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ABSORBING TIME ASYMPTOTICS IN THE ORIENTED SWAP PROCESS

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The oriented swap process is a natural directed random walk on the symmetric group that can be interpreted as a multispecies version of the totally asymmetric simple exclusion process (TASEP) on a finite interval. An open problem from a 2009 paper of Angel, Holroyd, and Romik asks for the limiting distribution of the absorbing time of the process as the number of particles goes to infinity. We resolve this question by proving that this random variable satisfies GOE Tracy–Widom asymptotics. As a central ingredient of our proof, we reexamine a distributional identity relating the behavior of the oriented swap process to last passage percolation, conjectured in a recent paper of Bisi, Cunden, Gibbons, and Romik. We use a shift-invariance principle for multispecies TASEPs, obtained by exploiting recent results of Borodin, Gorin, and Wheeler for the stochastic colored six-vertex model, to prove a weakened form of the Bisi et al. conjectural identity, that is nonetheless sufficient for proving the asymptotic result for the absorbing time.

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A COUPLING APPROACH FOR THE CONVERGENCE TO EQUILIBRIUM FOR A COLLISIONLESS GAS

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We use a probabilistic approach to study the rate of convergence to equilibrium for a collisionless (Knudsen) gas in dimension equal to or larger than 2. The use of a coupling between two stochastic processes allows us to extend and refine, in total variation distance, the polynomial rate of convergence given in (*Kinet. Relat. Models* **4** (2011) 87–107) and (*Comm. Math. Phys.* **318** (2013) 375–409). This is, to our knowledge, the first quantitative result in collisionless kinetic theory in dimension equal to or larger than 2 that does not require any symmetry of the domain, nor a monokinetic regime. Our study is also more general in terms of reflection at the boundary: we allow for rather general diffusive reflections and for a specular reflection component.

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Key words and phrases. Stochastic billiards, Markov process, collisionless gas, coupling, long-time behaviour, subexponential convergence to equilibrium.

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RANDOM CONCAVE FUNCTIONS

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Spaces of convex and concave functions appear naturally in theory and applications. For example, convex regression and log-concave density estimation are important topics in nonparametric statistics. In stochastic portfolio theory, concave functions on the unit simplex measure the concentration of capital, and their gradient maps define novel investment strategies. The gradient maps may also be regarded as optimal transport maps on the simplex. In this paper we construct and study probability measures supported on spaces of concave functions. These measures may serve as prior distributions in Bayesian statistics and Cover's universal portfolio, and induce distribution-valued random variables via optimal transport. The random concave functions are constructed on the unit simplex by taking a suitably scaled (mollified, or soft) minimum of random hyperplanes. Depending on the regime of the parameters, we show that as the number of hyperplanes tends to infinity there are several possible limiting behaviors. In particular, there is a transition from a deterministic almost sure limit to a nontrivial limiting distribution that can be characterized using convex duality and Poisson point processes.

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ARE AMERICAN OPTIONS EUROPEAN AFTER ALL?

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We call a given American option *representable* if there exists a European claim which dominates the American payoff at any time and such that the values of the two options coincide in the continuation region of the American option. This concept has interesting implications from a probabilistic, analytic, financial, and numeric point of view. Relying on methods from (*Math. Finance* **24** (2014) 156–172; *Ann. Inst. H. Poincaré Anal. Non Linéaire* **18** (2001) 1–17; *Ann. Appl. Probab.* **12** (2002) 196–223) and convex duality, we make a first step towards verifying representability of American options.

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A CLASSIFICATION OF THE DYNAMICS OF THREE-DIMENSIONAL STOCHASTIC ECOLOGICAL SYSTEMS

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The classification of the long-term behavior of dynamical systems is a fundamental problem in mathematics. For both deterministic and stochastic dynamics specific classes of models verify Palis' conjecture: the long-term behavior is determined by a finite number of stationary distributions. In this paper we consider the classification problem for stochastic models of interacting species. For a large class of three-species, stochastic differential equation models, we prove a variant of Palis' conjecture: the long-term statistical behavior is determined by a finite number of stationary distributions and, generically, three general types of behavior are possible: 1) convergence to a unique stationary distribution that supports all species, 2) convergence to one of a finite number of stationary distributions supporting two or fewer species, 3) convergence to convex combinations of single species, stationary distributions due to a rock-paper-scissors type of dynamic. Moreover, we prove that the classification reduces to computing Lyapunov exponents (external Lyapunov exponents) that correspond to the average per-capita growth rate of species when rare. Our results stand in contrast to the deterministic setting where the classification is incomplete even for three-dimensional, competitive Lotka–Volterra systems. For these SDE models, our results also provide a rigorous foundation for ecology's modern coexistence theory (MCT) which assumes the external Lyapunov exponents determine long-term ecological outcomes.

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A PROBABILISTIC APPROACH TO CONVEX (ϕ)-ENTROPY DECAY FOR MARKOV CHAINS

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We study the exponential dissipation of entropic functionals along the semigroup generated by a continuous time Markov chain and the associated convex Sobolev inequalities, including MLSI and Beckner inequalities. We propose a method that combines the Bakry–Émery approach and coupling arguments, which we use as a probabilistic alternative to the discrete Bochner identities. In particular, the validity of the method is not limited to the perturbative setting and we establish convex entropy decay for interacting random walks beyond the high temperature/weak interaction regime. In this framework, we show that the exponential contraction of the Wasserstein distance implies MLSI. We also revisit classical examples often obtaining new inequalities and sometimes improving on the best known constants. In particular, we analyse the zero range dynamics, hardcore and Bernoulli–Laplace models and the Glauber dynamics for the Curie–Weiss and Ising model.

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LOCAL GEOMETRY OF THE ROUGH-SMOOTH INTERFACE IN THE TWO-PERIODIC AZTEC DIAMOND

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Random tilings of the two-periodic Aztec diamond contain three macroscopic regions: frozen, where the tilings are deterministic; rough, where the correlations between dominoes decay polynomially; smooth, where the correlations between dominoes decay exponentially. In a previous paper, the authors found that a certain averaging of height function differences at the rough-smooth interface converged to the extended Airy kernel point process. In this paper, we augment the local geometrical picture at this interface by introducing well-defined lattice paths which are closely related to the level lines of the height function. We show, after suitable centering and rescaling, that a point process from these paths converge to the extended Airy kernel point process provided that the natural parameter associated to the two-periodic Aztec diamond is small enough.

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ENTROPY DECAY IN THE SWENDSEN–WANG DYNAMICS ON $\mathbb{Z}^d$

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We study the mixing time of the *Swendsen–Wang* dynamics for the ferromagnetic Ising and Potts models on the integer lattice $\mathbb{Z}^d$. This dynamics is a widely used Markov chain that has largely resisted sharp analysis because it is *nonlocal*, that is, it changes the entire configuration in one step. We prove that, whenever *strong spatial mixing (SSM)* holds, the mixing time on any n-vertex cube in $\mathbb{Z}^d$ is $O(\log n)$, and we prove this is tight by establishing a matching lower bound on the mixing time. The previous best known bound was $O(n)$. SSM is a standard condition corresponding to exponential decay of correlations with distance between spins on the lattice and is known to hold in $d = 2$ dimensions throughout the high-temperature (single phase) region. Our result follows from a *modified log-Sobolev inequality*, which expresses the fact that the dynamics contracts relative entropy at a constant rate at each step. The proof of this fact utilizes a new factorization of the entropy in the joint probability space over spins and edges that underlies the Swendsen–Wang dynamics, which extends to general bipartite graphs of bounded degree. This factorization leads to several additional results, including mixing time bounds for a number of natural local and nonlocal Markov chains on the joint space, as well as for the standard random-cluster dynamics.

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CORRELATED RANDOMLY GROWING GRAPHS

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We introduce a new model of correlated randomly growing graphs and study the fundamental questions of detecting correlation and estimating aspects of the correlated structure. The model is simple and starts with any model of randomly growing graphs, such as uniform attachment (UA) or preferential attachment (PA). Given such a model, a pair of graphs (G_1, G_2) is grown in two stages: until time $t_\star$ they are grown together (i.e., $G_1 = G_2$), after which they grow independently according to the underlying growth model.

We show that whenever the seed graph has an influence in the underlying graph growth model—this has been shown for PA and UA trees and is conjectured to hold broadly—then correlation can be detected in this model, even if the graphs are grown together for just a *single time step*. We also give a general sufficient condition (which holds for PA and UA trees) under which detection is possible with probability going to 1 as $t_\star \to \infty$. Finally, we show for PA and UA trees that the amount of correlation, measured by $t_\star$, can be estimated with vanishing relative error as $t_\star \to \infty$.

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RARE EVENT ASYMPTOTICS FOR EXPLORATION PROCESSES FOR RANDOM GRAPHS

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Large deviations for random graph models has been a topic of significant recent research activity. Much work in this area is focused on the class of *dense* random graph models (number of edges in the graph scale as n^2, where n is the number of vertices) where the theory of graphons has emerged as a principal tool in the study of large deviation properties. These tools do not give a good approach to large deviation problems for random graph models in the sparse regime. The aim of this paper is to study an approach for large deviation problems in this regime by establishing large deviation principles (LDP) on suitable path spaces for certain exploration processes of the associated random graph sequence. Exploration processes are an important tool in the study of sparse random graph models and have been used to understand detailed asymptotics of many functionals of sparse random graphs, such as component sizes, surplus, deviations from trees, etc. In the context of rare event asymptotics of interest here, the point of view of exploration process transforms a large deviation analysis of a static random combinatorial structure to the study of a small noise LDP for certain stochastic dynamical systems with jumps.

Our work focuses on one particular class of random graph models, namely the configuration model; however, the general approach of using exploration processes for studying large deviation properties of sparse random graph models has broader applicability. The goal is to study asymptotics of probabilities of nontypical behavior in the large network limit. The first key step for this is to establish a LDP for an exploration process associated with the configuration model. A suitable exploration process here turns out to be an infinite-dimensional Markov process with transition probability rates that diminish to zero in certain parts of the state space. Large deviation properties of such Markovian models is challenging due to poor regularity behavior of the associated local rate functions. Our proof of the LDP relies on a representation of the exploration process in terms of a system of stochastic differential equations driven by Poisson random measures and variational formulas for moments of nonnegative functionals of Poisson random measures. Uniqueness results for certain controlled systems of deterministic equations play a key role in the analysis. Next, using the rate function in the LDP for the exploration process we formulate a calculus of variations problem associated with the asymptotics of component degree distributions. The second key ingredient in our study is a careful analysis of the infinite-dimensional Euler–Lagrange equations associated with this calculus of variations problem. Exact solutions of these systems of nonlinear differential equations are identified which then provide explicit formulas for decay rates of probabilities of nontypical component degree distributions and related quantities.

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SCALING LIMIT OF THE HOMOGENIZATION COMMUTATOR FOR GAUSSIAN COEFFICIENT FIELDS

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Consider a linear elliptic partial differential equation in divergence form with a random coefficient field. The solution-operator displays fluctuations around its expectation. The recently-developed pathwise theory of fluctuations in stochastic homogenization reduces the characterization of these fluctuations to those of the so-called standard homogenization commutator. In this contribution, we investigate the scaling limit of this key quantity: starting from a Gaussian-like coefficient field with possibly strong correlations, we establish the convergence of the rescaled commutator to a fractional Gaussian field, depending on the decay of correlations of the coefficient field, and we investigate the (non)degeneracy of the limit. This extends to general dimension $d \geq 1$ previous results so far limited to dimension $d = 1$, and to the continuum setting with strong correlations recent results in the discrete i.i.d. case.

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HOUSEHOLD EPIDEMIC MODELS AND MCKEAN–VLASOV POISSON DRIVEN STOCHASTIC DIFFERENTIAL EQUATIONS

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This paper presents a new view of household epidemic models, where we exploit the fact that the interaction between the households is of mean field type. We prove the convergence, as the number of households tends to infinity, of the number of infectious individuals in a uniformly chosen household to a nonlinear Markov process solving a McKean–Vlasov Poisson driven stochastic differential equation, as well as a propagation of chaos result. We also define a basic reproduction number R_* and show that if $R_* > 1$, then the nonlinear Markov process has a unique nontrivial ergodic invariant probability measure, whereas if $R_* \leq 1$, it converges to 0 as t tends to infinity.

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A SINGULAR TOEPLITZ DETERMINANT AND THE DISCRETE TACNODE KERNEL FOR SKEW-AZTEC RECTANGLES

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Random tilings of geometrical shapes with dominos or lozenges have been a rich source of universal statistical distributions. This paper deals with domino tilings of checker board rectangular shapes such that the top two and bottom two adjacent squares have the same orientation and the two most left and two most right ones as well. It forces these so-called "*skew-Aztec rectangles*" to have cuts on either side. For large sizes of the domain and upon an appropriate scaling of the location of the cuts, one finds *split tacnodes* between liquid regions with two distinct adjacent frozen phases descending into the tacnode. Zooming about such split tacnodes, filaments appear between the liquid patches evolving in a bricklike sea of dimers of another type. This work shows that the random fluctuations in a neighborhood of the split tacnode are governed asymptotically by the *discrete tacnode kernel*, providing strong evidence that this kernel is a universal discrete-continuous limiting kernel occurring naturally whenever we have doubly interlacing patterns. The analysis involves the inversion of a singular Toeplitz matrix which leads to considerable difficulties.

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PERIODIC WORDS, COMMON SUBSEQUENCES AND FROGS

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Let $W^{(n)}$ be the n-letter word obtained by repeating a fixed word W, and let R_n be a random n-letter word over the same alphabet. We show several results about the length of the longest common subsequence (LCS) between $W^{(n)}$ and R_n; in particular, we show that its expectation is $\gamma_W n - O(\sqrt{n})$ for an efficiently-computable constant γ_W.

This is done by relating the problem to a new interacting particle system, which we dub "frog dynamics". In this system, the particles ("frogs") hop over one another in the order given by their labels. Stripped of the labeling, the frog dynamics reduces to a variant of the PushTASEP.

In the special case when all symbols of W are distinct, we obtain an explicit formula for the constant γ_W and a closed-form expression for the stationary distribution of the associated frog dynamics.

In addition, we propose new conjectures about the asymptotic of the LCS of a pair of random words. These conjectures are informed by computer experiments using a new heuristic algorithm to compute the LCS. Through our computations, we found periodic words that are more random-like than a random word, as measured by the LCS.

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ON EXPLICIT L^2-CONVERGENCE RATE ESTIMATE FOR PIECEWISE DETERMINISTIC MARKOV PROCESSES IN MCMC ALGORITHMS

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We establish L^2-exponential convergence rate for three popular piecewise deterministic Markov processes for sampling: the randomized Hamiltonian Monte Carlo method, the zigzag process and the bouncy particle sampler. Our analysis is based on a variational framework for hypocoercivity, which combines a Poincaré-type inequality in time-augmented state space and a standard L^2 energy estimate. Our analysis provides explicit convergence rate estimates, which are more quantitative than existing results.

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METASTABILITY IN LOSS NETWORKS WITH DYNAMIC ALTERNATIVE ROUTING

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Consider N stations interconnected with links, each of capacity K, forming a complete graph. Calls arrive to each link at rate λ and depart at rate 1. If a call arrives to a link xy, connecting stations x and y, which is at capacity, then a third station z is chosen uniformly at random and the call is attempted to be routed via z: if both links xz and zy have spare capacity, then the call is held simultaneously on these two; otherwise the call is lost.

We analyse an approximation of this model. We show rigorously that there are three phases according to the traffic intensity $\alpha := \lambda/K$: for $\alpha \in (0, \alpha_c) \cup (1, \infty)$, the system has mixing time logarithmic in the number of links $n := \binom{N}{2}$; for $\alpha \in (\alpha_c, 1)$ the system has mixing time exponential in n, the number of links. Here $\alpha_c := \frac{1}{3}(5\sqrt{10} - 13) \approx 0.937$ is an explicit critical threshold with a simple interpretation. We also consider allowing multiple rerouting attempts. This has little effect on the overall behaviour; it does not remove the metastability phase.

Finally, we add *trunk reservation*: in this, some number σ of circuits are reserved; a rerouting attempt is only accepted if at least $\sigma + 1$ circuits are available. We show that if σ is chosen sufficiently large, depending only on α, not K or n, then the metastability phase is removed.

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THE DIRECTIONAL OPTIMAL TRANSPORT

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We introduce a constrained optimal transport problem where origins x can only be transported to destinations $y \geq x$. Our statistical motivation is to describe the sharp upper bound for the variance of the treatment effect $Y - X$ given marginals when the effect is monotone, or $Y \geq X$. We thus focus on supermodular costs (or submodular rewards) and introduce a coupling P_* that is optimal for all such costs and yields the sharp bound. This coupling admits manifold characterizations—geometric, order-theoretic, as optimal transport, through the cdf, and via the transport kernel—that explain its structure and imply useful bounds. When the first marginal is atomless, P_* is concentrated on the graphs of two maps which can be described in terms of the marginals, the second map arising due to the binding constraint.

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MEAN-FIELD MARKOV DECISION PROCESSES WITH COMMON NOISE AND OPEN-LOOP CONTROLS

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We develop an exhaustive study of Markov decision process (MDP) under mean field interaction both on states and actions in the presence of common noise, and when optimization is performed over open-loop controls on infinite horizon. Such model, called CMKV-MDP for conditional McKean–Vlasov MDP, arises and is obtained here rigorously with a rate of convergence as the asymptotic problem of N-cooperative agents controlled by a social planner/influencer that observes the environment noises but not necessarily the individual states of the agents. We highlight the crucial role of relaxed controls and randomization hypothesis for this class of models with respect to classical MDP theory. We prove the correspondence between CMKV-MDP and a general lifted MDP on the space of probability measures, and establish the dynamic programming Bellman fixed point equation satisfied by the value function, as well as the existence of ϵ-optimal randomized feedback controls. The arguments of proof involve an original measurable optimal coupling for the Wasserstein distance. This provides a procedure for learning strategies in a large population of interacting collaborative agents.

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BERRY–ESSEEN BOUNDS FOR CHERNOFF-TYPE NONSTANDARD ASYMPTOTICS IN ISOTONIC REGRESSION

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A Chernoff-type distribution is a nonnormal distribution defined by the slope at zero of the greatest convex minorant of a two-sided Brownian motion with a polynomial drift. While a Chernoff-type distribution is known to appear as the distributional limit in many nonregular statistical estimation problems, the accuracy of Chernoff-type approximations has remained largely unknown. In the present paper, we tackle this problem and derive Berry–Esseen bounds for Chernoff-type limit distributions in the canonical nonregular statistical estimation problem of isotonic (or monotone) regression. The derived Berry–Esseen bounds match those of the oracle local average estimator with optimal bandwidth in each scenario of possibly different Chernoff-type asymptotics, up to multiplicative logarithmic factors. Our method of proof differs from standard techniques on Berry–Esseen bounds, and relies on new localization techniques in isotonic regression and an anti-concentration inequality for the supremum of a Brownian motion with a Lipschitz drift.

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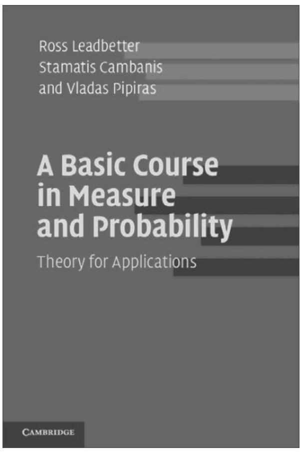

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