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# MIXING OF HAMILTONIAN MONTE CARLO ON STRONGLY LOG-CONCAVE DISTRIBUTIONS: CONTINUOUS DYNAMICS

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We obtain several quantitative bounds on the mixing properties of an "ideal" Hamiltonian Monte Carlo (HMC) Markov chain for a strongly log-concave target distribution $\pi$ on $\mathbb{R}^d$. Our main result says that the HMC Markov chain generates a sample with Wasserstein error $\epsilon$ in roughly $O(\kappa^2 \log(\frac{1}{\epsilon}))$ steps, where the condition number $\kappa = \frac{M_2}{m_2}$ is the ratio of the maximum $M_2$ and minimum $m_2$ eigenvalues of the Hessian of $-\log(\pi)$. In particular, this mixing bound does not depend explicitly on the dimension $d$. These results significantly extend and improve previous quantitative bounds on the mixing of ideal HMC, and can be used to analyze more realistic HMC algorithms. The main ingredient of our argument is a proof that initially "parallel" Hamiltonian trajectories contract over much longer steps than would be predicted by previous heuristics based on the Jacobi manifold.

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# EXCHANGEABLE COALESCENTS, ULTRAMETRIC SPACES, NESTED INTERVAL-PARTITIONS: A UNIFYING APPROACH

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Kingman's (1978) representation theorem (*J. Lond. Math. Soc. (2)* **18** (1978) 374–380) states that any exchangeable partition of $\mathbb{N}$ can be represented as a paintbox based on a random mass-partition. Similarly, any exchangeable composition (i.e., ordered partition of $\mathbb{N}$) can be represented as a paintbox based on an interval-partition (Gnedin (1997) *Ann. Probab.* **25** (1997) 1437–1450).

Our first main result is that any exchangeable coalescent process (not necessarily Markovian) can be represented as a paintbox based on a random nondecreasing process valued in interval-partitions, called nested interval-partition, generalizing the notion of comb metric space introduced in Lambert and Uribe Bravo (2017) (*p-Adic Numbers Ultrametric Anal. Appl.* **9** (2017) 22–38) to represent compact ultrametric spaces.

As a special case, we show that any $\Lambda$-coalescent can be obtained from a paintbox based on a unique random nested interval partition called $\Lambda$-comb, which is Markovian with explicit transitions. This nested interval-partition directly relates to the flow of bridges of Bertoin and Le Gall (2003) (*Probab. Theory Related Fields* **126** (2003) 261–288). We also display a particularly simple description of the so-called evolving coalescent (Pfaffelhuber and Wakolbinger (2006) *Stochastic Process. Appl.* **116** (2006) 1836–1859) by a comb-valued Markov process.

Next, we prove that any ultrametric measure space $U$, under mild measure-theoretic assumptions on $U$, is the leaf set of a tree composed of a separable subtree called the backbone, on which are grafted additional subtrees, which act as star-trees from the standpoint of sampling. Displaying this so-called weak isometry requires us to extend the Gromov-weak topology of Greven, Pfaffelhuber and Winter (2009) (*Probab. Theory Related Fields* **145** (2009) 285–322), that was initially designed for separable metric spaces, to nonseparable ultrametric spaces. It allows us to show that for any such ultrametric space $U$, there is a nested interval-partition which is (1) indistinguishable from $U$ in the Gromov-weak topology; (2) weakly isometric to $U$ if $U$ has a complete backbone; (3) isometric to $U$ if $U$ is complete and separable.

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# CENTRAL LIMIT THEOREMS FOR SIR EPIDEMICS AND PERCOLATION ON CONFIGURATION MODEL RANDOM GRAPHS

BY FRANK BALL

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We consider a stochastic SIR (susceptible → infective → recovered) epidemic defined on a configuration model random graph, in which infective individuals can infect only their neighbours in the graph during an infectious period which has an arbitrary but specified distribution. Central limit theorems for the final size (number of initial susceptibles that become infected) of such an epidemic as the population size $n$ tends to infinity, with explicit, easy to compute expressions for the asymptotic variance, are proved assuming that the degrees are bounded. The results are obtained for both the Molloy–Reed random graph, in which the degrees of individuals are deterministic, and the Newman–Strogatz–Watts random graph, in which the degrees are independent and identically distributed. The central limit theorems cover the cases when the number of initial infectives either (a) tends to infinity or (b) is held fixed as $n \to \infty$. In (a) it is assumed that the fraction of the population that is initially infected converges to a limit (which may be 0) as $n \to \infty$, while in (b) the central limit theorems are conditional upon the occurrence of a large outbreak (more precisely one of size at least $\log n$). Central limit theorems for the size of the largest cluster in bond percolation on Molloy–Reed and Newman–Strogatz–Watts random graphs follow immediately from our results, as do central limit theorems for the size of the giant component of those graphs. Corresponding central limit theorems for site percolation on those graphs are also proved.

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# APPROXIMATION OF FRACTIONAL LOCAL TIMES: ZERO ENERGY AND DERIVATIVES

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We consider empirical processes associated with high-frequency observations of a fractional Brownian motion (fBm) $X$ with Hurst parameter $H \in (0, 1)$, and derive conditions under which these processes verify a (possibly uniform) law of large numbers, as well as a second order (possibly uniform) limit theorem. We devote specific emphasis to the "zero energy" case, corresponding to a kernel whose integral on the real line equals zero. Our asymptotic results are associated with explicit rates of convergence, and are expressed either in terms of the local time of $X$ or of its derivatives: in particular, the full force of our finding applies to the "rough range" $0 < H < 1/3$, on which the previous literature has been mostly silent. The use of the derivatives of local times for studying the fluctuations of high-frequency observations of a fBm is new, and is the main technological breakthrough of the present paper. Our results are based on the use of Malliavin calculus and Fourier analysis, and extend and complete several findings in the literature, for example, by Jeganathan (*Ann. Probab.* **32** (2004) 1771–1795; (2006); (2008)) and Podolskij and Rosenbaum (*J. Financ. Econom.* **16** (2018) 588–598).

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# THE FRACTAL CYLINDER PROCESS: EXISTENCE AND CONNECTIVITY PHASE TRANSITIONS

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We consider a semi-scale invariant version of the Poisson cylinder model which in a natural way induces a random fractal set. We show that this random fractal exhibits an existence phase transition for any dimension $d \geq 2$, and a connectivity phase transition whenever $d \geq 4$. We determine the exact value of the critical point of the existence phase transition, and we show that the fractal set is almost surely empty at this critical point.

A key ingredient when analysing the connectivity phase transition is to consider a restriction of the full process onto a subspace. We show that this restriction results in a fractal ellipsoid model which we describe in detail, as it is key to obtaining our main results.

In addition we also determine the almost sure Hausdorff dimension of the fractal set.

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# LINEAR-QUADRATIC CONTROL FOR A CLASS OF STOCHASTIC VOLTERRA EQUATIONS: SOLVABILITY AND APPROXIMATION

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We provide an exhaustive treatment of linear-quadratic control problems for a class of stochastic Volterra equations of convolution type, whose kernels are Laplace transforms of certain signed matrix measures which are not necessarily finite. These equations are in general neither Markovian nor semimartingales, and include the fractional Brownian motion with Hurst index smaller than $1/2$ as a special case. We establish the correspondence of the initial problem with a possibly infinite dimensional Markovian one in a Banach space, which allows us to identify the Markovian controlled state variables. Using a refined martingale verification argument combined with a squares completion technique, we prove that the value function is of linear quadratic form in these state variables with a linear optimal feedback control, depending on nonstandard Banach space valued Riccati equations. Furthermore, we show that the value function of the stochastic Volterra optimization problem can be approximated by that of conventional finite dimensional Markovian linear-quadratic problems, which is of crucial importance for numerical implementation.

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# ENTROPY DISSIPATION ESTIMATES FOR INHOMOGENEOUS ZERO-RANGE PROCESSES

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Introduced by Lu and Yau (*Comm. Math. Phys.* **156** (1993) 399–433), the martingale decomposition method is a powerful recursive strategy that has produced sharp log-Sobolev inequalities for homogeneous particle systems. However, the intractability of certain covariance terms has so far precluded applications to heterogeneous models. Here we demonstrate that the existence of an appropriate coupling can be exploited to bypass this limitation effortlessly. Our main result is a dimension-free modified log-Sobolev inequality for zero-range processes on the complete graph, under the only requirement that all rate increments lie in a compact subset of $(0, \infty)$. This settles an open problem raised by Caputo and Posta (*Probab. Theory Related Fields* **139** (2007) 65–87) and reiterated by Caputo, Dai Pra and Posta (*Ann. Inst. Henri Poincaré Probab. Stat.* **45** (2009) 734–753). We believe that our approach is simple enough to be applicable to many systems.

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# COEXISTENCE OF LOCALIZED GIBBS MEASURES AND DELOCALIZED GRADIENT GIBBS MEASURES ON TREES

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We study gradient models for spins taking values in the integers (or an integer lattice), which interact via a general potential depending only on the differences of the spin values at neighboring sites, located on a regular tree with $d + 1$ neighbors. We first provide general conditions in terms of the relevant $p$-norms of the associated transfer operator $Q$ which ensure the existence of a countable family of proper Gibbs measures, describing localization at different heights. Next we prove existence of delocalized gradient Gibbs measures, under natural conditions on $Q$. We show that the two conditions can be fulfilled at the same time, which then implies coexistence of both types of measures for large classes of models including the SOS-model, and heavy-tailed models arising for instance for potentials of logarithmic growth.

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# NON-UNIVERSAL FLUCTUATIONS OF THE EMPIRICAL MEASURE FOR ISOTROPIC STATIONARY FIELDS ON $\mathbb{S}^2 \times \mathbb{R}$

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In this paper, we consider isotropic and stationary real Gaussian random fields defined on $\mathbb{S}^2 \times \mathbb{R}$ and we investigate the asymptotic behavior, as $T \to +\infty$, of the empirical measure (excursion area) in $\mathbb{S}^2 \times [0, T]$ at any threshold, covering both cases when the field exhibits short and long memory, that is, integrable and nonintegrable temporal covariance. It turns out that the limiting distribution is not universal, depending both on the memory parameters and the threshold. In particular, in the long memory case a form of Berry's cancellation phenomenon occurs at zero-level, inducing phase transitions for both variance rates and limiting laws.

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# BRANCHING DIFFUSION REPRESENTATION FOR NONLINEAR CAUCHY PROBLEMS AND MONTE CARLO APPROXIMATION

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We provide probabilistic representations of the solution of some semilinear hyperbolic and high-order PDEs based on branching diffusions. These representations pave the way for an approximation of the solution by the standard Monte Carlo method, whose error estimate is controlled by the standard central limit theorem, thus partly bypassing the curse of dimensionality. We illustrate the numerical implications in the context of some popular PDEs in physics such as nonlinear Klein–Gordon equation, a simplified scalar version of the Yang–Mills equation, a fourth-order nonlinear beam equation and the Gross–Pitaevskii PDE as an example of nonlinear Schrödinger equations.

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# MANY-SERVER ASYMPTOTICS FOR JOIN-THE-SHORTEST-QUEUE: LARGE DEVIATIONS AND RARE EVENTS

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The join-the-shortest-queue routing policy is studied in an asymptotic regime where the number of processors $n$ scales with the arrival rate. A large deviation principle (LDP) for the occupancy process is established, as $n \to \infty$, in a suitable infinite-dimensional path space. Model features that present technical challenges include, Markovian dynamics with discontinuous statistics, a diminishing rate property of the transition probability rates, and an infinite-dimensional state space. The difficulty is in the proof of the Laplace lower bound which requires establishing the uniqueness of solutions of certain infinite-dimensional systems of controlled ordinary differential equations. The LDP gives information on the rate of decay of probabilities of various types of rare events associated with the system. We illustrate this by establishing explicit exponential decay rates for probabilities of long queues. In particular, denoting by $E_j^n(T)$ the event that there is at least one queue with $j$ or more jobs at some time instant over $[0, T]$, we show that, in the critical case, for large $n$ and $T$, $\mathbb{P}(E_j^n(T)) \approx \exp[-\frac{n(j-2)^2}{4T}]$.

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# ASYMPTOTIC BEHAVIOUR OF THE ONE-DIMENSIONAL "ROCK–PAPER–SCISSORS" CYCLIC CELLULAR AUTOMATON

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The one-dimensional three-state cyclic cellular automaton is a simple spatial model with three states in a cyclic "rock–paper–scissors" prey–predator relationship. Starting from a random configuration, similar states gather in increasingly large clusters; asymptotically, any finite region is filled with a uniform state that is, after some time, driven out by its predator, each state taking its turn in dominating the region (heteroclinic cycles).

We consider the situation where each site in the initial configuration is chosen independently at random with a different probability for each state. We prove that the asymptotic probability that a state dominates a finite region corresponds to the initial probability of its prey. The proof methods are based on discrete probability tools, mainly particle systems and random walks.

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# CONVERGENCE OF METADYNAMICS: DISCUSSION OF THE ADIABATIC HYPOTHESIS

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By drawing a parallel between metadynamics and self interacting models for polymers, we study the longtime convergence of the original metadynamics algorithm in the adiabatic setting, namely when the dynamics along the collective variables decouples from the dynamics along the other degrees of freedom. We also discuss the bias which is introduced when the adiabatic assumption does not hold.

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# HYPOCOERCIVITY OF PIECEWISE DETERMINISTIC MARKOV PROCESS-MONTE CARLO

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In this work, we establish $\mathrm{L}^2$-exponential convergence for a broad class of piecewise deterministic Markov processes recently proposed in the context of Markov process Monte Carlo methods and covering in particular the randomized Hamiltonian Monte Carlo (*Trans. Amer. Math. Soc.* **367** (2015) 3807–3828; *Ann. Appl. Probab.* **27** (2017) 2159–2194), the zig-zag process (*Ann. Statist.* **47** (2019) 1288–1320) and the bouncy particle Sampler (*Phys. Rev. E* **85** (2012) 026703; *J. Amer. Statist. Assoc.* **113** (2018) 855–867). The kernel of the symmetric part of the generator of such processes is nontrivial, and we follow the ideas recently introduced in (*C. R. Math. Acad. Sci. Paris* **347** (2009) 511–516; *Trans. Amer. Math. Soc.* **367** (2015) 3807–3828) to develop a rigorous framework for hypocoercivity in a fairly general and unifying set-up, while deriving tractable estimates of the constants involved in terms of the parameters of the dynamics. As a by-product we characterize the scaling properties of these algorithms with respect to the dimension of classes of problems, therefore providing some theoretical evidence to support their practical relevance.

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