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ON THE ASYMPTOTIC OPTIMALITY OF THE COMB STRATEGY FOR PREDICTION WITH EXPERT ADVICE

BY ERHAN BAYRAKTAR[1,*], IBRAHIM EKREN[2] AND YILI ZHANG[1,†]

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For the problem of prediction with expert advice in the adversarial setting with geometric stopping, we compute the exact leading order expansion for the long time behavior of the value function. Then, we use this expansion to prove that as conjectured in Gravin, Peres and Sivan (In *Proceedings of the Twenty-Seventh Annual ACM-SIAM Symposium on Discrete Algorithms* (2016) 528–547, ACM), the comb strategies are indeed asymptotically optimal for the adversary in the case of 4 experts.

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MARKOV SELECTION FOR THE STOCHASTIC COMPRESSIBLE NAVIER–STOKES SYSTEM

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We analyze the Markov property of solutions to the compressible Navier–Stokes system perturbed by a general multiplicative stochastic forcing. We show the existence of an almost sure Markov selection to the associated martingale problem. Our proof is based on the abstract framework introduced in Flandoli and Romito (*Probab. Theory Related Fields* **40** (2008) 407–458). A major difficulty arises from the fact, different from the incompressible case, that the velocity field is not continuous in time. In addition, it cannot be recovered from the variables whose time evolution is described by the Navier–Stokes system, namely, the density and the momentum. We overcome this issue by introducing an auxiliary variable into the Markov selection procedure.

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STOCHASTIC METHODS FOR THE NEUTRON TRANSPORT EQUATION I: LINEAR SEMIGROUP ASYMPTOTICS

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The neutron transport equation (NTE) describes the flux of neutrons through an inhomogeneous fissile medium. In this paper, we reconnect the NTE to the physical model of the spatial Markov branching process which describes the process of nuclear fission, transport, scattering, and absorption. By reformulating the NTE in its mild form and identifying its solution as an expectation semigroup, we use modern techniques to develop a Perron–Fröbenius (PF) type decomposition, showing that growth is dominated by a leading eigenfunction and its associated left and right eigenfunctions. In the spirit of results for spatial branching and fragmentation processes, we use our PF decomposition to show the existence of an intrinsic martingale and associated spine decomposition. Moreover, we show how criticality in the PF decomposition dictates the convergence of the intrinsic martingale. The mathematical difficulties in this context come about through unusual piecewise linear motion of particles coupled with an infinite type-space which is taken as neutron velocity. The fundamental nature of our PF decomposition also plays out in accompanying work (Harris, Horton and Kyprianou (2020), Cox et al. (2020)).

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A SECOND ORDER ANALYSIS OF MCKEAN–VLASOV SEMIGROUPS

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We propose a second order differential calculus to analyze the regularity and the stability properties of the distribution semigroup associated with McKean–Vlasov diffusions. This methodology provides second order Taylor type expansions with remainder for both the evolution semigroup as well as the stochastic flow associated with this class of nonlinear diffusions. Bismut–Elworthy–Li formulae for the gradient and the Hessian of the integro-differential operators associated with these expansions are also presented.

The article also provides explicit Dyson–Phillips expansions and a refined analysis of the norm of these integro-differential operators. Under some natural and easily verifiable regularity conditions we derive a series of exponential decays inequalities with respect to the time horizon. We illustrate the impact of these results with a second order extension of the Alekseev–Gröbner lemma to nonlinear measure valued semigroups and interacting diffusion flows. This second order perturbation analysis provides direct proofs of several uniform propagation of chaos properties w.r.t. the time parameter, including bias, fluctuation error estimate as well as exponential concentration inequalities.

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UTILITY MAXIMIZATION VIA DECOUPLING FIELDS

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We consider the utility maximization problem for a general class of utility functions defined on the real line. We rely on existing results which reduce the problem to a coupled forward–backward stochastic differential equation (FBSDE) and concentrate on showing existence and uniqueness of solution processes to this FBSDE. We use the method of decoupling fields for strongly coupled, multi-dimensional and possibly non-Lipschitz systems as the central technique in conducting the proofs.

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SAMPLE PATH LARGE DEVIATIONS FOR LÉVY PROCESSES AND RANDOM WALKS WITH WEIBULL INCREMENTS

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We study sample path large deviations for Lévy processes and random walks with heavy-tailed jump-size distributions that are of Weibull type. The sharpness and applicability of these results are illustrated by a counterexample proving the nonexistence of a full LDP in the J_1 topology, and by an application to a first passage problem.

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FROM TICK DATA TO SEMIMARTINGALES

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Tick-by-tick asset price data exhibit a number of empirical regularities, including discreteness, long periods where prices are flat, periods of price moves of alternating plus and minus one tick, periods of rapid successive price moves of the same sign, and others. This paper proposes a framework to examine whether and how these microscopic features of the tick data are compatible with the typical macroscopic continuous-time models, based on Itô semimartingales, that are employed to represent asset prices. We construct in particular tick-by-tick models that deliver by scaling macroscopic semimartingale models with stochastic volatility and jumps.

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APPLICATIONS OF MESOSCOPIC CLTS IN RANDOM MATRIX THEORY

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We present some applications of central limit theorems on mesoscopic scales for random matrices. When combined with the recent theory of "homogenization" for Dyson Brownian motion, this yields the universality of quantities which depend on the behavior of single eigenvalues of Wigner matrices and β-ensembles. Among the results we obtain are the Gaussian fluctuations of single eigenvalues for Wigner matrices (without an assumption of 4 matching moments) and classical β-ensembles ($\beta = 1, 2, 4$), Gaussian fluctuations of the eigenvalue counting function, and an asymptotic expansion up to order $o(N^{-1})$ for the expected value of eigenvalues in the bulk of the spectrum. The latter result solves a conjecture of Tao and Vu.

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HOW FRAGILE ARE INFORMATION CASCADES?

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It is well known that sequential decision making may lead to information cascades. That is, when agents make decisions based on their private information, as well as observing the actions of those before them, then it might be rational to ignore their private signal and imitate the action of previous individuals. If the individuals are choosing between a right and a wrong state, and the initial actions are wrong, then the whole cascade will be wrong. This issue is due to the fact that cascades can be based on very little information.

We show that if agents occasionally disregard the actions of others and base their action only on their private information, then wrong cascades can be avoided. Moreover, we study the optimal asymptotic rate at which the error probability at time t can go to zero. The optimal policy is for the player at time t to follow their private information with probability $p_t = c/t$, leading to a learning rate of c'/t, where the constants c and c' are explicit.

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STOCHASTIC METHODS FOR THE NEUTRON TRANSPORT EQUATION II: ALMOST SURE GROWTH

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The neutron transport equation (NTE) describes the flux of neutrons across a planar cross-section in an inhomogeneous fissile medium when the process of nuclear fission is active. Classical work on the NTE emerges from the applied mathematics literature in the 1950s through the work of R. Dautray and collaborators (*Méthodes Probabilistes Pour les équations de la Physique* (1989) Eyrolles; *Mathematical Analysis and Numerical Methods for Science and Technology. Vol. 6: Evolution Problems. II* (1993) Springer; *Mathematical Topics in Neutron Transport Theory: New Aspects* (1997) World Scientific). The NTE also has a probabilistic representation through the semigroup of the underlying *physical process* when envisaged as a stochastic process (cf. *Méthodes Probabilistes pour les équations de la Physique* (1989) Eyrolles; *Introduction to Monte-Carlo Methods for Transport and Diffusion Equations* (2003) Oxford Univ. Press; *IMA J. Numer. Anal.* **26** (2006) 657–685; *Publ. Res. Inst. Math. Sci.* **7** (1971/72) 153–179). More recently, Cox et al. (*J. Stat. Phys.* **176** (2019) 425–455) and Cox et al. (2019) have continued the probabilistic analysis of the NTE, introducing more recent ideas from the theory of spatial branching processes and quasi-stationary distributions. In this paper, we continue in the same vein and look at a fundamental description of stochastic growth in the supercritical regime. Our main result provides a significant improvement on the last known contribution to growth properties of the physical process in (*Publ. Res. Inst. Math. Sci.* **7** (1971/72) 153–179), bringing neutron transport theory in line with modern branching process theory such as (*Ann. Probab.* **44** (2016) 235–275; *Ann. Probab.* **43** (2015) 2545–2610). An important aspect of the proofs focuses on the use of a skeletal path decomposition, which we derive for general branching particle systems in the new context of nonlocal branching generators.

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BULK EIGENVALUE FLUCTUATIONS OF SPARSE RANDOM MATRICES

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We consider a class of sparse random matrices, which includes the adjacency matrix of Erdős–Rényi graphs $\mathcal{G}(N, p)$ for $p \in [N^{\varepsilon-1}, N^{-\varepsilon}]$. We identify the joint limiting distributions of the eigenvalues away from 0 and the spectral edges. Our result indicates that unlike Wigner matrices, the eigenvalues of sparse matrices satisfy central limit theorems with normalization $N\sqrt{p}$. In addition, the eigenvalues fluctuate simultaneously: the correlation of two eigenvalues of the same/different sign is asymptotically 1/-1. We also prove CLTs for the eigenvalue counting function and trace of the resolvent at mesoscopic scales.

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EXTINCTION TIME FOR THE WEAKER OF TWO COMPETING SIS EPIDEMICS

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We consider a simple Markov model for the spread of a disease caused by two virus strains in a closed homogeneously mixing population of size N. The spread of each strain in the absence of the other one is described by the stochastic SIS logistic epidemic process, and we assume that there is perfect cross-immunity between the two strains, that is, individuals infected by one are temporarily immune to re-infections and infections by the other. For the case where one strain is strictly stronger than the other, and the stronger strain on its own is supercritical, we derive precise asymptotic results for the distribution of the time when the weaker strain disappears from the population. We further extend our results to certain parameter values where the difference between the basic reproductive ratios of the two strains may tend to 0 as $N \to \infty$.

In our proofs, we illustrate a new approach to a fluid limit approximation for a sequence of Markov chains in the vicinity of a stable fixed point of the limit differential equation, valid over long time intervals.

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MODELLING INFORMATION FLOWS BY MEYER-σ-FIELDS IN THE SINGULAR STOCHASTIC CONTROL PROBLEM OF IRREVERSIBLE INVESTMENT

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In stochastic control problems delicate issues arise when the controlled system can jump due to both exogenous shocks and endogenous controls. Here one has to specify what the controller knows when about the exogenous shocks and how and when she can act on this information. We propose to use Meyer-σ-fields as a flexible tool to model information flow in such situations. The possibilities of this approach are illustrated first in a very simple linear stochastic control problem and then in a fairly general formulation for the singular stochastic control problem of irreversible investment with inventory risk. For the latter, we illustrate in a first case study how different signals on exogenous jumps lead to different optimal controls, interpolating between the predictable and the optional case in a systematic manner.

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SPLITTING ALGORITHMS FOR RARE EVENT SIMULATION OVER LONG TIME INTERVALS

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In this paper we study the performance of splitting algorithms, and in particular the RESTART method, for the numerical approximation of the probability that a process leaves a neighborhood of a metastable point during some long time interval $[0, T]$. We show that, in contrast to alternatives such as importance sampling, the decay rate of the second moment does not degrade as $T \to \infty$. In the course of the analysis we develop some related large deviation estimates that apply when the time interval of interest depends on the large deviation parameter.

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HOMOGENEOUS MAPPINGS OF REGULARLY VARYING VECTORS

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It is well known that the product of two independent regularly varying random variables with the same tail index is again regularly varying with this index. In this paper, we provide sharp sufficient conditions for the regular variation property of product-type functions of regularly varying random vectors, generalizing and extending the univariate theory in various directions. The main result is then applied to characterize the regular variation property of products of i.i.d. regularly varying quadratic random matrices and of solutions to affine stochastic recurrence equations under nonstandard conditions.

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Key words and phrases. Products of random matrices, multivariate regular variation, Breiman lemma, random difference equation.

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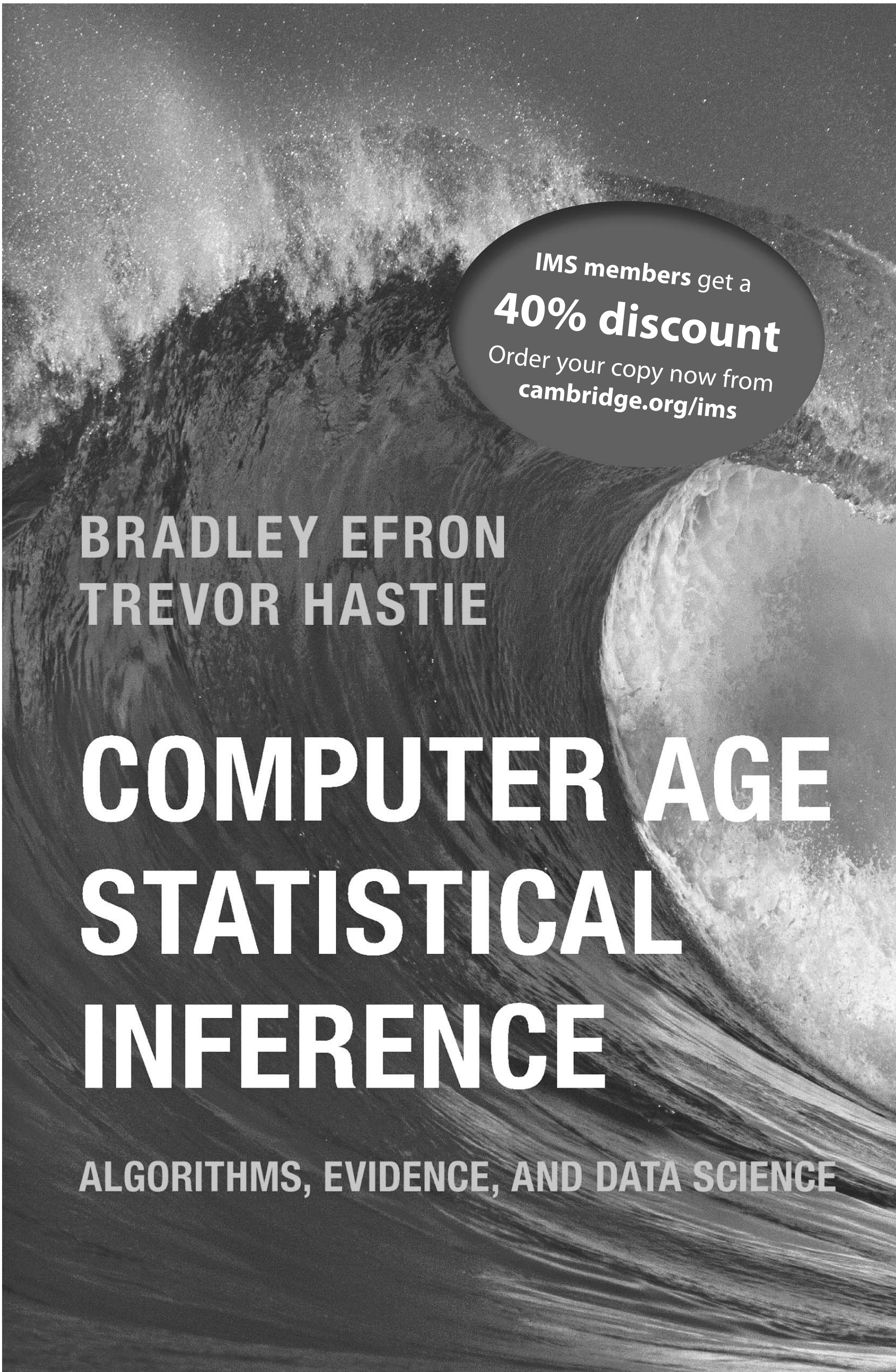
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