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GENERAL SELECTION MODELS: BERNSTEIN DUALITY AND MINIMAL ANCESTRAL STRUCTURES

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Λ-Wright–Fisher processes provide a robust framework to describe the type-frequency evolution of an infinite neutral population. We add a polynomial drift to the corresponding stochastic differential equation to incorporate frequency-dependent selection. A decomposition of the drift allows us to approximate the solution of the stochastic differential equation by a sequence of Moran models. The genealogical structure underlying the Moran model leads in the large population limit to a generalisation of the ancestral selection graph of Krone and Neuhauser. Building on this object, we construct a continuous-time Markov chain and relate it to the forward process via a new form of duality, which we call Bernstein duality. We adapt classical methods based on the moment duality to determine the time to absorption and criteria for the accessibility of the boundaries; this extends a recent result by González Casanova and Spanò. An intriguing feature of the construction is that the same forward process is compatible with multiple backward models. In this context we introduce suitable notions for minimality among the ancestral processes and characterise the corresponding parameter sets. In this way we recover classic ancestral structures as minimal ones.

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CAPACITY OF THE RANGE OF TREE-INDEXED RANDOM WALK

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By introducing a new measure for the infinite Galton–Watson process and providing estimates for (discrete) Green's functions on trees, we establish the asymptotic behavior of the capacity of critical branching random walks: in high dimensions $d \geq 7$, the capacity grows linearly; and in the critical dimension $d = 6$, it grows asymptotically proportional to $\frac{n}{\log n}$.

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UNIFORM POINCARÉ AND LOGARITHMIC SOBOLEV INEQUALITIES FOR MEAN FIELD PARTICLE SYSTEMS

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In this paper we consider a mean field particle systems whose confinement potentials have many local minima. We establish some explicit and sharp estimates of the spectral gap and logarithmic Sobolev constants uniform in the number of particles. The uniform Poincaré inequality is based on the work of Ledoux (In *Séminaire de Probabilités, XXXV* (2001) 167–194, Springer) and the uniform logarithmic Sobolev inequality is based on Zegarlinski's theorem for Gibbs measures, both combined with an explicit estimate of the Lipschitz norm of the Poisson operator for a single particle from (*J. Funct. Anal.* **257** (2009) 4015–4033). The logarithmic Sobolev inequality then implies the exponential convergence in entropy of the McKean–Vlasov equation with an explicit rate, We need here weaker conditions than the results of (*Rev. Mat. Iberoam.* **19** (2003) 971–1018) (by means of the displacement convexity approach), (*Stochastic Process. Appl.* **95** (2001) 109–132; *Ann. Appl. Probab.* **13** (2003) 540–560) (by Bakry–Emery's technique) or the recent work (*Arch. Ration. Mech. Anal.* **208** (2013) 429–445) (by dissipation of the Wasserstein distance).

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FUNCTIONAL LIMIT THEOREMS FOR NON-MARKOVIAN EPIDEMIC MODELS

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We study non-Markovian stochastic epidemic models (SIS, SIR, SIRS, and SEIR), in which the infectious (and latent/exposing, immune) periods have a general distribution. We provide a representation of the evolution dynamics using the time epochs of infection (and latency/exposure, immunity). Taking the limit as the size of the population tends to infinity, we prove both a functional law of large number (FLLN) and a functional central limit theorem (FCLT) for the processes of interest in these models. In the FLLN, the limits are a unique solution to a system of deterministic Volterra integral equations, while in the FCLT, the limit processes are multidimensional Gaussian solutions of linear Volterra stochastic integral equations. In the proof of the FCLT, we provide an important Poisson random measures representation of the diffusion-scaled processes converging to Gaussian components driving the limit process.

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LARGE DEVIATIONS OF MEAN-FIELD INTERACTING PARTICLE SYSTEMS IN A FAST VARYING ENVIRONMENT

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This paper studies large deviations of a "fully coupled" finite state mean-field interacting particle system in a fast varying environment. The empirical measure of the particles evolves in the slow time scale and the random environment evolves in the fast time scale. Our main result is the path-space large deviation principle for the joint law of the empirical measure process of the particles and the occupation measure process of the fast environment. This extends previous results known for two time scale diffusions to two time scale mean-field models with jumps. Our proof is based on the method of stochastic exponentials. We characterise the rate function by studying a certain variational problem associated with an exponential martingale.

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UNDERSTANDING THE DUAL FORMULATION FOR THE HEDGING OF PATH-DEPENDENT OPTIONS WITH PRICE IMPACT

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We consider a general path-dependent version of the hedging problem with price impact of Bouchard et al. (*SIAM J. Control Optim.* **57** (2019) 4125–4149), in which a dual formulation for the super-hedging price is obtained by means of PDE arguments, in a Markovian setting and under strong regularity conditions. Using only probabilistic arguments, we prove, in a path-dependent setting and under weak regularity conditions, that any solution to this dual problem actually allows one to construct explicitly a perfect hedging portfolio. From a pure probabilistic point of view, our approach also allows one to exhibit solutions to a specific class of second order forward backward stochastic differential equations, in the sense of Cheridito et al. (*Comm. Pure Appl. Math.* **60** (2007) 1081–1110). Existence of a solution to the dual optimal control problem is also addressed in particular settings. As a by-product of our arguments, we prove a version of Itô's lemma for path-dependent functionals that are only $C^{0,1}$ in the sense of Dupire.

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ON THE SPECTRUM OF DENSE RANDOM GEOMETRIC GRAPHS

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In this paper we study the spectrum of the random geometric graph $G(n, r)$, in a regime where the graph is dense and highly connected. In the Erdős–Rényi $G(n, p)$ random graph it is well known that upon connectivity the spectrum of the normalized graph Laplacian is concentrated around 1. We show that such concentration does not occur in the $G(n, r)$ case, even when the graph is dense and almost a complete graph. In particular, we show that the limiting spectral gap is strictly smaller than 1. In the special case where the vertices are distributed uniformly in the unit cube and $r = 1$, we show that for every $0 \le k \le d$ there are at least $\binom{d}{k}$ eigenvalues near $1 - 2^{-k}$, and the limiting spectral gap is exactly $1/2$. We also show that the corresponding eigenfunctions in this case are tightly related to the geometric configuration of the points.

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ON THE VALUE OF NON-MARKOVIAN DYNKIN GAMES WITH PARTIAL AND ASYMMETRIC INFORMATION

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We prove that zero-sum Dynkin games in continuous time with partial and asymmetric information admit a value in randomised stopping times when the stopping payoffs of the players are general càdlàg measurable processes. As a by-product of our method of proof we also obtain existence of optimal strategies for both players. The main novelties are that we do not assume a Markovian nature of the game nor a particular structure of the information available to the players. This allows us to go beyond the variational methods (based on PDEs) developed in the literature on Dynkin games in continuous time with partial/asymmetric information. Instead, we focus on a probabilistic and functional analytic approach based on the general theory of stochastic processes and Sion's min-max theorem (*Pacific J. Math.* **8** (1958) 171–176). Our framework encompasses examples found in the literature on continuous time Dynkin games with asymmetric information and we provide counterexamples to show that our assumptions cannot be further relaxed.

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CENTRAL LIMIT THEOREMS FOR STATIONARY RANDOM FIELDS UNDER WEAK DEPENDENCE WITH APPLICATION TO AMBIT AND MIXED MOVING AVERAGE FIELDS

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We obtain central limit theorems for stationary random fields employing a novel measure of dependence called θ-lex weak dependence. We show that this dependence notion is more general than strong mixing, that is, it applies to a broader class of models. Moreover, we discuss hereditary properties for θ-lex and η-weak dependence and illustrate the possible applications of the weak dependence notions to the study of the asymptotic properties of stationary random fields. Our general results apply to mixed moving average fields (MMAF) and ambit fields. We show general conditions such that MMAF and ambit fields, with the volatility field being an MMAF or a p-dependent random field, are weakly dependent. For all the models mentioned above, we give a complete characterization of their weak dependence coefficients and sufficient conditions to obtain the asymptotic normality of their sample moments. Finally, we give explicit computations of the weak dependence coefficients of MSTOU processes and analyze under which conditions the developed asymptotic theory applies to CARMA fields.

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ASYMPTOTIC BEHAVIOR OF A CRITICAL FLUID MODEL FOR BANDWIDTH SHARING WITH GENERAL FILE SIZE DISTRIBUTIONS

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This work concerns the asymptotic behavior of solutions to a critical fluid model for a data communication network, where file sizes are generally distributed and the network operates under a fair bandwidth sharing policy, chosen from the family of (weighted) α-fair policies introduced by Mo and Walrand (*IEEE/ACM Trans. Netw.* **8** (2000) 556–567). Solutions of the fluid model are measure-valued functions of time. Under law of large numbers scaling, Gromoll and Williams (*Ann. Appl. Probab.* **19** (2009) 243–280) proved that these solutions approximate dynamic solutions of a flow level model for congestion control in data communication networks, introduced by Massoulié and Roberts (*Telecommun. Syst.* **15** (2000) 185–201).

In a recent work (*Stoch. Syst.* **10** (2020) 251–273), we proved stability of the strictly subcritical version of this fluid model under mild assumptions. In the current work, we study the asymptotic behavior (as time goes to infinity) of solutions of the *critical* fluid model, in which the nominal load on each network resource is less than or equal to its capacity and at least one resource is fully loaded. For this we introduce a new Lyapunov function, inspired by the work of Kelly and Williams (*Ann. Appl. Probab.* **14** (2004) 1055–1083), Mulvany, Puha and Williams (*Queueing Syst.* **93** (2019) 351–397) and Paganini et al. (*IEEE Trans. Automat. Control* **57** (2012) 579–591). Using this, under moderate conditions on the file size distributions, we prove that critical fluid model solutions converge uniformly to the set of invariant states as time goes to infinity, when started in suitable relatively compact sets. We expect that this result will play a key role in developing a diffusion approximation for the critically loaded flow level model of Massoulié and Roberts (*Telecommun. Syst.* **15** (2000) 185–201). Furthermore, the techniques developed here may be useful for studying other stochastic network models with resource sharing.

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A DYNAMIC PROGRAMMING APPROACH TO DISTRIBUTION-CONSTRAINED OPTIMAL STOPPING

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We consider an optimal stopping problem where a constraint is placed on the distribution of the stopping time. Reformulating the problem in terms of so-called measure-valued martingales enables us to transform the distributional constraint into an initial condition and view the problem as a stochastic control problem; we establish the corresponding dynamic programming principle. The method offers a systematic approach for solving the problem for general constraints and under weak assumptions on the cost function. In addition, we provide certain continuity results for the value of the problem viewed as a function of its distributional constraint.

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WEAK QUANTITATIVE PROPAGATION OF CHAOS VIA DIFFERENTIAL CALCULUS ON THE SPACE OF MEASURES

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Consider the metric space $(\mathcal{P}_2(\mathbb{R}^d), W_2)$ of square integrable laws on $\mathbb{R}^d$ with the topology induced by the 2-Wasserstein distance W_2. Let $\Phi : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$. In this work, we consider (a) μ_N being the empirical measure of N-samples from μ, and the other case in which (b) μ_N is the empirical measure of marginal laws of the particle system of a McKean–Vlasov PDE $(\mu_t)_t$. The main result of this paper is to show that under suitable regularity conditions, we have

$$\left|\Phi(\mu) - \mathbb{E}\Phi(\mu_N)\right| = \sum_{j=1}^{k-1} \frac{C_j}{N^j} + O\left(\frac{1}{N^k}\right),$$

for some positive constants $C_1, \dots, C_{k-1}$ that do not depend on N, where k corresponds to the degree of smoothness. The case where the samples are i.i.d. is studied using functional derivatives on the space of measures. The case of particle systems relies on an Itô-type formula for the flow of probability measures and is intimately connected to PDEs on the space of measures, called the master equation in the literature of mean-field games. We state general regularity conditions required for each case and analyze the regularity in the case of functionals of the laws of McKean–Vlasov PDEs. Ultimately, this work reveals quantitative estimates of propagation of chaos for interacting particle systems. Furthermore, we are able to provide weak propagation of chaos estimates for ensembles of interacting particles and show that these may have some remarkable properties.

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CENTRAL LIMIT THEOREM FOR THE ANTITHETIC MULTILEVEL MONTE CARLO METHOD

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In this paper, we give a natural extension of the antithetic multilevel Monte Carlo (MLMC) estimator for a multidimensional diffusion introduced by Giles and Szpruch (*Ann. Appl. Probab.* **24** (2014) 1585–1620) by considering the permutation between m Brownian increments, $m \geq 2$, instead of using two increments as in the original paper. Our aim is to study the asymptotic behavior of the weak errors involved in this new algorithm. Among the obtained results, we prove that the error between on the one hand the average of the Milstein scheme without Lévy area and its σ-antithetic version build on the finer grid, and on the other hand, the coarse approximation stably converges in distribution with a rate of order 1. We also prove that the error between the Milstein scheme without Lévy area and its σ-antithetic version stably converges in distribution with a rate of order $1/2$. More precisely, we have a functional limit theorem on the asymptotic behavior of the joined distribution of these errors based on a triangular array approach (see, e.g., Jacod (In *Séminaire de Probabilités, XXXI* (1997) 232–246 Springer). Thanks to this result, we establish a central limit theorem of Lindeberg–Feller type for the antithetic MLMC estimator. The time complexity of the algorithm is analyzed.

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CENTRAL LIMIT THEOREM FOR GIBBS MEASURES ON PATH SPACES INCLUDING LONG RANGE AND SINGULAR INTERACTIONS AND HOMOGENIZATION OF THE STOCHASTIC HEAT EQUATION

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We consider a class of Gibbs measures defined with respect to increments $\{\omega(t)-\omega(s)\}_{s<t}$ of d-dimensional Wiener measure, with the underlying Hamiltonian carrying interactions of the form $H(t-s,\omega(t)-\omega(s))$ that are invariant under uniform translations of paths. In such interactions, we allow *long-range* dependence in the time variable (including power law decay up to $t\mapsto(1+t)^{-(2+\varepsilon)}$ for $\varepsilon>0$) and *unbounded (singular)* interactions (including singularities of the form $x\mapsto 1/|x|^p$ in $d\geq 3$ or $x\mapsto\delta_0(x)$ in $d=1$) attached to the space variables. These assumptions on the interaction seem to be sharp and cover quantum mechanical models like the Nelson model and the polaron problem with ultraviolet cut off (both carrying bounded spatial interactions with power law decay in time) as well as the Fröhlich polaron with a short range interaction in time but carrying Coulomb singularity in space. In this set up, we develop a unified approach for proving a central limit theorem for the rescaled process of increments for any coupling parameter and obtain an explicit expression for the limiting variance, which is strictly positive.

As a further application, we study the solution of the multiplicative-noise stochastic heat equation in spatial dimensions $d\geq 3$. When the noise is mollified both in time and space, we show that the averages of the diffusively rescaled solutions converge pointwise to the solution of a diffusion equation whose coefficients are homogenized in this limit.

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CORRELATION DECAY FOR HARD SPHERES VIA MARKOV CHAINS

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We improve upon all known lower bounds on the critical fugacity and critical density of the hard sphere model in dimensions three and higher. As the dimension tends to infinity, our improvements are by factors of 2 and 1.7, respectively. We make these improvements by utilizing techniques from theoretical computer science to show that a certain Markov chain for sampling from the hard sphere model mixes rapidly at low enough fugacities. We then prove an equivalence between optimal spatial and temporal mixing for hard spheres to deduce our results.

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NEAR EQUILIBRIUM FLUCTUATIONS FOR SUPERMARKET MODELS WITH GROWING CHOICES

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We consider the supermarket model in the usual Markovian setting where jobs arrive at rate $n\lambda_n$ for some $\lambda_n > 0$, with n parallel servers each processing jobs in its queue at rate 1. An arriving job joins the shortest among $d_n \le n$ randomly selected service queues. We show that when $d_n \to \infty$ and $\lambda_n \to \lambda \in (0, \infty)$, under natural conditions on the initial queues, the state occupancy process converges in probability, in a suitable path space, to the unique solution of an infinite system of constrained ordinary differential equations parametrized by λ. Our main interest is in the study of fluctuations of the state process about its near equilibrium state in the critical regime, namely when $\lambda_n \to 1$. Previous papers, for example, (*Stoch. Syst.* **8** (2018) 265–292) have considered the regime $\frac{d_n}{\sqrt{n}\log n} \to \infty$ while the objective of the current work is to develop diffusion approximations for the state occupancy process that allow for all possible rates of growth of d_n. In particular, we consider the three canonical regimes (a) $d_n/\sqrt{n} \to 0$; (b) $d_n/\sqrt{n} \to c \in (0, \infty)$ and, (c) $d_n/\sqrt{n} \to \infty$. In all three regimes, we show, by establishing suitable functional limit theorems, that (under conditions on λ_n) fluctuations of the state process about its near equilibrium are of order $n^{-1/2}$ and are governed asymptotically by a one-dimensional Brownian motion. The forms of the limit processes in the three regimes are quite different; in the first case, we get a linear diffusion; in the second case, we get a diffusion with an exponential drift; and in the third case we obtain a reflected diffusion in a half space. In the special case $d_n/(\sqrt{n}\log n) \to \infty$, our work gives alternative proofs for the universality results established in (*Stoch. Syst.* **8** (2018) 265–292).

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THE MULTIPLICATIVE CHAOS OF $H = 0$ FRACTIONAL BROWNIAN FIELDS

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We consider a family of fractional Brownian fields $\{B^H\}_{H\in(0,1)}$ on $\mathbb{R}^d$, where H denotes their Hurst parameter. We first define a rich class of normalizing kernels ψ and we rescale the normalised field by the square-root of the gamma function $\Gamma(H)$, such that the covariance of

$$X^H(x) = \Gamma(H)^{\frac{1}{2}}\left(B^H(x) - \int_{\mathbb{R}^d} B^H(u)\psi(u,x)\,du\right),$$

converges to the covariance of a log-correlated Gaussian field when $H \downarrow 0$. We then use Berestycki's "good points" approach (*Electron. Commun. Probab.* **22** (2017) Paper No. 27) in order to derive the convergence of the exponential measure of the fractional Brownian field

$$M_\gamma^H(dx) = e^{\gamma X^H(x) - \frac{\gamma^2}{2}E[X^H(x)^2]}\,dx,$$

towards a *Gaussian multiplicative chaos*, as $H \downarrow 0$ for all $\gamma \in (0, \gamma^*(d))$, where $\gamma^*(d) > \sqrt{\frac{7}{4}d}$. As a corollary we establish the L^2 convergence of M_γ^H over the sets of "good points", where the field X^H has a typical behaviour. As a by-product of the convergence result, we prove that for log-normal rough volatility models with small Hurst parameter, the volatility process is supported on the sets of "good points" with probability close to 1. Moreover, on these sets the volatility converges in L^2 to the volatility of multifractal random walks.

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ROOT FINDING ALGORITHMS AND PERSISTENCE OF JORDAN CENTRALITY IN GROWING RANDOM TREES

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We consider models of growing random trees $\{\mathcal{T}_f(n) : n \geq 1\}$ with model dynamics driven by an attachment function $f : \mathbb{Z}_+ \to \mathbb{R}_+$. At each stage a new vertex enters the system and connects to a vertex v in the current tree with probability proportional to $f(\text{degree}(v))$. The main goal of this study is to understand the performance of root finding algorithms. A large body of work (e.g., *Random Structures Algorithms* **50** (2017) 158–172; *IEEE Trans. Netw. Sci. Eng.* **4** (2017) 1–12; *Random Structures Algorithms* **52** (2018) 136–157) has emerged in the last few years in using techniques based on the Jordan centrality measure (*J. Reine Angew. Math.* **70** (1869) 185–190) and its variants to develop root finding algorithms. Given an unlabeled unrooted tree, one computes the Jordan centrality for each vertex in the tree and for a fixed budget K outputs the optimal K vertices (as measured by Jordan centrality). Under general conditions on the attachment function f, we derive necessary and sufficient bounds on the budget $K(\varepsilon)$ in order to recover the root with probability at least $1 - \varepsilon$. For canonical examples such as linear preferential attachment and uniform attachment, these general results give matching upper and lower bounds for the budget. We also prove persistence of the optimal K Jordan centers for any K, that is, the existence of an almost surely finite random time n^* such that for $n \geq n^*$ the identity of the K-optimal Jordan centers in $\{\mathcal{T}_f(n) : n \geq n^*\}$ does not change, thus describing robustness properties of this measure. Key technical ingredients in the proofs of independent interest include sufficient conditions for the existence of exponential moments for limits of (appropriately normalized) continuous time branching processes within which the models $\{\mathcal{T}_f(n) : n \geq 1\}$ can be embedded, as well as rates of convergence results to these limits.

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A TWO-PARAMETER FAMILY OF MEASURE-VALUED DIFFUSIONS WITH POISSON–DIRICHLET STATIONARY DISTRIBUTIONS

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We give a pathwise construction of a two-parameter family of purely-atomic-measure-valued diffusions in which ranked masses of atoms are stationary with the Poisson–Dirichlet(α,θ) distributions, for $\alpha \in (0,1)$ and $\theta \geq 0$. These processes resolve a conjecture of Feng and Sun (*Probab. Theory Related Fields* **148** (2010) 501–525). We build on our previous work on $(\alpha,0)$- and (α,α)-interval partition evolutions. The extension to general $\theta \geq 0$ is achieved by the construction of a σ-finite excursion measure of a new measure-valued branching diffusion. Our measure-valued processes are Hunt processes on an incomplete subspace of the space of all probability measures and do not possess an extension to a Feller process. In a companion paper, we use generators to show that ranked masses evolve according to a two-parameter family of diffusions introduced by Petrov (*Funktsional. Anal. i Prilozhen.* **43** (2009) 45–66), extending work of Ethier and Kurtz (*Adv. in Appl. Probab.* **13** (1981) 429–452).

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QUANTUM MEAN-FIELD GAMES

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In this paper we are merging the two new branches of game theory: quantum games and mean-field games (MFG). Building a quantum analog of MFGs requires the full reconstruction of its foundations and methodology, because in N-particle quantum evolution particles are not separated in individual dynamics and the key concept of the classical MFG theory, the empirical measure defined as the sum of Dirac masses of the positions of the players, is not applicable in quantum setting.

As a preliminary result we derive the new nonlinear stochastic Schrödinger equation, as the limit of the quantum filtering equation describing continuously observed and controlled system of a large number of interacting particles, the result that may have an independent value. We then show that to a control quantum system of interacting particles there corresponds a special system of classical interacting particles with the identical limiting MFG system, defined on an appropriate Riemanian manifold. Solutions of this system are shown to specify approximate Nash equilibria for N-agent quantum games.

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ON FIRST ORDER MEAN FIELD GAME SYSTEMS WITH A COMMON NOISE

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We consider mean field games without idiosyncratic but with Brownian type common noise. We introduce a notion of solutions of the associated backward-forward system of stochastic partial differential equations. We show that the solution exists and is unique for monotone coupling functions. We also use the solution to find approximate optimal strategies (Nash equilibria) for N-player differential games with common but no idiosyncratic noise. An important step in the analysis is the study of the well-posedness of a stochastic backward Hamilton–Jacobi equation.

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LARGE DEVIATION FOR UNIFORM GRAPHS WITH GIVEN DEGREES

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Consider the random graph sampled uniformly from the set of all simple graphs with a given degree sequence. Under mild conditions on the degrees, we establish a large deviation principle (LDP) for these random graphs, viewed as elements of the graphon space. As a corollary of our result, we obtain LDPs for functionals continuous with respect to the cut metric, and obtain an asymptotic enumeration formula for graphs with given degrees, subject to an additional constraint on the value of a continuous functional. Our assumptions on the degrees are identical to those of Chatterjee, Diaconis and Sly (*Ann. Appl. Probab.* **21** (2011) 1400–1435), who derived the almost sure graphon limit for these random graphs.

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