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CENTRAL MOMENTS OF THE FREE ENERGY OF THE STATIONARY O'CONNELL–YOR POLYMER

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Seppäläinen and Valkó showed in (*ALEA Lat. Am. J. Probab. Math. Stat.* **7** (2010) 451–476) that for a suitable choice of parameters, the variance growth of the free energy of the stationary O'Connell–Yor polymer is governed by the exponent $2/3$, characteristic of models in the KPZ universality class.

We develop exact formulas based on Gaussian integration by parts to relate the cumulants of the free energy, $\log Z^{\theta}_{n,t}$, to expectations of products of quenched cumulants of the time of the first jump from the boundary into the system, s_0. We then use these formulas to obtain estimates for the kth central moment of $\log Z^{\theta}_{n,t}$ as well as the kth annealed moment of s_0 for $k > 2$, with nearly optimal exponents $(1/3)k + \epsilon$ and $(2/3)k + \epsilon$, respectively.

As an application, we derive new high probability bounds for the distance between the polymer path and a straight line connecting the origin to the endpoint of the path.

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MULTISCALE ANALYSIS FOR TRAVELING-PULSE SOLUTIONS TO THE STOCHASTIC FITZHUGH–NAGUMO EQUATIONS

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We investigate the stability of traveling-pulse solutions to the stochastic FitzHugh–Nagumo equations with additive noise. Special attention is given to the effect of small noise on the classical deterministically stable fast traveling pulse. Our method is based on adapting the velocity of the traveling wave by solving a scalar stochastic ordinary differential equation (SODE) and tracking perturbations to the wave meeting a system of a scalar stochastic partial differential equation (SPDE) coupled to a scalar ordinary differential equation (ODE). This approach has been recently employed by Krüger and Stannat (*Nonlinear Anal.* **162** (2017) 197–223) for scalar stochastic bistable reaction–diffusion equations such as the Nagumo equation. A main difference in our situation of an SPDE coupled to an ODE is that the linearization has essential spectrum parallel to the imaginary axis and thus only generates a strongly continuous semigroup. Furthermore, the linearization around the traveling wave is not self-adjoint anymore, so that fluctuations around the wave cannot be expected to be orthogonal in a corresponding inner product. We demonstrate that this problem can be overcome by making use of Riesz instead of orthogonal spectral projections as recently employed in a series of papers by Hamster and Hupkes in case of analytic semigroups. We expect that our approach can also be applied to traveling waves and other patterns in more general situations such as systems of SPDEs with linearizations only generating a strongly continuous semigroup. This provides a relevant generalization as these systems are prevalent in many applications.

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WELL-POSEDNESS AND TAMED SCHEMES FOR MCKEAN–VLASOV EQUATIONS WITH COMMON NOISE

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In this paper, we first establish well-posedness of McKean–Vlasov stochastic differential equations (McKean–Vlasov SDEs) with common noise, possibly with coefficients of super-linear growth in the state variable. Second, we present stable time-stepping schemes for this class of McKean–Vlasov SDEs. Specifically, we propose an explicit tamed Euler and tamed Milstein scheme for an interacting particle system associated with the McKean–Vlasov equation. We prove stability and strong convergence of order $1/2$ and 1, respectively. To obtain our main results, we employ techniques from calculus on the Wasserstein space. The proof for the strong convergence of the tamed Milstein scheme only requires the coefficients to be once continuously differentiable in the state and measure component. To demonstrate our theoretical findings, we present several numerical examples, including mean-field versions of the stochastic $3/2$ volatility model and the stochastic double well dynamics with multiplicative noise.

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GROWTH OF STATIONARY HASTINGS–LEVITOV

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We construct and study a stationary version of the Hastings–Levitov(0) model. We prove that, unlike in the classical HL(0) model, in the stationary case the size of particles attaching to the aggregate is tight, and therefore SHL(0) is proposed as a potential candidate for a stationary off-lattice variant of diffusion limited aggregation (DLA). The stationary setting, together with a geometric interpretation of the harmonic measure, yields new geometric results such as stabilization, finiteness of arms and arm size distribution. We show that, under appropriate scaling, arms in SHL(0) converge to the graph of Brownian motion which has fractal dimension $3/2$. Moreover we show that trees with n particles reach a height of order $n^{2/3}$, corresponding to a numerical prediction of Meakin from 1983 for the gyration radius of DLA growing on a long line segment.

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HIGH-DIMENSIONAL SCALING LIMITS OF PIECEWISE DETERMINISTIC SAMPLING ALGORITHMS

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Piecewise deterministic Markov processes are an important new tool in the design of Markov chain Monte Carlo algorithms. Two examples of fundamental importance are the bouncy particle sampler (BPS) and the zig–zag process (ZZ). In this paper scaling limits for both algorithms are determined. Here the dimensionality of the space tends towards infinity and the target distribution is the multivariate standard normal distribution. For several quantities of interest (angular momentum, first coordinate and negative log-density) the scaling limits show qualitatively very different and rich behaviour. Based on these scaling limits the performance of the two algorithms in high dimensions can be compared. Although for angular momentum both processes require only a computational effort of $O(d)$ to obtain approximately independent samples, the computational effort for negative log-density and first coordinate differ: for these BPS requires $O(d^2)$ computational effort whereas ZZ requires $O(d)$. Finally we provide a criterion for the choice of the refreshment rate of BPS.

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METASTABILITY FOR EXPANDING BUBBLES ON A STICKY SUBSTRATE

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We study the dynamical behavior of a one dimensional interface interacting with a sticky impenetrable substrate or wall. The interface is subject to two effects going in opposite directions. Contacts between the interface and the substrate are given an energetic bonus while an external force with constant intensity pulls the interface away from the wall. Our interface is modeled by the graph of a one-dimensional nearest-neighbor path on $\mathbb{Z}_+$, starting at 0 and ending at 0 after $2N$ steps, with the wall being the horizontal axis. At equilibrium each path $\xi = (\xi_x)_{x=0}^{2N}$ is given a probability proportional to $\lambda^{H(\xi)} \exp(\frac{\sigma}{N} A(\xi))$, where $H(\xi) := \#\{x : \xi_x = 0\}$ and $A(\xi)$ is the area enclosed between the path ξ and the x-axis. We then consider the classical heat-bath dynamics which equilibrates the value of each ξ_x at a constant rate via corner-flip.

Investigating the statics of the model, we derive the full phase diagram in λ and σ of this model, and identify the critical line which separates a localized phase where the pinning force sticks the interface to the wall and a delocalized one, for which the external force stabilizes ξ around a deterministic shape at a macroscopic distance of the wall. On the dynamical side, we identify a second critical line, which separates a rapidly mixing phase (for which the system mixes in polynomial time) to a slow phase where the mixing time grows exponentially. In this slowly mixing regime, we obtain a sharp estimate of the mixing time on the log scale, and provide evidences of a metastable behavior.

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ON EIGENVALUE DISTRIBUTIONS OF LARGE AUTOCOVARIANCE MATRICES

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In this article, we establish a limiting distribution for eigenvalues of a class of autocovariance matrices. The same distribution has been found in the literature for a regularized version of these autocovariance matrices. The original nonregularized autocovariance matrices are noninvertible, thus introducing supplementary difficulties for the study of their eigenvalues through Girko's Hermitization scheme. The key result in this paper is a new polynomial lower bound for a specific family of least singular values associated to a rank-defective quadratic function of a random matrix with independent and identically distributed entries. Another innovation from the paper is that the lag of the autocovariance matrices can grow to infinity with the matrix dimension.

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RADNER EQUILIBRIUM AND SYSTEMS OF QUADRATIC BSDES WITH DISCONTINUOUS GENERATORS

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Motivated by an equilibrium problem, we establish the existence of a solution for a family of Markovian backward stochastic differential equations with quadratic nonlinearity and discontinuity in Z. Using unique continuation and backward uniqueness, we show that the set of discontinuity has measure zero. In a continuous-time stochastic model of an endowment economy, we prove the existence of an incomplete Radner equilibrium with nondegenerate endogenous volatility.

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INTERVAL FRAGMENTATIONS WITH CHOICE: EQUIDISTRIBUTION AND THE EVOLUTION OF TAGGED FRAGMENTS

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We consider a Markovian evolution on point processes, the Ψ-process, on the unit interval in which points are added according to a rule that depends only on the spacings of the existing point configuration. Having chosen a spacing, a new point is added uniformly within it. Building on previous work of the authors and of Junge, we show that the empirical distribution of points in such a process is always equidistributed under mild assumptions on the rule, generalizing work of Junge.

A major portion of this article is devoted to the study of a particular growth-fragmentation process, or cell process, which is a type of piecewise-deterministic Markov process (PDMP). This process represents a linearized version of a size-biased sampling from the Ψ-process. We show that this PDMP is ergodic and develop the semigroup theory of it, to show that it describes a linearized version of the Ψ-process. This PDMP has appeared in other contexts, and in some sense we develop its theory under minimal assumptions.

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DUALITY FOR OPTIMAL CONSUMPTION UNDER NO UNBOUNDED PROFIT WITH BOUNDED RISK

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We give a definitive treatment of duality for optimal consumption over the infinite horizon, in a semimartingale incomplete market satisfying no unbounded profit with bounded risk (NUPBR). Rather than base the dual domain on (local) martingale deflators, we use a class of supermartingale deflators such that deflated wealth plus cumulative deflated consumption is a supermartingale for all admissible consumption plans. This yields a strong duality, because the enlarged dual domain of processes dominated by deflators is naturally closed, without invoking its closure. In this way, we automatically reach the bipolar of the set of deflators. We complete this picture by proving that the set of processes dominated by local martingale deflators is dense in our dual domain, confirming that we have identified the natural dual space. In addition to the optimal consumption and deflator, we characterise the optimal wealth process. At the optimum, deflated wealth is a supermartingale and a potential, while deflated wealth plus cumulative deflated consumption is a uniformly integrable martingale. This is the natural generalisation of the corresponding feature in the terminal wealth problem, where deflated wealth at the optimum is a uniformly integrable martingale. We use no constructions involving equivalent local martingale measures. This is natural, given that such measures typically do not exist over the infinite horizon and that we are working under NUPBR, which does not require their existence. The structure of the duality proof reveals an interesting feature compared with the terminal wealth problem. There, the dual domain is L^1-bounded, but here the primal domain has this property, and hence many steps in the duality proof show a marked reversal of roles for the primal and dual domains, compared with the proofs of Kramkov and Schachermayer (*Ann. Appl. Probab.* **9** (1999) 904–950; *Ann. Appl. Probab.* **13** (2003) 1504–1516).

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SHOCK FLUCTUATIONS IN TASEP UNDER A VARIETY OF TIME SCALINGS

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We consider the totally asymmetric simple exclusion process (TASEP) with two different initial conditions with shock discontinuities formed by blocks of fully packed particles. Initially a second class particle is at the left of a shock discontinuity. Using multicolored TASEP we derive exact formulas for the distribution of the second class particle and colored height functions. These are given in terms of the height function at different positions of a single TASEP configuration. We study the limiting distributions of second class particles (and colored height functions). The result depends on how the width blocks of particles scale with the observation time; we study a variety of such scalings.

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CONSTRUCTION OF CONTINUOUS-STATE BRANCHING PROCESSES IN VARYING ENVIRONMENTS

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A continuous-state branching process in varying environments is constructed by the pathwise unique positive solution to a stochastic integral equation driven by time-space noises. The cumulant semigroup of the process is characterized in terms of a backward integral equation. We clarify the behavior of the process at its bottlenecks, which are the deterministic times when it arrives at zero almost surely by negative jumps. The process arises naturally as the scaling limit of Galton–Watson processes in varying environments.

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MEAN-FIELD GAMES OF FINITE-FUEL CAPACITY EXPANSION WITH SINGULAR CONTROLS

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We study Nash equilibria for a sequence of symmetric N-player stochastic games of finite-fuel capacity expansion with singular controls and their mean-field game (MFG) counterpart. We construct a solution of the MFG via a simple iterative scheme that produces an optimal control in terms of a Skorokhod reflection at a (state-dependent) surface that splits the state space into *action* and *inaction* regions. We then show that a solution of the MFG of capacity expansion induces approximate Nash equilibria for the N-player games with approximation error ε going to zero as N tends to infinity. Our analysis relies entirely on probabilistic methods and extends the well-known connection between singular stochastic control and optimal stopping to a mean-field framework.

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BELIEF PROPAGATION ON THE RANDOM k-SAT MODEL

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Corroborating a prediction from statistical physics, we prove that the belief propagation message passing algorithm approximates the partition function of the random k-SAT model well for all clause/variable densities and all inverse temperatures for which a modest absence of long-range correlations condition is satisfied. This condition is known as "replica symmetry" in physics language. From this result we deduce that a replica symmetry breaking phase transition occurs in the random k-SAT model at low temperature for clause/variable densities below but close to the satisfiability threshold.

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THE PHASE STRUCTURE OF ASYMMETRIC BALLISTIC ANNIHILATION

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Ballistic annihilation is an interacting system in which particles placed throughout the real line move at preassigned velocities and annihilate upon colliding. The longstanding conjecture that in the symmetric three-velocity setting there exists a phase transition for the survival of middle-velocity particles was recently resolved by Haslegrave, Sidoravicius, and Tournier. We develop a framework based on a mass transport principle to analyze three-velocity ballistic annihilation with asymmetric velocities assigned according to an asymmetric probability measure. We show the existence of a phase transition in all cases by deriving universal bounds. In particular, all middle-speed particles perish almost surely if their initial density is less than $1/5$, regardless of the velocities, relative densities, and spacing of initial particles. We additionally prove the continuity of several fundamental statistics as the probability measure is varied.

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CENTRAL LIMIT THEOREM FOR BIFURCATING MARKOV CHAINS UNDER POINTWISE ERGODIC CONDITIONS

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Bifurcating Markov chains (BMC) are Markov chains indexed by a full binary tree representing the evolution of a trait along a population where each individual has two children. We provide a central limit theorem for general additive functionals of BMC, and prove the existence of three regimes. This corresponds to a competition between the reproducing rate (each individual has two children) and the ergodicity rate for the evolution of the trait. This is in contrast with the work of Guyon (*Ann. Appl. Probab.* **17** (2007) 1538–1569), where the considered additive functionals are sums of martingale increments, and only one regime appears. Our result can be seen as a discrete time version, but with general trait evolution, of results in the time continuous setting of branching particle system from Adamczak and Miłoś (*Electron. J. Probab.* **20** (2015) 42), where the evolution of the trait is given by an Ornstein–Uhlenbeck process.

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ADAPTIVE FORCE BIASING ALGORITHMS: NEW CONVERGENCE RESULTS AND TENSOR APPROXIMATIONS OF THE BIAS

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We analyze and propose variants of the adaptive biasing force method. First, we prove the convergence of a version of the algorithm where the biasing force is estimated using a weighted occupation measure, with an explicit asymptotic variance. Second, we propose a new flavour of the algorithm adapted to high-dimensional reaction coordinates, for which the standard approaches suffer from the curse of dimensionality. More precisely, the free energy is approximated by a sum of tensor products of one-dimensional functions. The consistency of the tensor approximation is established. Numerical experiments on five-dimensional reaction coordinates demonstrate that the method is indeed able to capture correlations between them.

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FINDING GEODESICS ON GRAPHS USING REINFORCEMENT LEARNING

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It is well known in biology that ants are able to find shortest paths between their nest and the food by successive random explorations, without any mean of communication other than the pheromones they leave behind them. This striking phenomenon has been observed experimentally and modelled by different mean-field reinforcement-learning models in the biology literature.

In this paper, we introduce the first probabilistic reinforcement-learning model for this phenomenon. In this model, the ants explore a finite graph in which two nodes are distinguished as the nest and the source of food. The ants perform successive random walks on this graph, starting from the nest and stopping when they first reach the food; the transition probabilities of each random walk depend on the realizations of all previous walks through some dynamic weighting of the graph. We discuss different variants of this model based on different reinforcement rules and show that slight changes in this reinforcement rule can lead to drastically different outcomes.

We prove that the ants indeed *eventually find the shortest path(s)* between their nest and the food in two variants of this model and when the underlying graph is, respectively, any series-parallel graph and a five-edge nonseries-parallel *losange* graph. Both proofs rely on the electrical network method for random walks on weighted graphs and on Rubin's embedding in continuous time. The proof in the series-parallel cases uses the recursive nature of this family of graphs, while the proof in the seemingly simpler losange case turns out to be quite intricate: it relies on a fine analysis of some stochastic approximation, and on various couplings with standard and generalised Pólya urns.

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REGULARIZATION OF MULTIPLICATIVE SDES THROUGH ADDITIVE NOISE

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We investigate the regularizing effect of certain additive continuous perturbations on SDEs with multiplicative fractional Brownian motion (fBm). Traditionally, a Lipschitz requirement on the drift and diffusion coefficients is imposed to ensure existence and uniqueness of the SDE. We show that suitable perturbations restore existence, uniqueness and regularity of the flow for the resulting equation, even when both the drift and the diffusion coefficients are distributional, thus extending the program of regularization by noise to the case of multiplicative SDEs. Our method relies on a combination of the nonlinear Young formalism developed by Catellier and Gubinelli (*Stochastic Process. Appl.* **126** (2016) 2323–2366), and stochastic averaging estimates recently obtained by Hairer and Li (*Ann. Probab.* **48** (2020) 1826–1860).

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SLOW-FAST SYSTEMS WITH FRACTIONAL ENVIRONMENT AND DYNAMICS

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We prove a fractional averaging principle for interacting slow-fast systems. The mode of convergence is in Hölder norm in probability. The main technical result is a quenched ergodic theorem on the conditioned fractional dynamics. We also establish geometric ergodicity for a class of fractional-driven stochastic differential equations, improving a recent result of Panloup and Richard.

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COEXISTENCE FOR A POPULATION MODEL WITH FOREST FIRE EPIDEMICS

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We investigate the effect on survival and coexistence of introducing forest fire epidemics to a certain two-species competition model. The model is an extension of the one introduced by Durrett and Remenik (*Ann. Appl. Probab.* **19** (2009) 1656–1685), who studied a discrete time particle system running on a random 3-regular graph where occupied sites grow until they become sufficiently dense so that an epidemic wipes out large clusters. In our extension we let two species affected by independent epidemics compete for space, and we allow the epidemic to attack not only giant clusters, but also clusters of smaller order. Our main results show that there are explicit parameter regions where either one species dominates or there is coexistence; this contrasts with the behavior of the model without epidemics, where the fitter species always dominates. We also discuss the survival and extinction regimes for the model with a single species. In both cases we prove convergence to explicit dynamical systems; simulations suggest that their orbits present chaotic behavior.

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FAST AND MEMORY-OPTIMAL DIMENSION REDUCTION USING KAC'S WALK

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In this work, we analyze dimension reduction algorithms based on the Kac walk and discrete variants.

(1) For n points in $\mathbb{R}^d$, we design an optimal Johnson–Lindenstrauss (JL) transform based on the Kac walk which can be applied to any vector in time $O(d \log d)$ for essentially the same restriction on n as in the best-known transforms due to Ailon and Liberty, and Bamberger and Krahmer. Our algorithm is memory-optimal, and outperforms existing algorithms in regimes when n is sufficiently large and the distortion parameter is sufficiently small. In particular, this confirms a conjecture of Ailon and Chazelle, and of Oliveira, in a stronger form.

(2) The same construction gives a simple transform with optimal restricted isometry property (RIP) which can be applied in time $O(d \log d)$ for essentially the same range of sparsity as in the best-known such transform due to Ailon and Rauhut.

(3) We show that by fixing the angle in the Kac walk to be $\pi/4$ throughout, one obtains optimal JL and RIP transforms with almost the same running time, thereby confirming—up to a $\log \log d$ factor—a conjecture of Avron, Maymounkov, and Toledo. Our moment-based analysis of this modification of the Kac walk may also be of independent interest in connection with repeated averaging processes.

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