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RISK-SENSITIVE CREDIT PORTFOLIO OPTIMIZATION UNDER PARTIAL INFORMATION AND CONTAGION RISK

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This paper investigates the finite horizon risk-sensitive portfolio optimization in a regime-switching credit market with physical and information-induced default contagion. It is assumed that the underlying regime-switching process has countable states and is unobservable. The stochastic control problem is formulated under partial observations of asset prices and sequential default events. By establishing a martingale representation theorem based on incomplete and phasing out filtration, we connect the control problem to a quadratic BSDE with jumps, in which the driver term is nonstandard and carries the conditional filter as an infinite-dimensional parameter. By proposing some truncation techniques and proving uniform a priori estimates, we obtain the existence of a solution to the BSDE using the convergence of solutions associated to some truncated BSDEs. The verification theorem can be concluded with the aid of our BSDE results, which in turn yields the uniqueness of the solution to the BSDE.

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LINES OF DESCENT IN THE DETERMINISTIC MUTATION–SELECTION MODEL WITH PAIRWISE INTERACTION

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We consider the mutation–selection differential equation with pairwise interaction (or, equivalently, the diploid mutation–selection equation) and establish the corresponding ancestral process, which is a random tree and a variant of the ancestral selection graph. The formal relation to the forward model is given via duality. To make the tree tractable, we prune branches upon mutations, thus reducing it to its informative parts. The hierarchies inherent in the tree are encoded systematically via tripod trees with weighted leaves; this leads to the stratified ancestral selection graph. The latter also satisfies a duality relation with the mutation–selection equation. Each of the dualities provides a stochastic representation of the solution of the differential equation. This allows us to connect the equilibria and their bifurcations to the long-term behaviour of the ancestral process. Furthermore, with the help of the stratified ancestral selection graph, we obtain explicit results about the ancestral type distribution in the case of unidirectional mutation.

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LIMITS OF MULTIPLICATIVE INHOMOGENEOUS RANDOM GRAPHS AND LÉVY TREES: THE CONTINUUM GRAPHS

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Motivated by limits of critical inhomogeneous random graphs, we construct a family of measured metric spaces that we call *continuous multiplicative graphs*, that are expected to be the universal limit of graphs related to the multiplicative coalescent (the Erdős–Rényi random graph, more generally the so-called rank-one inhomogeneous random graphs of various types, and the configuration model). At the discrete level, the construction relies on a new point of view on (discrete) inhomogeneous random graphs that involves an embedding into a Galton–Watson forest. The new representation allows us to demonstrate that a process that was already present in the pioneering work of Aldous [*Ann. Probab.* **25** (1997) 812–854] and Aldous and Limic [*Electron. J. Probab.* **3** (1998) 1–59] about the multiplicative coalescent actually also essentially encodes the limiting metric. The discrete embedding of random graphs into a Galton–Watson forest is paralleled by an embedding of the encoding process into a Lévy process which is crucial in proving the very existence of the local time functionals on which the metric is based; it also yields a transparent approach to compactness and fractal dimensions of the continuous objects. In a companion paper, we show that the continuous multiplicative graphs are indeed the scaling limit of inhomogeneous random graphs.

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BARYCENTRIC BROWNIAN BEES

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We establish an invariance principle for the barycenter of a Brunet–Derrida particle system in d dimensions. The model consists of N particles undergoing dyadic branching Brownian motion with rate 1. At a branching event, the number of particles is kept equal to N by removing the particle located furthest away from the barycenter. To prove the invariance principle, a key step is to establish Harris recurrence for the process viewed from its barycenter.

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HAMILTON–JACOBI EQUATIONS FOR NONSYMMETRIC MATRIX INFERENCE

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We study the high-dimensional limit of the free energy associated with the inference problem of a rank-one nonsymmetric matrix. The matrix is expressed as the outer product of two vectors, not necessarily independent. The distributions of the two vectors are only assumed to have scaled bounded supports. We bound the difference between the free energy and the solution to a suitable Hamilton–Jacobi equation in terms of two much simpler quantities: concentration rate of this free energy, and the convergence rate of a simpler free energy in a decoupled system. To demonstrate the versatility of this approach, we apply our result to the i.i.d. case and the spherical case. By plugging in estimates of the two simpler quantities, we identify the limits and obtain convergence rates.

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GLOBAL WELL-POSEDNESS OF THE 3D NAVIER–STOKES EQUATIONS PERTURBED BY A DETERMINISTIC VECTOR FIELD

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We are concerned with the problem of global well-posedness of the 3D Navier–Stokes equations on the torus with unitary viscosity. While a full answer to this question seems to be out of reach of the current techniques, we establish a regularization by a deterministic vector field. More precisely, we consider the vorticity form of the system perturbed by an additional transport type term. Such a perturbation conserves the enstrophy and therefore a priori it does not imply any smoothing. Our main result is a construction of a deterministic vector field $v = v(t, x)$ which provides the desired regularization of the system and yields global well-posedness for large initial data outside arbitrary small sets. The proof relies on probabilistic arguments developed by Flandoli and Luo, tools from rough path theory by Hofmanová, Leahy and Nilssen and a new Wong–Zakai approximation result, which itself combines probabilistic and rough path techniques.

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HEAVY TRAFFIC SCALING LIMITS FOR SHORTEST REMAINING PROCESSING TIME QUEUES WITH HEAVY TAILED PROCESSING TIME DISTRIBUTIONS

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We study a single server queue operating under the shortest remaining processing time (SRPT) scheduling policy; that is, the server preemptively serves the job with the shortest remaining processing time first. Since one needs to keep track of the remaining processing times of all jobs in the system in order to describe the evolution, a natural state descriptor for an SRPT queue is a measure valued process in which the state of the system at a given time is the finite nonnegative Borel measure on the nonnegative real line that puts a unit atom at the remaining processing time of each job in system. In this work we are interested in studying the asymptotic behavior of the suitably scaled measure valued state descriptors for a sequence of SRPT queuing systems. Gromoll, Kruk and Puha (*Stoch. Syst.* **1** (2011) 1–16) have studied this problem under diffusive scaling (time is scaled by r^2 and the mass of the measure normalized by r, where r is a scaling parameter approaching infinity). In the setting where the processing time distributions have *bounded support*, under suitable conditions, they show that the measure valued state descriptors converge in distribution to the process that at any given time is a single atom located at the right edge of the support of the processing time distribution with the size of the atom fluctuating randomly in time. In the setting where the processing time distributions have *unbounded support*, under suitable conditions, they show that the diffusion scaled measure valued state descriptors converge in distribution to the process that is identically zero. In Puha (*Ann. Appl. Probab.* **25** (2015) 3381–3404) for the setting where the processing time distributions have *unbounded support and light tails*, a nonstandard scaling of the queue length process is shown to give rise to a form of state space collapse that results in a nonzero limit.

In the current work we consider the case where processing time distributions have finite second moments and regularly varying tails. Results of Puha (*Ann. Appl. Probab.* **25** (2015) 3381–3404) suggest that the right scaling for the measure valued process is governed by a parameter c^r that is given as a certain inverse function related to the tails of the first moment of the processing time distribution. Using this parameter we consider a novel scaling for the measure valued process in which the time is scaled by a factor of r^2, the mass is scaled by the factor c^r/r and the space (representing the remaining processing times) is scaled by the factor $1/c^r$. We show that the scaled measure valued process converges in distribution (in the space of paths of measures). In a sharp contrast to results for bounded support and light tailed service time distributions, this time there is no state space collapse and the limiting measures are not concentrated on a single atom. Nevertheless, the description of the limit is simple and given explicitly in terms of a certain $\mathbb{R}_+$ valued random field which is determined from a single Brownian motion. Along the way we establish convergence of suitably scaled workload

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and queue length processes. We also show that as the tail of the distribution of job processing times becomes lighter in an appropriate fashion, the difference between the limiting queue length process and the limiting workload process converges to zero, thereby approaching the behavior of state space collapse.

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QUICKEST REAL-TIME DETECTION OF A BROWNIAN COORDINATE DRIFT

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Consider the motion of a Brownian particle in two or more dimensions, whose coordinate processes are standard Brownian motions with zero drift initially, and then at some random/unobservable time, one of the coordinate processes gets a (known) nonzero drift permanently. Given that the position of the Brownian particle is being observed in real time, the problem is to detect the time at which a coordinate process gets the drift as accurately as possible. We solve this problem in the most uncertain scenario when the random/unobservable time is (i) exponentially distributed and (ii) independent from the initial motion without drift. The solution is expressed in terms of a stopping time that minimises the probability of a false early detection and the expected delay of a missed late detection. To our knowledge this is the first time that such a problem has been solved exactly in the literature.

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VERTEX-REINFORCED JUMP PROCESS ON THE INTEGERS WITH NONLINEAR REINFORCEMENT

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We consider a nonlinear vertex-reinforced jump process (VRJP(w)) on $\mathbb{Z}$ with an increasing measurable weight function $w : [1, \infty) \to [1, \infty)$ and initial weights equal to one. Our main goal is to study the asymptotic behaviour of VRJP(w) depending on the integrability of the reciprocal of w. In particular, we prove that if $\int_1^\infty \frac{\mathrm{d}u}{w(u)} = \infty$ then the process is recurrent, that is, it visits each vertex infinitely often and all local times are unbounded. On the other hand, if $\int_1^\infty \frac{\mathrm{d}u}{w(u)} < \infty$ and there exists a $\rho > 0$ such that $t \mapsto w(t)^\rho \int_t^\infty \frac{\mathrm{d}u}{w(u)}$ is nonincreasing then the process will eventually get stuck on exactly three vertices, and there is only one vertex with unbounded local time. We also show that if the initial weights are all the same, VRJP on $\mathbb{Z}$ cannot be transient, that is, there exists at least one vertex that is visited infinitely often. Our results extend the ones previously obtained by Davis and Volkov (*Probab. Theory Related Fields* **123** (2002) 281–300) who showed that VRJP with linear reinforcement on $\mathbb{Z}$ is recurrent.

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EIGENVECTOR CORRELATIONS IN THE COMPLEX GINIBRE ENSEMBLE

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The complex Ginibre ensemble is the distribution of an $N \times N$ non-Hermitian random matrix over $\mathbb{C}$ with i.i.d. complex Gaussian entries normalized to have mean zero and variance $1/N$. Unlike the Gaussian unitary ensemble, for which the eigenvectors are distributed according to Haar measure on the compact group $U(N)$, independently of the eigenvalues, the geometry of the eigenbases of the Ginibre ensemble are not particularly well understood. In this paper we systematically study properties of eigenvector correlations in this matrix ensemble. In particular, we uncover an extended algebraic structure which describes their asymptotic behavior (as N goes to infinity). Our work extends previous results of Chalker and Mehlig (*Phys. Rev. Lett.* **81** (1998) 3367–3370), in which the correlation for pairs of eigenvectors was computed.

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AN EFFICIENT ALGORITHM FOR SOLVING ELLIPTIC PROBLEMS ON PERCOLATION CLUSTERS

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We present an efficient algorithm to solve elliptic Dirichlet problems defined on the cluster of supercritical $\mathbb{Z}^d$-Bernoulli percolation, as a generalization of the iterative method proposed by S. Armstrong, A. Hannukainen, T. Kuusi and J.-C. Mourrat (*ESAIM Math. Model. Numer. Anal.* (2021) **55** 37–55). We also explore the two-scale expansion on the infinite cluster of percolation, and use it to give a rigorous analysis of the algorithm.

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STOCHASTIC FIXED-POINT EQUATION AND LOCAL DEPENDENCE MEASURE

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We study solutions to the stochastic fixed-point equation $X \stackrel{d}{=} AX + B$ where the coefficients A and B are nonnegative random variables. We introduce the "local dependence measure" (LDM) and its Legendre-type transform to analyze the left tail behavior of the distribution of X. We discuss the relationship of LDM with earlier results on the stochastic fixed-point equation and we apply LDM to prove a theorem on a Fleming–Viot-type process.

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COALESCING AND BRANCHING SIMPLE SYMMETRIC EXCLUSION PROCESS

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Motivated by kinetically constrained interacting particle systems (KCM), we consider a reversible coalescing and branching simple exclusion process on a general finite graph $G = (V, E)$ dual to the biased voter model on G. Our main goal is tight bounds on its logarithmic Sobolev constant and relaxation time, with particular focus on the delicate slightly supercritical regime in which the equilibrium density of particles tends to zero as $|V| \to \infty$. Our results allow us to recover very directly and improve to ℓ^p-mixing, $p \geq 2$, and to more general graphs, the mixing time results of Pillai and Smith for the Fredrickson–Andersen one spin facilitated (FA-1f) KCM on the discrete d-dimensional torus. In view of applications to the more complex FA-jf KCM, $j > 1$, we also extend part of the analysis to an analogous process with a more general product state space.

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DYNAMICAL MODELS FOR RANDOM SIMPLICIAL COMPLEXES

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We study a general model of random dynamical simplicial complexes and derive a formula for the asymptotic degree distribution. This asymptotic formula generalises results for a number of existing models, including random Apollonian networks and the weighted random recursive tree. It also confirms results on the scale-free nature of complex quantum network manifolds in dimensions $d > 2$, and special types of network geometry with Flavour models studied in the physics literature by Bianconi and Rahmede [*Sci. Rep.* **5** (2015) 13979 and *Phys. Rev. E* **93** (2016) 032315].

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CONVERGENCES OF THE RESCALED WHITTAKER STOCHASTIC DIFFERENTIAL EQUATIONS AND INDEPENDENT SUMS

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We study some SDEs derived from the $q \to 1$ limit of a 2D surface growth model called the q-Whittaker process. The fluctuations are proven to exhibit Gaussian characteristics that "come down from infinity": After rescaling and re-centering, convergences to the time-inverted stationary additive stochastic heat equation (SHE) hold. The point of view in this paper is a novel probabilistic representation of the SDEs by independent sums. By this connection, the normal and Poisson approximations, both in diverging integrated forms, explain the convergence of the re-centered covariance functions. The proof of the process-level convergence identifies additional divergent terms in the dynamics and considers nontrivial cancellations.

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TRACY–WIDOM AT EACH EDGE OF REAL COVARIANCE AND MANOVA ESTIMATORS

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We study the sample covariance matrix for real-valued data with general population covariance, as well as MANOVA-type covariance estimators in variance components models under null hypotheses of global sphericity. In the limit as matrix dimensions increase proportionally, the asymptotic spectra of such estimators may have multiple disjoint intervals of support, possibly intersecting the negative half line. We show that the distribution of the extremal eigenvalue at each regular edge of the support has a GOE Tracy–Widom limit. Our proof extends a comparison argument of Ji Oon Lee and Kevin Schnelli, replacing a continuous Green function flow by a discrete Lindeberg swapping scheme.

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ON THE MINIMAL DRIFT FOR RECURRENCE IN THE FROG MODEL ON d-ARY TREES

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We study the recurrence property of one-per-site frog model FM(d, p) on a d-ary tree with drift parameter $p \in [0, 1]$, which determines the bias of frogs' random walks. In this model, active frogs move toward the root with probability p or otherwise move to a uniformly chosen child vertex. Whenever a site is visited for the first time, a new active frog is introduced at the site. We are interested in the minimal drift p_d so that the frog model is recurrent. Using a coupling argument together with a recursive construction of two series of polynomials involved in the generating functions, we prove that for all $d \geq 2$, $p_d \leq 1/3$, achieving the best, universal upper bound predicted by the monotonicity conjecture.

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CORRECTION TERMS FOR THE HEIGHT OF WEIGHTED RECURSIVE TREES

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Weighted recursive trees are built by adding successively vertices with predetermined weights to a tree: each new vertex is attached to a parent chosen randomly proportionally to its weight. Under some assumptions on the sequence of weights, the first order for the height of such trees has been recently established by one of the authors. In this paper, we obtain the second and third orders in the asymptotic expansion of the height of weighted recursive trees, under similar assumptions. Our methods are inspired from those used to prove similar results for branching random walks. Our results also apply to a related model of growing trees, called the preferential attachment tree with additive fitnesses.

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PAGERANK ASYMPTOTICS ON DIRECTED PREFERENTIAL ATTACHMENT NETWORKS

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We characterize the tail behavior of the distribution of the PageRank of a uniformly chosen vertex in a directed preferential attachment graph and show that it decays as a power law with an explicit exponent that is described in terms of the model parameters. Interestingly, this power law is heavier than the tail of the limiting in-degree distribution, which goes against the commonly accepted *power law hypothesis*. This deviation from the power law hypothesis points at the structural differences between the inbound neighborhoods of typical vertices in a preferential attachment graph versus those in static random graph models where the power law hypothesis has been proven to hold (e.g., directed configuration models and inhomogeneous random digraphs). In addition to characterizing the PageRank distribution of a typical vertex, we also characterize the explicit growth rate of the PageRank of the oldest vertex as the network size grows.

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QUANTITATIVE TWO-SCALE STABILIZATION ON THE POISSON SPACE

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We establish inequalities for assessing the distance between the distribution of a (possibly multidimensional) functional of a Poisson random measure and that of a Gaussian element. Our bounds only involve add-one cost operators at the order one—that we evaluate and compare at two different scales—and are specifically tailored for studying the Gaussian fluctuations of sequences of geometric functionals displaying a form of weak stabilization—see Penrose and Yukich (*Ann. Appl. Probab.* **11** (2001) 1005–1041) and Penrose (*Ann. Probab.* **33** (2005) 1945–1991). Our main bounds extend the estimates recently exploited by Chatterjee and Sen (*Ann. Appl. Probab.* **27** (2017) 1588–1645) in the proof of a quantitative version of the central limit theorem (CLT) for the length of the Poisson-based Euclidean minimal spanning tree (MST). We develop in full detail three applications of our bounds, namely: (i) to a quantitative multidimensional spatial CLT for functionals of the on-line nearest neighbour graph, (ii) to a quantitative multidimensional CLT involving functionals of the empirical measure associated with the edge-length of the Euclidean MST, and (iii) to a collection of multidimensional CLTs for geometric functionals of the excursion set of heavy-tailed shot noise random fields. Application (i) is based on a collection of general probabilistic approximations for strongly stabilizing functionals, that is of independent interest.

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ON MONTE-CARLO METHODS IN CONVEX STOCHASTIC OPTIMIZATION

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We develop a novel procedure for estimating the optimizer of general convex stochastic optimization problems of the form $\min_{x\in\mathcal{X}} \mathbb{E}[F(x,\xi)]$, when the given data is a finite independent sample selected according to ξ. The procedure is based on a median-of-means tournament, and is the first procedure that exhibits the optimal statistical performance in heavy tailed situations: we recover the asymptotic rates dictated by the central limit theorem in a nonasymptotic manner once the sample size exceeds some explicitly computable threshold. Additionally, our results apply in the high-dimensional setup, as the threshold sample size exhibits the optimal dependence on the dimension (up to a logarithmic factor). The general setting allows us to recover recent results on multivariate mean estimation and linear regression in heavy-tailed situations and to prove the first sharp, nonasymptotic results for the portfolio optimization problem.

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ERRATUM: MALLIAVIN CALCULUS APPROACH TO LONG EXIT TIMES FROM AN UNSTABLE EQUILIBRIUM

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We close a gap in the above paper.

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ERRATUM TO "LOWER BOUNDS FOR TRACE RECONSTRUCTION"

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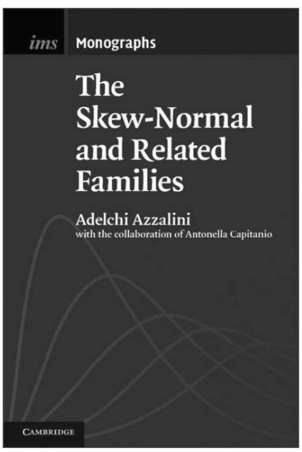

The Skew-Normal and Related Families

Adelchi Azzalini
in collaboration with Antonella Capitanio

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