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LARGE-SCALE BEHAVIOUR AND HYDRODYNAMIC LIMIT OF BETA COALESCENTS

BY LUKE MILLER[1,a] AND HELMUT H. PITTERS[1,2,b]

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We quantify the behaviour at large scales of the beta coalescent $\Pi = \{\Pi(t), t \geq 0\}$ with parameters $a, b > 0$. Specifically, we study the rescaled block size spectrum of $\Pi(t)$ and of its restriction $\Pi_n(t)$ to $\{1, \dots, n\}$. Our main result is a law of large numbers type of result if Π comes down from infinity. In the case of Kingman's coalescent the derivation of this so-called hydrodynamic limit has been known since the work of Smoluchowski (*Z. Phys.* **17** (1916) 557–585). We extend Smoluchowski's result to beta coalescents and show that if Π comes down from infinity both rescaled spectra

$$n^{-1}(\mathfrak{c}_1\Pi(t\tau_n), \dots, \mathfrak{c}_n\Pi(t\tau_n)), \quad \text{and} \quad n^{-1}(\mathfrak{c}_1\Pi_n(t\tau_n), \dots, \mathfrak{c}_n\Pi_n(t\tau_n)),$$

converge to (different) deterministic limits that we compute explicitly in terms of partial Bell polynomials. Here $\mathfrak{c}_i\pi$ counts the number of blocks of size i in a partition π, and (τ_n) is a sequence such that $\tau_n \sim n^{-(1-a)}$ as $n \to \infty$.

Along the way we study the nontrivial limits of the rescaled block counting processes $\{n^\alpha \#\Pi_n(t\tau_n), t \geq 0\}$, and $\{n^\alpha \#\Pi(t\tau_n), t \geq 0\}$, where $\alpha \in [-1, -2/(3-a))$, and $\tau_n \sim n^{\alpha(1-a)}$ if Π comes down from infinity.

REFERENCES

[1] ALDOUS, D. J. (1999). Deterministic and stochastic models for coalescence (aggregation and coagulation): A review of the mean-field theory for probabilists. *Bernoulli* **5** 3–48. MR1673235 https://doi.org/10.2307/3318611

[2] BAUR, E. and BERTOIN, J. (2015). The fragmentation process of an infinite recursive tree and Ornstein-Uhlenbeck type processes. *Electron. J. Probab.* **20** no. 98, 20. MR3399834 https://doi.org/10.1214/EJP.v20-3866

[3] BERESTYCKI, J., BERESTYCKI, N. and LIMIC, V. (2010). The Λ-coalescent speed of coming down from infinity. *Ann. Probab.* **38** 207–233. MR2599198 https://doi.org/10.1214/09-AOP475

[4] BERESTYCKI, J., BERESTYCKI, N. and SCHWEINSBERG, J. (2008). Small-time behavior of beta coalescents. *Ann. Inst. Henri Poincaré Probab. Stat.* **44** 214–238. MR2446321 https://doi.org/10.1214/07-AIHP103

[5] BERTOIN, J. and LE GALL, J.-F. (2006). Stochastic flows associated to coalescent processes. III. Limit theorems. *Illinois J. Math.* **50** 147–181. MR2247827

[6] BINGHAM, N. H. (1976). Continuous branching processes and spectral positivity. *Stochastic Process. Appl.* **4** 217–242. MR0410961 https://doi.org/10.1016/0304-4149(76)90011-9

[7] BIRKNER, M., BLATH, J. and ELDON, B. (2013). Statistical properties of the site-frequency spectrum associated with lambda-coalescents. *Genetics* **195** 1037–1053.

[8] DONNELLY, P. and KURTZ, T. G. (1999). Particle representations for measure-valued population models. *Ann. Probab.* **27** 166–205. MR1681126 https://doi.org/10.1214/aop/1022677258

[9] DREI, M. V. S. (1916). Vorträge über Diffusion, Brownsche Bewegung und Koagulation von Kolloidteilchen. *Z. Phys.* **17** 557–585.

[10] ETHIER, S. N. and KURTZ, T. G. (1986). *Markov Processes: Characterization and Convergence. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics*. Wiley, New York. MR0838085 https://doi.org/10.1002/9780470316658

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[11] EVANS, S. N. (2008). *Probability and Real Trees*. *Lecture Notes in Math*. **1920**. Springer, Berlin. MR2351587 https://doi.org/10.1007/978-3-540-74798-7

[12] GREY, D. R. (1974). Asymptotic behaviour of continuous time, continuous state-space branching processes. *J. Appl. Probab.* **11** 669–677. MR0408016 https://doi.org/10.2307/3212550

[13] JOHNSON, W. P. (2002). The curious history of Faà di Bruno's formula. *Amer. Math. Monthly* **109** 217–234. MR1903577 https://doi.org/10.2307/2695352

[14] KINGMAN, J. F. C. (1982). The coalescent. *Stochastic Process. Appl.* **13** 235–248. MR0671034 https://doi.org/10.1016/0304-4149(82)90011-4

[15] KINGMAN, J. F. C. (1982). Exchangeability and the evolution of large populations. In *Exchangeability in Probability and Statistics* (*Rome*, 1981) 97–112. North-Holland, Amsterdam. MR0675968

[16] KINGMAN, J. F. C. (1982). On the genealogy of large populations. *J. Appl. Probab.* **19A** 27–43. MR0633178

[17] KUKLA, J. and MÖHLE, M. (2018). On the block counting process and the fixation line of the Bolthausen-Sznitman coalescent. *Stochastic Process. Appl.* **128** 939–962. MR3758343 https://doi.org/10.1016/j.spa.2017.06.012

[18] LIMIC, V. and TALARCZYK, A. (2015). Diffusion limits at small times for Λ-coalescents with a Kingman component. *Electron. J. Probab.* **20** no. 45, 20. MR3339865 https://doi.org/10.1214/EJP.v20-3818

[19] LIMIC, V. and TALARCZYK, A. (2015). Second-order asymptotics for the block counting process in a class of regularly varying Λ-coalescents. *Ann. Probab.* **43** 1419–1455. MR3342667 https://doi.org/10.1214/13-AOP902

[20] LIN, Y. and MALLEIN, B. (2017). Second order behavior of the block counting process of beta coalescents. *Electron. Commun. Probab.* **22** Paper No. 61, 8. MR3724559 https://doi.org/10.1214/17-ECP93

[21] MÖHLE, M. (2015). The Mittag-Leffler process and a scaling limit for the block counting process of the Bolthausen-Sznitman coalescent. *ALEA Lat. Am. J. Probab. Math. Stat.* **12** 35–53. MR3333734

[22] PITMAN, J. (1999). Coalescents with multiple collisions. *Ann. Probab.* **27** 1870–1902. MR1742892 https://doi.org/10.1214/aop/1022677552

[23] PITMAN, J. (2006). *Combinatorial Stochastic Processes*. *Lecture Notes in Math*. **1875**. Springer, Berlin. MR2245368

[24] PITTERS, H. H. (2016). Lifting linear preferential attachment trees yields the arcsine coalescent. Preprint available at: https://doi.org/10.48550/arxiv.1611.05517.

[25] SAGITOV, S. (1999). The general coalescent with asynchronous mergers of ancestral lines. *J. Appl. Probab.* **36** 1116–1125. MR1742154 https://doi.org/10.1017/s0021900200017903

[26] SAMORODNITSKY, G. and TAQQU, M. S. (1994). *Stable Non-Gaussian Random Processes*: *Stochastic Models with Infinite Variance*. *Stochastic Modeling*. CRC Press, New York. MR1280932

[27] SCHWEINSBERG, J. (2000). A necessary and sufficient condition for the Λ-coalescent to come down from infinity. *Electron. Commun. Probab.* **5** 1–11. MR1736720 https://doi.org/10.1214/ECP.v5-1013

[28] SCHWEINSBERG, J. (2012). Dynamics of the evolving Bolthausen-Sznitman coalecent. *Electron. J. Probab.* **17** no. 91, 50. MR2988406 https://doi.org/10.1214/EJP.v17-2378

[29] SCHWEINSBERG, J. (2017). Rigorous results for a population model with selection II: Genealogy of the population. *Electron. J. Probab.* **22** Paper No. 38, 54. MR3646064 https://doi.org/10.1214/17-ejp58

[30] ULMET, M. G. (1992). Properties of semigroups generated by first order differential operators. *Results Math.* **22** 821–832. MR1189768 https://doi.org/10.1007/BF03323126

[31] WATTIS, J. A. D. (2006). An introduction to mathematical models of coagulation-fragmentation processes: A discrete deterministic mean-field approach. *Phys. D* **222** 1–20. MR2265763 https://doi.org/10.1016/j.physd.2006.07.024

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APPROXIMATION BOUNDS FOR RANDOM NEURAL NETWORKS AND RESERVOIR SYSTEMS

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This work studies approximation based on single-hidden-layer feedforward and recurrent neural networks with randomly generated internal weights. These methods, in which only the last layer of weights and a few hyperparameters are optimized, have been successfully applied in a wide range of static and dynamic learning problems. Despite the popularity of this approach in empirical tasks, important theoretical questions regarding the relation between the unknown function, the weight distribution, and the approximation rate have remained open. In this work it is proved that, as long as the unknown function, functional, or dynamical system is sufficiently regular, it is possible to draw the internal weights of the random (recurrent) neural network from a generic distribution (not depending on the unknown object) and quantify the error in terms of the number of neurons and the hyperparameters. In particular, this proves that echo state networks with randomly generated weights are capable of approximating a wide class of dynamical systems arbitrarily well and thus provides the first mathematical explanation for their empirically observed success at learning dynamical systems.

REFERENCES

[1] ALIPRANTIS, C. D. and BORDER, K. C. (2006). *Infinite Dimensional Analysis*: *A Hitchhiker's Guide*, 3rd ed. Springer, Berlin. MR2378491

[2] BARRON, A. R. (1992). Neural Net Approximation. In *Proceedings of the 7th Yale Workshop on Adaptive and Learning Systems* 69–72.

[3] BARRON, A. R. (1993). Universal approximation bounds for superpositions of a sigmoidal function. *IEEE Trans*. *Inf*. *Theory* **39** 930–945. MR1237720 https://doi.org/10.1109/18.256500

[4] BERGH, J. and LÖFSTRÖM, J. (1976). *Interpolation Spaces*. Springer.

[5] CUCHIERO, C., GONON, L., GRIGORYEVA, L., ORTEGA, J. P. and TEICHMANN, J. (2020). Discrete-time signatures and randomness in reservoir computing. *IEEE Trans. Neural Netw. Learn. Syst.* https://doi.org/10.1109/TNNLS.2021.3076777

[6] CUCHIERO, C., GONON, L., GRIGORYEVA, L., ORTEGA, J. P. and TEICHMANN, J. (2021). Expressive Power of Randomized Signature. NeurIPS 2021.

[7] FOLLAND, G. B. (1995). *Introduction to Partial Differential Equations*, 2nd ed. Princeton Univ. Press, Princeton, NJ. MR1357411

[8] FRIZ, P. K. and VICTOIR, N. B. (2010). *Multidimensional Stochastic Processes as Rough Paths*. *Cambridge Studies in Advanced Mathematics* **120**. Cambridge Univ. Press, Cambridge. MR2604669 https://doi.org/10.1017/CBO9780511845079

[9] GONON, L. (2021). Random feature neural networks learn Black–Scholes type PDEs without curse of dimensionality. Preprint. Available at arXiv:2106.08900.

[10] GONON, L., GRIGORYEVA, L. and ORTEGA, J.-P. (2020). Risk bounds for reservoir computing. *J. Mach*. *Learn*. *Res*. **21** Paper No. 240. MR4209526

[11] GONON, L. and ORTEGA, J.-P. (2020). Reservoir computing universality with stochastic inputs. *IEEE Trans*. *Neural Netw*. *Learn*. *Syst*. **31** 100–112. MR4056386

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Key words and phrases. Neural networks, approximation error, reservoir computing, echo state networks, random function approximation.

[12] GONON, L. and ORTEGA, J. P. (2022). Fading memory echo state networks are universal. *Neural Netw.* **138** 10–13.

[13] GRIGORYEVA, L. and ORTEGA, J.-P. (2018). Echo state networks are universal. *Neural Netw.* **108** 495–508. https://doi.org/10.1016/j.neunet.2018.08.025

[14] GRIGORYEVA, L. and ORTEGA, J.-P. (2018). Universal discrete-time reservoir computers with stochastic inputs and linear readouts using non-homogeneous state-affine systems. *J. Mach. Learn. Res.* **19** Paper No. 24. MR3862431

[15] GRIGORYEVA, L. and ORTEGA, J.-P. (2019). Differentiable reservoir computing. *J. Mach. Learn. Res.* **20** Paper No. 179. MR4048990

[16] HART, A. G., HOOK, J. L. and DAWES, J. H. P. (2020). Embedding and approximation theorems for echo state networks. *Neural Netw.* **128** 234–247.

[17] HORNIK, K. (1991). Approximation capabilities of muitilayer feedforward networks. *Neural Netw.* **4** 251–257.

[18] HUANG, G. B., ZHU, Q. Y. and SIEW, C. K. (2006). Extreme learning machine: Theory and applications. *Neurocomputing* **70** 489–501.

[19] JAEGER, H. and HAAS, H. (2004). Harnessing nonlinearity: Predicting chaotic systems and saving energy in wireless communication. *Science* **304** 78–80. https://doi.org/10.1126/science.1091277

[20] KALLENBERG, O. (2002). *Foundations of Modern Probability*, 2nd ed. *Probability and Its Applications* (*New York*). Springer, New York. MR1876169 https://doi.org/10.1007/978-1-4757-4015-8

[21] KLOEDEN, P. E. and PLATEN, E. (1992). *Numerical Solution of Stochastic Differential Equations*. *Applications of Mathematics* (*New York*) **23**. Springer, Berlin. MR1214374 https://doi.org/10.1007/978-3-662-12616-5

[22] KLUSOWSKI, J. M. and BARRON, A. R. (2018). Approximation by combinations of ReLU and squared ReLU ridge functions with ℓ^1 and ℓ^0 controls. *IEEE Trans. Inf. Theory* **64** 7649–7656. MR3883687 https://doi.org/10.1109/tit.2018.2874447

[23] LEDOUX, M. and TALAGRAND, M. (2011). *Probability in Banach Spaces*. *Classics in Mathematics*. Springer, Berlin. MR2814399

[24] LU, Z., HUNT, B. R. and OTT, E. (2018). Attractor reconstruction by machine learning. *Chaos* **28** 061104. MR3816936 https://doi.org/10.1063/1.5039508

[25] MAIOROV, V. E. and MEIR, R. (2000). On the near optimality of the stochastic approximation of smooth functions by neural networks. *Adv. Comput. Math.* **13** 79–103. MR1759189 https://doi.org/10.1023/A:1018993908478

[26] MATTHEWS, M. and MOSCHYTZ, G. (1994). The identification of nonlinear discrete-time fading-memory systems using neural network models. *IEEE Trans. Circuits Syst.*, 2 *Analog Digit. Signal Process.* **41** 740–751.

[27] MATTHEWS, M. B. (1992). On the Uniform Approximation of Nonlinear Discrete-Time Fading-Memory Systems Using Neural Network Models. Ph.D. thesis, ETH Zürich.

[28] MATTHEWS, M. B. (1993). Approximating nonlinear fading-memory operators using neural network models. *Circuits Systems Signal Process.* **12** 279–307.

[29] MHASKAR, N. H. (1996). Neural networks for optimal approximation of smooth and analytic functions. *Neural Comput.* **8** 164–177.

[30] PATHAK, J., HUNT, B., GIRVAN, M., LU, Z. and OTT, E. (2018). Model-free prediction of large spatiotemporally chaotic systems from data: A reservoir computing approach. *Phys. Rev. Lett.* **120** 24102.

[31] PATHAK, J., LU, Z., HUNT, B. R., GIRVAN, M. and OTT, E. (2017). Using machine learning to replicate chaotic attractors and calculate Lyapunov exponents from data. *Chaos* **27** 121102. MR3732799 https://doi.org/10.1063/1.5010300

[32] POGGIO, T., MHASKAR, H., ROSASCO, L., MIRANDA, B. and LIAO, Q. (2017). Why and when can deep-but not shallow-networks avoid the curse of dimensionality: A review. *Int. J. Autom. Comput.* **14** 503–519.

[33] RAHIMI, A. and RECHT, B. (2007). Random features for large-scale kernel machines. *Adv. Neural Inf. Process. Syst.*

[34] RAHIMI, A. and RECHT, B. (2008). Uniform approximation of functions with random bases. In 2008 46*th Annual Allerton Conference on Communication, Control, and Computing* 555–561.

[35] RAHIMI, A. and RECHT, B. (2009). Weighted sums of random kitchen sinks: Replacing minimization with randomization in learning. *Adv. Neural Inf. Process. Syst.*

[36] RUDIN, W. (1987). *Real and Complex Analysis*, 3rd ed. McGraw-Hill, New York. MR0924157

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CONVERGENCE IN WASSERSTEIN DISTANCE FOR EMPIRICAL MEASURES OF SEMILINEAR SPDES

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The convergence rate in Wasserstein distance is estimated for the empirical measures of symmetric semilinear SPDEs. Unlike in the finite-dimensional case that the convergence is of algebraic order in time, in the present situation the convergence is of log order with a power given by eigenvalues of the underlying linear operator.

REFERENCES

[1] AMBROSIO, L., STRA, F. and TREVISAN, D. (2019). A PDE approach to a 2-dimensional matching problem. *Probab. Theory Related Fields* **173** 433–477. MR3916111 https://doi.org/10.1007/s00440-018-0837-x

[2] DA PRATO, G. and ZABCZYK, J. (1992). *Stochastic Equations in Infinite Dimensions*. *Encyclopedia of Mathematics and Its Applications* **44**. Cambridge Univ. Press, Cambridge. MR1207136 https://doi.org/10.1017/CBO9780511666223

[3] DA PRATO, G. and ZABCZYK, J. (1996). *Ergodicity for Infinite-Dimensional Systems*. *London Mathematical Society Lecture Note Series* **229**. Cambridge Univ. Press, Cambridge. MR1417491 https://doi.org/10.1017/CBO9780511662829

[4] FOURNIER, N. and GUILLIN, A. (2015). On the rate of convergence in Wasserstein distance of the empirical measure. *Probab. Theory Related Fields* **162** 707–738. MR3383341 https://doi.org/10.1007/s00440-014-0583-7

[5] GONG, F.-Z. and WANG, F.-Y. (2002). Functional inequalities for uniformly integrable semigroups and application to essential spectrums. *Forum Math*. **14** 293–313. MR1880915 https://doi.org/10.1515/form.2002.013

[6] KLOECKNER, B. (2012). Approximation by finitely supported measures. *ESAIM Control Optim. Calc. Var.* **18** 343–359. MR2954629 https://doi.org/10.1051/cocv/2010100

[7] LEDOUX, M. (2017). On optimal matching of Gaussian samples. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov.* (*POMI*) **457** 226–264. MR3723584

[8] WANG, F.-Y. (2013). *Harnack Inequalities for Stochastic Partial Differential Equations*. *Springer Briefs in Mathematics*. Springer, New York. MR3099948 https://doi.org/10.1007/978-1-4614-7934-5

[9] WANG, F.-Y. (2021). Precise limit in Wasserstein distance for conditional empirical measures of Dirichlet diffusion processes. *J. Funct. Anal.* **280** Paper No. 108998, 23. MR4232671 https://doi.org/10.1016/j.jfa.2021.108998

[10] WANG, F.-Y. (2022). Convergence in Wasserstein distance for empirical measures of Dirichlet diffusion processes on manifolds. *J. Eur. Math. Soc.* (*JEMS*) To appear.

[11] WANG, F.-Y. (2022). Wasserstein convergence rate for empirical measures on noncompact manifolds. *Stochastic Process. Appl.* **144** 271–287. MR4347493 https://doi.org/10.1016/j.spa.2021.11.006

[12] WANG, F.-Y. and ZHU, J.-X. (2022). Limit theorems in Wasserstein distance for empirical measures of diffusion processes on Riemannian manifolds. *Ann. Inst. Henri Poincaré Probab. Stat.* To appear.

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THE SIZE OF t-CORES AND HOOK LENGTHS OF RANDOM CELLS IN RANDOM PARTITIONS

BY ARVIND AYYER[1,a] AND SHUBHAM SINHA[2,b]

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Fix $t \geq 2$. We first give an asymptotic formula for certain sums of the number of t-cores. We then use this result to compute the distribution of the size of the t-core of a uniformly random partition of an integer n. We show that this converges weakly to a gamma distribution after dividing by $\sqrt{n}$. As a consequence, we find that the size of the t-core is of the order of $\sqrt{n}$ in expectation. We then apply this result to show that the probability that t divides the hook length of a uniformly random cell in a uniformly random partition equals $1/t$ in the limit. Finally, we extend this result to all modulo classes of t using abacus representations for cores and quotients.

REFERENCES

[1] ANDERSON, J. (2008). An asymptotic formula for the t-core partition function and a conjecture of Stanton. *J. Number Theory* **128** 2591–2615. MR2444214 https://doi.org/10.1016/j.jnt.2007.10.006

[2] ANDREWS, G. E. (1976). *The Theory of Partitions*. *Encyclopedia of Mathematics and Its Applications*, *Vol*. 2. Addison-Wesley Co., Reading, MA–London–Amsterdam. MR0557013

[3] APOSTOL, T. M. (1976). *Introduction to Analytic Number Theory*. *Undergraduate Texts in Mathematics*. Springer, New York. MR0434929

[4] AYYER, A. and LINUSSON, S. (2017). Correlations in the multispecies TASEP and a conjecture by Lam. *Trans*. *Amer*. *Math*. *Soc*. **369** 1097–1125. MR3572266 https://doi.org/10.1090/tran/6806

[5] BILLINGSLEY, P. (1995). *Probability and Measure*, 3rd ed. *Wiley Series in Probability and Mathematical Statistics*. Wiley, New York. MR1324786

[6] THE SAGE DEVELOPERS (2017). SageMath, the Sage Mathematics Software System (Version 7.6) https://www.sagemath.org.

[7] GARVAN, F., KIM, D. and STANTON, D. (1990). Cranks and t-cores. *Invent*. *Math*. **101** 1–17. MR1055707 https://doi.org/10.1007/BF01231493

[8] GRANVILLE, A. and ONO, K. (1996). Defect zero p-blocks for finite simple groups. *Trans*. *Amer*. *Math*. *Soc*. **348** 331–347. MR1321575 https://doi.org/10.1090/S0002-9947-96-01481-X

[9] JAMES, G. and KERBER, A. (1981). *The Representation Theory of the Symmetric Group*. *Encyclopedia of Mathematics and Its Applications* **16**. Addison-Wesley, Reading, MA. MR0644144

[10] KLYACHKO, A. A. (1982). Modular forms and representations of symmetric groups. *Zap*. *Nauchn*. *Sem*. *Leningrad*. *Otdel*. *Mat*. *Inst*. *Steklov*. (*LOMI*) **116** 74–85, 162. MR0687842

[11] LAM, T. (2015). The shape of a random affine Weyl group element and random core partitions. *Ann*. *Probab*. **43** 1643–1662. MR3353811 https://doi.org/10.1214/14-AOP915

[12] LOEHR, N. A. (2011). *Bijective Combinatorics*. *Discrete Mathematics and Its Applications* (*Boca Raton*). CRC Press, Boca Raton, FL. MR2777360

[13] LULOV, N. and PITTEL, B. (1999). On the random Young diagrams and their cores. *J*. *Combin*. *Theory Ser*. *A* **86** 245–280. MR1685533 https://doi.org/10.1006/jcta.1998.2939

[14] OLSSON, J. B. (1993). *Combinatorics and Representations of Finite Groups*. *Vorlesungen aus dem Fachbereich Mathematik der Universität GH Essen* [*Lecture Notes in Mathematics at the University of Essen*] **20**. Universität Essen, Fachbereich Mathematik, Essen. MR1264418

[15] ROSTAM, S. (2021). Core size of a random partition for the Plancherel measure. Available at arXiv:2111.05970.

[16] VINOGRADOV, A. I. and SKRIGANOV, M. M. (1979). The number of lattice points inside the sphere with variable center. *Zap*. *Nauchn*. *Sem*. *Leningrad*. *Otdel*. *Mat*. *Inst*. *Steklov*. (*LOMI*) **91** 25–30, 180. MR0566505

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TRANSITIVE CLOSURE IN A POLLUTED ENVIRONMENT

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We introduce and study a new percolation model, inspired by recent works on jigsaw percolation, graph bootstrap percolation, and percolation in polluted environments. Start with an oriented graph G_0 of initially occupied edges on n vertices, and iteratively occupy additional (oriented) edges by transitivity, with the constraint that only open edges in a certain random set can ever be occupied. All other edges are closed, creating a set of obstacles for the spread of occupied edges. When G_0 is an unoriented linear graph, and leftward and rightward edges are open independently with possibly different probabilities, we identify three regimes in which the set of eventually occupied edges is either all open edges, the majority of open edges in one direction, or only a very small proportion of all open edges. In the more general setting where G_0 is a connected unoriented graph of bounded degree, we show that the transition between sparse and full occupation of open edges occurs when the probability of open edges is $(\log n)^{-1/2+o(1)}$. We conclude with several conjectures and open problems.

REFERENCES

[1] AIZENMAN, M. and LEBOWITZ, J. L. (1988). Metastability effects in bootstrap percolation. *J. Phys. A* **21** 3801–3813. MR0968311

[2] ANGEL, O. and KOLESNIK, B. (2018). Sharp thresholds for contagious sets in random graphs. *Ann. Appl. Probab.* **28** 1052–1098. MR3784495 https://doi.org/10.1214/17-AAP1325

[3] BALOGH, J., BOLLOBÁS, B., DUMINIL-COPIN, H. and MORRIS, R. (2012). The sharp threshold for bootstrap percolation in all dimensions. *Trans. Amer. Math. Soc.* **364** 2667–2701. MR2888224 https://doi.org/10.1090/S0002-9947-2011-05552-2

[4] BALOGH, J., BOLLOBÁS, B. and MORRIS, R. (2012). Graph bootstrap percolation. *Random Structures Algorithms* **41** 413–440. MR2993128 https://doi.org/10.1002/rsa.20458

[5] BALOGH, J., BOLLOBÁS, B., MORRIS, R. and RIORDAN, O. (2012). Linear algebra and bootstrap percolation. *J. Combin. Theory Ser. A* **119** 1328–1335. MR2915649 https://doi.org/10.1016/j.jcta.2012.03.005

[6] BARBOUR, A. D., HOLST, L. and JANSON, S. (1992). *Poisson Approximation*. *Oxford Studies in Probability* **2**. The Clarendon Press, Oxford University Press, New York. Oxford Science Publications. MR1163825

[7] BARTHA, Z. and KOLESNIK, B. (2022). Weakly saturated random graphs. *Random Structures Algorithms*. To appear.

[8] BOLLOBÁS, B. (1968). Weakly k-saturated graphs. In *Beiträge zur Graphentheorie* (*Kolloquium, Manebach*, 1967) 25–31. Teubner, Leipzig. MR0244077

[9] BOLLOBÁS, B., PRZYKUCKI, M., RIORDAN, O. and SAHASRABUDHE, J. (2017). On the maximum running time in graph bootstrap percolation. *Electron. J. Combin.* **24** 2.16. MR3650265 https://doi.org/10.37236/5771

[10] BOLLOBÁS, B., RIORDAN, O., SLIVKEN, E. and SMITH, P. (2017). The threshold for jigsaw percolation on random graphs. *Electron. J. Combin.* **24** 2.36. MR3665569 https://doi.org/10.37236/6102

[11] BRUMMITT, C. D., CHATTERJEE, S., DEY, P. S. and SIVAKOFF, D. (2015). Jigsaw percolation: What social networks can collaboratively solve a puzzle? *Ann. Appl. Probab.* **25** 2013–2038. MR3349000 https://doi.org/10.1214/14-AAP1041

[12] CHALUPA, J., LEATH, P. L. and REICH, G. R. (1979). Bootstrap percolation on a Bethe lattice. *J. Phys. C* **21** L31–L35.

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[13] COOLEY, O., KAPETANOPOULOS, T. and MAKAI, T. (2019). The sharp threshold for jigsaw percolation in random graphs. *Adv. in Appl. Probab.* **51** 378–407. MR3989519 https://doi.org/10.1017/apr.2019.24

[14] DURRETT, R. (1984). Oriented percolation in two dimensions. *Ann. Probab.* **12** 999–1040. MR0757768

[15] FORTUIN, C. M., KASTELEYN, P. W. and GINIBRE, J. (1971). Correlation inequalities on some partially ordered sets. *Comm. Math. Phys.* **22** 89–103. MR0309498

[16] FRIEDGUT, E. and KALAI, G. (1996). Every monotone graph property has a sharp threshold. *Proc. Amer. Math. Soc.* **124** 2993–3002. MR1371123 https://doi.org/10.1090/S0002-9939-96-03732-X

[17] GRAVNER, J. and HOLROYD, A. E. (2019). Polluted bootstrap percolation with threshold two in all dimensions. *Probab. Theory Related Fields* **175** 467–486. MR4009714 https://doi.org/10.1007/s00440-018-0892-3

[18] GRAVNER, J., HOLROYD, A. E. and SIVAKOFF, D. (2021). Polluted bootstrap percolation in three dimensions. *Ann. Appl. Probab.* **31** 218–246. MR4254479 https://doi.org/10.1214/20-aap1588

[19] GRAVNER, J. and MCDONALD, E. (1997). Bootstrap percolation in a polluted environment. *J. Stat. Phys.* **87** 915–927. MR1459046 https://doi.org/10.1007/BF02181252

[20] GRAVNER, J. and SIVAKOFF, D. (2017). Nucleation scaling in jigsaw percolation. *Ann. Appl. Probab.* **27** 395–438. MR3619791 https://doi.org/10.1214/16-AAP1206

[21] GUNDERSON, K., KOCH, S. and PRZYKUCKI, M. (2017). The time of graph bootstrap percolation. *Random Structures Algorithms* **51** 143–168. MR3668849 https://doi.org/10.1002/rsa.20660

[22] HOLROYD, A. E. (2003). Sharp metastability threshold for two-dimensional bootstrap percolation. *Probab. Theory Related Fields* **125** 195–224. MR1961342 https://doi.org/10.1007/s00440-002-0239-x

[23] JANSON, S., ŁUCZAK, T. and RUCINSKI, A. (2000). *Random Graphs. Wiley-Interscience Series in Discrete Mathematics and Optimization*. Wiley Interscience, New York. MR1782847 https://doi.org/10.1002/9781118032718

[24] KARP, R. M. (1990). The transitive closure of a random digraph. *Random Structures Algorithms* **1** 73–93. MR1068492 https://doi.org/10.1002/rsa.3240010106

[25] KOLESNIK, B. (2022). The sharp K_4-percolation threshold on the Erdős–Rényi random graph. *Electron. J. Probab.* **27** 13. MR4372097 https://doi.org/10.1214/21-ejp710

[26] POLLAK, M. and RIESS, I. (1975). Application of percolation theory to 2D-3D Heisenberg ferromagnets. *Phys. Status Solidi* (*b*) **69** K15–K18.

[27] VAN DEN BERG, J. and KESTEN, H. (1985). Inequalities with applications to percolation and reliability. *J. Appl. Probab.* **22** 556–569. MR0799280 https://doi.org/10.1017/s0021900200029326

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INVARIANCE PRINCIPLES FOR INTEGRATED RANDOM WALKS CONDITIONED TO STAY POSITIVE

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Let $S(n)$ be a centered random walk with finite second moment. We consider the integrated random walk $T(n) = S(0) + S(1) + \cdots + S(n)$. We prove invariance principles for the meander and for the bridge of this process, under the condition that the integrated random walk remains positive. Furthermore, we prove the functional convergence of its Doob's h-transform to the h-transform of the Kolmogorov diffusion conditioned to stay positive.

REFERENCES

[1] AURZADA, F., DEREICH, S. and LIFSHITS, M. (2014). Persistence probabilities for a bridge of an integrated simple random walk. *Probab. Math. Statist.* **34** 1–22. MR3225998

[2] AURZADA, F. and SIMON, T. (2015). Persistence probabilities and exponents. In *Lévy Matters. V. Lecture Notes in Math.* **2149** 183–224. Springer, Cham. MR3468226 https://doi.org/10.1007/978-3-319-23138-9_3

[3] BILLINGSLEY, P. (1968). *Convergence of Probability Measures*. Wiley, New York. MR0233396

[4] CARAVENNA, F. and CHAUMONT, L. (2013). An invariance principle for random walk bridges conditioned to stay positive. *Electron. J. Probab.* **18** no. 60, 32 pp. MR3068391 https://doi.org/10.1214/EJP.v18-2362

[5] CARAVENNA, F. and DEUSCHEL, J.-D. (2008). Pinning and wetting transition for $(1+1)$-dimensional fields with Laplacian interaction. *Ann. Probab.* **36** 2388–2433. MR2478687 https://doi.org/10.1214/08-AOP395

[6] DEMBO, A., DING, J. and GAO, F. (2013). Persistence of iterated partial sums. *Ann. Inst. Henri Poincaré Probab. Stat.* **49** 873–884. MR3112437 https://doi.org/10.1214/11-AIHP452

[7] DENISOV, D. and WACHTEL, V. (2015). Exit times for integrated random walks. *Ann. Inst. Henri Poincaré Probab. Stat.* **51** 167–193. MR3300967 https://doi.org/10.1214/13-AIHP577

[8] DENISOV, D. and WACHTEL, V. (2015). Random walks in cones. *Ann. Probab.* **43** 992–1044. MR3342657 https://doi.org/10.1214/13-AOP867

[9] DENISOV, D. and WACHTEL, V. (2019). Alternative constructions of a harmonic function for a random walk in a cone. *Electron. J. Probab.* **24** Paper No. 92, 26 pp. MR4003145 https://doi.org/10.1214/19-ejp349

[10] DOBRUSHIN, R. and HRYNIV, O. (1996). Fluctuations of shapes of large areas under paths of random walks. *Probab. Theory Related Fields* **105** 423–458. MR1402652 https://doi.org/10.1007/BF01191908

[11] DURAJ, J. and WACHTEL, V. (2020). Invariance principles for random walks in cones. *Stochastic Process. Appl.* **130** 3920–3942. MR4102254 https://doi.org/10.1016/j.spa.2019.11.004

[12] GAO, F., LIU, Z. and YANG, X. (2014). Conditional persistence of Gaussian random walks. *Electron. Commun. Probab.* **19** no. 70, 9 pp. MR3269170 https://doi.org/10.1214/ECP.v19-3587

[13] GROENEBOOM, P., JONGBLOED, G. and WELLNER, J. A. (1999). Integrated Brownian motion, conditioned to be positive. *Ann. Probab.* **27** 1283–1303. MR1733148 https://doi.org/10.1214/aop/1022677447

[14] IOFFE, D., SHLOSMAN, S. and VELENIK, Y. (2015). An invariance principle to Ferrari–Spohn diffusions. *Comm. Math. Phys.* **336** 905–932. MR3322390 https://doi.org/10.1007/s00220-014-2277-5

[15] LACHAL, A. (1991). Sur le premier instant de passage de l'intégrale du mouvement brownien. *Ann. Inst. Henri Poincaré Probab. Stat.* **27** 385–405. MR1131839

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[16] LACHAL, A. (2003). Application de la théorie des excursions à l'intégrale du mouvement brownien. In *Séminaire de Probabilités XXXVII* (J. Azéma, M. Émery, M. Ledoux and M. Yor, eds.). *Lecture Notes in Math*. **1832** 109–195. Springer, Berlin. MR2053045 https://doi.org/10.1007/978-3-540-40004-2_6
[17] MAJOR, P. (1976). The approximation of partial sums of independent RV's. *Z. Wahrsch. Verw. Gebiete* **35** 213–220. MR0415743 https://doi.org/10.1007/BF00532673
[18] MAJUMDAR, S. N. (1999). Persistence in non-equilibrium systems. *Current Sci*. **77** 370–375.
[19] MCKEAN, H. P. JR. (1963). A winding problem for a resonator driven by a white noise. *J. Math. Kyoto Univ.* **2** 227–235. MR0156389 https://doi.org/10.1215/kjm/1250524936
[20] OLVER, F. W. J. (1974). *Asymptotics and Special Functions*. *Computer Science and Applied Mathematics*. Academic Press, New York. MR0435697
[21] SINAI, Y. G. (1992). Distribution of some functionals of the integral of a random walk. *Theoret. Math. Phys*. **90** 219–241. MR1182301 https://doi.org/10.1007/BF01036528
[22] VYSOTSKY, V. (2010). On the probability that integrated random walks stay positive. *Stochastic Process. Appl*. **120** 1178–1193. MR2639743 https://doi.org/10.1016/j.spa.2010.03.005
[23] VYSOTSKY, V. (2014). Positivity of integrated random walks. *Ann. Inst. Henri Poincaré Probab. Stat*. **50** 195–213. MR3161528 https://doi.org/10.1214/12-AIHP487

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ASYMPTOTICALLY LINEAR ITERATED FUNCTION SYSTEMS ON THE REAL LINE

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Given a sequence of i.i.d. random functions $\Psi_n : \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}$, we consider the iterated function system and Markov chain, which is recursively defined by $X_0^x := x$ and $X_n^x := \Psi_{n-1}(X_{n-1}^x)$ for $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Under the two basic assumptions that the Ψ_n are a.s. continuous at any point in $\mathbb{R}$ and asymptotically linear at the "endpoints" $\pm\infty$, we study the tail behavior of the stationary laws of such Markov chains by means of Markov renewal theory. Our approach provides an extension of Goldie's implicit renewal theory (*Ann. Appl. Probab.* (1991) **1** 126–166) and can also be viewed as an adaptation of Kesten's work on products of random matrices (*Acta Math.* (1973) **131** 207–248) to one-dimensional function systems as described. Our results have applications in quite different areas of applied probability like queuing theory, econometrics, mathematical finance and population dynamics, for example, ARCH models and random logistic transforms.

REFERENCES

[1] ALDOUS, D. J. and BANDYOPADHYAY, A. (2005). A survey of max-type recursive distributional equations. *Ann. Appl. Probab.* **15** 1047–1110. MR2134098 https://doi.org/10.1214/105051605000000142

[2] ALSMEYER, G. (1994). On the Markov renewal theorem. *Stochastic Process. Appl.* **50** 37–56. MR1262329 https://doi.org/10.1016/0304-4149(94)90146-5

[3] ALSMEYER, G. (1997). The Markov renewal theorem and related results. *Markov Process. Related Fields* **3** 103–127. MR1446921

[4] ALSMEYER, G. (2014). Quasistochastic matrices and Markov renewal theory. *J. Appl. Probab.* **51A** 359–376. MR3317369 https://doi.org/10.1239/jap/1417528486

[5] ALSMEYER, G. (2016). On the stationary tail index of iterated random Lipschitz functions. *Stochastic Process. Appl.* **126** 209–233. MR3426517 https://doi.org/10.1016/j.spa.2015.08.004

[6] ALSMEYER, G. and MENTEMEIER, S. (2012). Tail behaviour of stationary solutions of random difference equations: The case of regular matrices. *J. Difference Equ. Appl.* **18** 1305–1332. MR2956047 https://doi.org/10.1080/10236198.2011.571383

[7] ATHREYA, K. B. and DAI, J. (2000). Random logistic maps. I. *J. Theoret. Probab.* **13** 595–608. MR1778589 https://doi.org/10.1023/A:1007828804691

[8] ATHREYA, K. B., MCDONALD, D. and NEY, P. (1978). Limit theorems for semi-Markov processes and renewal theory for Markov chains. *Ann. Probab.* **6** 788–797. MR0503952

[9] BORKOVEC, M. and KLÜPPELBERG, C. (2001). The tail of the stationary distribution of an autoregressive process with ARCH(1) errors. *Ann. Appl. Probab.* **11** 1220–1241. MR1878296 https://doi.org/10.1214/aoap/1015345401

[10] BROFFERIO, S. and BURACZEWSKI, D. (2015). On unbounded invariant measures of stochastic dynamical systems. *Ann. Probab.* **43** 1456–1492. MR3342668 https://doi.org/10.1214/13-AOP903

[11] BURACZEWSKI, D. and DAMEK, E. (2017). A simple proof of heavy tail estimates for affine type Lipschitz recursions. *Stochastic Process. Appl.* **127** 657–668. MR3583767 https://doi.org/10.1016/j.spa.2016.06.022

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[12] BURACZEWSKI, D., DAMEK, E., GUIVARC'H, Y. and MENTEMEIER, S. (2014). On multidimensional Mandelbrot cascades. *J. Difference Equ. Appl.* **20** 1523–1567. MR3268907 https://doi.org/10.1080/10236198.2014.950259

[13] BURACZEWSKI, D., DAMEK, E. and MIKOSCH, T. (2016). *Stochastic Models with Power-Law Tails: The Equation $X = AX + B$*. *Springer Series in Operations Research and Financial Engineering*. Springer, Cham. MR3497380 https://doi.org/10.1007/978-3-319-29679-1

[14] DA PRATO, G. and ZABCZYK, J. (1996). *Ergodicity for Infinite-Dimensional Systems*. *London Mathematical Society Lecture Note Series* **229**. Cambridge Univ. Press, Cambridge. MR1417491 https://doi.org/10.1017/CBO9780511662829

[15] DAMEK, E., MATSUI, M. and ŚWIĄTKOWSKI, W. (2019). Componentwise different tail solutions for bivariate stochastic recurrence equations with application to GARCH(1,1) processes. *Colloq. Math.* **155** 227–254. MR3897508 https://doi.org/10.4064/cm7313A-5-2018

[16] DAMEK, E. and ZIENKIEWICZ, J. (2018). Affine stochastic equation with triangular matrices. *J. Difference Equ. Appl.* **24** 520–542. MR3767029 https://doi.org/10.1080/10236198.2017.1422249

[17] DE SAPORTA, B. (2005). Tail of the stationary solution of the stochastic equation $Y_{n+1} = a_n Y_n + b_n$ with Markovian coefficients. *Stochastic Process. Appl.* **115** 1954–1978. MR2178503 https://doi.org/10.1016/j.spa.2005.06.009

[18] DIACONIS, P. and FREEDMAN, D. (1999). Iterated random functions. *SIAM Rev.* **41** 45–76. MR1669737 https://doi.org/10.1137/S0036144598338446

[19] EMBRECHTS, P., KLÜPPELBERG, C. and MIKOSCH, T. (1997). *Modelling Extremal Events: For Insurance and Finance*. *Applications of Mathematics (New York)* **33**. Springer, Berlin. MR1458613 https://doi.org/10.1007/978-3-642-33483-2

[20] GOLDIE, C. M. (1991). Implicit renewal theory and tails of solutions of random equations. *Ann. Appl. Probab.* **1** 126–166. MR1097468

[21] GUÉGAN, D. and DIEBOLT, J. (1994). Probabilistic properties of the β-ARCH-model. *Statist. Sinica* **4** 71–87. MR1282866

[22] GUIVARC'H, Y. and LE PAGE, É. (2016). Spectral gap properties for linear random walks and Pareto's asymptotics for affine stochastic recursions. *Ann. Inst. Henri Poincaré Probab. Stat.* **52** 503–574. MR3498000 https://doi.org/10.1214/15-AIHP668

[23] JACOD, J. (1971). Théorème de renouvellement et classification pour les chaînes semi-markoviennes. *Ann. Inst. Henri Poincaré B, Calc. Probab. Stat.* **7** 83–129. MR0305496

[24] KESTEN, H. (1973). Random difference equations and renewal theory for products of random matrices. *Acta Math.* **131** 207–248. MR0440724 https://doi.org/10.1007/BF02392040

[25] KESTEN, H. (1974). Renewal theory for functionals of a Markov chain with general state space. *Ann. Probab.* **2** 355–386. MR0365740 https://doi.org/10.1214/aop/1176996654

[26] MAERCKER, G. (1997). Statistical inference in conditional heteroscedastic autoregressive models Ph.D. thesis Technische Universität Braunschweig.

[27] MIREK, M. (2011). Heavy tail phenomenon and convergence to stable laws for iterated Lipschitz maps. *Probab. Theory Related Fields* **151** 705–734. MR2851697 https://doi.org/10.1007/s00440-010-0312-9

[28] SHURENKOV, V. M. (1984). On Markov renewal theory. *Teor. Veroyatn. Primen.* **29** 248–263. MR0749913

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ON A NONLINEAR SPDE DERIVED FROM A HYDRODYNAMIC LIMIT IN A SINAI-TYPE RANDOM ENVIRONMENT

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With the recent developments on nonlinear SPDEs, where smoothing of rough noises is needed, one is naturally led to study interacting particle systems whose macroscopic evolution is described by these equations and which possess an in-built smoothing. In this article, our main results are to derive regularized versions of the ill-posed one-dimensional SPDE

$$\partial_t \rho = \frac{1}{2}\Delta\Phi(\rho) - 2\nabla\big(W'\Phi(\rho)\big),$$

where the spatial white noise W' is replaced by a regularization W'_ε, as quenched and annealed hydrodynamic limits of zero-range interacting particle systems in ε-regularized Sinai-type random environments. Some computations are also made about annealed mean hydrodynamic limits in unregularized Sinai-type random environments with respect to independent particles.

REFERENCES

[1] ANDJEL, E. D. (1982). Invariant measures for the zero range processes. *Ann. Probab.* **10** 525–547. MR0659526

[2] ANDRIOPOULOS, G. (2021). Invariance principles for random walks in random environment on trees. *Electron. J. Probab.* **26** Paper No. 115. MR4309973 https://doi.org/10.1214/21-ejp687

[3] BAHADORAN, C., MOUNTFORD, T. S., RAVISHANKAR, K. and SAADA, E. (2021). Zero-range process in random environment. In *From Particle Systems to Partial Differential Equations. Springer Proc. Math. Stat.* **352** 51–77. Springer, Cham. MR4376180 https://doi.org/10.1007/978-3-030-69784-6_4

[4] BENOIS, O., KIPNIS, C. and LANDIM, C. (1995). Large deviations from the hydrodynamical limit of mean zero asymmetric zero range processes. *Stochastic Process. Appl.* **55** 65–89. MR1312149 https://doi.org/10.1016/0304-4149(95)91543-A

[5] BILLINGSLEY, P. (1968). *Convergence of Probability Measures*. Wiley, New York. MR0233396

[6] BROX, T. (1986). A one-dimensional diffusion process in a Wiener medium. *Ann. Probab.* **14** 1206–1218. MR0866343

[7] ETHIER, S. and KURTZ, T. (2005). *Markov Processes: Characterization and Convergence*. Wiley-Interscience, Hoboken, NJ.

[8] EVANS, L. C. (2010). *Partial Differential Equations*, 2nd ed. *Graduate Studies in Mathematics* **19**. Amer. Math. Soc., Providence, RI. MR2597943 https://doi.org/10.1090/gsm/019

[9] EVANS, M. R. and HANNEY, T. (2005). Nonequilibrium statistical mechanics of the zero-range process and related models. *J. Phys. A* **38** R195–R240. MR2145800 https://doi.org/10.1088/0305-4470/38/19/R01

[10] FAGGIONATO, A. (2007). Bulk diffusion of 1D exclusion process with bond disorder. *Markov Process. Related Fields* **13** 519–542. MR2357386

[11] FAGGIONATO, A. (2008). Random walks and exclusion processes among random conductances on random infinite clusters: Homogenization and hydrodynamic limit. *Electron. J. Probab.* **13** 2217–2247. MR2469609 https://doi.org/10.1214/EJP.v13-591

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[12] FAGGIONATO, A. (2010). Hydrodynamic limit of zero range processes among random conductances on the supercritical percolation cluster. *Electron. J. Probab.* **15** 259–291. MR2609588 https://doi.org/10.1214/EJP.v15-748

[13] FAGGIONATO, A., JARA, M. and LANDIM, C. (2009). Hydrodynamic behavior of 1D subdiffusive exclusion processes with random conductances. *Probab. Theory Related Fields* **144** 633–667. MR2496445 https://doi.org/10.1007/s00440-008-0157-7

[14] FATKULLIN, I., SETHURAMAN, S. and XUE, J. (2020). On hydrodynamic limits of Young diagrams. *Electron. J. Probab.* **25** Paper No. 58. MR4095054 https://doi.org/10.1214/20-ejp455

[15] FUNAKI, T., HOSHINO, M., SETHURAMAN, S. and XIE, B. (2021). Asymptotics of PDE in random environment by paracontrolled calculus. *Ann. Inst. Henri Poincaré Probab. Stat.* **57** 1702–1735. MR4291459 https://doi.org/10.1214/20-aihp1129

[16] GOLOSOV, A. O. (1986). Limit distributions for a random walk in a critical one-dimensional random environment. *Russian Math. Surveys* **41** 199–200.

[17] GONÇALVES, P. and JARA, M. (2008). Scaling limits for gradient systems in random environment. *J. Stat. Phys.* **131** 691–716. MR2398949 https://doi.org/10.1007/s10955-008-9509-z

[18] HU, Y., LÊ, K. and MYTNIK, L. (2017). Stochastic differential equation for Brox diffusion. *Stochastic Process. Appl.* **127** 2281–2315. MR3652414 https://doi.org/10.1016/j.spa.2016.10.010

[19] JARA, M., LANDIM, C. and SETHURAMAN, S. (2013). Nonequilibrium fluctuations for a tagged particle in one-dimensional sublinear zero-range processes. *Ann. Inst. Henri Poincaré Probab. Stat.* **49** 611–637. MR3112428 https://doi.org/10.1214/12-AIHP478

[20] JARA, M., LANDIM, C. and TEIXEIRA, A. (2011). Quenched scaling limits of trap models. *Ann. Probab.* **39** 176–223. MR2778800 https://doi.org/10.1214/10-AOP554

[21] JARA, M. and PETERSON, J. (2017). Hydrodynamic limit for a system of independent, sub-ballistic random walks in a common random environment. *Ann. Inst. Henri Poincaré Probab. Stat.* **53** 1747–1792. MR3729634 https://doi.org/10.1214/16-AIHP770

[22] JARA, M. D. and LANDIM, C. (2006). Nonequilibrium central limit theorem for a tagged particle in symmetric simple exclusion. *Ann. Inst. Henri Poincaré Probab. Stat.* **42** 567–577. MR2259975 https://doi.org/10.1016/j.anihpb.2005.04.007

[23] JARA, M. D., LANDIM, C. and SETHURAMAN, S. (2009). Nonequilibrium fluctuations for a tagged particle in mean-zero one-dimensional zero-range processes. *Probab. Theory Related Fields* **145** 565–590. MR2529439 https://doi.org/10.1007/s00440-008-0178-2

[24] KESTEN, H. (1986). The limit distribution of Sinaĭ's random walk in random environment. *Phys. A* **138** 299–309. MR0865247 https://doi.org/10.1016/0378-4371(86)90186-X

[25] KIPNIS, C. and LANDIM, C. (1999). *Scaling Limits of Interacting Particle Systems. Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **320**. Springer, Berlin. MR1707314 https://doi.org/10.1007/978-3-662-03752-2

[26] LANDIM, C., SETHURAMAN, S. and VARADHAN, S. (1996). Spectral gap for zero-range dynamics. *Ann. Probab.* **24** 1871–1902. MR1415232 https://doi.org/10.1214/aop/1041903209

[27] LIGGETT, T. M. (1985). *Interacting Particle Systems. Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **276**. Springer, New York. MR0776231 https://doi.org/10.1007/978-1-4613-8542-4

[28] MATZAVINOS, A., ROITERSHTEIN, A. and SEOL, Y. (2016). Random walks in a sparse random environment. *Electron. J. Probab.* **21** Paper No. 72. MR3592203 https://doi.org/10.1214/16-EJP16

[29] MORRIS, B. (2006). Spectral gap for the zero range process with constant rate. *Ann. Probab.* **34** 1645–1664. MR2271475 https://doi.org/10.1214/009117906000000304

[30] NAGAHATA, Y. (2010). Spectral gap for zero-range processes with jump rate $g(x) = x^{\gamma}$. *Stochastic Process. Appl.* **120** 949–958. MR2610333 https://doi.org/10.1016/j.spa.2010.01.019

[31] NAGY, K. (2002). Symmetric random walk in random environment in one dimension. *Period. Math. Hungar.* **45** 101–120. MR1955197 https://doi.org/10.1023/A:1022354131403

[32] PACHECO, C. G. (2018). From the Sinai's walk to the Brox diffusion using bilinear forms. Available at arXiv:1605.02826v1.

[33] PETERSON, J. (2010). Systems of one-dimensional random walks in a common random environment. *Electron. J. Probab.* **15** 1024–1040. MR2659756 https://doi.org/10.1214/EJP.v15-784

[34] QUASTEL, J. (2006). Bulk diffusion in a system with site disorder. *Ann. Probab.* **34** 1990–2036. MR2271489 https://doi.org/10.1214/009117906000000322

[35] SEIGNOUREL, P. (2000). Discrete schemes for processes in random media. *Probab. Theory Related Fields* **118** 293–322. MR1800534 https://doi.org/10.1007/PL00008743

[36] SINAĬ, Y. G. (1982). The limit behavior of a one-dimensional random walk in a random environment. *Teor. Veroyatn. Primen.* **27** 247–258. MR0657919

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OPTIMAL STOPPING WITH SIGNATURES

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We propose a new method for solving optimal stopping problems (such as American option pricing in finance) under minimal assumptions on the underlying stochastic process X. We consider classic and randomized stopping times represented by linear and nonlinear functionals of the *rough path signature* $\mathbb{X}^{<\infty}$ associated to X, and prove that maximizing over these classes of *signature stopping times*, in fact, solves the original optimal stopping problem. Using the algebraic properties of the signature, we can then recast the problem as a (deterministic) optimization problem depending only on the (truncated) expected signature $\mathbb{E}[\mathbb{X}^{\le N}_{0,T}]$. By applying a deep neural network approach to approximate the nonlinear signature functionals, we can efficiently solve the optimal stopping problem numerically. The only assumption on the process X is that it is a continuous (geometric) random rough path. Hence, the theory encompasses processes such as fractional Brownian motion, which fail to be either semimartingales or Markov processes, and can be used, in particular, for American-type option pricing in fractional models, for example, on financial or electricity markets.

REFERENCES

[1] ARRIBAS, I. P., GOODWIN, G. M., GEDDES, J. R., LYONS, T. and SAUNDERS, K. E. (2018). A signature-based machine learning model for distinguishing bipolar disorder and borderline personality disorder. *Transl. Psychiatry* **8** 1–7.

[2] BACKHOFF-VERAGUAS, J., BARTL, D., BEIGLBÖCK, M. and EDER, M. (2020). All adapted topologies are equal. *Probab. Theory Related Fields* **178** 1125–1172. MR4168395 https://doi.org/10.1007/s00440-020-00993-8

[3] BARNDORFF-NIELSEN, O. (2012). Notes on the gamma kernel Technical Report No. 03 T.N. Thiele Centre, Department of Mathematics, Aarhus Univ.

[4] BAYER, C., HARANG, F. A. and PIGATO, P. (2021). Log-modulated rough stochastic volatility models. *SIAM J. Financial Math.* **12** 1257–1284. MR4318211 https://doi.org/10.1137/20M135902X

[5] BAYER, C., TEMPONE, R. and WOLFERS, S. (2020). Pricing American options by exercise rate optimization. *Quant. Finance* **20** 1749–1760. MR4159590 https://doi.org/10.1080/14697688.2020.1750678

[6] BECKER, S., CHERIDITO, P. and JENTZEN, A. (2019). Deep optimal stopping. *J. Mach. Learn. Res.* **20** Paper No. 74, 25. MR3960928

[7] BENNEDSEN, M. (2017). A rough multi-factor model of electricity spot prices. *Energy Econ.* **63** 301–313. https://doi.org/10.1016/j.eneco.2017.02.007

[8] BENNEDSEN, M. (2020). Semiparametric estimation and inference on the fractal index of Gaussian and conditionally Gaussian time series data. *Econometric Rev.* **39** 875–903. MR4160432 https://doi.org/10.1080/07474938.2020.1721832

[9] CASAS, F. and MURUA, A. (2009). An efficient algorithm for computing the Baker–Campbell–Hausdorff series and some of its applications. *J. Math. Phys.* **50** 033513, 23. MR2510918 https://doi.org/10.1063/1.3078418

[10] CHEN, K.-T. (1957). Integration of paths, geometric invariants and a generalized Baker–Hausdorff formula. *Ann. of Math.* (2) **65** 163–178. MR0085251 https://doi.org/10.2307/1969671

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[11] CHEVYREV, I. and LYONS, T. (2016). Characteristic functions of measures on geometric rough paths. *Ann. Probab.* **44** 4049–4082. MR3572331 https://doi.org/10.1214/15-AOP1068

[12] CONT, R. and FOURNIE, D. (2010). A functional extension of the Ito formula. *C. R. Math. Acad. Sci. Paris* **348** 57–61. MR2586744 https://doi.org/10.1016/j.crma.2009.11.013

[13] DUPIRE, B. (2019). Functional Itô calculus. *Quant. Finance* **19** 721–729. MR3939653 https://doi.org/10.1080/14697688.2019.1575974

[14] FAWCETT, T. (2002). Problems in stochastic analysis: Connections between rough paths and non-commutative harmonic analysis Ph.D. thesis, Univ. Oxford.

[15] FRIZ, P. K. and HAIRER, M. (2014). *A Course on Rough Paths: With an Introduction to Regularity Structures. Universitext*. Springer, Cham. MR3289027 https://doi.org/10.1007/978-3-319-08332-2

[16] FRIZ, P. K. and VICTOIR, N. B. (2010). *Multidimensional Stochastic Processes as Rough Paths: Theory and Applications. Cambridge Studies in Advanced Mathematics* **120**. Cambridge Univ. Press, Cambridge. MR2604669 https://doi.org/10.1017/CBO9780511845079

[17] FYODOROV, Y. V., KHORUZHENKO, B. A. and SIMM, N. J. (2016). Fractional Brownian motion with Hurst index $H = 0$ and the Gaussian unitary ensemble. *Ann. Probab.* **44** 2980–3031. MR3531684 https://doi.org/10.1214/15-AOP1039

[18] HAGER, P. and NEUMAN, E. (2020). The multiplicative chaos of $H = 0$ fractional Brownian fields. E-prints. Available at arXiv:2008.01385.

[19] HAMBLY, B. and LYONS, T. (2010). Uniqueness for the signature of a path of bounded variation and the reduced path group. *Ann. of Math.* (2) **171** 109–167. MR2630037 https://doi.org/10.4007/annals.2010.171.109

[20] KALSI, J., LYONS, T. and ARRIBAS, I. P. (2020). Optimal execution with rough path signatures. *SIAM J. Financial Math.* **11** 470–493. MR4091169 https://doi.org/10.1137/19M1259778

[21] LESHNO, M., LIN, V. Y., PINKUS, A. and SCHOCKEN, S. (1993). Multilayer feedforward networks with a nonpolynomial activation function can approximate any function. *Neural Netw.* **6** 861–867. https://doi.org/10.1016/S0893-6080(05)80131-5

[22] LYONS, T. J., CARUANA, M. and LÉVY, T. (2007). *Differential Equations Driven by Rough Paths. Lecture Notes in Math.* **1908**. Springer, Berlin. MR2314753

[23] NEUMAN, E. and ROSENBAUM, M. (2018). Fractional Brownian motion with zero Hurst parameter: A rough volatility viewpoint. *Electron. Commun. Probab.* **23** Paper No. 61, 12. MR3863917 https://doi.org/10.1214/18-ECP158

[24] REIZENSTEIN, J. (2017). Calculation of Iterated-Integral Signatures and Log Signatures.

[25] REIZENSTEIN, J. F. and GRAHAM, B. (2020). Algorithm 1004: The iisignature library: Efficient calculation of iterated-integral signatures and log signatures. *ACM Trans. Math. Software* **46** Art. 8, 21. MR4095194 https://doi.org/10.1145/3371237

[26] REUTENAUER, C. (1993). *Free Lie Algebras. London Mathematical Society Monographs. New Series* **7**. The Clarendon Press, Oxford University Press, New York. MR1231799

[27] ROGERS, L. C. G. and WILLIAMS, D. (2000). *Diffusions, Markov Processes, and Martingales. Vol.* 1: *Foundations. Cambridge Mathematical Library*. Cambridge Univ. Press, Cambridge. MR1796539 https://doi.org/10.1017/CBO9781107590120

[28] SHIRYAEV, A. N. (2008). *Optimal Stopping Rules. Stochastic Modelling and Applied Probability* **8**. Springer, Berlin. MR2374974

[29] STOYANOV, J. M. (2013). *Counterexamples in Probability*, 3rd ed. Dover, Mineola, NY. MR3837562

[30] WIŚNIEWSKI, A. (1994). The structure of measurable mappings on metric spaces. *Proc. Amer. Math. Soc.* **122** 147–150. MR1201807 https://doi.org/10.2307/2160853

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CONVERGENCE OF A TIME-STEPPING SCHEME TO THE FREE BOUNDARY IN THE SUPERCOOLED STEFAN PROBLEM

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The supercooled Stefan problem and its variants describe the freezing of a supercooled liquid in physics, as well as the large system limits of systemic risk models in finance and of integrate-and-fire models in neuroscience. Adopting the physics terminology, the supercooled Stefan problem is known to feature a finite-time blow-up of the freezing rate for a wide range of initial temperature distributions in the liquid. Such a blow-up can result in a discontinuity of the liquid-solid boundary. In this paper, we prove that the natural Euler time-stepping scheme applied to a probabilistic formulation of the supercooled Stefan problem converges to the liquid-solid boundary of its physical solution globally in time, in the Skorokhod M1 topology. In the course of the proof, we give an explicit bound on the rate of local convergence for the time-stepping scheme. We also run numerical tests to compare our theoretical results to the practically observed convergence behavior.

REFERENCES

[1] CUCHIERO, C., RIGGER, S. and SVALUTO-FERRO, S. Propagation of minimality in the supercooled Stefan problem. arXiv:2010.03580.

[2] DELARUE, F., INGLIS, J., RUBENTHALER, S. and TANRÉ, E. (2015). Global solvability of a networked integrate-and-fire model of McKean-Vlasov type. *Ann. Appl. Probab.* **25** 2096–2133. MR3349003 https://doi.org/10.1214/14-AAP1044

[3] DELARUE, F., INGLIS, J., RUBENTHALER, S. and TANRÉ, E. (2015). Particle systems with a singular mean-field self-excitation. Application to neuronal networks. *Stochastic Process. Appl.* **125** 2451–2492. MR3322871 https://doi.org/10.1016/j.spa.2015.01.007

[4] DELARUE, F., NADTOCHIY, S. and SHKOLNIKOV, M. Global solutions to the supercooled Stefan problem with blow-ups: Regularity and uniqueness. *Probab. Math. Phys.* To appear. arXiv:1902.05174.

[5] DEWYNNE, J. N., HOWISON, S. D., OCKENDON, J. R. and XIE, W. Q. (1989). Asymptotic behavior of solutions to the Stefan problem with a kinetic condition at the free boundary. *J. Austral. Math. Soc. Ser. B* **31** 81–96. MR1002093 https://doi.org/10.1017/S0334270000006494

[6] DIBENEDETTO, E. and FRIEDMAN, A. (1984). The ill-posed Hele-Shaw model and the Stefan problem for supercooled water. *Trans. Amer. Math. Soc.* **282** 183–204. MR0728709 https://doi.org/10.2307/1999584

[7] FASANO, A. and PRIMICERIO, M. (1980/81). New results on some classical parabolic free-boundary problems. *Quart. Appl. Math.* **38** 439–460. MR0614552 https://doi.org/10.1090/qam/614552

[8] FASANO, A. and PRIMICERIO, M. (1983). A critical case for the solvability of Stefan-like problems. *Math. Methods Appl. Sci.* **5** 84–96. MR0690897 https://doi.org/10.1002/mma.1670050107

[9] FASANO, A., PRIMICERIO, M., HOWISON, S. D. and OCKENDON, J. R. (1989). On the singularities of one-dimensional Stefan problems with supercooling. In *Mathematical Models for Phase Change Problems (Óbidos, 1988). Internat. Ser. Numer. Math.* **88** 215–226. Birkhäuser, Basel. MR1038071

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[10] FISCHER, M. and NAPPO, G. (2010). On the moments of the modulus of continuity of Itô processes. *Stoch. Anal. Appl.* **28** 103–122. MR2597982 https://doi.org/10.1080/07362990903415825

[11] GOBET, E. (2000). Weak approximation of killed diffusion using Euler schemes. *Stochastic Process. Appl.* **87** 167–197. MR1757112 https://doi.org/10.1016/S0304-4149(99)00109-X

[12] HAMBLY, B., LEDGER, S. and SØJMARK, A. (2019). A McKean-Vlasov equation with positive feedback and blow-ups. *Ann. Appl. Probab.* **29** 2338–2373. MR3983340 https://doi.org/10.1214/18-AAP1455

[13] HOWISON, S. and XIE, W. Q. (1989). Kinetic undercooling regularization of supercooled Stefan problems. In *Mathematical Models for Phase Change Problems* (*Óbidos*, 1988). *Internat. Ser. Numer. Math.* **88** 227–237. Birkhäuser, Basel. MR1038072

[14] HOWISON, S. D., OCKENDON, J. R. and LACEY, A. A. (1985). Singularity development in moving-boundary problems. *Quart. J. Mech. Appl. Math.* **38** 343–360. MR0800769 https://doi.org/10.1093/qjmam/38.3.343

[15] KAUSHANSKY, V. and REISINGER, C. (2019). Simulation of a simple particle system interacting through hitting times. *Discrete Contin. Dyn. Syst. Ser. B* **24** 5481–5502. MR4026935 https://doi.org/10.3934/dcdsb.2019067

[16] LACEY, A. A. and OCKENDON, J. R. (1985). Ill-posed free boundary problems. *Control Cybernet.* **14** 275–296. MR0839524

[17] LAMÉ, G. and CLAPEYRON, B. P. (1831). Mémoire sur la solidification par réfroidissement d'un globe liquide. *Ann. Chim. Phys.* **47** 250–256.

[18] LEDGER, S. and SØJMARK, A. Uniqueness for contagious McKean-Vlasov systems in the weak feedback regime. arXiv:1811.12356.

[19] LEDGER, S. and SØJMARK, A. (2021). At the mercy of the common noise: Blow-ups in a conditional McKean-Vlasov problem. *Electron. J. Probab.* **26** Paper No. 35, 39. MR4235486 https://doi.org/10.1214/21-EJP597

[20] LIPTON, A., KAUSHANSKY, V. and REISINGER, C. (2021). Semi-analytical solution of a McKean-Vlasov equation with feedback through hitting a boundary. *European J. Appl. Math.* **32** 1035–1068. MR4335143 https://doi.org/10.1017/S0956792519000342

[21] NADTOCHIY, S. and SHKOLNIKOV, M. (2019). Particle systems with singular interaction through hitting times: Application in systemic risk modeling. *Ann. Appl. Probab.* **29** 89–129. MR3910001 https://doi.org/10.1214/18-AAP1403

[22] NADTOCHIY, S. and SHKOLNIKOV, M. (2020). Mean field systems on networks, with singular interaction through hitting times. *Ann. Probab.* **48** 1520–1556. MR4112723 https://doi.org/10.1214/19-AOP1403

[23] SHERMAN, B. (1970). A general one-phase Stefan problem. *Quart. Appl. Math.* **28** 377–382. MR0282082 https://doi.org/10.1090/qam/282082

[24] SKOROHOD, A. V. (1956). Limit theorems for stochastic processes. *Teor. Veroyatn. Primen.* **1** 289–319. MR0084897

[25] STEFAN, J. (1889). Über einige Probleme der Theorie der Wärmeleitung. *Sitzungber., Wien, Akad. Mat. Natur.* **98** 473–484.

[26] STEFAN, J. (1890). Über die Theorie der Eisbildung. *Monatsh. Math. Phys.* **1** 1–6. MR1546138 https://doi.org/10.1007/BF01692459

[27] STEFAN, J. (1890). Über die Verdampfung und die Auflösung als Vorgänge der Diffusion. *Ann. Phys.* **277** 725–747.

[28] STEFAN, J. (1891). Über die Theorie der Eisbildung, insbesondere über die Eisbildung im Polarmeere. *Ann. Physik Chemie* **42** 269–286.

[29] VISINTIN, A. (1987). Stefan problem with a kinetic condition at the free boundary. *Ann. Mat. Pura Appl.* (4) **146** 97–122. MR0916689 https://doi.org/10.1007/BF01762361

[30] WHITT, W. (2002). *Stochastic-Process Limits*: *An Introduction to Stochastic-Process Limits and Their Application to Queues*. *Springer Series in Operations Research*. Springer, New York. MR1876437

[31] XIE, W. Q. (1990). The Stefan problem with a kinetic condition at the free boundary. *SIAM J. Math. Anal.* **21** 362–373. MR1038897 https://doi.org/10.1137/0521020

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SPREAD OF PREMALIGNANT MUTANT CLONES AND CANCER INITIATION IN MULTILAYERED TISSUE

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Over 80% of human cancers originate from the epithelium, which covers the outer and inner surfaces of organs and blood vessels. In stratified epithelium, the bottom layers are occupied by stem and stem-like cells that continually divide and replenish the upper layers. In this work, we study the spread of premalignant mutant clones and cancer initiation in stratified epithelium, using the biased voter model on stacked two-dimensional lattices. Our main result is an estimate of the propagation speed of a premalignant mutant clone, which is asymptotically precise in the cancer-relevant weak-selection limit. We use our main result to study cancer initiation under a two-step mutational model of cancer, which includes computing the distributions of the time of cancer initiation and the size of the premalignant clone giving rise to cancer. Our work quantifies the effect of epithelial tissue thickness on the process of carcinogenesis, thereby contributing to an emerging understanding of the spatial evolutionary dynamics of cancer.

REFERENCES

[1] ARMITAGE, P. and DOLL, R. (1954). The age distribution of cancer and a multi-stage theory of carcinogenesis. *Br. J. Cancer* **8** 1–12.

[2] ARMITAGE, P. and DOLL, R. (1957). A two-stage theory of carcinogenesis in relation to the age distribution of human cancer. *Br. J. Cancer* **11** 161–169.

[3] BOZIC, I., ANTAL, T., OHTSUKI, H., CARTER, H., KIM, D., CHEN, S., KARCHIN, R., KINZLER, K. W., VOGELSTEIN, B. et al. (2010). Accumulation of driver and passenger mutations during tumor progression. *Proc. Natl. Acad. Sci. USA* **107** 18545–18550.

[4] BRAAKHUIS, B., TABOR, M., KUMMER, J., LEEMANS, C. and BRAKENHOFF, R. (2003). A genetic explanation of Slaughter's concept of field cancerization evidence and clinical implications. *Cancer Res.* **63** 1727–1730.

[5] BRAMSON, M. and GRIFFEATH, D. (1980). On the Williams–Bjerknes tumour growth model. II. *Math. Proc. Cambridge Philos. Soc.* **88** 339–357. MR0578279 https://doi.org/10.1017/S0305004100057650

[6] BRAMSON, M. and GRIFFEATH, D. (1981). On the Williams–Bjerknes tumour growth model. I. *Ann. Probab.* **9** 173–185. MR0606980

[7] CANNONE, C. (2017). A short note on Poisson tail bounds. Available at http://www.cs.columbia.edu/~ccanonne/files/misc/2017-poissonconcentration.pdf.

[8] CHAI, H. and BROWN, R. (2009). Field effect in cancer–an update. *Annals of Clinical & Laboratory Science* **39** 331–337.

[9] CHANDRASOMA, P. T. (2018). Chapter 4—histologic definition and diagnosis of epithelia in the esophagus and proximal stomach. In *GERD* (P. T. Chandrasoma, ed.) 73–107. Academic Press, San Diego.

[10] CURTIUS, K., WRIGHT, N. and GRAHAM, T. (2018). An evolutionary perspective on field cancerization. *Nat. Rev. Cancer* **18** 19–32.

[11] DURRETT, R. (1984). Oriented percolation in two dimensions. *Ann. Probab.* **12** 999–1040. MR0757768

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[12] DURRETT, R. (1988). *Lecture Notes on Particle Systems and Percolation*. *The Wadsworth & Brooks/Cole Statistics/Probability Series*. Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, CA. MR0940469

[13] DURRETT, R. (1995). Ten lectures on particle systems. In *Lectures on Probability Theory* (*Saint-Flour*, 1993). *Lecture Notes in Math*. **1608** 97–201. Springer, Berlin. MR1383122 https://doi.org/10.1007/BFb0095747

[14] DURRETT, R., FOO, J. and LEDER, K. (2016). Spatial Moran models, II: Cancer initiation in spatially structured tissue. *J. Math. Biol*. **72** 1369–1400. MR3464206 https://doi.org/10.1007/s00285-015-0912-1

[15] DURRETT, R. and MOSELEY, S. (2015). Spatial Moran models I. Stochastic tunneling in the neutral case. *Ann. Appl. Probab*. **25** 104–115. MR3297767 https://doi.org/10.1214/13-AAP989

[16] DURRETT, R. and ZÄHLE, I. (2007). On the width of hybrid zones. *Stochastic Process. Appl*. **117** 1751–1763. MR2437727 https://doi.org/10.1016/j.spa.2006.05.017

[17] DVORETZKY, A. and ERDÖS, P. (1951). Some problems on random walk in space. In *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, 1950 353–367. Univ. California Press, Berkeley-Los Angeles, CA. MR0047272

[18] FOO, J., LEDER, K. and RYSER, M. D. (2014). Multifocality and recurrence risk: A quantitative model of field cancerization. *J. Theoret. Biol*. **355** 170–184. MR3217192 https://doi.org/10.1016/j.jtbi.2014.02.042

[19] FOO, J., LEDER, K. and SCHWEINSBERG, J. (2020). Mutation timing in a spatial model of evolution. *Stochastic Process. Appl*. **130** 6388–6413. MR4140038 https://doi.org/10.1016/j.spa.2020.05.015

[20] FOO, J., LEDER, K. and ZHU, J. (2014). Escape times for branching processes with random mutational fitness effects. *Stochastic Process. Appl*. **124** 3661–3697. MR3249351 https://doi.org/10.1016/j.spa.2014.06.003

[21] GEBOES, K. (1994). Squamous mucosa and reflux. http://www.hon.ch/OESO/books/Vol_3_Eso_Mucosa/Articles/ART004.HTML.

[22] INSTITUTE, N. N. C. SEER Training Cancer classification. https://training.seer.cancer.gov/disease/categories/classification.html.

[23] KNUDSON, A. (2001). Two genetic hits (more or less) to cancer. *Nat. Rev. Cancer* **1** 157–161.

[24] KNUDSON, A. G. (1971). Mutation and cancer: Statistical study of retinoblastoma. *Proc. Natl. Acad. Sci. USA* **68** 820–823.

[25] KOMAROVA, N. L. (2006). Spatial stochastic models for cancer initiation and progression. *Bull. Math. Biol*. **68** 1573–1599. MR2257717 https://doi.org/10.1007/s11538-005-9046-8

[26] LAWLER, G. F. and LIMIC, V. (2010). *Random Walk*: *A Modern Introduction*. *Cambridge Studies in Advanced Mathematics* **123**. Cambridge Univ. Press, Cambridge. MR2677157 https://doi.org/10.1017/CBO9780511750854

[27] LEVIN, D. A., PERES, Y. and WILMER, E. L. (2009). *Markov Chains and Mixing Times*. Amer. Math. Soc., Providence, RI. MR2466937 https://doi.org/10.1090/mbk/058

[28] LIGGETT, T. M. (2005). *Interacting Particle Systems*. *Classics in Mathematics*. Springer, Berlin. MR2108619 https://doi.org/10.1007/b138374

[29] NOWAK, M. A. (2006). *Evolutionary Dynamics*: *Exploring the Equations of Life*. The Belknap Press of Harvard Univ. Press, Cambridge, MA. MR2252879

[30] RICHARDSON, D. (1973). Random growth in a tessellation. *Proc. Camb. Philos. Soc*. **74** 515–528. MR0329079 https://doi.org/10.1017/s0305004100077288

[31] WILLIAMS, T. and BJERKNES, R. (1972). Stochastic model for abnormal clone spread through epithelial basal layer. *Nature* **236** 19–21. https://doi.org/10.1038/236019a0

[32] WODARZ, D. and KOMAROVA, N. L. (2014). *Dynamics of Cancer*: *Mathematical Foundations of Oncology*. World Scientific Co. Pte. Ltd., Hackensack, NJ. MR3309233 https://doi.org/10.1142/8973

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DARWINIAN EVOLUTION AS BROWNIAN MOTION ON THE SIMPLEX: A GEOMETRIC PERSPECTIVE ON STOCHASTIC REPLICATOR DYNAMICS

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We prove that stochastic replicator dynamics can be interpreted as intrinsic Brownian motion on the simplex equipped with the Aitchison geometry. As an immediate consequence, we derive three approximation results in the spirit of Wong–Zakai approximation, Donsker's invariance principle and a JKO-scheme. Using the Fokker–Planck equation and Wasserstein-contraction estimates, we also study the long time behavior of the stochastic replicator equation, as an example of a nongradient drift diffusion on the Aitchison simplex.

REFERENCES

[1] AITCHISON, J. (1982). The statistical analysis of compositional data. *J. Roy. Statist. Soc. Ser. B* **44** 139–177. MR0676206

[2] AITCHISON, J. (1984). The statistical analysis of geochemical compositions. *J. Int. Assoc. Math. Geol.* **16** 531–564.

[3] AMBROSIO, L., GIGLI, N. and SAVARÉ, G. (2008). *Gradient Flows in Metric Spaces and in the Space of Probability Measures*, 2nd ed. *Lectures in Mathematics ETH Zürich*. Birkhäuser, Basel. MR2401600

[4] AY, N., JOST, J., LÊ, H. V. and SCHWACHHÖFER, L. (2017). *Information Geometry*. *Ergebnisse der Mathematik und Ihrer Grenzgebiete*. 3. *Folge/A Series of Modern Surveys in Mathematics*. Springer, Berlin.

[5] BAKRY, D., GENTIL, I. and LEDOUX, M. (2013). *Analysis and Geometry of Markov Diffusion Operators*. *Grundlehren der Mathematischen Wissenschaften*. Springer, Berlin.

[6] BARCELÓ-VIDAL, C., MARTÍN-FERNÁNDEZ, J. A. and MATEU-FIGUERAS, G. (2011). Compositional differential calculus on the simplex. In *Compositional Data Analysis* 176–190. Wiley, Chichester. MR2953384 https://doi.org/10.1002/9781119976462.ch13

[7] BILLHEIMER, D., GUTTORP, P. and FAGAN, W. F. (2001). Statistical interpretation of species composition. *J. Amer. Statist. Assoc.* **96** 1205–1214. MR1946574 https://doi.org/10.1198/016214501753381850

[8] BOGACHEV, V. I., KRYLOV, N. V., RÖCKNER, M. and SHAPOSHNIKOV, S. V. (2015). *Fokker–Planck–Kolmogorov Equations*. *Mathematical Surveys and Monographs* **207**. Amer. Math. Soc., Providence, RI. MR3443169 https://doi.org/10.1090/surv/207

[9] BOLLEY, F., GENTIL, I. and GUILLIN, A. (2012). Convergence to equilibrium in Wasserstein distance for Fokker–Planck equations. *J. Funct. Anal.* **263** 2430–2457. MR2964689 https://doi.org/10.1016/j.jfa.2012.07.007

[10] DA PRATO, G. (2006). *An Introduction to Infinite-Dimensional Analysis*. *Universitext*. Springer, Berlin. MR2244975 https://doi.org/10.1007/3-540-29021-4

[11] DA PRATO, G. and ZABCZYK, J. (1996). *Ergodicity for Infinite-Dimensional Systems*. *London Mathematical Society Lecture Note Series* **229**. Cambridge Univ. Press, Cambridge. MR1417491 https://doi.org/10.1017/CBO9780511662829

[12] EGOZCUE, J. J., PAWLOWSKY-GLAHN, V., MATEU-FIGUERAS, G. and BARCELÓ-VIDAL, C. (2003). Isometric logratio transformations for compositional data analysis. *Math. Geol.* **35** 279–300. MR1986165 https://doi.org/10.1023/A:1023818214614

[13] EIGEN, M. (1971). Selforganization of matter and the evolution of biological macromolecules. *Naturwissenschaften* **58** 465–523. https://doi.org/10.1007/BF00623322

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[14] EIGEN, M. and SCHUSTER, P. (2012). *The Hypercycle*: *A Principle of Natural Self-Organization*. Springer, Berlin.

[15] ERB, I. and AY, N. (2020). The information-geometric perspective of Compositional Data Analysis.

[16] ERBAR, M. (2010). The heat equation on manifolds as a gradient flow in the Wasserstein space. *Ann*. *Inst*. *Henri Poincaré Probab*. *Stat*. **46** 1–23. MR2641767 https://doi.org/10.1214/08-AIHP306

[17] FERNHOLZ, E. R. (2002). Stochastic portfolio theory. In *Stochastic Portfolio Theory* 1–24. Springer, Berlin.

[18] FRIZ, P. K. and HAIRER, M. (2014). *A Course on Rough Paths*. *Universitext*. Springer, Cham. MR3289027 https://doi.org/10.1007/978-3-319-08332-2

[19] FUDENBERG, D. and HARRIS, C. (1992). Evolutionary dynamics with aggregate shocks. *J*. *Econom*. *Theory* **57** 420–441. MR1180005 https://doi.org/10.1016/0022-0531(92)90044-I

[20] GAO, B. and PAVEL, L. (2017). On the properties of the softmax function with application in game theory and reinforcement learning. ArXiv preprint. Available at arXiv:1704.00805.

[21] HARPER, M. (2009). Information geometry and evolutionary game theory. ArXiv preprint. Available at arXiv:0911.1383.

[22] HOFBAUER, J. and IMHOF, L. A. (2009). Time averages, recurrence and transience in the stochastic replicator dynamics. *Ann*. *Appl*. *Probab*. **19** 1347–1368. MR2538073 https://doi.org/10.1214/08-AAP577

[23] HOFBAUER, J. and SIGMUND, K. (1998). *Evolutionary Games and Population Dynamics*. Cambridge Univ. Press, Cambridge. MR1635735 https://doi.org/10.1017/CBO9781139173179

[24] HOFRICHTER, J., JOST, J. and TRAN, T. D. (2017). *Information Geometry and Population Genetics*. *Understanding Complex Systems*. Springer, Cham. MR3444602 https://doi.org/10.1007/978-3-319-52045-2

[25] IKEDA, N. and WATANABE, S. (2014). *Stochastic Differential Equations and Diffusion Processes* **24**. Elsevier, Amsterdam.

[26] IMHOF, L. A. (2005). The long-run behavior of the stochastic replicator dynamics. *Ann*. *Appl*. *Probab*. **15** 1019–1045. MR2114999 https://doi.org/10.1214/105051604000000837

[27] ITÔ, S. (1952). Brownian motions in a topological group and in its covering group. *Rend*. *Circ*. *Mat*. *Palermo* (2) **1** 40–48. MR0049507 https://doi.org/10.1007/BF02843718

[28] JORDAN, R., KINDERLEHRER, D. and OTTO, F. (1998). The variational formulation of the Fokker–Planck equation. *SIAM J*. *Math*. *Anal*. **29** 1–17. MR1617171 https://doi.org/10.1137/S0036141096303359

[29] KRISLOCK, N. and WOLKOWICZ, H. (2012). Euclidean distance matrices and applications. In *Handbook on Semidefinite*, *Conic and Polynomial Optimization*. *Internat*. *Ser*. *Oper*. *Res*. *Management Sci*. **166** 879–914. Springer, New York. MR2894673 https://doi.org/10.1007/978-1-4614-0769-0_30

[30] LYONS, T. J., CARUANA, M. and LÉVY, T. (2007). *Differential Equations Driven by Rough Paths*. *Lecture Notes in Math*. **1908**. Springer, Berlin. MR2314753

[31] NATILE, L., PELETIER, M. A. and SAVARÉ, G. (2011). Contraction of general transportation costs along solutions to Fokker–Planck equations with monotone drifts. *J*. *Math*. *Pures Appl*. (9) **95** 18–35. MR2746434 https://doi.org/10.1016/j.matpur.2010.07.003

[32] NEWMAN, C. M. and WRIGHT, A. L. (1981). An invariance principle for certain dependent sequences. *Ann*. *Probab*. **9** 671–675. MR0624694

[33] NOWAK, M. A. (2006). *Evolutionary Dynamics*. Harvard Univ. Press, Cambridge.

[34] OTTO, F. (2001). The geometry of dissipative evolution equations: The porous medium equation. *Comm*. *Partial Differential Equations* **26** 101–174. MR1842429 https://doi.org/10.1081/PDE-100002243

[35] PAVLIOTIS, G. A. (2014). *Stochastic Processes and Applications*: *Diffusion Processes*, *the Fokker–Planck and Langevin Equations*. *Texts in Applied Mathematics* **60**. Springer, New York. MR3288096 https://doi.org/10.1007/978-1-4939-1323-7

[36] PAWLOWSKY-GLAHN, V. and EGOZCUE, J. J. (2001). Geometric approach to statistical analysis on the simplex. *Stoch*. *Environ*. *Res*. *Risk Assess*. **15** 384–398.

[37] PAWLOWSKY-GLAHN, V., EGOZCUE, J. J. and TOLOSANA DELGADO, R. (2007). Lecture notes on compositional data analysis.

[38] PAWLOWSKY-GLAHN, V. et al. (2003). Statistical modeling on coordinates.

[39] PINSKY, R. G. (1995). *Positive Harmonic Functions and Diffusion*. *Cambridge Studies in Advanced Mathematics* **45**. Cambridge Univ. Press, Cambridge. MR1326606 https://doi.org/10.1017/CBO9780511526244

[40] SANDHOLM, W. H., DOKUMACI, E. and FRANCHETTI, F. (2012). Dynamo: Diagrams for Evolutionary Game Dynamics.

[41] SHAHSHAHANI, S. (1979). *A New Mathematical Framework for the Study of Linkage and Selection*. American Mathematical Soc., Providence.

[42] SHALIZI, C. R. (2009). Dynamics of Bayesian updating with dependent data and misspecified models. *Electron*. *J*. *Stat*. **3** 1039–1074. MR2557128 https://doi.org/10.1214/09-EJS485

[43] Sigmund, K. (1986). A survey of replicator equations. In *Complexity, Language, and Life: Mathematical Approaches*. *Biomathematics* **16** 88–104. Springer, Berlin. MR0828280

[44] Smith, J. M. and Price, G. R. (1973). The logic of animal conflict. *Nature* **246** 15.

[45] Stroock, D. W. (2003). *Markov Processes from K. Itô's Perspective*. *Annals of Mathematics Studies* **155**. Princeton Univ. Press, Princeton, NJ. MR1980149 https://doi.org/10.1515/9781400835577

[46] Taylor, P. D. and Jonker, L. B. (1978). Evolutionarily stable strategies and game dynamics. *Math. Biosci.* **40** 145–156. MR0489983 https://doi.org/10.1016/0025-5564(78)90077-9

[47] van den Boogaart, K. G. and Tolosana-Delgado, R. (2013). *Analyzing Compositional Data with R*. *Use R!* Springer, Heidelberg. MR3099409 https://doi.org/10.1007/978-3-642-36809-7

[48] von Renesse, M.-K. and Sturm, K.-T. (2005). Transport inequalities, gradient estimates, entropy, and Ricci curvature. *Comm. Pure Appl. Math.* **58** 923–940. MR2142879 https://doi.org/10.1002/cpa.20060

[49] Wong, E. and Zakai, M. (1965). On the relation between ordinary and stochastic differential equations. *Internat. J. Engrg. Sci.* **3** 213–229. MR0183023 https://doi.org/10.1016/0020-7225(65)90045-5

[50] Wu, B., Gokhale, C. S., van Veelen, M., Wang, L. and Traulsen, A. (2013). Interpretations arising from Wrightian and Malthusian fitness under strong frequency dependent selection. *Ecol. Evol.* **3** 1276–1280.

[51] http://www.compositionaldata.com/.

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DIMENSION-FREE LOCAL CONVERGENCE AND PERTURBATIONS FOR REFLECTED BROWNIAN MOTIONS

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We describe and analyze a class of positive recurrent reflected Brownian motions (RBMs) in $\mathbb{R}_+^d$ for which local statistics converge to equilibrium at a rate independent of the dimension d. Under suitable assumptions on the reflection matrix, drift and diffusivity coefficients, dimension-independent stretched exponential convergence rates are obtained by estimating contractions in an underlying weighted distance between synchronously coupled RBMs. We also study the symmetric Atlas model as a first step in obtaining dimension-independent convergence rates for RBMs not satisfying the above assumptions. By analyzing a pathwise derivative process and connecting it to a random walk in a random environment, we obtain polynomial convergence rates for the gap process of the symmetric Atlas model started from appropriate perturbations of stationarity.

REFERENCES

[1] ANDRES, S. (2009). Diffusion processes with reflection. PhD thesis, TU Berlin.

[2] ATAR, R., BUDHIRAJA, A. and DUPUIS, P. (2001). On positive recurrence of constrained diffusion processes. *Ann. Probab.* **29** 979–1000. MR1849184 https://doi.org/10.1214/aop/1008956699

[3] BANERJEE, S. and BUDHIRAJA, A. (2020). Parameter and dimension dependence of convergence rates to stationarity for reflecting Brownian motions. *Ann. Appl. Probab.* **30** 2005–2029. MR4149521 https://doi.org/10.1214/19-AAP1550

[4] BANERJEE, S. and BUDHIRAJA, A. (2022). Domains of attraction of invariant distributions of the infinite Atlas model. *Ann. Probab.* **50** 1610–1646. MR4420428 https://doi.org/10.1214/22-aop1570

[5] BANERJEE, S. and BURDZY, K. (2021). Rates of convergence to equilibrium for potlatch and smoothing processes. *Ann. Probab.* **49** 1129–1163. MR4255140 https://doi.org/10.1214/20-aop1473

[6] BANERJEE, S. and SANKARARAMAN, A. (2021). Ergodicity and steady state analysis for interference queueing networks. In *Stochastic Processes and Functional Analysis—New Perspectives*. *Contemp. Math.* **774** 9–23. Amer. Math. Soc., Providence, RI. MR4344291 https://doi.org/10.1090/conm/774/15565

[7] BLANCHET, J. and CHEN, X. (2020). Rates of convergence to stationarity for reflected Brownian motion. *Math. Oper. Res.* **45** 660–681. MR4100279 https://doi.org/10.1287/moor.2019.1006

[8] BLANCHET, J., CHEN, X., SI, N. and GLYNN, P. W. (2021). Efficient steady-state simulation of high-dimensional stochastic networks. *Stoch. Syst.* **11** 174–192. MR4318724

[9] BUDHIRAJA, A. and LEE, C. (2007). Long time asymptotics for constrained diffusions in polyhedral domains. *Stochastic Process. Appl.* **117** 1014–1036. MR2340877 https://doi.org/10.1016/j.spa.2006.11.007

[10] CABEZAS, M., DEMBO, A., SARANTSEV, A. and SIDORAVICIUS, V. (2019). Brownian particles with rank-dependent drifts: Out-of-equilibrium behavior. *Comm. Pure Appl. Math.* **72** 1424–1458. MR3957396 https://doi.org/10.1002/cpa.21825

[11] DEMBO, A., GANTERT, N. and ZEITOUNI, O. (2004). Large deviations for random walk in random environment with holding times. *Ann. Probab.* **32** 996–1029. MR2044672 https://doi.org/10.1214/aop/1079021470

[12] DEMBO, A., JARA, M. and OLLA, S. (2019). The infinite Atlas process: Convergence to equilibrium. *Ann. Inst. Henri Poincaré Probab. Stat.* **55** 607–619. MR3949947 https://doi.org/10.1214/17-aihp875

[13] DEMBO, A. and TSAI, L.-C. (2017). Equilibrium fluctuation of the Atlas model. *Ann. Probab.* **45** 4529–4560. MR3737917 https://doi.org/10.1214/16-AOP1171

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Key words and phrases. Diffusion processes, reflected Brownian motion, Atlas model, Wasserstein distance, coupling, dimension-free convergence, weighted distance, derivative process.

[14] FERNHOLZ, E. R. (2002). Stochastic portfolio theory. In *Stochastic Portfolio Theory* 1–24. Springer, New York.

[15] HARRISON, J. M. and REIMAN, M. I. (1981). Reflected Brownian motion on an orthant. *Ann. Probab.* **9** 302–308. MR0606992

[16] HARRISON, J. M. and WILLIAMS, R. J. (1987). Brownian models of open queueing networks with homogeneous customer populations. *Stochastics* **22** 77–115. MR0912049 https://doi.org/10.1080/17442508708833469

[17] HARRISON, J. M. and WILLIAMS, R. J. (1987). Multidimensional reflected Brownian motions having exponential stationary distributions. *Ann. Probab.* **15** 115–137. MR0877593

[18] ICHIBA, T., KARATZAS, I. and SHKOLNIKOV, M. (2013). Strong solutions of stochastic equations with rank-based coefficients. *Probab. Theory Related Fields* **156** 229–248. MR3055258 https://doi.org/10.1007/s00440-012-0426-3

[19] ICHIBA, T., PAL, S. and SHKOLNIKOV, M. (2013). Convergence rates for rank-based models with applications to portfolio theory. *Probab. Theory Related Fields* **156** 415–448. MR3055264 https://doi.org/10.1007/s00440-012-0432-5

[20] ICHIBA, T., PAPATHANAKOS, V., BANNER, A., KARATZAS, I. and FERNHOLZ, R. (2011). Hybrid Atlas models. *Ann. Appl. Probab.* **21** 609–644. MR2807968 https://doi.org/10.1214/10-AAP706

[21] JOURDAIN, B. and MALRIEU, F. (2008). Propagation of chaos and Poincaré inequalities for a system of particles interacting through their CDF. *Ann. Appl. Probab.* **18** 1706–1736. MR2462546 https://doi.org/10.1214/07-AAP513

[22] JOURDAIN, B. and REYGNER, J. (2013). Propogation of chaos for rank-based interacting diffusions and long time behaviour of a scalar quasilinear parabolic equation. *Stoch. Partial Differ. Equ. Anal. Comput.* **1** 455–506. MR3327514 https://doi.org/10.1007/s40072-013-0014-2

[23] KARATZAS, I., PAL, S. and SHKOLNIKOV, M. (2016). Systems of Brownian particles with asymmetric collisions. *Ann. Inst. Henri Poincaré Probab. Stat.* **52** 323–354. MR3449305 https://doi.org/10.1214/14-AIHP646

[24] KELLA, O. and RAMASUBRAMANIAN, S. (2012). Asymptotic irrelevance of initial conditions for Skorohod reflection mapping on the nonnegative orthant. *Math. Oper. Res.* **37** 301–312. MR2931282 https://doi.org/10.1287/moor.1120.0538

[25] LEVIN, D. A. and PERES, Y. (2017). *Markov Chains and Mixing Times*. Amer. Math. Soc., Providence, RI. MR3726904 https://doi.org/10.1090/mbk/107

[26] MANDELBAUM, A. and RAMANAN, K. (2010). Directional derivatives of oblique reflection maps. *Math. Oper. Res.* **35** 527–558. MR2724063 https://doi.org/10.1287/moor.1100.0453

[27] MEYN, S. P. and TWEEDIE, R. L. (2012). *Markov Chains and Stochastic Stability*. *Communications and Control Engineering Series*. Springer London, Ltd., London. MR1287609 https://doi.org/10.1007/978-1-4471-3267-7

[28] PAL, S. and PITMAN, J. (2008). One-dimensional Brownian particle systems with rank-dependent drifts. *Ann. Appl. Probab.* **18** 2179–2207. MR2473654 https://doi.org/10.1214/08-AAP516

[29] PAL, S. and SARANTSEV, A. (2019). A note on transportation cost inequalities for diffusions with reflections. *Electron. Commun. Probab.* **24** Paper No. 21, 11 pp. MR3940196 https://doi.org/10.1214/19-ECP223

[30] REIMAN, M. I. (1984). Open queueing networks in heavy traffic. *Math. Oper. Res.* **9** 441–458. MR0757317 https://doi.org/10.1287/moor.9.3.441

[31] SARANTSEV, A. (2015). Triple and simultaneous collisions of competing Brownian particles. *Electron. J. Probab.* **20** no. 29, 28 pp. MR3325099 https://doi.org/10.1214/EJP.v20-3279

[32] SARANTSEV, A. (2017). Reflected Brownian motion in a convex polyhedral cone: Tail estimates for the stationary distribution. *J. Theoret. Probab.* **30** 1200–1223. MR3687255 https://doi.org/10.1007/s10959-016-0674-8

[33] SARANTSEV, A. (2017). Two-sided infinite systems of competing Brownian particles. *ESAIM Probab. Stat.* **21** 317–349. MR3743917 https://doi.org/10.1051/ps/2017013

[34] SARANTSEV, A. (2017). Infinite systems of competing Brownian particles. *Ann. Inst. Henri Poincaré Probab. Stat.* **53** 2279–2315. MR3729655 https://doi.org/10.1214/16-AIHP791

[35] SARANTSEV, A. (2019). Comparison techniques for competing Brownian particles. *J. Theoret. Probab.* **32** 545–585. MR3959619 https://doi.org/10.1007/s10959-019-00887-z

[36] SARANTSEV, A. and TSAI, L.-C. (2017). Stationary gap distributions for infinite systems of competing Brownian particles. *Electron. J. Probab.* **22** Paper No. 56, 20 pp. MR3672832 https://doi.org/10.1214/17-EJP78

[37] SZNITMAN, A.-S. (2004). Topics in random walks in random environment. In *School and Conference on Probability Theory*. *ICTP Lect. Notes*, *XVII* 203–266. Abdus Salam Int. Cent. Theoret. Phys., Trieste. MR2198849

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SPIKING AND COLLAPSING IN LARGE NOISE LIMITS OF SDES

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We analyze the strong noise limit of one-dimensional stochastic differential equations (SDEs).

Our initial motivation comes from continuous measurements of open quantum systems. In this context, Bauer, Bernard and Tilloy pointed out an intriguing behavior. As the noise grows larger, the solutions exhibit locally a collapsing, that is to say, converge to pure jump processes very reminiscent of a metastability phenomenon. But surprisingly the limiting jump process is decorated by a spike process.

We give a precise meaning to the convergence and completely prove these statements for a large class of one-dimensional diffusions, thanks to a robust strategy of proof.

REFERENCES

[1] BALLESTEROS, M., CRAWFORD, N., FRAAS, M., FRÖHLICH, J. and SCHUBNEL, B. (2019). Perturbation theory for weak measurements in quantum mechanics, systems with finite-dimensional state space. *Ann. Henri Poincaré* **20** 299–335. MR3900810 https://doi.org/10.1007/s00023-018-0741-z

[2] BAUER, M. and BERNARD, D. (2014). Real time imaging of quantum and thermal fluctuations: The case of a two-level system. *Lett. Math. Phys.* **104** 707–729. MR3200936 https://doi.org/10.1007/s11005-014-0688-z

[3] BAUER, M. and BERNARD, D. (2018). Stochastic spikes and strong noise limits of stochastic differential equations. *Ann. Henri Poincaré* **19** 653–693. MR3769243 https://doi.org/10.1007/s00023-018-0645-y

[4] BAUER, M., BERNARD, D. and TILLOY, A. (2015). Computing the rates of measurement-induced quantum jumps. *J. Phys. A* **48** 25FT02, 15. MR3355272 https://doi.org/10.1088/1751-8113/48/25/25FT02

[5] BAUER, M., BERNARD, D. and TILLOY, A. (2016). Zooming in on quantum trajectories. *J. Phys. A* **49** 10LT01, 9. MR3462320 https://doi.org/10.1088/1751-8113/49/10/10LT01

[6] BENOIST, T., BERNARDIN, C., CHETRITE, R., CHHAIBI, R., NAJNUDEL, J. and PELLEGRINI, C. (2021). Emergence of jumps in quantum trajectories via homogenization. *Comm. Math. Phys.* **387** 1821–1867. MR4324392 https://doi.org/10.1007/s00220-021-04179-8

[7] BILLINGSLEY, P. (2013). *Convergence of Probability Measures*. John Wiley & Sons. MR0233396

[8] FREIDLIN, M. I. and WENTZELL, A. D. (2012). *Random Perturbations of Dynamical Systems*, 3rd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **260**. Springer, Heidelberg. Translated from the 1979 Russian original by Joseph Szücs. MR2953753 https://doi.org/10.1007/978-3-642-25847-3

[9] HALL, B. C. (2013). *Quantum Theory for Mathematicians*. *Graduate Texts in Mathematics* **267**. Springer, New York. MR3112817 https://doi.org/10.1007/978-1-4614-7116-5

[10] HAROCHE, S. and RAIMOND, J.-M. (2006). *Exploring the Quantum*: *Atoms*, *Cavities and Photons*. *Oxford Graduate Texts*. Oxford Univ. Press, Oxford. MR2271425 https://doi.org/10.1093/acprof:oso/9780198509141.001.0001

[11] KARLIN, S. and TAYLOR, H. M. (1981). *A Second Course in Stochastic Processes*. Academic Press [Harcourt Brace Jovanovich, Publishers], New York–London. MR0611513

[12] KLEBANER, F. C. (2005). *Introduction to Stochastic Calculus with Applications*. World Scientific Publishing Company. MR1832481

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[13] KOLB, M. and LIESENFELD, M. (2019). Stochastic spikes and Poisson approximation of one-dimensional stochastic differential equations with applications to continuously measured quantum systems. *Ann. Henri Poincaré* **20** 1753–1783. MR3956159 https://doi.org/10.1007/s00023-019-00772-9

[14] LE BELLAC, M. (2011). *Quantum Physics*. Cambridge Univ. Press.

[15] MISRA, B. and SUDARSHAN, E. C. G. (1977). The Zeno's paradox in quantum theory. *J. Math. Phys.* **18** 756–763. MR0432085 https://doi.org/10.1063/1.523304

[16] MUNKRES, J. R. (2000). *Topology*. Prentice Hall, Inc., Upper Saddle River, NJ. Second edition of [MR0464128]. MR3728284

[17] REVUZ, D. and YOR, M. (1999). *Continuous Martingales and Brownian Motion*, 3rd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **293**. Springer, Berlin. MR1725357 https://doi.org/10.1007/978-3-662-06400-9

[18] TILLOY, A., BAUER, M. and BERNARD, D. (2015). Spikes in quantum trajectories. *Phys. Rev. A* **92** 052111.

[19] WHITT, W. (2002). *Stochastic-Process Limits*: *An Introduction to Stochastic-Process Limits and Their Application to Queues*. *Springer Series in Operations Research*. Springer, New York. MR1876437

[20] WISEMAN, H. M. and MILBURN, G. J. (2010). *Quantum Measurement and Control*. Cambridge Univ. Press, Cambridge. MR2761009

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FUNCTIONAL CENTRAL LIMIT THEOREMS FOR WIGNER MATRICES

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We consider the fluctuations of regular functions f of a Wigner matrix W viewed as an entire matrix $f(W)$. Going beyond the well-studied tracial mode, $\operatorname{Tr} f(W)$, which is equivalent to the customary linear statistics of eigenvalues, we show that $\operatorname{Tr} f(W)A$ is asymptotically normal for any nontrivial bounded deterministic matrix A. We identify three different and asymptotically independent modes of this fluctuation, corresponding to the tracial part, the traceless diagonal part and the off-diagonal part of $f(W)$ in the entire mesoscopic regime, where we find that the off-diagonal modes fluctuate on a much smaller scale than the tracial mode. As a main motivation to study CLT in such generality on small mesoscopic scales, we determine the fluctuations in the eigenstate thermalization hypothesis (*Phys. Rev. A* **43** (1991) 2046–2049), that is, prove that the eigenfunction overlaps with any deterministic matrix are asymptotically Gaussian after a small spectral averaging. Finally, in the macroscopic regime our result also generalizes (*Zh. Mat. Fiz. Anal. Geom.* **9** (2013) 536–581, 611, 615) to complex W and to all crossover ensembles in between. The main technical inputs are the recent multiresolvent local laws with traceless deterministic matrices from the companion paper (*Comm. Math. Phys.* **388** (2021) 1005–1048).

REFERENCES

[1] ADHIKARI, A. and HUANG, J. (2020). Dyson Brownian motion for general β and potential at the edge. *Probab. Theory Related Fields* **178** 893–950. MR4168391 https://doi.org/10.1007/s00440-020-00992-9

[2] BAI, Z., WANG, X. and ZHOU, W. (2009). CLT for linear spectral statistics of Wigner matrices. *Electron. J. Probab.* **14** 2391–2417. MR2556016 https://doi.org/10.1214/EJP.v14-705

[3] BAI, Z. D. and YAO, J. (2005). On the convergence of the spectral empirical process of Wigner matrices. *Bernoulli* **11** 1059–1092. MR2189081 https://doi.org/10.3150/bj/1137421640

[4] BAO, Z., SCHNELLI, K. and XU, Y. (2022). Central limit theorem for mesoscopic eigenvalue statistics of the free sum of matrices. *Int. Math. Res. Not. IMRN* **7** 5320–5382. MR4403963 https://doi.org/10.1093/imrn/rnaa210

[5] BAO, Z. and XIE, J. (2016). CLT for linear spectral statistics of Hermitian Wigner matrices with general moment conditions. *Theory Probab. Appl.* **60** 187–206. MR3568772 https://doi.org/10.1137/S0040585X97T987624

[6] BEKERMAN, F. and LODHIA, A. (2018). Mesoscopic central limit theorem for general β-ensembles. *Ann. Inst. Henri Poincaré Probab. Stat.* **54** 1917–1938. MR3865662 https://doi.org/10.1214/17-AIHP860

[7] BLOEMENDAL, A., ERDŐS, L., KNOWLES, A., YAU, H.-T. and YIN, J. (2014). Isotropic local laws for sample covariance and generalized Wigner matrices. *Electron. J. Probab.* **19** 33. MR3183577 https://doi.org/10.1214/ejp.v19-3054

[8] BOROT, G. and GUIONNET, A. (2013). Asymptotic expansion of beta matrix models in the multi-cut regime. Preprint. Available at arXiv:1303.1045.

[9] BOURGADE, P. and YAU, H.-T. (2017). The eigenvector moment flow and local quantum unique ergodicity. *Comm. Math. Phys.* **350** 231–278. MR3606475 https://doi.org/10.1007/s00220-016-2627-6

[10] BOURGADE, P., YAU, H.-T. and YIN, J. (2020). Random band matrices in the delocalized phase I: Quantum unique ergodicity and universality. *Comm. Pure Appl. Math.* **73** 1526–1596. MR4156609 https://doi.org/10.1002/cpa.21895

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[11] BOUTET DE MONVEL, A. and KHORUNZHY, A. (1999). Asymptotic distribution of smoothed eigenvalue density. I. Gaussian random matrices. *Random Oper. Stoch. Equ.* **7** 1–22. MR1678012 https://doi.org/10.1515/rose.1999.7.1.1

[12] BOUTET DE MONVEL, A. and KHORUNZHY, A. (1999). Asymptotic distribution of smoothed eigenvalue density. II. Wigner random matrices. *Random Oper. Stoch. Equ.* **7** 149–168. MR1689027 https://doi.org/10.1515/rose.1999.7.2.149

[13] BREUER, J. and DUITS, M. (2016). Universality of mesoscopic fluctuations for orthogonal polynomial ensembles. *Comm. Math. Phys.* **342** 491–531. MR3459158 https://doi.org/10.1007/s00220-015-2514-6

[14] CIPOLLONI, G., ERDŐS, L. and SCHRÖDER, D. (2023). Supplement to "Functional central limit theorems for Wigner matrices." https://doi.org/10.1214/22-AAP1820SUPP

[15] CIPOLLONI, G., ERDŐS, L. and SCHRÖDER, D. (2021). Edge universality for non-Hermitian random matrices. *Probab. Theory Related Fields* **179** 1–28. MR4221653 https://doi.org/10.1007/s00440-020-01003-7

[16] CIPOLLONI, G., ERDŐS, L. and SCHRÖDER, D. (2021). Eigenstate thermalization hypothesis for Wigner matrices. *Comm. Math. Phys.* **388** 1005–1048. MR4334253 https://doi.org/10.1007/s00220-021-04239-z

[17] COLLINS, B., MINGO, J. A., ŚNIADY, P. and SPEICHER, R. (2007). Second order freeness and fluctuations of random matrices. III. Higher order freeness and free cumulants. *Doc. Math.* **12** 1–70. MR2302524

[18] DEUTSCH, J. M. (1991). Quantum statistical mechanics in a closed system. *Phys. Rev. A* **43** 2046–2049. https://doi.org/10.1103/physreva.43.2046

[19] DUITS, M. and JOHANSSON, K. (2018). On mesoscopic equilibrium for linear statistics in Dyson's Brownian motion. *Mem. Amer. Math. Soc.* **255** v+118. MR3852256 https://doi.org/10.1090/memo/1222

[20] ERDŐS, L. and KNOWLES, A. (2015). The Altshuler–Shklovskii formulas for random band matrices I: The unimodular case. *Comm. Math. Phys.* **333** 1365–1416. MR3302637 https://doi.org/10.1007/s00220-014-2119-5

[21] ERDŐS, L. and KNOWLES, A. (2015). The Altshuler–Shklovskii formulas for random band matrices II: The general case. *Ann. Henri Poincaré* **16** 709–799. MR3311888 https://doi.org/10.1007/s00023-014-0333-5

[22] ERDŐS, L., KNOWLES, A., YAU, H.-T. and YIN, J. (2013). The local semicircle law for a general class of random matrices. *Electron. J. Probab.* **18** 59. MR3068390 https://doi.org/10.1214/EJP.v18-2473

[23] ERDŐS, L., KRÜGER, T. and SCHRÖDER, D. (2019). Random matrices with slow correlation decay. *Forum Math. Sigma* **7** e8. MR3941370 https://doi.org/10.1017/fms.2019.2

[24] ERDŐS, L. and SCHRÖDER, D. (2016). Fluctuations of functions of Wigner matrices. *Electron. Commun. Probab.* **21** 86. MR3600514 https://doi.org/10.1214/16-ECP38

[25] ERDŐS, L. and SCHRÖDER, D. (2018). Fluctuations of rectangular Young diagrams of interlacing Wigner eigenvalues. *Int. Math. Res. Not. IMRN* **10** 3255–3298. MR3805203 https://doi.org/10.1093/imrn/rnw330

[26] ERDŐS, L., YAU, H.-T. and YIN, J. (2012). Rigidity of eigenvalues of generalized Wigner matrices. *Adv. Math.* **229** 1435–1515. MR2871147 https://doi.org/10.1016/j.aim.2011.12.010

[27] GUIONNET, A. (2002). Large deviations upper bounds and central limit theorems for non-commutative functionals of Gaussian large random matrices. *Ann. Inst. Henri Poincaré Probab. Stat.* **38** 341–384. MR1899457 https://doi.org/10.1016/S0246-0203(01)01093-7

[28] HE, Y. (2019). Mesoscopic linear statistics of Wigner matrices of mixed symmetry class. *J. Stat. Phys.* **175** 932–959. MR3959983 https://doi.org/10.1007/s10955-019-02266-8

[29] HE, Y. and KNOWLES, A. (2017). Mesoscopic eigenvalue statistics of Wigner matrices. *Ann. Appl. Probab.* **27** 1510–1550. MR3678478 https://doi.org/10.1214/16-AAP1237

[30] HE, Y. and KNOWLES, A. (2020). Mesoscopic eigenvalue density correlations of Wigner matrices. *Probab. Theory Related Fields* **177** 147–216. MR4095015 https://doi.org/10.1007/s00440-019-00946-w

[31] HUANG, J. and LANDON, B. (2019). Rigidity and a mesoscopic central limit theorem for Dyson Brownian motion for general β and potentials. *Probab. Theory Related Fields* **175** 209–253. MR4009708 https://doi.org/10.1007/s00440-018-0889-y

[32] JOHANSSON, K. (1998). On fluctuations of eigenvalues of random Hermitian matrices. *Duke Math. J.* **91** 151–204. MR1487983 https://doi.org/10.1215/S0012-7094-98-09108-6

[33] KHORUNZHY, A. M., KHORUZHENKO, B. A. and PASTUR, L. A. (1996). Asymptotic properties of large random matrices with independent entries. *J. Math. Phys.* **37** 5033–5060. MR1411619 https://doi.org/10.1063/1.531589

[34] KNOWLES, A. and YIN, J. (2013). The isotropic semicircle law and deformation of Wigner matrices. *Comm. Pure Appl. Math.* **66** 1663–1750. MR3103909 https://doi.org/10.1002/cpa.21450

[35] LANDON, B., SOSOE, P. and YAU, H.-T. (2019). Fixed energy universality of Dyson Brownian motion. *Adv. Math.* **346** 1137–1332. MR3914908 https://doi.org/10.1016/j.aim.2019.02.010

[36] LI, Y., SCHNELLI, K. and XU, Y. (2021). Central limit theorem for mesoscopic eigenvalue statistics of deformed Wigner matrices and sample covariance matrices. *Ann. Inst. Henri Poincaré Probab. Stat.* **57** 506–546. MR4255183 https://doi.org/10.1214/20-aihp1086

[37] LI, Y. and XU, Y. (2021). On fluctuations of global and mesoscopic linear statistics of generalized Wigner matrices. *Bernoulli* **27** 1057–1076. MR4255226 https://doi.org/10.3150/20-bej1265

[38] LYTOVA, A. (2013). On non-Gaussian limiting laws for certain statistics of Wigner matrices. *Zh. Mat. Fiz. Anal. Geom.* **9** 536–581, 611, 615. MR3155024

[39] LYTOVA, A. and PASTUR, L. (2009). Central limit theorem for linear eigenvalue statistics of random matrices with independent entries. *Ann. Probab.* **37** 1778–1840. MR2561434 https://doi.org/10.1214/09-AOP452

[40] LYTOVA, A. and PASTUR, L. (2009). Fluctuations of matrix elements of regular functions of Gaussian random matrices. *J. Stat. Phys.* **134** 147–159. MR2489497 https://doi.org/10.1007/s10955-008-9665-1

[41] LYTOVA, A. and PASTUR, L. (2011). Non-Gaussian limiting laws for the entries of regular functions of the Wigner matrices. Preprint. Available at arXiv:1103.2345.

[42] MALE, C., MINGO, J. A., PÉCHÉ, S. and SPEICHER, R. (2022). Joint global fluctuations of complex Wigner and deterministic matrices. *Random Matrices Theory Appl.* **11** 2250015. MR4414193 https://doi.org/10.1142/S2010326322500150

[43] MARCINEK, J. (2020). *High Dimensional Normality of Noisy Eigenvectors*. ProQuest LLC, Ann Arbor, MI. Thesis (Ph.D.)—Harvard University. MR4272266

[44] MINGO, J. A. and SPEICHER, R. (2017). *Free Probability and Random Matrices*. *Fields Institute Monographs* **35**. Springer, New York. MR3585560 https://doi.org/10.1007/978-1-4939-6942-5

[45] NAGEL, J. (2021). A functional CLT for partial traces of random matrices. *J. Theoret. Probab.* **34** 953–974. MR4259455 https://doi.org/10.1007/s10959-019-00982-1

[46] O'ROURKE, S., RENFREW, D. and SOSHNIKOV, A. (2013). On fluctuations of matrix entries of regular functions of Wigner matrices with non-identically distributed entries. *J. Theoret. Probab.* **26** 750–780. MR3090549 https://doi.org/10.1007/s10959-011-0396-x

[47] PIZZO, A., RENFREW, D. and SOSHNIKOV, A. (2012). Fluctuations of matrix entries of regular functions of Wigner matrices. *J. Stat. Phys.* **146** 550–591. MR2880032 https://doi.org/10.1007/s10955-011-0404-7

[48] PLEIJEL, Å. (1963). On a theorem by P. Malliavin. *Israel J. Math.* **1** 166–168. MR0167751 https://doi.org/10.1007/BF02759705

[49] RUDNICK, Z. and SARNAK, P. (1994). The behaviour of eigenstates of arithmetic hyperbolic manifolds. *Comm. Math. Phys.* **161** 195–213. MR1266075

[50] SHAMIS, M. (2013). Density of states for Gaussian unitary ensemble, Gaussian orthogonal ensemble, and interpolating ensembles through supersymmetric approach. *J. Math. Phys.* **54** 113505. MR3137043 https://doi.org/10.1063/1.4830013

[51] SHCHERBINA, M. (2011). Central limit theorem for linear eigenvalue statistics of the Wigner and sample covariance random matrices. *Zh. Mat. Fiz. Anal. Geom.* **7** 176–192, 197, 199. MR2829615

[52] SILVERSTEIN, J. W. (1990). Weak convergence of random functions defined by the eigenvectors of sample covariance matrices. *Ann. Probab.* **18** 1174–1194. MR1062064

[53] SILVERSTEIN, J. W. (2020). Weak convergence of a collection of random functions defined by the eigenvectors of large dimensional random matrices. Preprint. Available at arXiv:2012.12950.

[54] SINAĬ, Y. G. and SOSHNIKOV, A. B. (1998). A refinement of Wigner's semicircle law in a neighborhood of the spectrum edge for random symmetric matrices. *Funktsional. Anal. i Prilozhen.* **32** 56–79, 96. MR1647832 https://doi.org/10.1007/BF02482597

[55] SOSOE, P. and WONG, P. (2013). Regularity conditions in the CLT for linear eigenvalue statistics of Wigner matrices. *Adv. Math.* **249** 37–87. MR3116567 https://doi.org/10.1016/j.aim.2013.09.004

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DIRECTED POLYMER FOR VERY HEAVY TAILED RANDOM WALKS

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In the present work, we investigate the case of directed polymer in a random environment (DPRE), when the increments of the one-dimensional random walk are heavy-tailed with tail-exponent equal to zero ($\mathbf{P}[|X_1| \geq n]$ decays slower than any power of n). This case has not yet been studied in the context of directed polymers and presents key differences with the simple symmetric random walk case and the cases where the increments belong to the domain of attraction of an α-stable law, where $\alpha \in (0, 2]$. We establish the absence of a very strong disorder regime—that is, the free energy equals zero at every temperature—for every disorder distribution. We also prove that a strong disorder regime (partition function converging to zero at low temperature) may exist or not depending on finer properties of the random walk: we establish nonmatching necessary and sufficient conditions for having a phase transition from weak to strong disorder. In particular our results imply that for this directed polymer model, very strong disorder is not equivalent to strong disorder, shedding a new light on a long standing conjecture concerning the original nearest-neighbor DPRE.

REFERENCES

[1] ALEXANDER, K. and ZYGOURAS, N. (2010). Equality of critical points for polymer depinning transitions with loop exponent one. *Ann. Appl. Probab.* **20** 356–366. https://doi.org/10.1214/09-AAP621

[2] BATES, E. (2018). Localization of directed polymers with general reference walk. *Electron. J. Probab.* **23** Paper No. 30, 45 pp. MR3785400 https://doi.org/10.1214/18-EJP158

[3] BINGHAM, N. H., GOLDIE, C. M. and TEUGELS, J. L. (1989). *Regular Variation. Encyclopedia of Mathematics and Its Applications* **27**. Cambridge Univ. Press, Cambridge. MR1015093

[4] BOLTHAUSEN, E. (1989). A note on the diffusion of directed polymers in a random environment. *Comm. Math. Phys.* **123** 529–534. MR1006293

[5] COMETS, F. (2007). Weak disorder for low dimensional polymers: The model of stable laws. *Markov Process. Related Fields* **13** 681–696. MR2381596

[6] COMETS, F. (2017). *Directed Polymers in Random Environments. Lecture Notes in Math.* **2175**. Springer, Cham. MR3444835 https://doi.org/10.1007/978-3-319-50487-2

[7] COMETS, F., SHIGA, T. and YOSHIDA, N. (2003). Directed polymers in a random environment: Path localization and strong disorder. *Bernoulli* **9** 705–723. MR1996276 https://doi.org/10.3150/bj/1066223275

[8] COMETS, F., SHIGA, T. and YOSHIDA, N. (2004). Probabilistic analysis of directed polymers in a random environment: A review. In *Stochastic Analysis on Large Scale Interacting Systems. Adv. Stud. Pure Math.* **39** 115–142. Math. Soc. Japan, Tokyo. MR2073332 https://doi.org/10.2969/aspm/03910115

[9] COMETS, F. and VARGAS, V. (2006). Majorizing multiplicative cascades for directed polymers in random media. *ALEA Lat. Am. J. Probab. Math. Stat.* **2** 267–277. MR2249671

[10] COMETS, F. and YOSHIDA, N. (2006). Directed polymers in random environment are diffusive at weak disorder. *Ann. Probab.* **34** 1746–1770. MR2271480 https://doi.org/10.1214/009117905000000828

[11] DAVID, H. A. and NAGARAJA, H. N. (2003). *Order Statistics*, 3rd ed. *Wiley Series in Probability and Statistics*. Wiley Interscience, Hoboken, NJ. MR1994955 https://doi.org/10.1002/0471722162

[12] HUSE, D. A. and HENLEY, C. L. (1985). Pinning and roughening of domain walls in Ising systems due to random impurities. *Phys. Rev. Lett.* **54** 2708–2711.

[13] LACOIN, H. (2010). New bounds for the free energy of directed polymers in dimension $1+1$ and $1+2$. *Comm. Math. Phys.* **294** 471–503. MR2579463 https://doi.org/10.1007/s00220-009-0957-3

[14] LACOIN, H. (2012). Existence of an intermediate phase for oriented percolation. *Electron. J. Probab.* **17** Paper No. 41, 17 pp. MR2928724 https://doi.org/10.1214/EJP.v17-1761

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[15] LACOIN, H. and SOHIER, J. (2017). Disorder relevance without Harris criterion: The case of pinning model with γ-stable environment. *Electron. J. Probab.* **22** Paper No. 50, 26 pp. MR3666013 https://doi.org/10.1214/17-EJP66

[16] MIURA, M., TAWARA, Y. and TSUCHIDA, K. (2008). Strong and weak disorder for Lévy directed polymers in random environment. *Stoch. Anal. Appl.* **26** 1000–1012. MR2440912 https://doi.org/10.1080/07362990802286418

[17] PURKAYASTHA, S. (1998). Simple proofs of two results on convolutions of unimodal distributions. *Statist. Probab. Lett.* **39** 97–100. MR1652520 https://doi.org/10.1016/S0167-7152(98)00013-3

[18] SOHIER, J. Hierarchical pinning model: Low disorder relevance in the b = s case. Preprint. Available at arXiv:1408.3208.

[19] VIVEROS, R. (2021). Directed polymer in γ-stable random environments. *Ann. Inst. Henri Poincaré Probab. Stat.* **57** 1081–1102. MR4260496 https://doi.org/10.1214/20-aihp1108

[20] WEI, R. (2016). On the long-range directed polymer model. *J. Stat. Phys.* **165** 320–350. MR3554384 https://doi.org/10.1007/s10955-016-1612-y

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RATES OF MULTIVARIATE NORMAL APPROXIMATION FOR STATISTICS IN GEOMETRIC PROBABILITY

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We employ stabilization methods and second order Poincaré inequalities to establish rates of multivariate normal convergence for a large class of vectors $(H_s^{(1)}, \dots, H_s^{(m)})$, $s \geq 1$, of statistics of marked Poisson processes on $\mathbb{R}^d$, $d \geq 2$, as the intensity parameter s tends to infinity. Our results are applicable whenever the functionals $H_s^{(i)}$, $i \in \{1, \dots, m\}$, are expressible as sums of exponentially stabilizing score functions satisfying a moment condition. The rates are for the $\boldsymbol{d}_2$-, $\boldsymbol{d}_3$-, and $\boldsymbol{d}_{\text{convex}}$-distances and are in general unimprovable. When we compare with a centered Gaussian random vector, whose covariance matrix is given by the asymptotic covariances, the rates are governed by the rate of convergence of $s^{-1}\operatorname{Cov}(H_s^{(i)}, H_s^{(j)})$, $i, j \in \{1, \dots, m\}$, to the limiting covariance, shown to be at most of order $s^{-1/d}$. We use the general results to deduce rates of multivariate normal convergence for statistics arising in random graphs and topological data analysis as well as for multivariate statistics used to test equality of distributions. Some of our results hold for stabilizing functionals of Poisson input on suitable metric spaces.

REFERENCES

[1] AKINWANDE, G. and REITZNER, M. (2020). Multivariate central limit theorems for random simplicial complexes. *Adv. in Appl. Math.* **121** 102076, 27. MR4126722 https://doi.org/10.1016/j.aam.2020.102076
[2] BARBOUR, A. D. and XIA, A. (2006). Normal approximation for random sums. *Adv. in Appl. Probab.* **38** 693–728. MR2256874 https://doi.org/10.1239/aap/1158684998
[3] BARYSHNIKOV, YU. and YUKICH, J. E. (2005). Gaussian limits for random measures in geometric probability. *Ann. Appl. Probab.* **15** 213–253. MR2115042 https://doi.org/10.1214/105051604000000594
[4] BHATTACHARYA, R. N. and HOLMES, S. (2010). An exposition of Götze's estimation of the rate of convergence in the multivariate central limit theorem. In *Normal Approximation and Asymptotic Expansions* SIAM, Philadelphia, PA.
[5] BOBROWSKI, O. and MUKHERJEE, S. (2015). The topology of probability distributions on manifolds. *Probab. Theory Related Fields* **161** 651–686. MR3334278 https://doi.org/10.1007/s00440-014-0556-x
[6] CHEN, L. H. Y., GOLDSTEIN, L. and SHAO, Q.-M. (2011). *Normal Approximation by Stein's Method. Probability and Its Applications* (*New York*). Springer, Heidelberg. MR2732624 https://doi.org/10.1007/978-3-642-15007-4
[7] FANG, X. (2011). Multivariate, combinatorial and discretized normal approximations by Stein's method Ph.D. thesis, National Univ. Singapore. https://scholarbank.nus.edu.sg/handle/10635/32447.
[8] FANG, X. (2016). A multivariate CLT for bounded decomposable random vectors with the best known rate. *J. Theoret. Probab.* **29** 1510–1523. MR3571252 https://doi.org/10.1007/s10959-015-0619-7
[9] FANG, X. and RÖLLIN, A. (2015). Rates of convergence for multivariate normal approximation with applications to dense graphs and doubly indexed permutation statistics. *Bernoulli* **21** 2157–2189. MR3378463 https://doi.org/10.3150/14-BEJ639
[10] GOLDSTEIN, L. and RINOTT, Y. (1996). Multivariate normal approximations by Stein's method and size bias couplings. *J. Appl. Probab.* **33** 1–17. MR1371949 https://doi.org/10.1017/s0021900200103675
[11] GÖTZE, F. (1991). On the rate of convergence in the multivariate CLT. *Ann. Probab.* **19** 724–739. MR1106283

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[12] HENZE, N. (1988). A multivariate two-sample test based on the number of nearest neighbor type coincidences. *Ann. Statist.* **16** 772–783. MR0947577 https://doi.org/10.1214/aos/1176350835

[13] HUG, D., LAST, G. and SCHULTE, M. (2016). Second-order properties and central limit theorems for geometric functionals of Boolean models. *Ann. Appl. Probab.* **26** 73–135. MR3449314 https://doi.org/10.1214/14-AAP1086

[14] KRANTZ, S. G. and PARKS, H. R. (2008). *Geometric Integration Theory*. *Cornerstones*. Birkhäuser, Inc., Boston, MA. MR2427002 https://doi.org/10.1007/978-0-8176-4679-0

[15] LACHIÈZE-REY, R. and PECCATI, G. (2017). New Berry–Esseen bounds for functionals of binomial point processes. *Ann. Appl. Probab.* **27** 1992–2031. MR3693518 https://doi.org/10.1214/16-AAP1218

[16] LACHIÈZE-REY, R., PECCATI, G. and YANG, X. (2020). Quantitative two-scale stabilization on the Poisson space. *Ann. Appl. Probab.* To appear. Available at arXiv:2010.13362.

[17] LACHIÈZE-REY, R., SCHULTE, M. and YUKICH, J. E. (2019). Normal approximation for stabilizing functionals. *Ann. Appl. Probab.* **29** 931–993. MR3910021 https://doi.org/10.1214/18-AAP1405

[18] LAST, G., PECCATI, G. and SCHULTE, M. (2016). Normal approximation on Poisson spaces: Mehler's formula, second order Poincaré inequalities and stabilization. *Probab. Theory Related Fields* **165** 667–723. MR3520016 https://doi.org/10.1007/s00440-015-0643-7

[19] LEONENKO, N., PRONZATO, L. and SAVANI, V. (2008). A class of Rényi information estimators for multidimensional densities. *Ann. Statist.* **36** 2153–2182. MR2458183 https://doi.org/10.1214/07-AOS539

[20] PECCATI, G. and ZHENG, C. (2010). Multi-dimensional Gaussian fluctuations on the Poisson space. *Electron. J. Probab.* **15** 1487–1527. MR2727319 https://doi.org/10.1214/EJP.v15-813

[21] PENROSE, M. (2003). *Random Geometric Graphs*. *Oxford Studies in Probability* **5**. Oxford Univ. Press, Oxford. MR1986198 https://doi.org/10.1093/acprof:oso/9780198506263.001.0001

[22] PENROSE, M. D. (2007). Gaussian limits for random geometric measures. *Electron. J. Probab.* **12** 989–1035. MR2336596 https://doi.org/10.1214/EJP.v12-429

[23] PENROSE, M. D. and WADE, A. R. (2008). Multivariate normal approximation in geometric probability. *J. Stat. Theory Pract.* **2** 293–326. MR2524467 https://doi.org/10.1080/15598608.2008.10411876

[24] PENROSE, M. D. and YUKICH, J. E. (2001). Central limit theorems for some graphs in computational geometry. *Ann. Appl. Probab.* **11** 1005–1041. MR1878288 https://doi.org/10.1214/aoap/1015345393

[25] PENROSE, M. D. and YUKICH, J. E. (2003). Weak laws of large numbers in geometric probability. *Ann. Appl. Probab.* **13** 277–303. MR1952000 https://doi.org/10.1214/aoap/1042765669

[26] PENROSE, M. D. and YUKICH, J. E. (2005). Normal approximation in geometric probability. In *Stein's Method and Applications*. *Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap.* **5** 37–58. Singapore Univ. Press, Singapore. MR2201885 https://doi.org/10.1142/9789812567673_0003

[27] PENROSE, M. D. and YUKICH, J. E. (2011). Laws of large numbers and nearest neighbor distances. In *Advances in Directional and Linear Statistics* 189–199. Physica-Verlag/Springer, Heidelberg. MR2767541 https://doi.org/10.1007/978-3-7908-2628-9_13

[28] PENROSE, M. D. and YUKICH, J. E. (2013). Limit theory for point processes in manifolds. *Ann. Appl. Probab.* **23** 2161–2211. MR3127932 https://doi.org/10.1214/12-AAP897

[29] REDMOND, C. and YUKICH, J. E. (1996). Asymptotics for Euclidean functionals with power-weighted edges. *Stochastic Process. Appl.* **61** 289–304. MR1386178 https://doi.org/10.1016/0304-4149(95)00075-5

[30] REINERT, G. and RÖLLIN, A. (2009). Multivariate normal approximation with Stein's method of exchangeable pairs under a general linearity condition. *Ann. Probab.* **37** 2150–2173. MR2573554 https://doi.org/10.1214/09-AOP467

[31] REITZNER, M., SCHULTE, M. and THÄLE, C. (2017). Limit theory for the Gilbert graph. *Adv. in Appl. Math.* **88** 26–61. MR3641808 https://doi.org/10.1016/j.aam.2016.12.006

[32] RINOTT, Y. and ROTAR, V. (1996). A multivariate CLT for local dependence with $n^{-1/2}\log n$ rate and applications to multivariate graph related statistics. *J. Multivariate Anal.* **56** 333–350. MR1379533 https://doi.org/10.1006/jmva.1996.0017

[33] SCHILLING, M. F. (1986). Multivariate two-sample tests based on nearest neighbors. *J. Amer. Statist. Assoc.* **81** 799–806. MR0860514

[34] SCHREIBER, T. (2010). Limit theorems in stochastic geometry. In *New Perspectives in Stochastic Geometry* 111–144. Oxford Univ. Press, Oxford. MR2654677

[35] SCHULTE, M. (2013). Malliavin-Stein Method in Stochastic Geometry Ph.D. thesis, Univ. Osnabrück. https://repositorium.ub.uni-osnabrueck.de/bitstream/urn:nbn:de:gbv:700-2013031910717/1/thesis_schulte.pdf.

[36] SCHULTE, M. and YUKICH, J. E. (2019). Multivariate second order Poincaré inequalities for Poisson functionals. *Electron. J. Probab.* **24** Paper No. 130, 42. MR4040990 https://doi.org/10.1214/19-ejp386

[37] YUKICH, J. E. (1998). *Probability Theory of Classical Euclidean Optimization Problems*. *Lecture Notes in Math.* **1675**. Springer, Berlin. MR1632875 https://doi.org/10.1007/BFb0093472

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STRONG APPROXIMATION OF GAUSSIAN β ENSEMBLE CHARACTERISTIC POLYNOMIALS: THE HYPERBOLIC REGIME

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We investigate the characteristic polynomials φ_N of the Gaussian β-ensemble for general $\beta > 0$ through its transfer matrix recurrence. Our motivation is to obtain a (probabilistic) approximation for φ_N in terms of a Gaussian log-correlated field. We distinguish between different types of transfer matrices and analyze completely the *hyperbolic part* of the recurrence. As a result, we obtain a new coupling between φ_N and a Gaussian analytic function with an error which is uniform away from the support of the semicircle law. We use this as input to give the almost sure scaling limit of the characteristic polynomial at the edge in (Lambert and Paquette (2020)). This is also required to obtain analogous strong approximations inside of the bulk of the semicircle law. Our analysis relies on moderate deviation estimates for the product of transfer matrices and this approach might also be useful in different contexts.

REFERENCES

ALBERTS, T. and NORMAND, R. (2019). Dimension results for the spectral measure of the circular beta ensembles. Preprint. Available at arXiv:1912.07788.

ARGUIN, L.-P., BELIUS, D. and BOURGADE, P. (2017). Maximum of the characteristic polynomial of random unitary matrices. *Comm. Math. Phys.* **349** 703–751. MR3594368 https://doi.org/10.1007/s00220-016-2740-6

BEKERMAN, F., LEBLÉ, T. and SERFATY, S. (2018). CLT for fluctuations of β-ensembles with general potential. *Electron. J. Probab.* **23** Paper no. 115, 31 pp. MR3885548 https://doi.org/10.1214/18-EJP209

BERESTYCKI, N. (2017). An elementary approach to Gaussian multiplicative chaos. *Electron. Commun. Probab.* **22** Paper No. 27, 12 pp. MR3652040 https://doi.org/10.1214/17-ECP58

BERESTYCKI, N., WEBB, C. and WONG, M. D. (2018). Random Hermitian matrices and Gaussian multiplicative chaos. *Probab. Theory Related Fields* **172** 103–189. MR3851831 https://doi.org/10.1007/s00440-017-0806-9

BHATTACHARJEE, C. and GOLDSTEIN, L. (2016). On strong embeddings by Stein's method. *Electron. J. Probab.* **21** Paper No. 15, 30 pp. MR3485357 https://doi.org/10.1214/16-EJP4299

BOROT, G. and GUIONNET, A. (2013). Asymptotic expansion of β matrix models in the one-cut regime. *Comm. Math. Phys.* **317** 447–483. MR3010191 https://doi.org/10.1007/s00220-012-1619-4

BOURGADE, P., MODY, K. and PAIN, M. (2022). Optimal local law and central limit theorem for β-ensembles. *Comm. Math. Phys.* **390** 1017–1079. MR4389077 https://doi.org/10.1007/s00220-022-04311-2

BOUTET DE MONVEL, A., PASTUR, L. and SHCHERBINA, M. (1995). On the statistical mechanics approach in the random matrix theory: Integrated density of states. *J. Stat. Phys.* **79** 585–611. MR1327898 https://doi.org/10.1007/BF02184872

CHATTERJEE, S. (2012). A new approach to strong embeddings. *Probab. Theory Related Fields* **152** 231–264. MR2875758 https://doi.org/10.1007/s00440-010-0321-8

CHHAIBI, R., MADAULE, T. and NAJNUDEL, J. (2018). On the maximum of the CβE field. *Duke Math. J.* **167** 2243–2345. MR3848391 https://doi.org/10.1215/00127094-2018-0016

CHHAIBI, R. and NAJNUDEL, J. (2019). On the circle, $\mathrm{GMC}^{\gamma} = \varprojlim \mathrm{C}\beta\mathrm{E}_n$ for $\gamma = \sqrt{\frac{2}{\beta}}$, $(\gamma \leq 1)$. Preprint. Available at arXiv:1904.00578.

CHHAIBI, R., NAJNUDEL, J. and NIKEGHBALI, A. (2017). The circular unitary ensemble and the Riemann zeta function: The microscopic landscape and a new approach to ratios. *Invent. Math.* **207** 23–113. MR3592756 https://doi.org/10.1007/s00222-016-0669-1

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CHHAIBI, R., HOVHANNISYAN, E., NAJNUDEL, J., NIKEGHBALI, A. and RODGERS, B. (2019). The limiting characteristic polynomial of classical random matrix ensembles. *Ann. Henri Poincaré* **20** 1093–1119. MR3928373 https://doi.org/10.1007/s00023-019-00769-4

CLAEYS, T., FAHS, B., LAMBERT, G. and WEBB, C. (2021). How much can the eigenvalues of a random Hermitian matrix fluctuate? *Duke Math. J.* **170** 2085–2235. MR4278668 https://doi.org/10.1215/00127094-2020-0070

DEIFT, P. and MCLAUGHLIN, K. T.-R. (1998). A continuum limit of the Toda lattice. *Mem. Amer. Math. Soc.* **131** x+216. MR1407901 https://doi.org/10.1090/memo/0624

DEIFT, P., KRIECHERBAUER, T., MCLAUGHLIN, K. T.-R., VENAKIDES, S. and ZHOU, X. (1999a). Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory. *Comm. Pure Appl. Math.* **52** 1335–1425. MR1702716 https://doi.org/10.1002/(SICI)1097-0312(199911)52:11<1335::AID-CPA1>3.0.CO;2-1

DEIFT, P., KRIECHERBAUER, T., MCLAUGHLIN, K. T.-R., VENAKIDES, S. and ZHOU, X. (1999b). Strong asymptotics of orthogonal polynomials with respect to exponential weights. *Comm. Pure Appl. Math.* **52** 1491–1552. MR1711036 https://doi.org/10.1002/(SICI)1097-0312(199912)52:12<1491::AID-CPA2>3.3.CO;2-R

DUBOIS, L. (2008). Real cone contractions and analyticity properties of the characteristic exponents. *Nonlinearity* **21** 2519–2536. MR2448229 https://doi.org/10.1088/0951-7715/21/11/003

DUBOIS, L. (2009). Projective metrics and contraction principles for complex cones. *J. Lond. Math. Soc.* (2) **79** 719–737. MR2506695 https://doi.org/10.1112/jlms/jdp008

DUMITRIU, I. and EDELMAN, A. (2002). Matrix models for beta ensembles. *J. Math. Phys.* **43** 5830–5847. MR1936554 https://doi.org/10.1063/1.1507823

FORRESTER, P. J. (2010). *Log-Gases and Random Matrices*. *London Mathematical Society Monographs Series* **34**. Princeton Univ. Press, Princeton, NJ. MR2641363 https://doi.org/10.1515/9781400835416

FREEDMAN, D. A. (1975). On tail probabilities for martingales. *Ann. Probab.* **3** 100–118. MR0380971 https://doi.org/10.1214/aop/1176996452

FURSTENBERG, H. and KESTEN, H. (1960). Products of random matrices. *Ann. Math. Stat.* **31** 457–469. MR0121828 https://doi.org/10.1214/aoms/1177705909

FYODOROV, Y. V. and DOUSSAL, P. L. (2016). Moments of the position of the maximum for GUE characteristic polynomials and for log-correlated Gaussian processes. *J. Stat. Phys.* **164** 190–240. MR3509054 https://doi.org/10.1007/s10955-016-1536-6

FYODOROV, Y. V. and SIMM, N. J. (2016). On the distribution of the maximum value of the characteristic polynomial of GUE random matrices. *Nonlinearity* **29** 2837–2855. MR3544809 https://doi.org/10.1088/0951-7715/29/9/2837

GÖTZE, F. and ZAITSEV, A. Y. (2008). Bounds for the rate of strong approximation in the multidimensional invariance principle. *Teor. Veroyatn. Primen.* **53** 100–123. MR2760567 https://doi.org/10.1137/S0040585X9798350X

HAFOUTA, Y. and KIFER, Y. (2018). *Nonconventional Limit Theorems and Random Dynamics*. World Scientific Co. Pte. Ltd., Hackensack, NJ. MR3793181 https://doi.org/10.1142/10849

JANSON, S. (2004). Large deviations for sums of partly dependent random variables. *Random Structures Algorithms* **24** 234–248. MR2068873 https://doi.org/10.1002/rsa.20008

JOHANSSON, K. (1998). On fluctuations of eigenvalues of random Hermitian matrices. *Duke Math. J.* **91** 151–204. MR1487983 https://doi.org/10.1215/S0012-7094-98-09108-6

JUNNILA, J. and SAKSMAN, E. (2017). Uniqueness of critical Gaussian chaos. *Electron. J. Probab.* **22** Paper No. 11, 31 pp. MR3613704 https://doi.org/10.1214/17-EJP28

JUNNILA, J., SAKSMAN, E. and WEBB, C. (2019). Decompositions of log-correlated fields with applications. *Ann. Appl. Probab.* **29** 3786–3820. MR4047992 https://doi.org/10.1214/19-AAP1492

KAHANE, J.-P. (1985). Sur le chaos multiplicatif. *Ann. Sci. Math. Québec* **9** 105–150. MR0829798

KIVIMAE, P. (2020). Gaussian multiplicative chaos for Gaussian orthogonal and symplectic ensembles. Preprint. Available at arXiv:2012.09969.

KOMLÓS, J., MAJOR, P. and TUSNÁDY, G. (1976). An approximation of partial sums of independent RV's, and the sample DF. II. *Z. Wahrsch. Verw. Gebiete* **34** 33–58. MR0402883 https://doi.org/10.1007/BF00532688

LAMBERT, G. (2021). Mesoscopic central limit theorem for the circular β-ensembles and applications. *Electron. J. Probab.* **26** Paper No. 7, 33 pp. MR4216520 https://doi.org/10.1214/20-ejp559

LAMBERT, G., LEDOUX, M. and WEBB, C. (2019). Quantitative normal approximation of linear statistics of β-ensembles. *Ann. Probab.* **47** 2619–2685. MR4021234 https://doi.org/10.1214/18-AOP1314

LAMBERT, G. and PAQUETTE, E. (2019). The law of large numbers for the maximum of almost Gaussian log-correlated fields coming from random matrices. *Probab. Theory Related Fields* **173** 157–209. MR3916106 https://doi.org/10.1007/s00440-018-0832-2

LAMBERT, G. and PAQUETTE, E. (2020). Strong approximation of Gaussian β-ensemble characteristic polynomials: The edge regime and the stochastic Airy function. Preprint. Available at arXiv:2009.05003.

MADAULE, T., RHODES, R. and VARGAS, V. (2016). Glassy phase and freezing of log-correlated Gaussian potentials. *Ann. Appl. Probab.* **26** 643–690. MR3476621 https://doi.org/10.1214/14-AAP1071

NIKULA, M., SAKSMAN, E. and WEBB, C. (2020). Multiplicative chaos and the characteristic polynomial of the CUE: The L^1-phase. *Trans. Amer. Math. Soc.* **373** 3905–3965. MR4105514 https://doi.org/10.1090/tran/8020

PAQUETTE, E. and ZEITOUNI, O. (2018). The maximum of the CUE field. *Int. Math. Res. Not. IMRN* **16** 5028–5119. MR3848227 https://doi.org/10.1093/imrn/rnx033

RAMÍREZ, J. A., RIDER, B. and VIRÁG, B. (2011). Beta ensembles, stochastic Airy spectrum, and a diffusion. *J. Amer. Math. Soc.* **24** 919–944. MR2813333 https://doi.org/10.1090/S0894-0347-2011-00703-0

SAKHANENKO, A. I. (1982). Estimates of the rate of convergence in the invariance principle. In *Limit Theorems of Probability Theory and Related Questions. Trudy Inst. Mat.* **1** 72–78, 206. “Nauka” Sibirsk. Otdel., Novosibirsk. MR0669045

SHAMOV, A. (2016). On Gaussian multiplicative chaos. *J. Funct. Anal.* **270** 3224–3261. MR3475456 https://doi.org/10.1016/j.jfa.2016.03.001

SHAO, Q. M. (1995). Strong approximation theorems for independent random variables and their applications. *J. Multivariate Anal.* **52** 107–130. MR1325373 https://doi.org/10.1006/jmva.1995.1006

SHCHERBINA, M. (2013). Fluctuations of linear eigenvalue statistics of β matrix models in the multi-cut regime. *J. Stat. Phys.* **151** 1004–1034. MR3063494 https://doi.org/10.1007/s10955-013-0740-x

STURMAN, R. and THIFFEAULT, J.-L. (2019). Lyapunov exponents for the random product of two shears. *J. Nonlinear Sci.* **29** 593–620. MR3927107 https://doi.org/10.1007/s00332-018-9497-3

TROPP, J. A. (2011). Freedman’s inequality for matrix martingales. *Electron. Commun. Probab.* **16** 262–270. MR2802042 https://doi.org/10.1214/ECP.v16-1624

VALKÓ, B. and VIRÁG, B. (2009). Continuum limits of random matrices and the Brownian carousel. *Invent. Math.* **177** 463–508. MR2534097 https://doi.org/10.1007/s00222-009-0180-z

VALKÓ, B. and VIRÁG, B. (2020). The many faces of the stochastic zeta function. Preprint. Available at arXiv:2009.04670.

VERSHYNIN, R. (2018). *High-Dimensional Probability: An Introduction with Applications in Data Science. Cambridge Series in Statistical and Probabilistic Mathematics* **47**. Cambridge Univ. Press, Cambridge. MR3837109 https://doi.org/10.1017/9781108231596

WEBB, C. (2015). The characteristic polynomial of a random unitary matrix and Gaussian multiplicative chaos—The L^2-phase. *Electron. J. Probab.* **20** no. 104, 21 pp. MR3407221 https://doi.org/10.1214/EJP.v20-4296

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EXTENDED L-ENSEMBLES: A NEW REPRESENTATION FOR DETERMINANTAL POINT PROCESSES

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Determinantal point processes (DPPs) are a class of repulsive point processes, popular for their relative simplicity. They are traditionally defined via their marginal distributions, but a subset of DPPs called "L-ensembles" have tractable likelihoods and are thus particularly easy to work with. Indeed, in many applications, DPPs are more naturally defined based on the L-ensemble formulation rather than through the marginal kernel.

The fact that not all DPPs are L-ensembles is unfortunate, but there is a unifying description. We introduce here *extended* L-ensembles, and show that all DPPs are extended L-ensembles (and vice versa). Extended L-ensembles have very simple likelihood functions, contain L-ensembles and projection DPPs as special cases. From a theoretical standpoint, they fix some pathologies in the usual formalism of DPPs, for instance, the fact that projection DPPs are not L-ensembles. From a practical standpoint, they extend the set of kernel functions that may be used to define DPPs: we show that conditional positive definite kernels are good candidates for defining DPPs, including DPPs that need no spatial scale parameter.

Finally, extended L-ensembles are based on so-called "saddle-point matrices", and we prove an extension of the Cauchy–Binet theorem for such matrices that may be of independent interest.

REFERENCES

[1] ANARI, N., VINZANT, C., LIU, K. and VUONG, T.-D. (2021). Log-concave polynomials IV: Approximate exchange, tight mixing times, and near-optimal sampling of forests. In *STOC '21—Proceedings of the 53rd Annual ACM SIGACT Symposium on Theory of Computing* 408–420. ACM, New York. MR4398852 https://doi.org/10.1145/3406325.3451091

[2] AVENA, L. and GAUDILLIERE, A. (2013). On some random forests with determinantal roots. ArXiv preprint. Available at arXiv:1310.1723.

[3] AVENA, L. and GAUDILLIÈRE, A. (2018). Two applications of random spanning forests. *J. Theoret. Probab.* **31** 1975–2004. MR3866604 https://doi.org/10.1007/s10959-017-0771-3

[4] BARDENET, R. and HARDY, A. (2020). Monte Carlo with determinantal point processes. *Ann. Appl. Probab.* **30** 368–417. MR4068314 https://doi.org/10.1214/19-AAP1504

[5] BARTHELMÉ, S., AMBLARD, P.-O. and TREMBLAY, N. (2019). Asymptotic equivalence of fixed-size and varying-size determinantal point processes. *Bernoulli* **25** 3555–3589. MR4010965 https://doi.org/10.3150/18-bej1102

[6] BARTHELMÉ, S., TREMBLAY, N., USEVICH, K. and AMBLARD, P.-O. (2023). Determinantal point processes in the flat limit. *Bernoulli*. To appear. Availiable at arXiv:2107.07213.

[7] BARTHELMÉ, S. and USEVICH, K. (2021). Spectral properties of kernel matrices in the flat limit. *SIAM J. Matrix Anal. Appl.* **42** 17–57. MR4199248 https://doi.org/10.1137/19M129677X

[8] BELHADJI, A., BARDENET, R. and CHAINAIS, P. (2020). A determinantal point process for column subset selection. *J. Mach. Learn. Res.* **21** Paper No. 197. MR4209483

[9] BISCIO, C. A. N. and LAVANCIER, F. (2016). Quantifying repulsiveness of determinantal point processes. *Bernoulli* **22** 2001–2028. MR3498021 https://doi.org/10.3150/15-BEJ718

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Key words and phrases. Determinantal point process, L-ensemble.

[10] BORODIN, A. and RAINS, E. M. (2005). Eynard-Mehta theorem, Schur process, and their Pfaffian analogs. *J. Stat. Phys.* **121** 291–317. MR2185331 https://doi.org/10.1007/s10955-005-7583-z

[11] BURTON, R. and PEMANTLE, R. (1993). Local characteristics, entropy and limit theorems for spanning trees and domino tilings via transfer-impedances. *Ann. Probab.* **21** 1329–1371. MR1235419

[12] CHEN, X.-H., DEMPSTER, A. P. and LIU, J. S. (1994). Weighted finite population sampling to maximize entropy. *Biometrika* **81** 457–469. MR1311090 https://doi.org/10.1093/biomet/81.3.457

[13] COEURJOLLY, J.-F., MAZOYER, A. and AMBLARD, P.-O. (2021). Monte Carlo integration of non-differentiable functions on $[0,1]^\iota$, $\iota = 1, \dots, d$, using a single determinantal point pattern defined on $[0,1]^d$. *Electron. J. Stat.* **15** 6228–6280. MR4355707 https://doi.org/10.1214/21-ejs1929

[14] DEREZIŃSKI, M. and MAHONEY, M. W. (2021). Determinantal point processes in randomized numerical linear algebra. *Notices Amer. Math. Soc.* **68** 34–45. MR4202314 https://doi.org/10.1090/noti2202

[15] FANUEL, M., SCHREURS, J. and SUYKENS, J. A. (2020). Determinantal point processes implicitly regularize semi-parametric regression problems. ArXiv preprint. Available at arXiv:2011.06964.

[16] GAUTIER, G. (2020). On sampling determinantal point processes. Ph.D. thesis, Ecole Centrale de Lille.

[17] HORN, R. A. and JOHNSON, C. R. (1990). *Matrix Analysis*. Cambridge Univ. Press, Cambridge. MR1084815

[18] HOUGH, J. B., KRISHNAPUR, M., PERES, Y. and VIRÁG, B. (2006). Determinantal processes and independence. *Probab. Surv.* **3** 206–229. MR2216966 https://doi.org/10.1214/154957806000000078

[19] KLEIN, D. J. and RANDIĆ, M. (1993). Resistance distance. *J. Math. Chem.* **12** 81–95. MR1219566 https://doi.org/10.1007/BF01164627

[20] KULESZA, A. and TASKAR, B. (2011). k-DPPs: Fixed-size determinantal point processes. In *Proceedings of the* 28*th International Conference on Machine Learning* (*ICML*-11) 1193–1200.

[21] KULESZA, A., TASKAR, B. et al. (2012). Determinantal point processes for machine learning. *Found. Trends Mach. Learn.* **5** 123–286.

[22] MACCHI, O. (1975). The coincidence approach to stochastic point processes. *Adv. in Appl. Probab.* **7** 83–122. MR0380979 https://doi.org/10.2307/1425855

[23] MICCHELLI, C. A. (1986). Interpolation of scattered data: Distance matrices and conditionally positive definite functions. *Constr. Approx.* **2** 11–22. MR0891767 https://doi.org/10.1007/BF01893414

[24] TREMBLAY, N., BARTHELME, S. and AMBLARD, P.-O. (2018). Optimized algorithms to sample determinantal point processes. Available at arXiv:1802.08471 [cs, Stat].

[25] TREMBLAY, N., BARTHELMÉ, S. and AMBLARD, P.-O. (2019). Determinantal point processes for coresets. *J. Mach. Learn. Res.* **20** Paper No. 168. MR4048979

[26] WENDLAND, H. (2005). *Scattered Data Approximation*. *Cambridge Monographs on Applied and Computational Mathematics* **17**. Cambridge Univ. Press, Cambridge. MR2131724

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CUTOFF FOR REWIRING DYNAMICS ON PERFECT MATCHINGS

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We establish cutoff for a natural random walk (RW) on the set of perfect matchings (PMs), based on "rewiring". An n-PM is a pairing of $2n$ objects. The k-PM RW selects k pairs uniformly at random, disassociates the corresponding $2k$ objects, then chooses a new pairing on these $2k$ objects uniformly at random. The equilibrium distribution is uniform over all n-PMs.

The 2-PM RW was first introduced by Diaconis and Holmes (*Proc. Natl. Acad. Sci. USA* **95** (1998) 14600–14602; *Electron. J. Probab.* **7** (2002) no. 6), seen as a RW on phylogenetic trees. They established cutoff in this case. We establish cutoff for the k-PM RW whenever $2 \le k \ll n$. If $k \gg 1$, then the mixing time is $\frac{n}{k} \log n$ to leading order.

Diaconis and Holmes (*Electron. J. Probab.* **7** (2002) no. 6) relate the 2-PM RW to the random transpositions card shuffle. Ceccherini-Silberstein, Scarabotti and Tolli (*J. Math. Sci.* **141** (2007) 1182–1229; *Harmonic Analysis on Finite Groups: Representation Theory, Gelfand Pairs and Markov Chains* (2008) Cambridge Univ. Press) establish the same result using representation theory. We are the first to handle $k > 2$. We relate the PM RW to conjugacy-invariant RWs on the permutation group by introducing a "cycle structure" for PMs, then build on work of Berestycki, Schramm, Şengül and Zeitouni (*Israel J. Math.* **147** (2005) 221–243; *Ann. Probab.* **39** (2011) 1815–1843; *Probab. Theory Related Fields* **173** (2019) 1197–1241) on such RWs.

REFERENCES

[1] AMANATIDIS, G. and KLEER, P. (2019). Rapid mixing of the switch Markov chain for strongly stable degree sequences and 2-class joint degree matrices. In *Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms* 966–985. SIAM, Philadelphia, PA. MR3909529 https://doi.org/10.1137/1.9781611975482.60

[2] AMANATIDIS, G. and KLEER, P. (2020). Rapid mixing of the switch Markov chain for strongly stable degree sequences. *Random Structures Algorithms* **57** 637–657. MR4144078 https://doi.org/10.1002/rsa.20949

[3] ARRATIA, R. and TAVARÉ, S. (1992). The cycle structure of random permutations. *Ann. Probab.* **20** 1567–1591. MR1175278

[4] AVENA, L., GÜLDAŞ, H., VAN DER HOFSTAD, R. and DEN HOLLANDER, F. (2018). Mixing times of random walks on dynamic configuration models. *Ann. Appl. Probab.* **28** 1977–2002. MR3843821 https://doi.org/10.1214/17-AAP1289

[5] AVENA, L., GÜLDAŞ, H., VAN DER HOFSTAD, R. and DEN HOLLANDER, F. (2019). Random walks on dynamic configuration models: A trichotomy. *Stochastic Process. Appl.* **129** 3360–3375. MR3985565 https://doi.org/10.1016/j.spa.2018.09.010

[6] BARBOUR, A. D. (1990). Poisson approximation and the Chen–Stein method: Comment. *Statist. Sci.* **5** 425–427. https://doi.org/10.1214/ss/1177012017

[7] BENDER, E. A. and CANFIELD, E. R. (1978). The asymptotic number of labeled graphs with given degree sequences. *J. Combin. Theory Ser. A* **24** 296–307. MR0505796 https://doi.org/10.1016/0097-3165(78)90059-6

[8] BERESTYCKI, N., SCHRAMM, O. and ZEITOUNI, O. (2011). Mixing times for random k-cycles and coalescence-fragmentation chains. *Ann. Probab.* **39** 1815–1843. MR2884874 https://doi.org/10.1214/10-AOP634

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[9] BERESTYCKI, N. and ŞENGÜL, B. (2019). Cutoff for conjugacy-invariant random walks on the permutation group. *Probab. Theory Related Fields* **173** 1197–1241. MR3936154 https://doi.org/10.1007/s00440-018-0844-y

[10] BJÖRNBERG, J. E., KOTOWSKI, M., LEES, B. and MIŁOŚ, P. (2019). The interchange process with reversals on the complete graph. *Electron. J. Probab.* **24** Paper No. 108, 43. MR4017126 https://doi.org/10.1214/19-ejp366

[11] BOLLOBÁS, B. (1980). A probabilistic proof of an asymptotic formula for the number of labelled regular graphs. *European J. Combin.* **1** 311–316. MR0595929 https://doi.org/10.1016/S0195-6698(80)80030-8

[12] BOLLOBÁS, B. (2001). *Random Graphs*, 2nd ed. *Cambridge Studies in Advanced Mathematics* **73**. Cambridge Univ. Press, Cambridge. MR1864966 https://doi.org/10.1017/CBO9780511814068

[13] BORMASHENKO, O. (2011). A coupling argument for the random transposition walk. arXiv:1109.3915.

[14] BUBLEY, R. and DYER, M. (1997). Path coupling: A technique for proving rapid mixing in Markov chains. In *Proceedings of the* 38*th Annual Symposium on Foundations of Computer Science*, *FOCS* '97 223. IEEE Computer Society, Washington, DC. https://doi.org/10.1109/SFCS.1997.646111

[15] CAPUTO, P., LIGGETT, T. M. and RICHTHAMMER, T. (2010). Proof of Aldous' spectral gap conjecture. *J. Amer. Math. Soc.* **23** 831–851. MR2629990 https://doi.org/10.1090/S0894-0347-10-00659-4

[16] CECCHERINI-SILBERSTEIN, T., SCARABOTTI, F. and TOLLI, F. (2007). Finite Gel'fand pairs and their applications to probability and statistics. *J. Math. Sci.* **141** 1182–1229. MR2462083 https://doi.org/10.1007/s10958-007-0041-5

[17] CECCHERINI-SILBERSTEIN, T., SCARABOTTI, F. and TOLLI, F. (2008). *Harmonic Analysis on Finite Groups*: *Representation Theory*, *Gelfand Pairs and Markov Chains*. *Cambridge Studies in Advanced Mathematics* **108**. Cambridge Univ. Press, Cambridge. MR2389056 https://doi.org/10.1017/CBO9780511619823

[18] DIACONIS, P. and HOLMES, S. P. (2002). Random walks on trees and matchings. *Electron. J. Probab.* **7** no. 6, 17. MR1887626 https://doi.org/10.1214/EJP.v7-105

[19] DIACONIS, P. and PITMAN, J. (1986). Permutations, record values and random measures. Unpublished Lecture Notes, Statistics Dept., Univ. California, Berkeley.

[20] DIACONIS, P. W. and HOLMES, S. P. (1998). Matchings and phylogenetic trees. *Proc. Natl. Acad. Sci. USA* **95** 14600–14602. MR1665632 https://doi.org/10.1073/pnas.95.25.14600

[21] ERDŐS, P. L., GREENHILL, C. S., MEZEI, T. R., MIKLÓS, I., SOLTÉSZ, D. and SOUKUP, L. (2019). The mixing time of the switch markov chains: A unified approach. arXiv:1903.06600.

[22] GOLDSCHMIDT, C., UELTSCHI, D. and WINDRIDGE, P. (2011). Quantum Heisenberg models and their probabilistic representations. In *Entropy and the Quantum II*. *Contemp. Math.* **552** 177–224. Amer. Math. Soc., Providence, RI. MR2868048 https://doi.org/10.1090/conm/552/10917

[23] HOUGH, B. (2016). The random k cycle walk on the symmetric group. *Probab. Theory Related Fields* **165** 447–482. MR3500276 https://doi.org/10.1007/s00440-015-0636-6

[24] KANNAN, R., TETALI, P. and VEMPALA, S. (1997). Simple Markov-chain algorithms for generating bipartite graphs and tournaments (extended abstract). In *Proceedings of the Eighth Annual ACM-SIAM Symposium on Discrete Algorithms* (*New Orleans*, *LA*, 1997) 193–200. ACM, New York. MR1447665

[25] KANNAN, R., TETALI, P. and VEMPALA, S. (1999). Simple Markov-chain algorithms for generating bipartite graphs and tournaments. *Random Structures Algorithms* **14** 293–308. MR1691976 https://doi.org/10.1002/(SICI)1098-2418(199907)14:4<293::AID-RSA1>3.3.CO;2-7

[26] LEVIN, D. A. and PERES, Y. (2017). *Markov Chains and Mixing Times*. Amer. Math. Soc., Providence, RI. Second edition of [MR2466937], With contributions by Elizabeth L. Wilmer, With a chapter on "Coupling from the past" by James G. Propp and David B. Wilson. MR3726904 https://doi.org/10.1090/mbk/107

[27] NESTORIDI, E. and OLESKER-TAYLOR, S. (2022). Limit profiles for reversible Markov chains. *Probab. Theory Related Fields* **182** 157–188. MR4367947 https://doi.org/10.1007/s00440-021-01061-5

[28] PITMAN, J. (2002). Poisson–Dirichlet and GEM invariant distributions for split-and-merge transformation of an interval partition. *Combin. Probab. Comput.* **11** 501–514. MR1930355 https://doi.org/10.1017/S0963548302005163

[29] SCHRAMM, O. (2005). Compositions of random transpositions. *Israel J. Math.* **147** 221–243. MR2166362 https://doi.org/10.1007/BF02785366

[30] TEYSSIER, L. (2020). Limit profile for random transpositions. *Ann. Probab.* **48** 2323–2343. MR4152644 https://doi.org/10.1214/20-AOP1424

[31] TIKHOMIROV, K. and YOUSSEF, P. (2020). Sharp Poincaré and log-Sobolev inequalities for the switch chain on regular bipartite graphs. arXiv:2007.02729.

[32] TSILEVICH, N. V. (1999). Stationary random partitions of a natural series. *Teor. Veroyatn. Primen.* **44** 55–73. MR1751188 https://doi.org/10.1137/S0040585X97977331

[33] VAN DER HOFSTAD, R. (2017). *Random Graphs and Complex Networks*. *Vol*. 1. *Cambridge Series in Statistical and Probabilistic Mathematics* **43**. Cambridge Univ. Press, Cambridge. MR3617364 https://doi.org/10.1017/9781316779422

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CONVERGENCE RATE TO THE TRACY–WIDOM LAWS FOR THE LARGEST EIGENVALUE OF SAMPLE COVARIANCE MATRICES

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We establish a quantitative version of the Tracy–Widom law for the largest eigenvalue of high-dimensional sample covariance matrices. To be precise, we show that the fluctuations of the largest eigenvalue of a sample covariance matrix X^*X converge to its Tracy–Widom limit at a rate nearly $N^{-1/3}$, where X is an $M \times N$ random matrix whose entries are independent real or complex random variables, assuming that both M and N tend to infinity at a constant rate. This result improves the previous estimate $N^{-2/9}$ obtained by Wang (2019). Our proof relies on a Green function comparison method (*Adv. Math.* **229** (2012) 1435–1515) using iterative cumulant expansions, the local laws for the Green function and asymptotic properties of the correlation kernel of the white Wishart ensemble.

REFERENCES

[1] ADLER, M., FORRESTER, P. J., NAGAO, T. and VAN MOERBEKE, P. (2000). Classical skew orthogonal polynomials and random matrices. *J. Stat. Phys.* **99** 141–170. MR1762659 https://doi.org/10.1023/A:1018644606835

[2] ALT, J., ERDŐS, L. and KRÜGER, T. (2017). Local law for random Gram matrices. *Electron. J. Probab.* **22** Paper No. 25. MR3622895 https://doi.org/10.1214/17-EJP42

[3] ALT, J., ERDŐS, L., KRÜGER, T. and SCHRÖDER, D. (2020). Correlated random matrices: Band rigidity and edge universality. *Ann. Probab.* **48** 963–1001. MR4089499 https://doi.org/10.1214/19-AOP1379

[4] ANDERSON, G. W., GUIONNET, A. and ZEITOUNI, O. (2010). *An Introduction to Random Matrices*. *Cambridge Studies in Advanced Mathematics* **118**. Cambridge Univ. Press, Cambridge. MR2760897

[5] ANDERSON, T. W. (1984). *An Introduction to Multivariate Statistical Analysis*, 2nd ed. *Wiley Publications in Statistics*. Wiley, New York. MR0091588

[6] BAO, Z. (2019). Tracy–Widom limit for Kendall's tau. *Ann. Statist.* **47** 3504–3532. MR4025750 https://doi.org/10.1214/18-AOS1786

[7] BAO, Z., HU, J., PAN, G. and ZHOU, W. (2019). Canonical correlation coefficients of high-dimensional Gaussian vectors: Finite rank case. *Ann. Statist.* **47** 612–640. MR3909944 https://doi.org/10.1214/18-AOS1704

[8] BAO, Z., PAN, G. and ZHOU, W. (2015). Universality for the largest eigenvalue of sample covariance matrices with general population. *Ann. Statist.* **43** 382–421. MR3311864 https://doi.org/10.1214/14-AOS1281

[9] BEN AROUS, G. and PÉCHÉ, S. (2005). Universality of local eigenvalue statistics for some sample covariance matrices. *Comm. Pure Appl. Math.* **58** 1316–1357. MR2162782 https://doi.org/10.1002/cpa.20070

[10] BIANCHI, P., DEBBAH, M., MAIDA, M. and NAJIM, J. (2011). Performance of statistical tests for single-source detection using random matrix theory. *IEEE Trans. Inf. Theory* **57** 2400–2419. MR2809098 https://doi.org/10.1109/TIT.2011.2111710

[11] BLOEMENDAL, A., ERDŐS, L., KNOWLES, A., YAU, H.-T. and YIN, J. (2014). Isotropic local laws for sample covariance and generalized Wigner matrices. *Electron. J. Probab.* **19** no. 33. MR3183577 https://doi.org/10.1214/ejp.v19-3054

[12] BOURGADE, P. (2022). Extreme gaps between eigenvalues of Wigner matrices. *J. Eur. Math. Soc.* (*JEMS*) **24** 2823–2873. MR4416591 https://doi.org/10.4171/jems/1141

[13] BOURGADE, P., ERDÖS, L. and YAU, H.-T. (2014). Edge universality of beta ensembles. *Comm. Math. Phys.* **332** 261–353. MR3253704 https://doi.org/10.1007/s00220-014-2120-z

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[14] BOURGADE, P., ERDŐS, L., YAU, H.-T. and YIN, J. (2016). Fixed energy universality for generalized Wigner matrices. *Comm. Pure Appl. Math.* **69** 1815–1881. MR3541852 https://doi.org/10.1002/cpa.21624

[15] BOUTET DE MONVEL, A. and KHORUNZHY, A. (1999). Asymptotic distribution of smoothed eigenvalue density. II. Wigner random matrices. *Random Oper. Stoch. Equ.* **7** 149–168. MR1689027 https://doi.org/10.1515/rose.1999.7.2.149

[16] DEIFT, P., GIOEV, D., KRIECHERBAUER, T. and VANLESSEN, M. (2007). Universality for orthogonal and symplectic Laguerre-type ensembles. *J. Stat. Phys.* **129** 949–1053. MR2363388 https://doi.org/10.1007/s10955-007-9325-x

[17] DING, X. and YANG, F. (2018). A necessary and sufficient condition for edge universality at the largest singular values of covariance matrices. *Ann. Appl. Probab.* **28** 1679–1738. MR3809475 https://doi.org/10.1214/17-AAP1341

[18] DING, X. and YANG, F. (2020). Tracy–Widom distribution for the edge eigenvalues of Gram type random matrices. Available at arXiv:2008.04166.

[19] EDELMAN, A. (1991). The distribution and moments of the smallest eigenvalue of a random matrix of Wishart type. *Linear Algebra Appl.* **159** 55–80. MR1133335 https://doi.org/10.1016/0024-3795(91)90076-9

[20] EL KAROUI, N. (2006). A rate of convergence result for the largest eigenvalue of complex white Wishart matrices. *Ann. Probab.* **34** 2077–2117. MR2294977 https://doi.org/10.1214/009117906000000502

[21] EL KAROUI, N. (2007). Tracy–Widom limit for the largest eigenvalue of a large class of complex sample covariance matrices. *Ann. Probab.* **35** 663–714. MR2308592 https://doi.org/10.1214/009117906000000917

[22] ERDŐS, L., KNOWLES, A. and YAU, H.-T. (2013). Averaging fluctuations in resolvents of random band matrices. *Ann. Henri Poincaré* **14** 1837–1926. MR3119922 https://doi.org/10.1007/s00023-013-0235-y

[23] ERDŐS, L., KNOWLES, A., YAU, H.-T. and YIN, J. (2012). Spectral statistics of Erdős–Rényi Graphs II: Eigenvalue spacing and the extreme eigenvalues. *Comm. Math. Phys.* **314** 587–640. MR2964770 https://doi.org/10.1007/s00220-012-1527-7

[24] ERDŐS, L., KRÜGER, T. and SCHRÖDER, D. (2019). Random matrices with slow correlation decay. *Forum Math. Sigma* **7** Paper No. e8. MR3941370 https://doi.org/10.1017/fms.2019.2

[25] ERDŐS, L. and YAU, H.-T. (2017). *A Dynamical Approach to Random Matrix Theory*. *Courant Lecture Notes in Mathematics* **28**. Amer. Math. Soc., Providence, RI. MR3699468

[26] ERDŐS, L., YAU, H.-T. and YIN, J. (2012). Rigidity of eigenvalues of generalized Wigner matrices. *Adv. Math.* **229** 1435–1515. MR2871147 https://doi.org/10.1016/j.aim.2011.12.010

[27] FAN, Z. and JOHNSTONE, I. M. (2017). Tracy–Widom at each edge of real covariance and MANOVA estimators. Available at arXiv:1707.02352.

[28] FELDHEIM, O. N. and SODIN, S. (2010). A universality result for the smallest eigenvalues of certain sample covariance matrices. *Geom. Funct. Anal.* **20** 88–123. MR2647136 https://doi.org/10.1007/s00039-010-0055-x

[29] FORRESTER, P. J. (1993). The spectrum edge of random matrix ensembles. *Nuclear Phys. B* **402** 709–728. MR1236195 https://doi.org/10.1016/0550-3213(93)90126-A

[30] FORRESTER, P. J. and TRINH, A. K. (2018). Functional form for the leading correction to the distribution of the largest eigenvalue in the GUE and LUE. *J. Math. Phys.* **59** 053302. MR3802426 https://doi.org/10.1063/1.5016347

[31] GEMAN, S. (1980). A limit theorem for the norm of random matrices. *Ann. Probab.* **8** 252–261. MR0566592

[32] HACHEM, W., HARDY, A. and NAJIM, J. (2016). Large complex correlated Wishart matrices: Fluctuations and asymptotic independence at the edges. *Ann. Probab.* **44** 2264–2348. MR3502605 https://doi.org/10.1214/15-AOP1022

[33] HAN, X., PAN, G. and YANG, Q. (2018). A unified matrix model including both CCA and F matrices in multivariate analysis: The largest eigenvalue and its applications. *Bernoulli* **24** 3447–3468. MR3788178 https://doi.org/10.3150/17-BEJ965

[34] HAN, X., PAN, G. and ZHANG, B. (2016). The Tracy–Widom law for the largest eigenvalue of F type matrices. *Ann. Statist.* **44** 1564–1592. MR3519933 https://doi.org/10.1214/15-AOS1427

[35] HE, Y. and KNOWLES, A. (2017). Mesoscopic eigenvalue statistics of Wigner matrices. *Ann. Appl. Probab.* **27** 1510–1550. MR3678478 https://doi.org/10.1214/16-AAP1237

[36] HE, Y. and KNOWLES, A. (2021). Fluctuations of extreme eigenvalues of sparse Erdős–Rényi graphs. *Probab. Theory Related Fields* **180** 985–1056. MR4288336 https://doi.org/10.1007/s00440-021-01054-4

[37] JOHANSSON, K. (2000). Shape fluctuations and random matrices. *Comm. Math. Phys.* **209** 437–476. MR1737991 https://doi.org/10.1007/s002200050027

[38] JOHANSSON, K. (2006). Random matrices and determinantal processes. In *Mathematical Statistical Physics* 1–55. Elsevier, Amsterdam. MR2581882 https://doi.org/10.1016/S0924-8099(06)80038-7

[39] JOHNSTONE, I. M. (2001). On the distribution of the largest eigenvalue in principal components analysis. *Ann. Statist.* **29** 295–327. MR1863961 https://doi.org/10.1214/aos/1009210544

[40] JOHNSTONE, I. M. (2007). High dimensional statistical inference and random matrices. In *International Congress of Mathematicians. Vol. I* 307–333. Eur. Math. Soc., Zürich. MR2334195 https://doi.org/10.4171/022-1/13

[41] JOHNSTONE, I. M. (2008). Multivariate analysis and Jacobi ensembles: Largest eigenvalue, Tracy–Widom limits and rates of convergence. *Ann. Statist.* **36** 2638–2716. MR2485010 https://doi.org/10.1214/08-AOS605

[42] JOHNSTONE, I. M. and PAUL, D. (2018). PCA in high dimensions: An orientation. *Proc. IEEE Inst. Electr. Electron. Eng.* **106** 1277–1292. https://doi.org/10.1109/JPROC.2018.2846730

[43] JOLLIFFE, I. T. (2002). *Principal Component Analysis*, 2nd ed. *Springer Series in Statistics*. Springer, New York. MR2036084

[44] KHORUNZHY, A. M., KHORUZHENKO, B. A. and PASTUR, L. A. (1996). Asymptotic properties of large random matrices with independent entries. *J. Math. Phys.* **37** 5033–5060. MR1411619 https://doi.org/10.1063/1.531589

[45] KNOWLES, A. and YIN, J. (2017). Anisotropic local laws for random matrices. *Probab. Theory Related Fields* **169** 257–352. MR3704770 https://doi.org/10.1007/s00440-016-0730-4

[46] LANDON, B., SOSOE, P. and YAU, H.-T. (2019). Fixed energy universality of Dyson Brownian motion. *Adv. Math.* **346** 1137–1332. MR3914908 https://doi.org/10.1016/j.aim.2019.02.010

[47] LANDON, B. and YAU, H.-T. (2017). *Edge statistics of Dyson Brownian motion*. Available at arXiv:1712.03881.

[48] LEE, J. O. and SCHNELLI, K. (2016). Tracy–Widom distribution for the largest eigenvalue of real sample covariance matrices with general population. *Ann. Appl. Probab.* **26** 3786–3839. MR3582818 https://doi.org/10.1214/16-AAP1193

[49] LEE, J. O. and SCHNELLI, K. (2018). Local law and Tracy–Widom limit for sparse random matrices. *Probab. Theory Related Fields* **171** 543–616. MR3800840 https://doi.org/10.1007/s00440-017-0787-8

[50] LEE, J. O. and YIN, J. (2014). A necessary and sufficient condition for edge universality of Wigner matrices. *Duke Math. J.* **163** 117–173. MR3161313 https://doi.org/10.1215/00127094-2414767

[51] LYTOVA, A. and PASTUR, L. (2009). Central limit theorem for linear eigenvalue statistics of random matrices with independent entries. *Ann. Probab.* **37** 1778–1840. MR2561434 https://doi.org/10.1214/09-AOP452

[52] MA, Z. (2012). Accuracy of the Tracy–Widom limits for the extreme eigenvalues in white Wishart matrices. *Bernoulli* **18** 322–359. MR2888709 https://doi.org/10.3150/10-BEJ334

[53] MARČENKO, V. A. and PASTUR, L. A. (1967). Distribution of eigenvalues in certain sets of random matrices. *Mat. Sb.* (*N.S.*) **72** 507–536. MR0208649

[54] MEHTA, M. L. (2004). *Random Matrices*, 3rd ed. *Pure and Applied Mathematics* (*Amsterdam*) **142**. Elsevier/Academic Press, Amsterdam. MR2129906

[55] MUIRHEAD, R. J. (1982). *Aspects of Multivariate Statistical Theory*. *Wiley Series in Probability and Mathematical Statistics*. Wiley, New York. MR0652932

[56] ONATSKI, A. (2008). The Tracy–Widom limit for the largest eigenvalues of singular complex Wishart matrices. *Ann. Appl. Probab.* **18** 470–490. MR2398763 https://doi.org/10.1214/07-AAP454

[57] ONATSKI, A. (2009). Testing hypotheses about the numbers of factors in large factor models. *Econometrica* **77** 1447–1479. MR2561070 https://doi.org/10.3982/ECTA6964

[58] PAUL, D. and AUE, A. (2014). Random matrix theory in statistics: A review. *J. Statist. Plann. Inference* **150** 1–29. MR3206718 https://doi.org/10.1016/j.jspi.2013.09.005

[59] PÉCHÉ, S. (2009). Universality results for the largest eigenvalues of some sample covariance matrix ensembles. *Probab. Theory Related Fields* **143** 481–516. MR2475670 https://doi.org/10.1007/s00440-007-0133-7

[60] PILLAI, N. S. and YIN, J. (2014). Universality of covariance matrices. *Ann. Appl. Probab.* **24** 935–1001. MR3199978 https://doi.org/10.1214/13-AAP939

[61] RAHMAN, A. A. and FORRESTER, P. J. (2021). Linear differential equations for the resolvents of the classical matrix ensembles. *Random Matrices Theory Appl.* **10** Paper No. 2250003. MR4302288 https://doi.org/10.1142/S2010326322500034

[62] ROY, S. N. (1953). On a heuristic method of test construction and its use in multivariate analysis. *Ann. Math. Stat.* **24** 220–238. MR0057519 https://doi.org/10.1214/aoms/1177729029

[63] SCHNELLI, K. and XU, Y. (2021). Convergence rate to the Tracy–Widom laws for the largest eigenvalue of Wigner matrices. Available at arXiv:2102.04330.

[64] SILVERSTEIN, J. W. (1989). On the weak limit of the largest eigenvalue of a large-dimensional sample covariance matrix. *J. Multivariate Anal.* **30** 307–311. MR1015375 https://doi.org/10.1016/0047-259X(89)90042-0

[65] SOSHNIKOV, A. (1999). Universality at the edge of the spectrum in Wigner random matrices. *Comm. Math. Phys.* **207** 697–733. MR1727234 https://doi.org/10.1007/s002200050743

[66] SOSHNIKOV, A. (2000). Determinantal random point fields. *Uspekhi Mat. Nauk* **55** 107–160. MR1799012 https://doi.org/10.1070/rm2000v055n05ABEH000321

[67] SOSHNIKOV, A. (2002). A note on universality of the distribution of the largest eigenvalues in certain sample covariance matrices *J. Stat. Phys.* **108** 1033–1056. MR1933444 https://doi.org/10.1023/A:1019739414239

[68] SOSHNIKOV, A. (2003). Janossy densities. II. Pfaffian ensembles. *J. Stat. Phys.* **113** 611–622. MR2013699 https://doi.org/10.1023/A:1026077020147

[69] SZEGÖ, G. (1939). *Orthogonal Polynomials. American Mathematical Society Colloquium Publications, Vol.* 23. Amer. Math. Soc., New York. MR0000077

[70] TAO, T. and VU, V. (2010). Random matrices: Universality of local eigenvalue statistics up to the edge. *Comm. Math. Phys.* **298** 549–572. MR2669449 https://doi.org/10.1007/s00220-010-1044-5

[71] TAO, T. and VU, V. (2010). Random matrices: The distribution of the smallest singular values. *Geom. Funct. Anal.* **20** 260–297. MR2647142 https://doi.org/10.1007/s00039-010-0057-8

[72] TRACY, C. A. and WIDOM, H. (1994). Level-spacing distributions and the Airy kernel. *Comm. Math. Phys.* **159** 151–174. MR1257246

[73] TRACY, C. A. and WIDOM, H. (1996). On orthogonal and symplectic matrix ensembles. *Comm. Math. Phys.* **177** 727–754. MR1385083

[74] WANG, H. (2019). Quantitative Universality for the Largest Eigenvalue of Sample Covariance Matrices. Available at arXiv:1912.05473.

[75] WANG, K. (2012). Random covariance matrices: Universality of local statistics of eigenvalues up to the edge. *Random Matrices Theory Appl.* **1** 1150005. MR2930383 https://doi.org/10.1142/S2010326311500055

[76] WIDOM, H. (1999). On the relation between orthogonal, symplectic and unitary matrix ensembles. *J. Stat. Phys.* **94** 347–363. MR1675356 https://doi.org/10.1023/A:1004516918143

[77] YANG, F. (2019). Edge universality of separable covariance matrices. *Electron. J. Probab.* **24** Paper No. 123. MR4029426 https://doi.org/10.1214/19-ejp381

[78] YANG, F. (2022). Sample canonical correlation coefficients of high-dimensional random vectors: Local law and Tracy–Widom limit. *Random Matrices Theory Appl.* **11** Paper No. 2250007. MR4391830 https://doi.org/10.1142/S2010326322500071

[79] YAO, J., ZHENG, S. and BAI, Z. (2015). *Large Sample Covariance Matrices and High-Dimensional Data Analysis. Cambridge Series in Statistical and Probabilistic Mathematics* **39**. Cambridge Univ. Press, New York. MR3468554 https://doi.org/10.1017/CBO9781107588080

[80] YIN, Y. Q., BAI, Z. D. and KRISHNAIAH, P. R. (1988). On the limit of the largest eigenvalue of the large-dimensional sample covariance matrix. *Probab. Theory Related Fields* **78** 509–521. MR0950344 https://doi.org/10.1007/BF00353874

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UNADJUSTED LANGEVIN ALGORITHM WITH MULTIPLICATIVE NOISE: TOTAL VARIATION AND WASSERSTEIN BOUNDS

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In this paper, we focus on nonasymptotic bounds related to the Euler scheme of an ergodic diffusion with a possibly multiplicative diffusion term (nonconstant diffusion coefficient). More precisely, the objective of this paper is to control the distance of the standard Euler scheme with decreasing step (usually called unadjusted Langevin algorithm in the Monte Carlo literature) to the invariant distribution of such an ergodic diffusion. In an appropriate Lyapunov setting and under uniform ellipticity assumptions on the diffusion coefficient, we establish (or improve) such bounds for total variation and L^1-Wasserstein distances in both multiplicative and additive and frameworks. These bounds rely on weak error expansions using stochastic analysis adapted to decreasing step setting.

REFERENCES

[1] BAKRY, D., GENTIL, I. and LEDOUX, M. (2014). *Analysis and Geometry of Markov Diffusion Operators*. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **348**. Springer, Cham. MR3155209 https://doi.org/10.1007/978-3-319-00227-9

[2] BALLY, V. and CARAMELLINO, L. (2019). Total variation distance between stochastic polynomials and invariance principles. *Ann. Probab.* **47** 3762–3811. MR4038042 https://doi.org/10.1214/19-aop1346

[3] BALLY, V., CARAMELLINO, L. and POLY, G. (2020). Regularization lemmas and convergence in total variation. *Electron. J. Probab.* **25** 74. MR4119120 https://doi.org/10.1214/20-ejp481

[4] BALLY, V. and TALAY, D. (1996). The law of the Euler scheme for stochastic differential equations. I. Convergence rate of the distribution function. *Probab. Theory Related Fields* **104** 43–60. MR1367666 https://doi.org/10.1007/BF01303802

[5] BISMUT, J.-M. (1984). *Large Deviations and the Malliavin Calculus*. *Progress in Mathematics* **45**. Birkhäuser, Inc., Boston, MA. MR0755001

[6] BONNEFONT, M., JOULIN, A. and MA, Y. (2016). Spectral gap for spherically symmetric log-concave probability measures, and beyond. *J. Funct. Anal.* **270** 2456–2482. MR3464047 https://doi.org/10.1016/j.jfa.2016.02.007

[7] CASS, T., CRISAN, D., DOBSON, P. and OTTOBRE, M. (2021). Long-time behaviour of degenerate diffusions: UFG-type SDEs and time-inhomogeneous hypoelliptic processes. *Electron. J. Probab.* **26** 22. MR4235473 https://doi.org/10.1214/20-EJP577

[8] CERRAI, S. (2000). Analytic semigroups and degenerate elliptic operators with unbounded coefficients: A probabilistic approach. *J. Differential Equations* **166** 151–174. MR1779259 https://doi.org/10.1006/jdeq.2000.3788

[9] CESA-BIANCHI, N. and LUGOSI, G. (2006). *Prediction, Learning, and Games*. Cambridge Univ. Press, Cambridge. MR2409394 https://doi.org/10.1017/CBO9780511546921

[10] CRISAN, D., DOBSON, P. and OTTOBRE, M. (2021). Uniform in time estimates for the weak error of the Euler method for SDEs and a pathwise approach to derivative estimates for diffusion semigroups. *Trans. Amer. Math. Soc.* **374** 3289–3330. MR4237949 https://doi.org/10.1090/tran/8301

[11] DALALYAN, A. S. (2017). Theoretical guarantees for approximate sampling from smooth and log-concave densities. *J. R. Stat. Soc. Ser. B. Stat. Methodol.* **79** 651–676. MR3641401 https://doi.org/10.1111/rssb.12183

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[12] DEL MORAL, P. and SINGH, S. S. (2020). A backward Itô–Ventzell formula with an application to stochastic interpolation. *C. R. Math. Acad. Sci. Paris* **358** 881–886. MR4174820 https://doi.org/10.5802/crmath.110

[13] DEVROYE, L., MEHRABIAN, A. and REDDAD, T. (2018). The total variation distance between high-dimensional Gaussians. ArXiv E-prints. Available at arXiv:1810.08693.

[14] DRAGONI, F., KONTIS, V. and ZEGARLIŃSKI, B. (2012). Ergodicity of Markov semigroups with Hörmander type generators in infinite dimensions. *Potential Anal.* **37** 199–227. MR2969300 https://doi.org/10.1007/s11118-011-9253-x

[15] DURMUS, A. and MOULINES, É. (2017). Nonasymptotic convergence analysis for the unadjusted Langevin algorithm. *Ann. Appl. Probab.* **27** 1551–1587. MR3678479 https://doi.org/10.1214/16-AAP1238

[16] DURMUS, A. and MOULINES, É. (2019). High-dimensional Bayesian inference via the unadjusted Langevin algorithm. *Bernoulli* **25** 2854–2882. MR4003567 https://doi.org/10.3150/18-BEJ1073

[17] EBERLE, A., GUILLIN, A. and ZIMMER, R. (2019). Quantitative Harris-type theorems for diffusions and McKean–Vlasov processes. *Trans. Amer. Math. Soc.* **371** 7135–7173. MR3939573 https://doi.org/10.1090/tran/7576

[18] ELWORTHY, K. D. and LI, X.-M. (1994). Formulae for the derivatives of heat semigroups. *J. Funct. Anal.* **125** 252–286. MR1297021 https://doi.org/10.1006/jfan.1994.1124

[19] ETHIER, S. N. and KURTZ, T. G. (1986). *Markov Processes. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics*. Wiley, New York. Characterization and convergence. MR0838085 https://doi.org/10.1002/9780470316658

[20] GADAT, S., PANLOUP, F. and PELLEGRINI, C. (2020). On the cost of Bayesian posterior mean strategy for log-concave models.

[21] GELFAND, S. B. and MITTER, S. K. (1991). Recursive stochastic algorithms for global optimization in $\mathbf{R}^d$. *SIAM J. Control Optim.* **29** 999–1018. MR1110084 https://doi.org/10.1137/0329055

[22] GELFAND, S. B. and MITTER, S. K. (1993). Metropolis-type annealing algorithms for global optimization in $\mathbf{R}^d$. *SIAM J. Control Optim.* **31** 111–131. MR1200226 https://doi.org/10.1137/0331009

[23] GUYON, J. (2006). Euler scheme and tempered distributions. *Stochastic Process. Appl.* **116** 877–904. MR2254663 https://doi.org/10.1016/j.spa.2005.11.011

[24] HONORÉ, I., MENOZZI, S. and PAGÈS, G. (2020). Non-asymptotic Gaussian estimates for the recursive approximation of the invariant distribution of a diffusion. *Ann. Inst. Henri Poincaré Probab. Stat.* **56** 1559–1605. MR4116702 https://doi.org/10.1214/19-AIHP985

[25] KONAKOV, V. and MAMMEN, E. (2002). Edgeworth type expansions for Euler schemes for stochastic differential equations. *Monte Carlo Methods Appl.* **8** 271–285. MR1931967 https://doi.org/10.1515/mcma.2002.8.3.271

[26] KUNITA, H. (1997). *Stochastic Flows and Stochastic Differential Equations. Cambridge Studies in Advanced Mathematics* **24**. Cambridge Univ. Press, Cambridge. Reprint of the 1990 original. MR1472487

[27] LAMBERTON, D. and PAGÈS, G. (2002). Recursive computation of the invariant distribution of a diffusion. *Bernoulli* **8** 367–405. MR1913112 https://doi.org/10.1142/S0219493703000838

[28] LAMBERTON, D. and PAGÈS, G. (2003). Recursive computation of the invariant distribution of a diffusion: The case of a weakly mean reverting drift. *Stoch. Dyn.* **3** 435–451. MR2030742 https://doi.org/10.1142/S0219493703000838

[29] LEMAIRE, V. (2005). Estimation récursive de la mesure invariante d'un processus de diffusion Thèse Université de Marne la Vallée.

[30] LEMAIRE, V. (2007). An adaptive scheme for the approximation of dissipative systems. *Stochastic Process. Appl.* **117** 1491–1518. MR2353037 https://doi.org/10.1016/j.spa.2007.02.004

[31] LI, C., CHEN, C., CARLSON, D. and CARIN, L. (2015). Preconditioned stochastic gradient Langevin dynamics for deep neural networks. ArXiv E-prints. Available at arXiv:1512.07666.

[32] LUO, D. and WANG, J. (2016). Exponential convergence in L^p-Wasserstein distance for diffusion processes without uniformly dissipative drift. *Math. Nachr.* **289** 1909–1926. MR3563910 https://doi.org/10.1002/mana.201500351

[33] MA, Y.-A., CHEN, T. and FOX, E. B. (2015). A complete recipe for stochastic gradient MCMC. ArXiv E-prints. Available at arXiv:1506.04696.

[34] MÁRQUEZ, D. and PELLETIER, M. (1996). Sur la vitesse de convergence en loi du recuit simulé. *C. R. Acad. Sci. Paris Sér. I Math.* **322** 391–394. MR1378518

[35] MOU, W., FLAMMARION, N., WAINWRIGHT, M. J. and BARTLETT, P. L. (2022). Improved bounds for discretization of Langevin diffusions: Near-optimal rates without convexity. *Bernoulli* **28** 1577–1601. MR4411503 https://doi.org/10.3150/21-bej1343

[36] NUALART, D. (2006). *The Malliavin Calculus and Related Topics*, 2nd ed. *Probability and Its Applications* (*New York*). Springer, Berlin. MR2200233

[37] PAGÈS, G. (2001). Sur quelques algorithmes récursifs pour les probabilités numériques. *ESAIM Probab. Stat.* **5** 141–170. MR1875668 https://doi.org/10.1051/ps:2001106

[38] PAGÈS, G. (2018). *Numerical Probability*. *Universitext*. Springer, Cham. An introduction with applications to finance. MR3823785 https://doi.org/10.1007/978-3-319-90276-0

[39] PAGÈS, G. and PANLOUP, F. (2014). A mixed-step algorithm for the approximation of the stationary regime of a diffusion. *Stochastic Process. Appl.* **124** 522–565. MR3131304 https://doi.org/10.1016/j.spa.2013.07.011

[40] PANLOUP, F. (2008). Recursive computation of the invariant measure of a stochastic differential equation driven by a Lévy process. *Ann. Appl. Probab.* **18** 379–426. MR2398761 https://doi.org/10.1214/105051607000000285

[41] PANLOUP, F. (2008). Computation of the invariant measure for a Lévy driven SDE: Rate of convergence. *Stochastic Process. Appl.* **118** 1351–1384. MR2427043 https://doi.org/10.1016/j.spa.2007.09.007

[42] PELLETIER, M. (1996). Sur le comportement asymptotique des algorithmes stochastiques Ph.D. thesis Université Paris Sud.

[43] REVUZ, D. and YOR, M. (1999). *Continuous Martingales and Brownian Motion*, 3rd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **293**. Springer, Berlin. MR1725357 https://doi.org/10.1007/978-3-662-06400-9

[44] TALAY, D. and TUBARO, L. (1990). Expansion of the global error for numerical schemes solving stochastic differential equations. *Stoch. Anal. Appl.* **8** 483–509. MR1091544 https://doi.org/10.1080/07362999008809220

[45] VILLANI, C. (2009). *Optimal Transport*. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **338**. Springer, Berlin. Old and new. MR2459454 https://doi.org/10.1007/978-3-540-71050-9

[46] WANG, F.-Y. (2020). Exponential contraction in Wasserstein distances for diffusion semigroups with negative curvature. *Potential Anal.* **53** 1123–1144. MR4140091 https://doi.org/10.1007/s11118-019-09800-z

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STOCHASTIC APPROXIMATION WITH DISCONTINUOUS DYNAMICS, DIFFERENTIAL INCLUSIONS, AND APPLICATIONS

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This work develops new results for stochastic approximation algorithms. The emphases are on treating algorithms and limits with discontinuities. The main ingredients include the use of differential inclusions, set-valued analysis, nonsmooth analysis, and stochastic differential inclusions. Under broad conditions, it is shown that a suitably scaled sequence of the iterates has a differential inclusion limit. In addition, it is shown for the first time that a centered and scaled sequence of the iterates converges weakly to a stochastic differential inclusion limit. The results are then used to treat several application examples including Markov decision processes, Lasso algorithms, Pegasos algorithms, support vector machine classification, and learning. Some numerical demonstrations are also provided.

REFERENCES

[1] AUBIN, J.-P. and CELLINA, A. (1984). *Differential Inclusions*. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **264**. Springer, Berlin. MR0755330 https://doi.org/10.1007/978-3-642-69512-4
[2] BACCIOTTI, A. and CERAGIOLI, F. (1999). Stability and stabilization of discontinuous systems and nonsmooth Lyapunov functions. *ESAIM Control Optim*. *Calc*. *Var.* **4** 361–376. MR1693045 https://doi.org/10.1051/cocv:1999113
[3] BENAIM, M. (1996). A dynamical system approach to stochastic approximations. *SIAM J. Control Optim*. **34** 437–472. MR1377706 https://doi.org/10.1137/S0363012993253534
[4] BENAÏM, M. and FAURE, M. (2012). Stochastic approximation, cooperative dynamics and supermodular games. *Ann*. *Appl*. *Probab*. **22** 2133–2164. MR3025692 https://doi.org/10.1214/11-AAP816
[5] BENAÏM, M., HOFBAUER, J. and SORIN, S. (2005). Stochastic approximations and differential inclusions. *SIAM J*. *Control Optim*. **44** 328–348. MR2177159 https://doi.org/10.1137/S0363012904439301
[6] BENAÏM, M., HOFBAUER, J. and SORIN, S. (2012). Perturbations of set-valued dynamical systems, with applications to game theory. *Dyn*. *Games Appl*. **2** 195–205. MR2922840 https://doi.org/10.1007/s13235-012-0040-0
[7] BENVENISTE, A., MÉTIVIER, M. and PRIOURET, P. (1990). *Stochastic Approximations and Adaptive Algorithms*. Springer, New York.
[8] BIANCHI, P., HACHEM, W. and SCHECHTMAN, S. Convergence of constant step stochastic gradient descent for non-smooth non-convex functions. Available at arXiv:2005.08513.
[9] BOLTE, J. and PAUWELS, E. (2021). Conservative set valued fields, automatic differentiation, stochastic gradient methods and deep learning. *Math*. *Program*. **188** 19–51. MR4276581 https://doi.org/10.1007/s10107-020-01501-5
[10] BURKE, J., CURTIS, F., LEWIS, A., OVERTON, M. and SIMOES, L. (2020). Gradient sampling methods for nonsmooth optimization. In *Numerical Nonsmooth Optimization* (A. Bagirov, M. Gaudioso, N. Karmitsa, M. Mäkelä and S. Taheri, eds.) Springer, Cham.
[11] CHEN, X. (2012). Smoothing methods for nonsmooth, nonconvex minimization. *Math*. *Program*. **134** 71–99. MR2947553 https://doi.org/10.1007/s10107-012-0569-0
[12] CHONG, E. K. P. and ZAK, S. H. (2014). *An Introduction to Optimization*. Wiley, New York.
[13] CLARKE, F. H. (1983). *Optimization and Nonsmooth Analysis*. *Canadian Mathematical Society Series of Monographs and Advanced Texts*. Wiley, New York. MR0709590

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[14] CLARKE, F. H., LEDYAEV, Y. S. and STERN, R. J. (1998). Asymptotic stability and smooth Lyapunov functions. *J. Differential Equations* **149** 69–114. MR1643670 https://doi.org/10.1006/jdeq.1998.3476

[15] CRISTIANINI, N. and SHAWE-TAYLOR, J. (2000). *An Introduction to Support Vector Machines*. Cambridge Univ. Press, Cambridge.

[16] DAVIS, D., DRUSVYATSKIY, D., KAKADE, S. and LEE, J. D. (2020). Stochastic subgradient method converges on tame functions. *Found. Comput. Math.* **20** 119–154. MR4056927 https://doi.org/10.1007/s10208-018-09409-5

[17] ETHIER, S. N. and KURTZ, T. G. (1986). *Markov Processes: Characterization and Convergence*. *Wiley Series in Probability and Mathematical Statistics*: *Probability and Mathematical Statistics*. Wiley, New York. MR0838085 https://doi.org/10.1002/9780470316658

[18] EWEDA, E. and MACCHI, O. (1983). Quadratic mean and almost-sure convergence of unbounded stochastic approximation algorithms with correlated observations. *Ann. Inst. Henri Poincaré Sect. B* (*N. S.*) **19** 235–255. MR0725558

[19] FRIKHA, N. (2016). Multi-level stochastic approximation algorithms. *Ann. Appl. Probab.* **26** 933–985. MR3476630 https://doi.org/10.1214/15-AAP1109

[20] GIRAUD, C. (2015). *Introduction to High-Dimensional Statistics*. *Monographs on Statistics and Applied Probability* **139**. CRC Press, Boca Raton, FL. MR3307991

[21] GRIMMER, B. (2019). Convergence rates for deterministic and stochastic subgradient methods without Lipschitz continuity. *SIAM J. Optim.* **29** 1350–1365. MR3949713 https://doi.org/10.1137/18M117306X

[22] HÁJEK, O. (1979). Discontinuous differential equations. I, II. *J. Differential Equations* **32** 149–170, 171–185. MR0534546 https://doi.org/10.1016/0022-0396(79)90056-1

[23] HAJINEZHAD, D. and HONG, M. (2019). Perturbed proximal primal-dual algorithm for nonconvex nonsmooth optimization. *Math. Program.* **176** 207–245. MR3960809 https://doi.org/10.1007/s10107-019-01365-4

[24] HELOU NETO, E. S. and DE PIERRO, Á. R. (2009). Incremental subgradients for constrained convex optimization: A unified framework and new methods. *SIAM J. Optim.* **20** 1547–1572. MR2587740 https://doi.org/10.1137/070711712

[25] HIGHAM, D. J., MAO, X., ROJ, M., SONG, Q. and YIN, G. (2013). Mean exit times and the multilevel Monte Carlo method. *SIAM/ASA J. Uncertain. Quantificat.* **1** 2–18. MR3283875 https://doi.org/10.1137/120883803

[26] HILL, R. W. and HOLLAND, P. W. (1977). Two robust alternatives to least squares regression. *J. Amer. Statist. Assoc.* **72** 828–833.

[27] HURLEY, M. (1995). Chain recurrence, semiflows, and gradients. *J. Dynam. Differential Equations* **7** 437–456. MR1348735 https://doi.org/10.1007/BF02219371

[28] JASSO-FUENTES, H. and YIN, G. (2013). *Advanced Criteria for Controlled Markov-Modulated Diffusions in an Infinite Horizon*: *Overtaking*, *Bias*, *and Blackwell Optimality*. Science Press, Beijing, China.

[29] KAMALAPURKAR, R., DIXON, W. E. and TEEL, A. R. (2020). On reduction of differential inclusions and Lyapunov stability. *ESAIM Control Optim. Calc. Var.* **26** 24. MR4071314 https://doi.org/10.1051/cocv/2019074

[30] KHAMARU, K. and WAINWRIGHT, M. J. (2019). Convergence guarantees for a class of non-convex and non-smooth optimization problems. *J. Mach. Learn. Res.* **20** 154. MR4030168

[31] KISIELEWICZ, M. (2001). Weak compactness of solution sets to stochastic differential inclusions with convex right-hand sides. *Topol. Methods Nonlinear Anal.* **18** 149–169.

[32] KISIELEWICZ, M. (2005). Weak compactness of solution sets to stochastic differential inclusions with non-convex right-hand sides. *Stoch. Anal. Appl.* **23** 871–901. MR2158883 https://doi.org/10.1080/SAP-200044414

[33] KISIELEWICZ, M. (2013). *Stochastic Differential Inclusions and Applications*. *Springer Optimization and Its Applications* **80**. Springer, New York. MR3097668 https://doi.org/10.1007/978-1-4614-6756-4

[34] KIWIEL, K. C. (2003). Convergence of approximate and incremental subgradient methods for convex optimization. *SIAM J. Optim.* **14** 807–840. MR2085944 https://doi.org/10.1137/S1052623400376366

[35] KUSHNER, H. J. (1983). An averaging method for stochastic approximations with discontinuous dynamics, constraints, and state-dependent noise. In *Recent Advances in Stochastics* (M. H. Rizri, J. S. Rustagi and D. Siegmund, eds.) Academic Press, New York.

[36] KUSHNER, H. J. and CLARK, D. S. (1978). *Stochastic Approximation Methods for Constrained and Unconstrained Systems*. *Applied Mathematical Sciences* **26**. Springer, New York-Berlin. MR0499560

[37] KUSHNER, H. J. and YIN, G. (2003). *Stochastic Approximation and Recursive Algorithms and Applications*, 2nd ed. *Applications of Mathematics* **35**. Springer, New York. MR1993642

[38] LANGFORD, J., LI, L. and ZHANG, T. (2009). Sparse online learning via truncated gradient. *Adv. Neural Inf. Process. Syst.* **10** 905–912.

[39] LJUNG, L. (1977). Analysis of recursive stochastic algorithms. *IEEE Trans. Automat. Control* **AC-22** 551–575. MR0465458 https://doi.org/10.1109/tac.1977.1101561

[40] MÉTIVIER, M. and PRIOURET, P. (1987). Théorèmes de convergence presque sure pour une classe d'algorithmes stochastiques à pas décroissant. *Probab. Theory Related Fields* **74** 403–428. MR0873887 https://doi.org/10.1007/BF00699098

[41] NEDIĆ, A. and LEE, S. (2014). On stochastic subgradient mirror-descent algorithm with weighted averaging. *SIAM J. Optim.* **24** 84–107. MR3150179 https://doi.org/10.1137/120894464

[42] NITANDA, A. (2014). Stochastic proximal gradient descent with acceleration techniques. *Adv. Neural Inf. Process. Syst.* 1574–1582.

[43] PADEN, B. E. and SASTRY, S. S. (1987). A calculus for computing Filippov's differential inclusion with application to the variable structure control of robot manipulators. *IEEE Trans. Circuits Syst.* **34** 73–82. MR0871547 https://doi.org/10.1109/TCS.1987.1086038

[44] PANG, C. H. J. (2011). Generalized differentiation with positively homogeneous maps: Applications in set-valued analysis and metric regularity. *Math. Oper. Res.* **36** 377–397. MR2832397 https://doi.org/10.1287/moor.1110.0497

[45] PERKINS, S. and LESLIE, D. S. (2012). Asynchronous stochastic approximation with differential inclusions. *Stoch. Syst.* **2** 409–446. MR3354772 https://doi.org/10.1214/11-SSY056

[46] RAM, S. S., NEDIĆ, A. and VEERAVALLI, V. V. (2009). Incremental stochastic subgradient algorithms for convex optimization. *SIAM J. Optim.* **20** 691–717. MR2515792 https://doi.org/10.1137/080726380

[47] ROBBINS, H. and MONRO, S. (1951). A stochastic approximation method. *Ann. Math. Stat.* **22** 400–407. MR0042668 https://doi.org/10.1214/aoms/1177729586

[48] ROCKAFELLAR, R. T. and WETS, R. JB. (1997). *Variational Analysis*. Springer, Berlin.

[49] SHALEV-SHWARTZ, S., SINGER, Y., SREBRO, N. and COTTER, A. (2011). Pegasos: Primal estimated sub-gradient solver for SVM. *Math. Program.* **127** 3–30. MR2776708 https://doi.org/10.1007/s10107-010-0420-4

[50] SHALEV-SHWARTZ, S. and SREBRO, N. (2018). SVM optimization: Inverse dependence on training set size. In *Proceedings of the* 25*th International Conference on Machine Learning* 928–935.

[51] SHALEV-SHWARTZ, S. and TEWARI, A. (2011). Stochastic methods for ℓ_1-regularized loss minimization. *J. Mach. Learn. Res.* **12** 1865–1892. MR2819020

[52] SHAMIR, O. and ZHANG, T. (2013). Stochastic gradient descent for non-smooth optimization: Convergence results and optimal averaging schemes. *Proc. Mach. Learn. Res.* **28** 71–79.

[53] TADIĆ, V. B. and DOUCET, A. (2017). Asymptotic bias of stochastic gradient search. *Ann. Appl. Probab.* **27** 3255–3304. MR3737925 https://doi.org/10.1214/16-AAP1272

[54] TIBSHIRANI, R. (1996). Regression shrinkage and selection via the lasso. *J. Roy. Statist. Soc. Ser. B* **58** 267–288. MR1379242

[55] WANG, H., LI, G. and JIANG, G. (2007). Robust regression shrinkage and consistent variable selection through the LAD-Lasso. *J. Bus. Econom. Statist.* **25** 347–355. MR2380753 https://doi.org/10.1198/073500106000000251

[56] WIDROW, B. and STEARNS, S. D. (1985). *Adaptive Signal Processing*. Prentice-Hall, Upper Saddle River, NJ.

[57] YIN, G. (1991). On extensions of Polyak's averaging approach to stochastic approximation. *Stoch. Stoch. Rep.* **36** 245–264. MR1128497 https://doi.org/10.1080/17442509108833721

[58] YIN, G., KRISHNAMURTHY, V. and ION, C. (2003). Iterate-averaging sign algorithms for adaptive filtering with applications to blind multiuser detection. *IEEE Trans. Inf. Theory* **49** 657–671. MR1967190 https://doi.org/10.1109/TIT.2002.808100