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OPTIMAL SUBGRAPH STRUCTURES IN SCALE-FREE CONFIGURATION MODELS

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Subgraphs reveal information about the geometry and functionalities of complex networks. For scale-free networks with unbounded degree fluctuations, we obtain the asymptotics of the number of times a small connected graph occurs as a subgraph or as an induced subgraph. We obtain these results by analyzing the configuration model with degree exponent $\tau \in (2,3)$ and introducing a novel class of optimization problems. For any given subgraph, the unique optimizer describes the degrees of the vertices that together span the subgraph. We find that subgraphs typically occur between vertices with specific degree ranges. In this way, we can count and characterize *all* subgraphs. We refrain from double counting in the case of multi-edges, essentially counting the subgraphs in the *erased* configuration model.

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POINT PROCESS CONVERGENCE FOR THE OFF-DIAGONAL ENTRIES OF SAMPLE COVARIANCE MATRICES

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We study point process convergence for sequences of i.i.d. random walks. The objective is to derive asymptotic theory for the extremes of these random walks. We show convergence of the maximum random walk to the Gumbel distribution under the existence of a $(2+\delta)$th moment. We make heavy use of precise large deviation results for sums of i.i.d. random variables. As a consequence, we derive the joint convergence of the off-diagonal entries in sample covariance and correlation matrices of a high-dimensional sample whose dimension increases with the sample size. This generalizes known results on the asymptotic Gumbel property of the largest entry.

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PERIODICITY INDUCED BY NOISE AND INTERACTION IN THE KINETIC MEAN-FIELD FITZHUGH–NAGUMO MODEL

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We consider the long-time behavior of a population of mean-field oscillators modeling the activity of interacting excitable neurons in a large population. Each neuron is represented by its voltage and recovery variables, which are the solution to a FitzHugh–Nagumo system, and interacts with the rest of the population through a mean-field linear coupling, in the presence of noise. The aim of the paper is to study the emergence of collective oscillatory behaviors induced by noise and interaction on such a system. The main difficulty of the present analysis is that we consider the kinetic case, where interaction and noise are only imposed on the voltage variable. We prove the existence of a stable cycle for the infinite population system, in a regime where the local dynamics is small.

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LONG EXIT TIMES NEAR A REPELLING EQUILIBRIUM

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For a smooth vector field in a neighborhood of a critical point with all positive eigenvalues of the linearization, we consider the associated dynamics perturbed by white noise. Using Malliavin calculus tools, we obtain polynomial asymptotics for probabilities of atypically long exit times in the vanishing noise limit.

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STEIN'S METHOD FOR THE POISSON–DIRICHLET DISTRIBUTION AND THE EWENS SAMPLING FORMULA, WITH APPLICATIONS TO WRIGHT–FISHER MODELS

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We provide a general theorem bounding the error in the approximation of a random measure of interest—for example, the empirical population measure of types in a Wright–Fisher model—and a Dirichlet process, which is a measure having Poisson–Dirichlet distributed atoms with i.i.d. labels from a diffuse distribution. The implicit metric of the approximation theorem captures the sizes and locations of the masses, and so also yields bounds on the approximation between the masses of the measure of interest and the Poisson–Dirichlet distribution. We apply the result to bound the error in the approximation of the stationary distribution of types in the finite Wright–Fisher model with infinite-alleles mutation structure (not necessarily parent independent) by the Poisson–Dirichlet distribution. An important consequence of our result is an explicit upper bound on the total variation distance between the random partition generated by sampling from a finite Wright–Fisher stationary distribution, and the Ewens sampling formula. The bound is small if the sample size n is much smaller than $N^{1/6}\log(N)^{-1/2}$, where N is the total population size. Our analysis requires a result of separate interest, giving an explicit bound on the second moment of the number of types of a finite Wright–Fisher stationary distribution. The general approximation result follows from a new development of Stein's method for the Dirichlet process, which follows by viewing the Dirichlet process as the stationary distribution of a Fleming–Viot process, and then applying Barbour's generator approach.

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CUTOFF FOR THE SQUARE PLAQUETTE MODEL ON A CRITICAL LENGTH SCALE

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Plaquette models are short range ferromagnetic spin models that play a key role in the dynamic facilitation approach to the liquid glass transition. In this paper we study the dynamics of the square plaquette model at the smallest of the three critical length scales discovered in (*J. Stat. Phys.* **169** (2017) 441–471). Our main result is that the plaquette model with *periodic* boundary conditions, on this length scale, exhibits a sharp transition in the convergence to equilibrium, known as cutoff. This substantially refines our coarse understanding of mixing from previous work (Chleboun and Smith (2018)). The basic approach is to reduce the problem to an analysis of the trace process on certain "metastable" states, which may be useful in proving cutoff in other situations.

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REGENERATION-ENRICHED MARKOV PROCESSES WITH APPLICATION TO MONTE CARLO

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We study a class of Markov processes that combine local dynamics, arising from a fixed Markov process, with regenerations arising at a state-dependent rate. We give conditions under which such processes possess a given target distribution as their invariant measures, thus making them amenable for use within Monte Carlo methodologies. Since the regeneration mechanism can compensate the choice of local dynamics, while retaining the same invariant distribution, great flexibility can be achieved in selecting local dynamics, and the mathematical analysis is simplified. We give straightforward conditions for the process to possess a central limit theorem, and additional conditions for uniform ergodicity and for a coupling from the past construction to hold, enabling exact sampling from the invariant distribution. We further consider and analyse a natural approximation of the process which may arise in the practical simulation of some classes of continuous-time dynamics.

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ON A ROUGH PERTURBATION OF THE NAVIER–STOKES SYSTEM AND ITS VORTICITY FORMULATION

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We introduce a rough perturbation of the Navier–Stokes system and justify its physical relevance from balance of momentum and conservation of circulation in the inviscid limit. We present a framework for a well-posedness analysis of the system. In particular, we define an intrinsic notion of strong solution based on ideas from the rough path theory and study the system in an equivalent vorticity formulation. In two space dimensions, we prove that well-posedness and enstrophy balance holds. Moreover, we derive rough path continuity of the equation, which yields a Wong–Zakai result for Brownian driving paths, and show that for a large class of driving signals, the system generates a continuous random dynamical system. In dimension three, the noise is not enstrophy balanced, and we establish the existence of local in time solutions.

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A CHARACTERIZATION OF MARTINGALE-EQUIVALENT MIXED COMPOUND POISSON PROCESSES

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If a given aggregate process S is a mixed compound Poisson process under a probability measure P, we provide a characterization of all probability measures Q on the domain of P, such that P and Q are progressively equivalent and S remains a mixed compound Poisson process with improved properties. This result generalizes earlier work of Delbaen and Haezendonck (*Insurance Math. Econom.* **8** (1989) 269–277). Implications related to the computation of premium calculation principles in an insurance market possessing the property of no free lunch with vanishing risk are also discussed.

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QUANTITATIVE SPECTRAL GAP ESTIMATE AND WASSERSTEIN CONTRACTION OF SIMPLE SLICE SAMPLING

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We prove Wasserstein contraction of simple slice sampling for approximate sampling w.r.t. distributions with log-concave and rotational invariant Lebesgue densities. This yields, in particular, an explicit quantitative lower bound of the spectral gap of simple slice sampling. Moreover, this lower bound carries over to more general target distributions depending only on the volume of the (super-)level sets of their unnormalized density.

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CHROMOSOME PAINTING: HOW RECOMBINATION MIXES ANCESTRAL COLORS

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We consider a Moran model with recombination in a haploid population of size N. At each birth event, with probability $1 - \rho_N R$ the offspring copies one parent's chromosome, and with probability $\rho_N R$ she inherits a chromosome that is a mosaic of both parental chromosomes. We assume that at time 0 each individual has her chromosome painted in a different color and we study the color partition of the chromosome that is asymptotically fixed in a large population, when we look at a portion of the chromosome such that $\rho := \lim_{N\to\infty} \frac{\rho_N N}{2} \to \infty$. To do so, we follow backwards in time the ancestry of the chromosome of a randomly sampled individual. This yields a Markov process valued in the color partitions of the half-line, that was introduced by Esser, Probst and Baake (*J. Math. Biol.* **73** (2016) 161–197), in which blocks can merge and split, called the partitioning process. Its stationary distribution is closely related to the fixed chromosome in our Moran model with recombination. We are able to provide an approximation of this stationary distribution when $\rho \gg 1$ and an error bound. This allows us to show that the distribution of the (renormalised) length of the leftmost block of the partition (i.e., the region of the chromosome that carries the same color as 0) converges to an exponential distribution. In addition, the geometry of this block can be described in terms of a Poisson point process with an explicit intensity measure.

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GAMBLER'S RUIN ESTIMATES ON FINITE INNER UNIFORM DOMAINS

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Gambler's ruin estimates can be viewed as harmonic measure estimates for finite Markov chains which are absorbed (or killed) at boundary points. We relate such estimates to properties of the underlying chain and its Doob transform. Precisely, we show that gambler's ruin estimates reduce to a good understanding of the Perron–Frobenius eigenfunction and eigenvalue whenever the underlying chain and its Doob transform are Harnack Markov chains. Finite inner-uniform domains (say, in the square grid $\mathbb{Z}^n$) provide a large class of examples where these ideas apply and lead to detailed estimates. In general, understanding the behavior of the Perron–Frobenius eigenfunction remains a challenge.

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PRECISE ASYMPTOTICS: ROBUST STOCHASTIC VOLATILITY MODELS

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We present a new methodology to analyze large classes of (classical and rough) stochastic volatility models, with special regard to short-time and small noise formulae for option prices. Our main tool is the theory of regularity structures, which we use in the form of Bayer et al. (*Math. Finance* **30** (2020) 782–832) In essence, we implement a Laplace method on the space of models (in the sense of Hairer), which generalizes classical works of Azencott and Ben Arous on path space and then Aida, Inahama–Kawabi on rough path space. When applied to rough volatility models, for example, in the setting of Bayer, Friz and Gatheral (*Quant. Finance* **16** (2016) 887–904) and Forde–Zhang (*SIAM J. Financial Math.* **8** (2017) 114–145), one obtains precise asymptotics for European options which refine known large deviation asymptotics.

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INDUCED IDLENESS LEADS TO DETERMINISTIC HEAVY TRAFFIC LIMITS FOR QUEUE-BASED RANDOM-ACCESS ALGORITHMS

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We examine a queue-based random-access algorithm where activation and deactivation rates are adapted as functions of queue lengths. We establish its heavy traffic behavior on a complete interference graph, which turns out to be nonstandard in two respects: (1) the scaling depends on some parameter of the algorithm and is not the N/N^2 scaling usually found in functional central limit theorems; (2) the heavy traffic limit is deterministic. We discuss how this nonstandard behavior arises from the idleness induced by the distributed nature of the algorithm. In order to prove our main result, we develop a new method for obtaining a fully coupled stochastic averaging principle.

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COUNTEREXAMPLES FOR OPTIMAL SCALING OF METROPOLIS–HASTINGS CHAINS WITH ROUGH TARGET DENSITIES

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For sufficiently smooth targets of product form it is known that the variance of a single coordinate of the proposal in RWM (random walk Metropolis) and MALA (Metropolis adjusted Langevin algorithm) should optimally scale as n^{-1} and as $n^{-\frac{1}{3}}$ with dimension n, and that the acceptance rates should be tuned to 0.234 and 0.574. We establish counterexamples to demonstrate that smoothness assumptions of the order of $\mathcal{C}^1(\mathbb{R})$ for RWM and $\mathcal{C}^3(\mathbb{R})$ for MALA are indeed required if these scaling rates are to hold. The counterexamples identify classes of marginal targets for which these guidelines are violated, obtained by perturbing a standard normal density (at the level of the potential for RWM and the second derivative of the potential for MALA) using roughness generated by a path of fractional Brownian motion with Hurst exponent H. For such targets there is strong evidence that RWM and MALA proposal variances should optimally be scaled as $n^{-\frac{1}{H}}$ and as $n^{-\frac{1}{2+H}}$ and will then obey anomalous acceptance rate guidelines. Useful heuristics resulting from this theory are discussed. The paper develops a framework capable of tackling optimal scaling results for quite general Metropolis–Hastings algorithms (possibly depending on a random environment).

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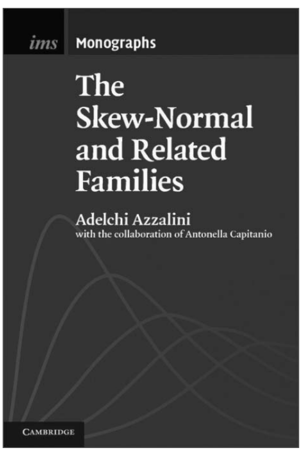

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