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VARIANCE ESTIMATION IN ADAPTIVE SEQUENTIAL MONTE CARLO

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Sequential Monte Carlo (SMC) methods represent a classical set of techniques to simulate a sequence of probability measures through a simple selection/mutation mechanism. However, the associated selection functions and mutation kernels usually depend on tuning parameters that are of first importance for the efficiency of the algorithm. A standard way to address this problem is to apply adaptive sequential Monte Carlo (ASMC) methods, which consist in exploiting the information given by the history of the sample to tune the parameters. This article is concerned with variance estimation in such ASMC methods. Specifically, we focus on the case where the asymptotic variance coincides with the one of the "limiting" sequential Monte Carlo algorithm as defined by Beskos et al. (*Ann. Appl. Probab.* **26** (2016) 1111–1146). We prove that, under natural assumptions, the estimator introduced by Lee and Whiteley (*Biometrika* **105** (2018) 609–625) in the nonadaptive case (i.e., SMC) is also a consistent estimator of the asymptotic variance for ASMC methods. To do this, we introduce a new estimator that is expressed in terms of coalescent tree-based measures, and explain its connection with the previous one. Our estimator is constructed by tracing the genealogy of the associated interacting particle system. The tools we use connect the study of particle Markov chain Monte Carlo methods and the variance estimation problem in SMC methods. As such, they may give some new insights when dealing with complex genealogy-involved problems of interacting particle systems in more general scenarios.

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ROBUST BOUNDS AND OPTIMIZATION AT THE LARGE DEVIATIONS SCALE FOR QUEUEING MODELS VIA RÉNYI DIVERGENCE

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This paper develops tools to obtain robust probabilistic estimates for queueing models at the large deviations (LD) scale. These tools are based on the recently introduced robust Rényi bounds, which provide LD estimates (and more generally risk-sensitive (RS) cost estimates) that hold uniformly over an uncertainty class of models, provided that the class is defined in terms of Rényi divergence with respect to a reference model and that estimates are available for the reference model. One very attractive quality of the approach is that the class to which the estimates apply may consist of hard models, such as highly non-Markovian models and ones for which the LD principle is not available. Our treatment provides exact expressions as well as bounds on the Rényi divergence rate on families of marked point processes, including as a special case renewal processes. Another contribution is a general result that translates robust RS control problems, where robustness is formulated via Rényi divergence, to finite dimensional convex optimization problems, when the control set is a finite dimensional convex set. The implications to queueing are vast, as they apply in great generality. This is demonstrated on two non-Markovian queueing models. One is the multiclass single-server queue considered as a RS control problem, with scheduling as the control process and exponential weighted queue length as cost. The second is the many-server queue with reneging, with the probability of atypically large reneging count as performance criterion. As far as LD analysis is concerned, no robust estimates or non-Markovian treatment were previously available for either of these models.

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ANTITHETIC MULTILEVEL SAMPLING METHOD FOR NONLINEAR FUNCTIONALS OF MEASURE

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Let $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, where $\mathcal{P}_2(\mathbb{R}^d)$ denotes the space of square integrable probability measures, and consider a Borel-measurable function $\Phi : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$. In this paper we develop an antithetic Monte Carlo estimator (A-MLMC) for $\Phi(\mu)$, which achieves sharp error bound under mild regularity assumptions. The estimator takes as input the empirical laws $\mu^N = \frac{1}{N}\sum_{i=1}^N \delta_{X_i}$, where (a) $(X_i)_{i=1}^N$ is a sequence of i.i.d. samples from μ or (b) $(X_i)_{i=1}^N$ is a system of interacting particles (diffusions) corresponding to a McKean–Vlasov stochastic differential equation (McKV-SDE). Each case requires a separate analysis. For a mean-field particle system, we also consider the empirical law induced by its Euler discretisation which gives a fully implementable algorithm. As by-products of our analysis, we establish a dimension-independent rate of uniform *strong propagation of chaos*, as well as an L^2 estimate of the antithetic difference for i.i.d. random variables corresponding to general functionals defined on the space of probability measures.

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"REGRESSION ANYTIME" WITH BRUTE-FORCE SVD TRUNCATION

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We propose a new least-squares Monte Carlo algorithm for the approximation of conditional expectations in the presence of stochastic derivative weights. The algorithm can serve as a building block for solving dynamic programming equations, which arise, for example, in nonlinear option pricing problems or in probabilistic discretization schemes for fully nonlinear parabolic partial differential equations. Our algorithm can be generically applied when the underlying dynamics stem from an Euler approximation to a stochastic differential equation. A built-in variance reduction ensures that the convergence in the number of samples to the true regression function takes place at an arbitrarily fast polynomial rate, if the problem under consideration is smooth enough.

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RANDOM CONDUCTANCE MODELS WITH STABLE-LIKE JUMPS: QUENCHED INVARIANCE PRINCIPLE

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We study the quenched invariance principle for random conductance models with long range jumps on $\mathbb{Z}^d$, where the transition probability from x to y is, on average, comparable to $|x-y|^{-(d+\alpha)}$ with $\alpha \in (0,2)$ but is allowed to be degenerate. Under some moment conditions on the conductance, we prove that the scaling limit of the Markov process is a symmetric α-stable Lévy process on $\mathbb{R}^d$. The well-known corrector method in homogenization theory does not seem to work in this setting. Instead, we utilize probabilistic potential theory for the corresponding jump processes. Two essential ingredients of our proof are the tightness estimate and the Hölder regularity of caloric functions for nonelliptic α-stable-like processes on graphs. Our method is robust enough to apply not only for $\mathbb{Z}^d$ but also for more general graphs whose scaling limits are nice metric measure spaces.

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PATH DEPENDENT OPTIMAL TRANSPORT AND MODEL CALIBRATION ON EXOTIC DERIVATIVES

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In this paper, we introduce and develop the theory of semimartingale optimal transport in a path dependent setting. Instead of the classical constraints on marginal distributions, we consider a general framework of path dependent constraints. Duality results are established, representing the solution in terms of path dependent partial differential equations (PPDEs). Moreover, we provide a dimension reduction result based on the new notion of "semifiltrations", which identifies appropriate Markovian state variables based on the constraints and the cost function. Our technique is then applied to the exact calibration of volatility models to the prices of general path dependent derivatives.

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STATISTICAL INFERENCE FOR BURES–WASSERSTEIN BARYCENTERS

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In this work we introduce the concept of Bures–Wasserstein barycenter Q_*, that is essentially a Fréchet mean of some distribution $\mathbb{P}$ supported on a subspace of positive semi-definite d-dimensional Hermitian operators $\mathbb{H}_+(d)$. We allow a barycenter to be constrained to some affine subspace of $\mathbb{H}_+(d)$, and we provide conditions ensuring its existence and uniqueness. We also investigate convergence and concentration properties of an empirical counterpart of Q_* in both Frobenius norm and Bures–Wasserstein distance, and explain, how the obtained results are connected to optimal transportation theory and can be applied to statistical inference in quantum mechanics.

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DIFFUSION-APPROXIMATION FOR A KINETIC EQUATION WITH PERTURBED VELOCITY REDISTRIBUTION PROCESS

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We derive the hydrodynamic limit of a kinetic equation with a stochastic, short range perturbation of the velocity operator. Under some mixing hypotheses on the stochastic perturbation, we establish a diffusion-approximation result: the limit we obtain is a parabolic stochastic partial differential equation on the macroscopic parameter, the density here.

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CONSERVATIVE STOCHASTIC TWO-DIMENSIONAL CAHN–HILLIARD EQUATION

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We consider the stochastic two-dimensional Cahn–Hilliard equation which is driven by the derivative in space of a space-time white noise. We use two different approaches to study this equation. First we prove that there exists a unique solution Y to the shifted equation (1.4). Then $X := Y + Z$ is the unique solution to the stochastic Cahn–Hilliard equation, where Z is the corresponding O-U process. Moreover, we use the Dirichlet form approach in (*Probab. Theory Related Fields* **89** (1991) 347–386) to construct a probabilistically weak solution to the original equation (1.1) below. By clarifying the precise relation between the two solutions, we also get the restricted Markov uniqueness of the generator and the uniqueness of the martingale solutions to the equation (1.1). Furthermore, we also obtain exponential ergodicity of the solutions.

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SEMIMARTINGALES AND SHRINKAGE OF FILTRATION

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We consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which is endowed with two filtrations, $\mathbb{G}$ and $\mathbb{F}$, assumed to satisfy the usual conditions and such that $\mathbb{F} \subset \mathbb{G}$. On this probability space we consider a real valued $\mathbb{G}$-semimartingale X.

The purpose of this work is to study the following two problems:

A. If X is $\mathbb{F}$-adapted, compute the $\mathbb{F}$-semimartingale characteristics of X in terms of the $\mathbb{G}$-semimartingale characteristics of X.

B. If X is a special $\mathbb{G}$-semimartingale but not $\mathbb{F}$-adapted, compute the $\mathbb{F}$-semimartingale characteristics of the $\mathbb{F}$-optional projection of X in terms of the $\mathbb{G}$-canonical decomposition and the $\mathbb{G}$-semimartingale characteristics of X.

In this paper problem B is solved under the assumption that the filtration $\mathbb{F}$ is immersed in $\mathbb{G}$. Beyond the obvious mathematical interest, our study is motivated by important practical applications in areas such as finance and insurance (cf. *Structured Dependence Between Stochastic Processes* (2020) Cambridge Univ. Press).

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PAGERANK'S BEHAVIOR UNDER DEGREE CORRELATIONS

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The focus of this work is the asymptotic analysis of the tail distribution of Google's PageRank algorithm on large scale-free directed networks. In particular, the main theorem provides the convergence, in the Kantorovich–Rubinstein metric, of the rank of a randomly chosen vertex in graphs generated via either a directed configuration model or an inhomogeneous random digraph. The theorem fully characterizes the limiting distribution by expressing it as a random sum of i.i.d. copies of the attracting endogenous solution to a branching distributional fixed-point equation. In addition, we provide the asymptotic tail behavior of the limit and use it to explain the effect that in-degree/out-degree correlations in the underlying graph can have on the qualitative performance of PageRank.

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FAST APPROXIMATE SIMULATION OF FINITE LONG-RANGE SPIN SYSTEMS

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Tau leaping is a popular method for performing fast approximate simulation of certain continuous time Markov chain models typically found in chemistry and biochemistry. This method is known to perform well when the transition rates satisfy some form of scaling behaviour. In a similar spirit to tau leaping, we propose a new method for approximate simulation of spin systems which approximates the evolution of spin at each site between sampling epochs as an independent two-state Markov chain. When combined with fast summation methods, our method offers considerable improvement in speed over the standard Doob–Gillespie algorithm. We provide a detailed analysis of the error incurred for both the number of sites incorrectly labelled and for linear functions of the state.

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COMPUTING THE PARTITION FUNCTION OF THE SHERRINGTON–KIRKPATRICK MODEL IS HARD ON AVERAGE

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We establish the average-case hardness of the algorithmic problem of exact computation of the partition function associated with the Sherrington–Kirkpatrick model of spin glasses with Gaussian couplings and random external field. In particular, we establish that unless $P = \#P$, there does not exist a polynomial-time algorithm to exactly compute the partition function on average. This is done by showing that if there exists a polynomial time algorithm, which exactly computes the partition function for inverse polynomial fraction $(1/n^{O(1)})$ of all inputs, then there is a polynomial time algorithm, which exactly computes the partition function for all inputs, with high probability, yielding $P = \#P$. The computational model that we adopt is *finite-precision arithmetic*, where the algorithmic inputs are truncated first to a certain level N of digital precision. The ingredients of our proof include the random and downward self-reducibility of the partition function with random external field; an argument of Cai et al. (In *STACS 99 (Trier)* (1999) 90–99 Springer) for establishing the average-case hardness of computing the permanent of a matrix; a list-decoding algorithm of Sudan (In *37th Annual Symposium on Foundations of Computer Science (Burlington, VT, 1996)* (1996) 164–172 IEEE Comput. Soc. Press), for reconstructing polynomials intersecting a given list of numbers at sufficiently many points; and near-uniformity of the log-normal distribution, modulo a large prime p. To the best of our knowledge, our result is the first one establishing a provable hardness of a model arising in the field of spin glasses.

Furthermore, we extend our result to the same problem under a different *real-valued* computational model, for example, using a Blum–Shub–Smale machine (In *[Proceedings 1988] 29th Annual Symposium on Foundations of Computer Science* (1988) 387–397 IEEE) operating over real-valued inputs. We establish that, if there exists a polynomial time algorithm which exactly computes the partition function for $\frac{3}{4} + \frac{1}{n^{O(1)}}$ fraction of all inputs, then there exists a polynomial time algorithm, which exactly computes the partition function for all inputs, with high probability, yielding $P = \#P$. Our proof uses the random self-reducibility of the partition function, together with a control over the total variation distance for log-normal random variables in presence of a convex perturbation, and the Berlekamp–Welch algorithm.

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CORRIGENDUM FOR "SECOND-ORDER REFLECTED BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS" AND "SECOND-ORDER BSDES WITH GENERAL REFLECTION AND GAME OPTIONS UNDER UNCERTAINTY"

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The aim of this short note is to fill in a gap in our earlier paper (*Ann. Appl. Probab.* **23** (2013) 2420–2457) on 2BSDEs with reflections, and to explain how to correct the subsequent results in the second paper (*Stochastic Process. Appl.* **124** (2014) 2281–2321). We also provide more insight on the properties of 2RBSDEs, in the light of the recent contributions (Li and Peng (2017); Soumana Hima (2017)) in the so-called G-framework.

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