.2.2.93

[5] BARNDORFF-NIELSEN, O. E., GRAVERSEN, S. E., JACOD, J., PODOLSKIJ, M. and SHEPHARD, N. (2006). A central limit theorem for realised power and bipower variations of continuous semimartingales. In *From Stochastic Calculus to Mathematical Finance* 33–68. Springer, Berlin. MR2233534 https://doi.org/10.1007/978-3-540-30788-4_3

[6] CHAN, K. S. and STRAMER, O. (1998). Weak consistency of the Euler method for numerically solving stochastic differential equations with discontinuous coefficients. *Stochastic Process. Appl.* **76** 33–44. MR1638015 https://doi.org/10.1016/S0304-4149(98)00020-9

[7] DALALYAN, A. and YOSHIDA, N. (2011). Second-order asymptotic expansion for a non-synchronous covariation estimator. *Ann. Inst. Henri Poincaré Probab. Stat.* **47** 748–789. MR2841074 https://doi.org/10.1214/10-AIHP383

[8] HALIDIAS, N. and KLOEDEN, P. E. (2006). A note on strong solutions of stochastic differential equations with a discontinuous drift coefficient. *J. Appl. Math. Stoch. Anal.* Art. ID 73257. MR2221001 https://doi.org/10.1155/JAMSA/2006/73257

[9] IKEDA, N. and WATANABE, S. (1989). *Stochastic Differential Equations and Diffusion Processes*, 2nd ed. *North-Holland Mathematical Library* **24**. North-Holland, Amsterdam; Kodansha, Ltd., Tokyo. MR0637061

[10] JACOD, J. and PROTTER, P. (1998). Asymptotic error distributions for the Euler method for stochastic differential equations. *Ann. Probab.* **26** 267–307. MR1617049 https://doi.org/10.1214/aop/1022855419

[11] JACOD, J. and SHIRYAEV, A. N. (2003). *Limit Theorems for Stochastic Processes*, 2nd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **288**. Springer, Berlin. MR1943877 https://doi.org/10.1007/978-3-662-05265-5

[12] KLOEDEN, P. E. and PLATEN, E. (1992). *Numerical Solution of Stochastic Differential Equations*. *Applications of Mathematics* (*New York*) **23**. Springer, Berlin. MR1214374 https://doi.org/10.1007/978-3-662-12616-5

MSC2020 subject classifications. 60F05, 60H10, 65C30.
Key words and phrases. Edgeworth expansion, diffusion processes, limit theorems, Euler scheme.

[13] KURTZ, T. G. and PROTTER, P. (1991). Wong–Zakai corrections, random evolutions, and simulation schemes for SDEs. In *Stochastic Analysis* 331–346. Academic Press, Boston, MA. MR1119837

[14] MIKULEVIČIUS, R. and PLATEN, E. (1991). Rate of convergence of the Euler approximation for diffusion processes. *Math. Nachr.* **151** 233–239. MR1121206 https://doi.org/10.1002/mana.19911510114

[15] NUALART, D. (2006). *The Malliavin Calculus and Related Topics*, 2nd ed. *Probability and Its Applications* (*New York*). Springer, Berlin. MR2200233

[16] PODOLSKIJ, M., VELIYEV, B. and YOSHIDA, N. (2017). Edgeworth expansion for the pre-averaging estimator. *Stochastic Process. Appl.* **127** 3558–3595. MR3707238 https://doi.org/10.1016/j.spa.2017.03.001

[17] PODOLSKIJ, M. and YOSHIDA, N. (2016). Edgeworth expansion for functionals of continuous diffusion processes. *Ann. Appl. Probab.* **26** 3415–3455. MR3582807 https://doi.org/10.1214/16-AAP1179

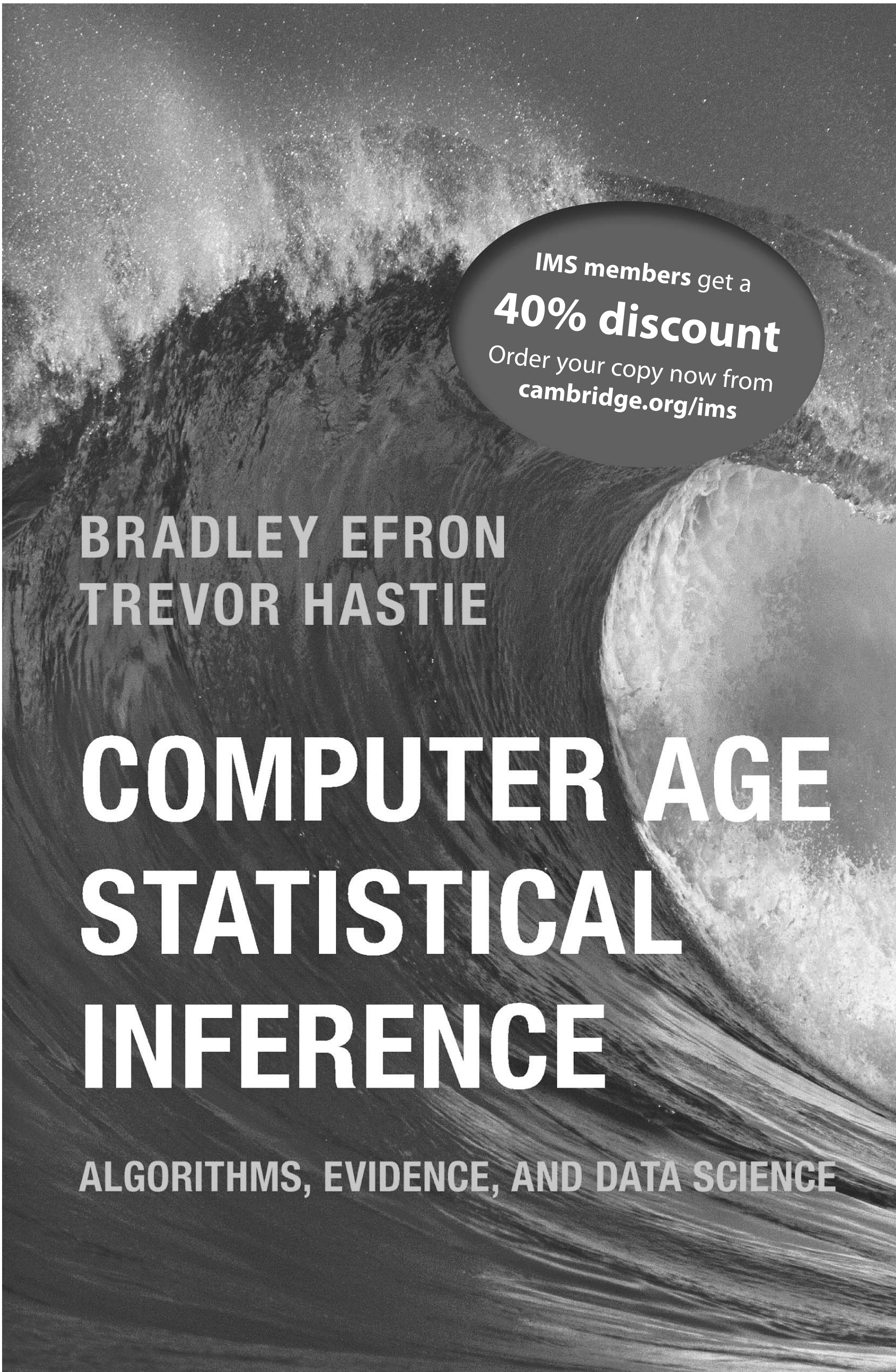
[18] RÉNYI, A. (1963). On stable sequences of events. *Sankhyā Ser. A* **25** 293–302. MR0170385

[19] YAN, L. (2002). The Euler scheme with irregular coefficients. *Ann. Probab.* **30** 1172–1194. MR1920104 https://doi.org/10.1214/aop/1029867124

[20] YOSHIDA, N. (1997). Malliavin calculus and asymptotic expansion for martingales. *Probab. Theory Related Fields* **109** 301–342. MR1481124 https://doi.org/10.1007/s004400050134

[21] YOSHIDA, N. (2012). Asymptotic expansion for the quadratic form of the diffusion process. Working paper. Available at arXiv:1212.5845.

[22] YOSHIDA, N. (2013). Martingale expansion in mixed normal limit. *Stochastic Process. Appl.* **123** 887–933. MR3005009 https://doi.org/10.1016/j.spa.2012.10.007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