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PARAMETER AND DIMENSION DEPENDENCE OF CONVERGENCE RATES TO STATIONARITY FOR REFLECTING BROWNIAN MOTIONS

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We obtain rates of convergence to stationarity in L^1 -Wasserstein distance for a d -dimensional reflected Brownian motion (RBM) in the nonnegative orthant that are explicit in the dimension and the system parameters. The results are then applied to a class of RBMs considered in (Blanchet and Xinyun (2016)) and to rank-based diffusions including the Atlas model. In both cases, we obtain explicit rates and bounds on relaxation times. In the first case we improve the relaxation time estimates of $O(d^4(\log d)^2)$ obtained in (Blanchet and Xinyun (2016)) to $O((\log d)^2)$. In the latter case, we give the first results on explicit parameter and dimension dependent rates under the Wasserstein distance. The proofs do not require an explicit form for the stationary measure or reversibility of the process with respect to this measure, and cover settings where these properties are not available. In the special case of the standard Atlas model (In *Stochastic Portfolio Theory* (2002) 1–24 Springer), we obtain a bound on the relaxation time of $O(d^6(\log d)^2)$.

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A LIMIT THEOREM FOR THE SURVIVAL PROBABILITY OF A SIMPLE RANDOM WALK AMONG POWER-LAW RENEWAL OBSTACLES

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We consider a one-dimensional simple random walk surviving among a field of static soft obstacles: each time it meets an obstacle the walk is killed with probability $1 - e^{-\beta}$, where β is a positive and fixed parameter. The positions of the obstacles are sampled independently from the walk and according to a renewal process. The increments between consecutive obstacles, or gaps, are assumed to have a power-law decaying tail with exponent $\gamma > 0$. We prove convergence in law for the properly rescaled logarithm of the quenched survival probability as time goes to infinity. The normalization exponent is $\gamma/(\gamma + 2)$, while the limiting law writes as a variational formula with both universal and nonuniversal features. The latter involves (i) a Poisson point process that emerges as the universal scaling limit of the properly rescaled gaps and (ii) a function of the parameter β that we call *asymptotic cost of crossing per obstacle* and that may, in principle, depend on the details of the gap distribution. Our proof suggests a confinement strategy of the walk in a single large gap. This model may also be seen as a $(1 + 1)$ -directed polymer among many repulsive interfaces, in which case β corresponds to the strength of repulsion, the survival probability to the partition function and its logarithm to the finite-volume free energy.

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GEOMETRIC ERGODICITY OF THE BOUNCY PARTICLE SAMPLER

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The Bouncy Particle Sampler (BPS) is a Monte Carlo Markov chain algorithm to sample from a target density known up to a multiplicative constant. This method is based on a kinetic piecewise deterministic Markov process for which the target measure is invariant. This paper deals with theoretical properties of BPS. First, we establish geometric ergodicity of the associated semi-group under weaker conditions than in (*Ann. Statist.* **47** (2019) 1268–1287) both on the target distribution and the velocity probability distribution. This result is based on a new coupling of the process which gives a quantitative minorization condition and yields more insights on the convergence. In addition, we study on a toy model the dependency of the convergence rates on the dimension of the state space. Finally, we apply our results to the analysis of simulated annealing algorithms based on BPS.

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IMAGINARY MULTIPLICATIVE CHAOS: MOMENTS, REGULARITY AND CONNECTIONS TO THE ISING MODEL

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In this article we study imaginary Gaussian multiplicative chaos—namely a family of random generalized functions which can formally be written as $e^{iX(x)}$, where X is a log-correlated real-valued Gaussian field on $\mathbf{R}^d$, that is, it has a logarithmic singularity on the diagonal of its covariance. We study basic analytic properties of these random generalized functions, such as what spaces of distributions these objects live in, along with their basic stochastic properties, such as moment and tail estimates.

After this, we discuss connections between imaginary multiplicative chaos and the critical planar Ising model, namely that the scaling limit of the spin field of the critical planar XOR-Ising model can be expressed in terms of the cosine of the Gaussian free field, that is, the real part of an imaginary multiplicative chaos distribution. Moreover, if one adds a magnetic perturbation to the XOR-Ising model, then the scaling limit of the spin field can be expressed in terms of the cosine of the sine-Gordon field, which can also be viewed as the real part of an imaginary multiplicative chaos distribution.

The first sections of the article have been written in the style of a review, and we hope that the text will also serve as an introduction to imaginary chaos for an uninitiated reader.

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STOCHASTIC EQUATION AND EXPONENTIAL ERGODICITY IN WASSERSTEIN DISTANCES FOR AFFINE PROCESSES

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This work is devoted to the study of conservative affine processes on the canonical state space $D = \mathbb{R}_+^m \times \mathbb{R}^n$, where $m + n > 0$. We show that each affine process can be obtained as the pathwise unique strong solution to a stochastic equation driven by Brownian motions and Poisson random measures. Then we study the long-time behavior of affine processes, that is, we show that under first moment condition on the state-dependent and log-moment conditions on the state-independent jump measures, respectively, each subcritical affine process is exponentially ergodic in a suitably chosen Wasserstein distance. Moments of affine processes are studied as well.

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SQUARE PERMUTATIONS ARE TYPICALLY RECTANGULAR

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We describe the limit (for two topologies) of large uniform random square permutations, that is, permutations where every point is a record. The starting point for all our results is a sampling procedure for asymptotically uniform square permutations. Building on that, we first describe the global behavior by showing that these permutations have a permuton limit which can be described by a random rectangle. We also explore fluctuations about this random rectangle, which we can describe through coupled Brownian motions. Second, we consider the limiting behavior of the neighborhood of a point in the permutation through local limits. As a byproduct, we also determine the random limits of the proportion of occurrences (and consecutive occurrences) of any given pattern in a uniform random square permutation.

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HAMILTON–JACOBI EQUATIONS FOR FINITE-RANK MATRIX INFERENCE

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We compute the large-scale limit of the free energy associated with the problem of inference of a finite-rank matrix. The method follows the principle put forward in Mourrat (2018) which consists in identifying a suitable Hamilton–Jacobi equation satisfied by the limit free energy. We simplify the approach of Mourrat (2018) using a notion of weak solution of the Hamilton–Jacobi equation which is more convenient to work with and is applicable whenever the nonlinearity in the equation is convex.

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THE TWO-TYPE RICHARDSON MODEL IN THE HALF-PLANE

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The two-type Richardson model describes the growth of two competing infection types on the two or higher dimensional integer lattice. For types that spread with the same intensity, it is known that there is a positive probability for infinite coexistence, while for types with different intensities, it is conjectured that infinite coexistence is not possible. In this paper we study the two-type Richardson model in the upper half-plane $\mathbb{Z} \times \mathbb{Z}_+$, and prove that coexistence of two types starting on the horizontal axis has positive probability if and only if the types have the same intensity.

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PATHWISE STOCHASTIC CONTROL WITH APPLICATIONS TO ROBUST FILTERING

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We study the problem of pathwise stochastic optimal control, where the optimization is performed for each fixed realisation of the driving noise, by phrasing the problem in terms of the optimal control of rough differential equations. We investigate the degeneracy phenomenon induced by directly controlling the coefficient of the noise term, and propose a simple procedure to resolve this degeneracy whilst retaining dynamic programming. As an application, we use pathwise stochastic control in the context of stochastic filtering to construct filters which are robust to parameter uncertainty, demonstrating an original application of rough path theory to statistics.

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PROPAGATION OF CHAOS AND THE MANY-DEMES LIMIT FOR WEAKLY INTERACTING DIFFUSIONS IN THE SPARSE REGIME

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Propagation of chaos is a well-studied phenomenon and shows that weakly interacting diffusions may become independent as the system size converges to infinity. Most of the literature focuses on the case of exchangeable systems where all involved diffusions have the same distribution and are “of the same size”. In this paper, we analyze the case where only a few diffusions start outside of an accessible trap. Our main result shows that in this “sparse regime” the system of weakly interacting diffusions converges in distribution to a forest of excursions from the trap. In particular, initial independence propagates in the limit and results in a forest of independent trees.

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PATHWISE MCKEAN–VLASOV THEORY WITH ADDITIVE NOISE

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We take a pathwise approach to classical McKean–Vlasov stochastic differential equations with additive noise, as for example, exposed in Sznitmann (In *École D'Été de Probabilités de Saint-Flour XIX—1989* (1991) 165–251, Springer). Our study was prompted by some concrete problems in battery modelling (*Contin. Mech. Thermodyn.* **30** (2018) 593–628), and also by recent progress on rough-pathwise McKean–Vlasov theory, notably Cass–Lyons (*Proc. Lond. Math. Soc.* (3) **110** (2015) 83–107), and then Bailleul, Catellier and Delarue (Bailleul, Catellier and Delarue (2018)). Such a “pathwise McKean–Vlasov theory” can be traced back to Tanaka (In *Stochastic Analysis (Katata/Kyoto, 1982)* (1984) 469–488, North-Holland). This paper can be seen as an attempt to advertize the ideas, power and simplicity of the pathwise approach, not so easily extracted from (Bailleul, Catellier and Delarue (2018); *Proc. Lond. Math. Soc.* (3) **110** (2015) 83–107; In *Stochastic Analysis (Katata/Kyoto, 1982)* (1984) 469–488, North-Holland), together with a number of novel applications. These include mean field convergence without a priori independence and exchangeability assumption; common noise, càdlàg noise, and reflecting boundaries. Last not least, we generalize Dawson–Gärtner large deviations and the central limit theorem to a non-Brownian noise setting.

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STOCHASTIC APPROXIMATION ON NONCOMPACT MEASURE SPACES AND APPLICATION TO MEASURE-VALUED PÓLYA PROCESSES

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Our main result is to prove almost-sure convergence of a stochastic approximation algorithm defined on the space of measures on a noncompact space. Our motivation is to apply this result to measure-valued Pólya processes (MVPPs, also known as infinitely-many Pólya urns). Our main idea is to use Foster–Lyapunov type criteria in a novel way to generalize stochastic approximation methods to measure-valued Markov processes with a noncompact underlying space, overcoming in a fairly general context one of the major difficulties of existing studies on this subject.

From the MVPPs point of view, our result implies almost-sure convergence of a large class of MVPPs; this convergence was only obtained until now for specific examples, with only convergence in probability established for general classes. Furthermore, our approach allows us to extend the definition of MVPPs by adding “weights” to the different colors of the infinitely-many-color urn. We also exhibit a link between non-“balanced” MVPPs and quasi-stationary distributions of Markovian processes, which allows us to treat, for the first time in the literature, the nonbalanced case.

Finally, we show how our result can be applied to designing stochastic approximation algorithms for the approximation of quasi-stationary distributions of discrete- and continuous-time Markov processes on noncompact spaces.

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RARITY OF EXTREMAL EDGES IN RANDOM SURFACES AND OTHER THEORETICAL APPLICATIONS OF CLUSTER ALGORITHMS

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Motivated by questions on the delocalization of random surfaces, we prove that random surfaces satisfying a Lipschitz constraint rarely develop extremal gradients. Previous proofs of this fact relied on reflection positivity and were thus limited to random surfaces defined on highly symmetric graphs, whereas our argument applies to general graphs. Our proof makes use of a cluster algorithm and reflection transformation for random surfaces of the type introduced by Swendsen–Wang, Wolff and Evertz et al. We discuss the general framework for such cluster algorithms, reviewing several particular cases with emphasis on their use in obtaining theoretical results. Two additional applications are presented: A reflection principle for random surfaces and a proof that pair correlations in the spin $O(n)$ model have monotone densities, strengthening Griffiths’ first inequality for such correlations.

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FUNCTIONAL LARGE DEVIATIONS FOR COX PROCESSES AND $Cox/G/\infty$ QUEUES, WITH A BIOLOGICAL APPLICATION

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We consider an infinite-server queue into which customers arrive according to a Cox process and have independent service times with a general distribution. We prove a functional large deviations principle for the equilibrium queue length process. The model is motivated by a linear feed-forward gene regulatory network, in which the rate of protein synthesis is modulated by the number of RNA molecules present in a cell. The system can be modelled as a nonstandard tandem of infinite-server queues, in which the number of customers present in a queue modulates the arrival rate into the next queue in the tandem. We establish large deviation principles for this queueing system in the asymptotic regime in which the arrival process is sped up, while the service process is not scaled.

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ASYMPTOTIC THEORY OF SPARSE BRADLEY–TERRY MODEL

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The Bradley–Terry model is a fundamental model in the analysis of network data involving paired comparison. Assuming every pair of subjects in the network have an equal number of comparisons, Simons and Yao (*Ann. Statist.* **27** (1999) 1041–1060) established an asymptotic theory for statistical estimation in the Bradley–Terry model. In practice, when the size of the network becomes large, the paired comparisons are generally sparse. The sparsity can be characterized by the probability p_n that a pair of subjects have at least one comparison, which tends to zero as the size of the network n goes to infinity. In this paper, the asymptotic properties of the maximum likelihood estimate of the Bradley–Terry model are shown under minimal conditions of the sparsity. Specifically, the uniform consistency is proved when p_n is as small as the order of $(\log n)^3/n$, which is near the theoretical lower bound $\log n/n$ by the theory of the Erdős–Rényi graph. Asymptotic normality and inference are also provided. Evidence in support of the theory is presented in simulation results, along with an application to the analysis of the ATP data.

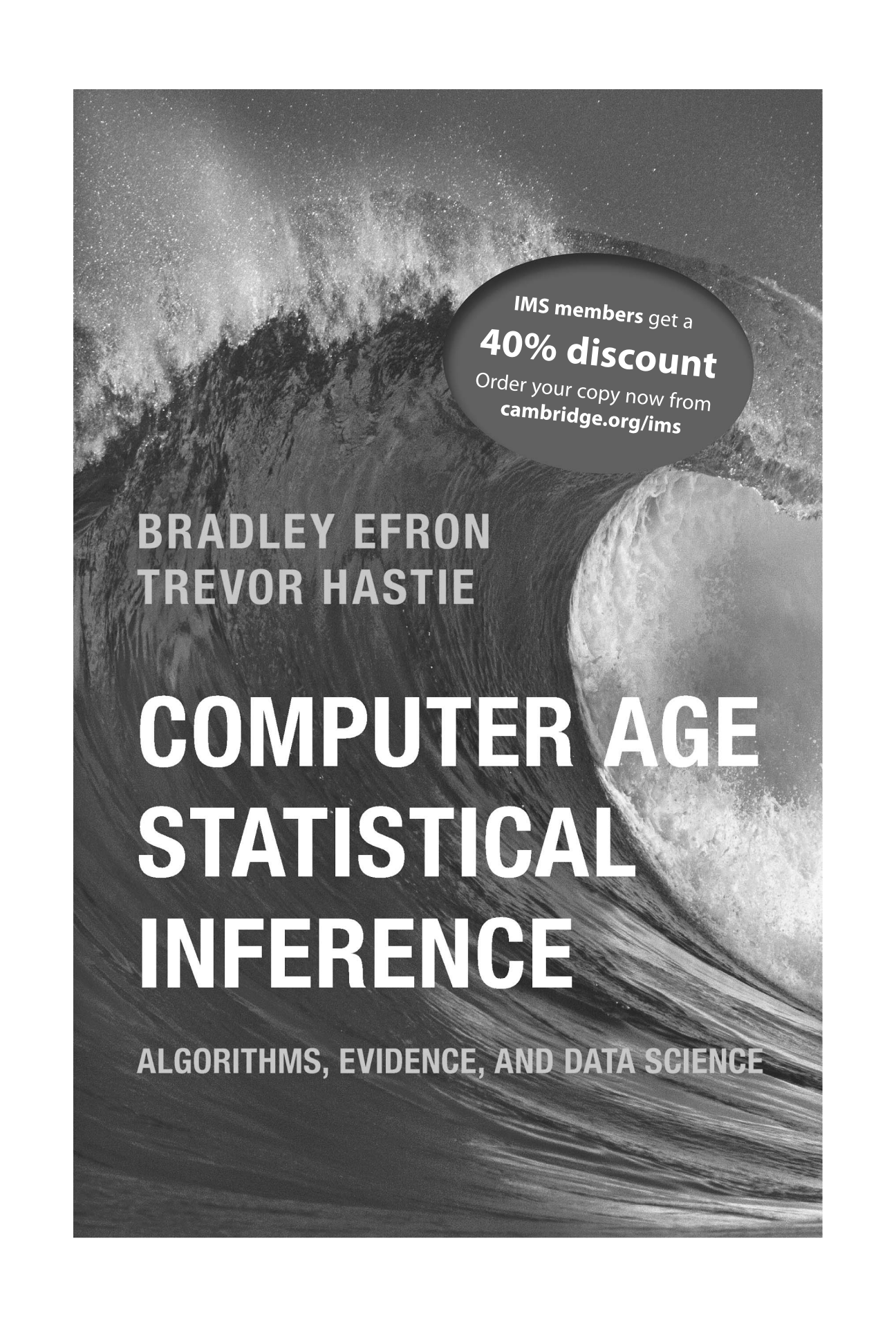
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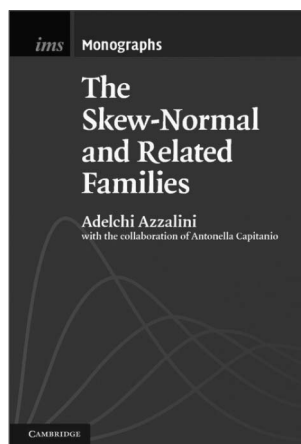
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