ISSN 1050-5164 (print)
ISSN 2168-8737 (online)

THE ANNALS *of* APPLIED PROBABILITY

AN OFFICIAL JOURNAL OF THE
INSTITUTE OF MATHEMATICAL STATISTICS

Articles

continued

Vol. 33, No. 2—April 2023

ISSN 1050-5164 (print)
ISSN 2168-8737 (online)

THE ANNALS *of* APPLIED PROBABILITY

AN OFFICIAL JOURNAL OF THE
INSTITUTE OF MATHEMATICAL STATISTICS

Articles—*Continued from front cover*

Vol. 33, No. 2—April 2023

THE ANNALS OF APPLIED PROBABILITY Vol. 33, No. 2, pp. 825–1659 April 2023

The Annals of Applied Probability **[ISSN 1050-5164 (print); ISSN 2168-8737 (online)],** Volume 33, Number 2, April 2023. Published bimonthly by the Institute of Mathematical Statistics, 9760 Smith Road, Waite Hill, Ohio 44094, USA. Periodicals postage paid at Cleveland, Ohio, and at additional mailing offices.

POSTMASTER: Send address changes to *The Annals of Applied Probability*, Institute of Mathematical Statistics, Dues and Subscriptions Office, PO Box 729, Middletown, Maryland 21769, USA.

The Annals of Applied Probability
2023, Vol. 33, No. 2, 825–842
https://doi.org/10.1214/21-AAP1747

OPTIMAL BOUNDS IN NORMAL APPROXIMATION FOR MANY INTERACTING WORLDS

BY LOUIS H. Y. CHEN[1,a] AND LÊ VĂN THÀNH[2,b]

[1]*Department of Mathematics, National University of Singapore,* [a]*matchyl@nus.edu.sg*
[2]*Department of Mathematics, Vinh University,* [b]*levt@vinhuni.edu.vn*

In this paper, we use Stein's method to obtain optimal bounds, both in Kolmogorov and in Wasserstein distance, in the normal approximation for the empirical distribution of the ground state of a many-interacting-worlds harmonic oscillator proposed by Hall, Deckert and Wiseman (*Phys. Rev. X* **4** (2014) 041013). Our bounds on the Wasserstein distance solve a conjecture of McKeague and Levin (*Ann. Appl. Probab.* **26** (2016) 2540–2555).

REFERENCES

[1] BOHM, D. (1952). A suggested interpretation of the quantum theory in terms of "hidden" variables. I. *Phys. Rev.* (2) **85** 166–179. MR0046287

[2] BOHM, D. (1952). A suggested interpretation of the quantum theory in terms of "hidden" variables. II. *Phys. Rev.* (2) **85** 180–193. MR0046288

[3] BONIS, T. (2020). Stein's method for normal approximation in Wasserstein distances with application to the multivariate central limit theorem. *Probab. Theory Related Fields* **178** 827–860. MR4168389 https://doi.org/10.1007/s00440-020-00989-4

[4] CHEN, L. H. Y., GOLDSTEIN, L. and SHAO, Q.-M. (2011). *Normal Approximation by Stein's Method*. *Probability and Its Applications* (*New York*). Springer, Heidelberg. MR2732624 https://doi.org/10.1007/978-3-642-15007-4

[5] CHEN, L. H. Y. and SHAO, Q.-M. (2001). A non-uniform Berry–Esseen bound via Stein's method. *Probab. Theory Related Fields* **120** 236–254. MR1841329 https://doi.org/10.1007/PL00008782

[6] COURTADE, T. A., FATHI, M. and PANANJADY, A. (2019). Existence of Stein kernels under a spectral gap, and discrepancy bounds. *Ann. Inst. Henri Poincaré Probab. Stat.* **55** 777–790. MR3949953 https://doi.org/10.1214/18-aihp898

[7] FANG, X. (2019). Wasserstein-2 bounds in normal approximation under local dependence. *Electron. J. Probab.* **24** Paper No. 35, 14 pp. MR3940765 https://doi.org/10.1214/19-EJP301

[8] GOLDSTEIN, L. (2007). L^1 bounds in normal approximation. *Ann. Probab.* **35** 1888–1930. MR2349578 https://doi.org/10.1214/009117906000001123

[9] GOLDSTEIN, L. and REINERT, G. (1997). Stein's method and the zero bias transformation with application to simple random sampling. *Ann. Appl. Probab.* **7** 935–952. MR1484792 https://doi.org/10.1214/aoap/1043862419

[10] HALL, M. J. W., DECKERT, D. A. and WISEMAN, H. M. (2014). Quantum phenomena modeled by interactions between many classical worlds. *Phys. Rev. X* **4** 041013.

[11] LEDOUX, M., NOURDIN, I. and PECCATI, G. (2015). Stein's method, logarithmic Sobolev and transport inequalities. *Geom. Funct. Anal.* **25** 256–306. MR3320893 https://doi.org/10.1007/s00039-015-0312-0

[12] MCKEAGUE, I. W. and LEVIN, B. (2016). Convergence of empirical distributions in an interpretation of quantum mechanics. *Ann. Appl. Probab.* **26** 2540–2555. MR3543904 https://doi.org/10.1214/15-AAP1154

[13] MCKEAGUE, I. W., PEKÖZ, E. A. and SWAN, Y. (2019). Stein's method and approximating the quantum harmonic oscillator. *Bernoulli* **25** 89–111. MR3892313 https://doi.org/10.3150/17-bej960

MSC2020 subject classifications. 60F05, 81Q65.
Key words and phrases. Stein's method, normal approximation, many interacting worlds, harmonic oscillator.

The Annals of Applied Probability
2023, Vol. 33, No. 2, 843–888
https://doi.org/10.1214/22-AAP1830

LOCAL WEAK CONVERGENCE FOR SPARSE NETWORKS OF INTERACTING PROCESSES

BY DANIEL LACKER[1,a], KAVITA RAMANAN[2,b] AND RUOYU WU[3,c]

[1]*Department of Industrial Engineering and Operations Research, Columbia University,* [a]*daniel.lacker@columbia.edu*
[2]*Division of Applied Mathematics, Brown University,* [b]*kavita_ramanan@brown.edu*
[3]*Department of Mathematics, Iowa State University,* [c]*ruoyu@iastate.edu*

We study the limiting behavior of interacting particle systems indexed by large sparse graphs, which evolve either according to a discrete time Markov chain or a diffusion, in which particles interact directly only with their nearest neighbors in the graph. To encode sparsity we work in the framework of local weak convergence of marked (random) graphs. We show that the joint law of the particle system varies continuously with respect to local weak convergence of the underlying graph marked with the initial conditions. In addition, we show that the global empirical measure converges to a nonrandom limit for a large class of graph sequences including sparse Erdős–Rényi graphs and configuration models, whereas the empirical measure of the connected component of a uniformly random vertex converges to a random limit. Along the way, we develop some related results on the time-propagation of ergodicity and empirical field convergence, as well as some general results on local weak convergence of Gibbs measures in the uniqueness regime which appear to be new. The results obtained here are also useful for obtaining autonomous descriptions of marginal dynamics of interacting diffusions and Markov chains on sparse graphs. While limits of interacting particle systems on dense graphs have been extensively studied, there are relatively few works that have studied the sparse regime in generality.

REFERENCES

[1] ALDOUS, D. and STEELE, J. M. (2004). The objective method: Probabilistic combinatorial optimization and local weak convergence. In *Probability on Discrete Structures*. *Encyclopaedia Math*. *Sci*. **110** 1–72. Springer, Berlin. MR2023650 https://doi.org/10.1007/978-3-662-09444-0_1

[2] BALADRON, J., FASOLI, D., FAUGERAS, O. and TOUBOUL, J. (2012). Mean-field description and propagation of chaos in networks of Hodgkin–Huxley and FitzHugh–Nagumo neurons. *J. Math. Neurosci*. **2** Art. 10, 50. MR2974499 https://doi.org/10.1186/2190-8567-2-10

[3] BAYRAKTAR, E. and WU, R. (2021). Mean field interaction on random graphs with dynamically changing multi-color edges. *Stochastic Process*. *Appl*. **141** 197–244. MR4297892 https://doi.org/10.1016/j.spa.2021.07.005

[4] BENJAMINI, I. and SCHRAMM, O. (2001). Recurrence of distributional limits of finite planar graphs. *Electron*. *J. Probab*. **6** no. 23, 13. MR1873300 https://doi.org/10.1214/EJP.v6-96

[5] BHAMIDI, S., BUDHIRAJA, A. and WU, R. (2019). Weakly interacting particle systems on inhomogeneous random graphs. *Stochastic Process*. *Appl*. **129** 2174–2206. MR3958427 https://doi.org/10.1016/j.spa.2018.06.014

[6] BOLLOBÁS, B. (1980). A probabilistic proof of an asymptotic formula for the number of labelled regular graphs. *European J. Combin*. **1** 311–316. MR0595929 https://doi.org/10.1016/S0195-6698(80)80030-8

[7] BORDENAVE, C. (2016). Lecture notes on random graphs and probabilistic combinatorial optimization.

[8] COPPINI, F., DIETERT, H. and GIACOMIN, G. (2020). A law of large numbers and large deviations for interacting diffusions on Erdős–Rényi graphs. *Stoch*. *Dyn*. **20** 2050010, 19. MR4080158 https://doi.org/10.1142/S0219493720500100

MSC2020 subject classifications. Primary 60K35, 60J05, 60J60, 60J80; secondary 60F17, 60B10, 82C22.
Key words and phrases. Interacting diffusions, probabilistic cellular automata, discrete-time Markov chains, sparse graphs, random graphs, local weak convergence, mean-field limits, nonlinear Markov processes, Erdős–Rényi graphs, configuration model, unimodularity, Gibbs measures, Markov random fields.

[9] DEL MORAL, P. (2004). *Feynman–Kac Formulae*: *Genealogical and Interacting Particle Systems with Applications*. *Probability and Its Applications* (*New York*). Springer, New York. MR2044973 https://doi.org/10.1007/978-1-4684-9393-1

[10] DELATTRE, S., GIACOMIN, G. and LUÇON, E. (2016). A note on dynamical models on random graphs and Fokker–Planck equations. *J*. *Stat*. *Phys*. **165** 785–798. MR3568168 https://doi.org/10.1007/s10955-016-1652-3

[11] DEMBO, A. and MONTANARI, A. (2010). Gibbs measures and phase transitions on sparse random graphs. *Braz*. *J*. *Probab*. *Stat*. **24** 137–211. MR2643563 https://doi.org/10.1214/09-BJPS027

[12] DEMBO, A., MONTANARI, A. and SUN, N. (2013). Factor models on locally tree-like graphs. *Ann*. *Probab*. **41** 4162–4213. MR3161472 https://doi.org/10.1214/12-AOP828

[13] DEREUDRE, D. (2003). Interacting Brownian particles and Gibbs fields on pathspaces. *ESAIM Probab*. *Stat*. **7** 251–277. MR1987789 https://doi.org/10.1051/ps:2003012

[14] DEREUDRE, D. and RŒLLY, S. (2017). Path-dependent infinite-dimensional SDE with non-regular drift: An existence result. *Ann*. *Inst*. *Henri Poincaré Probab*. *Stat*. **53** 641–657. MR3634268 https://doi.org/10.1214/15-AIHP728

[15] DURRETT, R. (2007). *Random Graph Dynamics*. *Cambridge Series in Statistical and Probabilistic Mathematics* **20**. Cambridge Univ. Press, Cambridge. MR2271734

[16] DURRETT, R. and LEVIN, S. A. (1994). Stochastic spatial models: A user's guide to ecological applications. *Philos*. *Trans*. *R*. *Soc*. *Lond*. *B*, *Biol*. *Sci*. **343** 329–350.

[17] EINSIEDLER, M. and WARD, T. (2013). *Ergodic Theory*. Springer.

[18] FÖLLMER, H. (1994). Stock price fluctuation as a diffusion in a random environment. *Philos*. *Trans*. *R*. *Soc*. *Lond*. *Ser*. *A* **347** 471–483. With a discussion by W. Cheung, M. A. H. Dempster, and H. Föllmer. MR1407254 https://doi.org/10.1098/rsta.1994.0057

[19] GEORGII, H.-O. (2011). *Gibbs Measures and Phase Transitions*, 2nd ed. *De Gruyter Studies in Mathematics* **9**. de Gruyter, Berlin. MR2807681 https://doi.org/10.1515/9783110250329

[20] GRASSBERGER, P. (1983). On the critical behavior of the general epidemic process and dynamical percolation. *Math*. *Biosci*. **63** 157–172.

[21] KOLOKOLTSOV, V. N. (2010). *Nonlinear Markov Processes and Kinetic Equations*. *Cambridge Tracts in Mathematics* **182**. Cambridge Univ. Press, Cambridge. MR2680971 https://doi.org/10.1017/CBO9780511760303

[22] LACKER, D., RAMANAN, K. and WU, R. (2019). Large sparse networks of interacting diffusions. arXiv preprint, arXiv:1904.02585v1.

[23] LACKER, D., RAMANAN, K. and WU, R. (2020). Marginal dynamics of interacting diffusions on unimodular Galton–Watson trees. arXiv preprint, arXiv:2009.11667.

[24] LACKER, D., RAMANAN, K. and WU, R. (2021). Locally interacting diffusions as Markov random fields on path space. *Stochastic Process*. *Appl*. **140** 81–114. MR4276494 https://doi.org/10.1016/j.spa.2021.06.007

[25] LACKER, D., RAMANAN, K. and WU, R. (2021). Marginal dynamics of probabilistic cellular automata on trees. Preprint.

[26] LEBOWITZ, J. L., MAES, C. and SPEER, E. R. (1990). Statistical mechanics of probabilistic cellular automata. *J*. *Stat*. *Phys*. **59** 117–170. MR1049965 https://doi.org/10.1007/BF01015566

[27] LOUIS, P.-Y. and NARDI, F. R., eds. (2018). *Probabilistic Cellular Automata*: *Theory*, *Applications and Future Perspectives*. *Emergence*, *Complexity and Computation* **27**. Springer, Cham. Papers based on the workshop held at EURANDOM, TU Eindhoven, June 2013. MR3791805

[28] LUÇON, E. (2020). Quenched asymptotics for interacting diffusions on inhomogeneous random graphs. *Stochastic Process*. *Appl*. **130** 6783–6842. MR4158803 https://doi.org/10.1016/j.spa.2020.06.010

[29] LUÇON, E. and STANNAT, W. (2014). Mean field limit for disordered diffusions with singular interactions. *Ann*. *Appl*. *Probab*. **24** 1946–1993. MR3226169 https://doi.org/10.1214/13-AAP968

[30] MCKEAN, H. P. JR. (1967). Propagation of chaos for a class of non-linear parabolic equations. In *Stochastic Differential Equations* (*Lecture Series in Differential Equations*, *Session* 7, *Catholic Univ*., 1967) 41–57. Air Force Office Sci. Res., Arlington, VA. MR0233437

[31] MEDVEDEV, G. S. (2019). The continuum limit of the Kuramoto model on sparse random graphs. *Commun*. *Math*. *Sci*. **17** 883–898. MR4030504 https://doi.org/10.4310/CMS.2019.v17.n4.a1

[32] MOLLOY, M. and REED, B. (1998). The size of the giant component of a random graph with a given degree sequence. *Combin*. *Probab*. *Comput*. **7** 295–305. MR1664335 https://doi.org/10.1017/S0963548398003526

[33] MONTANARI, A., MOSSEL, E. and SLY, A. (2012). The weak limit of Ising models on locally tree-like graphs. *Probab*. *Theory Related Fields* **152** 31–51. MR2875752 https://doi.org/10.1007/s00440-010-0315-6

[34] NADTOCHIY, S. and SHKOLNIKOV, M. (2020). Mean field systems on networks, with singular interaction through hitting times. *Ann*. *Probab*. **48** 1520–1556. MR4112723 https://doi.org/10.1214/19-AOP1403

[35] OLIVEIRA, R. I. and REIS, G. H. (2019). Interacting diffusions on random graphs with diverging average degrees: Hydrodynamics and large deviations. *J*. *Stat*. *Phys*. **176** 1057–1087. MR3999471 https://doi.org/10.1007/s10955-019-02332-1

[36] OLIVEIRA, R. I., REIS, G. H. and STOLERMAN, L. M. (2020). Interacting diffusions on sparse graphs: Hydrodynamics from local weak limits. *Electron*. *J*. *Probab*. **25** Paper No. 110, 35. MR4150522 https://doi.org/10.1214/20-ejp505

[37] REDIG, F., RŒLLY, S. and RUSZEL, W. (2010). Short-time Gibbsianness for infinite-dimensional diffusions with space–time interaction. *J*. *Stat*. *Phys*. **138** 1124–1144. MR2601426 https://doi.org/10.1007/s10955-010-9926-7

[38] SPILIOPOULOS, K. and YANG, J. (2019). Network effects in default clustering for large systems. *Appl*. *Math*. *Finance* **26** 523–582. MR4070271 https://doi.org/10.1080/1350486X.2020.1724804

[39] SZNITMAN, A.-S. (1991). Topics in propagation of chaos. In *École D'Été de Probabilités de Saint-Flour XIX*—1989. *Lecture Notes in Math*. **1464** 165–251. Springer, Berlin. MR1108185 https://doi.org/10.1007/BFb0085169

[40] VAN DER HOFSTAD, R. (2017). *Random Graphs and Complex Networks*. *Vol*. 1. *Cambridge Series in Statistical and Probabilistic Mathematics* **43**. Cambridge Univ. Press, Cambridge. MR3617364 https://doi.org/10.1017/9781316779422

[41] VAN DER HOFSTAD, R. (2020). Random graphs and complex networks, Volume 2. Available at https://www.win.tue.nl/~rhofstad/NotesRGCNII_11_07_2020.pdf.

The Annals of Applied Probability
2023, Vol. 33, No. 2, 889–930
https://doi.org/10.1214/22-AAP1831

STOCHASTIC DIFFERENTIAL GAMES WITH RANDOM COEFFICIENTS AND STOCHASTIC HAMILTON–JACOBI–BELLMAN–ISAACS EQUATIONS

BY JINNIAO QIU[1,a] AND JING ZHANG[2,b]

[1]*Department of Mathematics and Statistics, University of Calgary,* [a]*jinniao.qiu@ucalgary.ca*
[2]*School of Mathematical Sciences, Fudan University,* [b]*zhang_jing@fudan.edu.cn*

In this paper, we study a class of zero-sum two-player stochastic differential games with the controlled stochastic differential equations and the payoff/cost functionals of recursive type. As opposed to the pioneering work by Fleming and Souganidis [*Indiana Univ. Math. J.* **38** (1989) 293–314] and the seminal work by Buckdahn and Li [*SIAM J. Control Optim.* **47** (2008) 444–475], the involved coefficients may be random, going beyond the Markovian framework and leading to the random upper and lower value functions. We first prove the dynamic programming principle for the game, and then under the standard Lipschitz continuity assumptions on the coefficients, the upper and lower value functions are shown to be the viscosity solutions of the upper and the lower fully nonlinear stochastic Hamilton–Jacobi–Bellman–Isaacs (HJBI) equations, respectively. A stability property of viscosity solutions is also proved. Under certain additional regularity assumptions on the diffusion coefficient, the uniqueness of the viscosity solution is addressed as well.

REFERENCES

[1] BAYRAKTAR, E. and QIU, J. (2019). Controlled reflected SDEs and Neumann problem for backward SPDEs. *Ann. Appl. Probab.* **29** 2819–2848. MR4019876 https://doi.org/10.1214/19-AAP1465

[2] BRIAND, PH., DELYON, B., HU, Y., PARDOUX, E. and STOICA, L. (2003). L^p solutions of backward stochastic differential equations. *Stochastic Process. Appl.* **108** 109–129. MR2008603 https://doi.org/10.1016/S0304-4149(03)00089-9

[3] BUCKDAHN, R., CARDALIAGUET, P. and RAINER, C. (2004). Nash equilibrium payoffs for nonzero-sum stochastic differential games. *SIAM J. Control Optim.* **43** 624–642. MR2086176 https://doi.org/10.1137/S0363012902411556

[4] BUCKDAHN, R. and LI, J. (2008). Stochastic differential games and viscosity solutions of Hamilton–Jacobi–Bellman–Isaacs equations. *SIAM J. Control Optim.* **47** 444–475. MR2373477 https://doi.org/10.1137/060671954

[5] CARDALIAGUET, P., DELARUE, F., LASRY, J.-M. and LIONS, P.-L. (2019). *The Master Equation and the Convergence Problem in Mean Field Games*. *Annals of Mathematics Studies* **201**. Princeton Univ. Press, Princeton, NJ. MR3967062 https://doi.org/10.2307/j.ctvckq7qf

[6] CARMONA, R., DELARUE, F. and LACKER, D. (2016). Mean field games with common noise. *Ann. Probab.* **44** 3740–3803. MR3572323 https://doi.org/10.1214/15-AOP1060

[7] CONT, R. and FOURNIÉ, D.-A. (2013). Functional Itô calculus and stochastic integral representation of martingales. *Ann. Probab.* **41** 109–133. MR3059194 https://doi.org/10.1214/11-AOP721

[8] DU, K., QIU, J. and TANG, S. (2012). L^p theory for super-parabolic backward stochastic partial differential equations in the whole space. *Appl. Math. Optim.* **65** 175–219. MR2891221 https://doi.org/10.1007/s00245-011-9154-9

[9] EKREN, I., KELLER, C., TOUZI, N. and ZHANG, J. (2014). On viscosity solutions of path dependent PDEs. *Ann. Probab.* **42** 204–236. MR3161485 https://doi.org/10.1214/12-AOP788

[10] EKREN, I., TOUZI, N. and ZHANG, J. (2016). Viscosity solutions of fully nonlinear parabolic path dependent PDEs: Part I. *Ann. Probab.* **44** 1212–1253. MR3474470 https://doi.org/10.1214/14-AOP999

MSC2020 subject classifications. 49L12, 49L20, 49L25, 93E20, 35D40, 60H15.
Key words and phrases. Stochastic Hamilton–Jacobi–Bellman–Isaacs equation, stochastic differential game, backward stochastic partial differential equation, viscosity solution.

[11] El Karoui, N., Kapoudjian, C., Pardoux, E., Peng, S. and Quenez, M. C. (1997). Reflected solutions of backward SDE's, and related obstacle problems for PDE's. *Ann. Probab.* **25** 702–737. MR1434123 https://doi.org/10.1214/aop/1024404416

[12] El Karoui, N., Peng, S. and Quenez, M. C. (1997). Backward stochastic differential equations in finance. *Math. Finance* **7** 1–71. MR1434407 https://doi.org/10.1111/1467-9965.00022

[13] El Karoui, N., Peng, S. and Quenez, M. C. (2001). A dynamic maximum principle for the optimization of recursive utilities under constraints. *Ann. Appl. Probab.* **11** 664–693. MR1865020 https://doi.org/10.1214/aoap/1015345345

[14] Elliott, R. (1976). The existence of value in stochastic differential games. *SIAM J. Control Optim.* **14** 85–94. MR0462730 https://doi.org/10.1137/0314006

[15] Elliott, R. J. and Davis, M. H. A. (1981). Optimal play in a stochastic differential game. *SIAM J. Control Optim.* **19** 543–554. MR0618244 https://doi.org/10.1137/0319033

[16] Fleming, W. H. and Souganidis, P. E. (1989). On the existence of value functions of two-player, zero-sum stochastic differential games. *Indiana Univ. Math. J.* **38** 293–314. MR0997385 https://doi.org/10.1512/iumj.1989.38.38015

[17] Friedman, A. (1972). Stochastic differential games. *J. Differential Equations* **11** 79–108. MR0292528 https://doi.org/10.1016/0022-0396(72)90082-4

[18] Hamadène, S. (1998). Backward–forward SDE's and stochastic differential games. *Stochastic Process. Appl.* **77** 1–15. MR1644598 https://doi.org/10.1016/S0304-4149(98)00038-6

[19] Hamadene, S., Lepeltier, J.-P. and Peng, S. (1997). BSDEs with continuous coefficients and stochastic differential games. In *Backward Stochastic Differential Equations* (*Paris*, 1995–1996). *Pitman Res. Notes Math. Ser.* **364** 115–128. Longman, Harlow. MR1752678

[20] Horst, U., Qiu, J. and Zhang, Q. (2016). A constrained control problem with degenerate coefficients and degenerate backward SPDEs with singular terminal condition. *SIAM J. Control Optim.* **54** 946–963. MR3482391 https://doi.org/10.1137/15M1031060

[21] Hu, Y., Ma, J. and Yong, J. (2002). On semi-linear degenerate backward stochastic partial differential equations. *Probab. Theory Related Fields* **123** 381–411. MR1918539 https://doi.org/10.1007/s004400100193

[22] Krylov, N. V. (1982). Boundedly inhomogeneous elliptic and parabolic equations. *Izv. Ross. Akad. Nauk Ser. Mat.* **46** 487–523, 670. MR0661144

[23] Leão, D., Ohashi, A. and Simas, A. B. (2018). A weak version of path-dependent functional Itô calculus. *Ann. Probab.* **46** 3399–3441. MR3857859 https://doi.org/10.1214/17-AOP1250

[24] Ma, J., Yin, H. and Zhang, J. (2012). On non-Markovian forward-backward SDEs and backward stochastic PDEs. *Stochastic Process. Appl.* **122** 3980–4004. MR2971722 https://doi.org/10.1016/j.spa.2012.08.002

[25] Pardoux, E. (1980). Stochastic partial differential equations and filtering of diffusion processes. *Stochastics* **3** 127–167. MR0553909 https://doi.org/10.1080/17442507908833142

[26] Pardoux, É. and Peng, S. G. (1990). Adapted solution of a backward stochastic differential equation. *Systems Control Lett.* **14** 55–61. MR1037747 https://doi.org/10.1016/0167-6911(90)90082-6

[27] Peng, S. (1997). Backward stochastic differential equations: Stochastic optimization theory and viscosity solution for HJB equations. In *Topics on Stochastic Analysis* (J.-A. Yan, S. Peng, S. Fang and L. Wu, eds.) 85–138. Science Press, Beijing (in Chinese).

[28] Peng, S. (2010). Backward stochastic differential equation, nonlinear expectation and their applications. In *Proceedings of the International Congress of Mathematicians. Volume I* 393–432. Hindustan Book Agency, New Delhi. MR2827899

[29] Peng, S. G. (1992). Stochastic Hamilton–Jacobi–Bellman equations. *SIAM J. Control Optim.* **30** 284–304. MR1149069 https://doi.org/10.1137/0330018

[30] Pham, T. and Zhang, J. (2014). Two person zero-sum game in weak formulation and path dependent Bellman–Isaacs equation. *SIAM J. Control Optim.* **52** 2090–2121. MR3227460 https://doi.org/10.1137/120894907

[31] Possamaï, D., Touzi, N. and Zhang, J. (2020). Zero-sum path-dependent stochastic differential games in weak formulation. *Ann. Appl. Probab.* **30** 1415–1457. MR4133377 https://doi.org/10.1214/19-AAP1533

[32] Qiu, J. (2017). Weak solution for a class of fully nonlinear stochastic Hamilton–Jacobi–Bellman equations. *Stochastic Process. Appl.* **127** 1926–1959. MR3646436 https://doi.org/10.1016/j.spa.2016.09.010

[33] Qiu, J. (2018). Viscosity solutions of stochastic Hamilton–Jacobi–Bellman equations. *SIAM J. Control Optim.* **56** 3708–3730. MR3864678 https://doi.org/10.1137/17M1148232

[34] Qiu, J. and Tang, S. (2012). Maximum principle for quasi-linear backward stochastic partial differential equations. *J. Funct. Anal.* **262** 2436–2480. MR2876411 https://doi.org/10.1016/j.jfa.2011.12.002

[35] QIU, J. and WEI, W. (2019). Uniqueness of viscosity solutions of stochastic Hamilton–Jacobi equations. *Acta Math. Sci. Ser. B Engl. Ed.* **39** 857–873. MR4066509 https://doi.org/10.1007/s10473-019-0314-3

[36] REN, Z., TOUZI, N. and ZHANG, J. (2014). An overview of viscosity solutions of path-dependent PDEs. In *Stochastic Analysis and Applications* 2014. *Springer Proc. Math. Stat.* **100** 397–453. Springer, Cham. MR3332721 https://doi.org/10.1007/978-3-319-11292-3_15

[37] TANG, S. and WEI, W. (2016). On the Cauchy problem for backward stochastic partial differential equations in Hölder spaces. *Ann. Probab.* **44** 360–398. MR3456341 https://doi.org/10.1214/14-AOP976

[38] WAGNER, D. H. (1977). Survey of measurable selection theorems. *SIAM J. Control Optim.* **15** 859–903. MR0486391 https://doi.org/10.1137/0315056

[39] WANG, L. (1992). On the regularity theory of fully nonlinear parabolic equations. I. *Comm. Pure Appl. Math.* **45** 27–76. MR1135923 https://doi.org/10.1002/cpa.3160450103

[40] YONG, J. and ZHOU, X. Y. (1999). *Stochastic Controls: Hamiltonian Systems and HJB Equations. Applications of Mathematics* (*New York*) **43**. Springer, New York. MR1696772 https://doi.org/10.1007/978-1-4612-1466-3

[41] ZHANG, F. (2017). The existence of game value for path-dependent stochastic differential game. *SIAM J. Control Optim.* **55** 2519–2542. MR3686785 https://doi.org/10.1137/15M1015042

The Annals of Applied Probability
2023, Vol. 33, No. 2, 931–953
https://doi.org/10.1214/22-AAP1832

PROBABILISTIC AND ANALYTICAL PROPERTIES OF THE LAST PASSAGE PERCOLATION CONSTANT IN A WEIGHTED RANDOM DIRECTED GRAPH

BY SERGEY FOSS[1,a], TAKIS KONSTANTOPOULOS[2,b] AND ARTEM PYATKIN[3,c]

[1]*School of Mathematical Sciences, Heriot-Watt University,* [a]*s.foss@hw.ac.uk*
[2]*Department of Mathematical Sciences, The University of Liverpool,* [b]*takiskonst@gmail.com*
[3]*Laboratory of Discrete Optimization in Operations Research, Sobolev Institute of Mathematics,* [c]*artempyatkin@gmail.com*

To each edge (i, j), $i < j$, of the complete directed graph on the integers we assign unit weight with probability p or weight x with probability $1 - p$, independently from edge to edge, and give to each path weight equal to the sum of its edge weights. If $W^x_{0,n}$ is the maximum weight of all paths from 0 to n then $W^x_{0,n}/n \to C_p(x)$, as $n \to \infty$, almost surely, where $C_p(x)$ is positive and deterministic. We study $C_p(x)$ as a function of x, for fixed $0 < p < 1$, and show that it is a strictly increasing convex function that is not differentiable if and only if x is a nonpositive rational or a positive integer except 1 or the reciprocal of it. We allow x to be any real number, even negative, or, possibly, $-\infty$. The case $x = -\infty$ corresponds to the well-studied directed version of the Erdős–Rényi random graph (known as Barak–Erdős graph) for which $C_p(-\infty) = \lim_{x\to-\infty} C_p(x)$ has been studied as a function of p in a number of papers.

REFERENCES

[1] ALON, N., BOLLOBÁS, B., BRIGHTWELL, G. and JANSON, S. (1994). Linear extensions of a random partial order. *Ann. Appl. Probab.* **4** 108–123. MR1258175

[2] BARAK, A. B. and ERDŐS, P. (1984). On the maximal number of strongly independent vertices in a random acyclic directed graph. *SIAM J. Algebr. Discrete Methods* **5** 508–514. MR0763980 https://doi.org/10.1137/0605049

[3] BODINEAU, T. and MARTIN, J. (2005). A universality property for last-passage percolation paths close to the axis. *Electron. Commun. Probab.* **10** 105–112. MR2150699 https://doi.org/10.1214/ECP.v10-1139

[4] BOLLOBÁS, B. and BRIGHTWELL, G. (1997). The structure of random graph orders. *SIAM J. Discrete Math.* **10** 318–335. MR1445040 https://doi.org/10.1137/S0895480194281215

[5] DENISOV, D., FOSS, S. and KONSTANTOPOULOS, T. (2012). Limit theorems for a random directed slab graph. *Ann. Appl. Probab.* **22** 702–733. MR2953567 https://doi.org/10.1214/11-AAP783

[6] FOSS, S. and KONSTANTOPOULOS, T. (2003). Extended renovation theory and limit theorems for stochastic ordered graphs. *Markov Process. Related Fields* **9** 413–468. MR2028221

[7] FOSS, S. and KONSTANTOPOULOS, T. (2018). Limiting properties of random graph models with vertex and edge weights. *J. Stat. Phys.* **173** 626–643. MR3876901 https://doi.org/10.1007/s10955-018-2080-3

[8] FOSS, S., MARTIN, J. B. and SCHMIDT, P. (2014). Long-range last-passage percolation on the line. *Ann. Appl. Probab.* **24** 198–234. MR3161646 https://doi.org/10.1214/13-AAP920

[9] GELENBE, E., NELSON, R., PHILIPS, T. and TANTAWI, A. (1986). An approximation of the processing time for a random graph model of parallel computation. In *Proceedings of* 1986 *ACM Fall Joint Computer Conference*, *ACM*86 691–697. IEEE Computer Society Press, Los Alamitos, CA.

[10] ISOPI, M. and NEWMAN, C. M. (1994). Speed of parallel processing for random task graphs. *Comm. Pure Appl. Math.* **47** 361–376. MR1266246 https://doi.org/10.1002/cpa.3160470307

[11] ITOH, Y. and KRAPIVSKY, P. L. (2012). Continuum cascade model of directed random graphs: Traveling wave analysis. *J. Phys. A* **45** 455002. MR2990512 https://doi.org/10.1088/1751-8113/45/45/455002

[12] KALLENBERG, O. (2002). *Foundations of Modern Probability*, 2nd ed. *Probability and Its Applications* (*New York*). Springer, New York. MR1876169 https://doi.org/10.1007/978-1-4757-4015-8

MSC2020 subject classifications. 60K35, 05C80, 60K05.
Key words and phrases. Random graph, maximal path, last-passage percolation, skeleton point, critical point, regenerative structure.

[13] KONSTANTOPOULOS, T. and TRINAJSTIĆ, K. (2013). Convergence to the Tracy–Widom distribution for longest paths in a directed random graph. *ALEA Lat*. *Am*. *J*. *Probab*. *Math*. *Stat*. **10** 711–730. MR3108812

[14] KRISHNAN, A., RASSOUL-AGHA, F. and SEPPÄLÄINEN, T. (2021). Geodesic length and shifted weights in first-passage percolation. Available at https://arxiv.org/abs/2101.12324.

[15] LOTHAIRE, M. (2002). *Algebraic Combinatorics on Words*. Cambridge Univ. Press, Cambridge.

[16] MALLEIN, B. and RAMASSAMY, S. (2016). Barak–Erdős graphs and the infinite-bin model. *Ann*. *Inst*. *Henri Poincaré Probab*. *Stat*.

[17] MALLEIN, B. and RAMASSAMY, S. (2019). Two-sided infinite-bin models and analyticity for Barak–Erdős graphs. *Bernoulli* **25** 3479–3495. MR4010962 https://doi.org/10.3150/18-bej1097

[18] NEWMAN, C. M. (1992). Chain lengths in certain random directed graphs. *Random Structures Algorithms* **3** 243–253. MR1164838 https://doi.org/10.1002/rsa.3240030304

[19] NEWMAN, C. M. and COHEN, J. E. (1986). A stochastic theory of community food webs: IV; theory of food chains in large webs. *Proc*. *R*. *Soc*. *Lond*. *Ser*. *B* **228** 355–377.

[20] STEELE, J. M. and ZHANG, Y. (2003). Nondifferentiability of the time constants of first-passage percolation. *Ann*. *Probab*. **31** 1028–1051. MR1964957 https://doi.org/10.1214/aop/1048516544

The Annals of Applied Probability
2023, Vol. 33, No. 2, 954–998
https://doi.org/10.1214/22-AAP1833

GLOBAL-IN-TIME MEAN-FIELD CONVERGENCE FOR SINGULAR RIESZ-TYPE DIFFUSIVE FLOWS

BY MATTHEW ROSENZWEIG[1,a] AND SYLVIA SERFATY[2,b]

[1]*Department of Mathematics, Massachusetts Institute of Technology,* [a]*mrosenzw@mit.edu*
[2]*Department of Mathematics, Courant Institute of Mathematical Sciences, New York University,* [b]*serfaty@cims.nyu.edu*

We consider the mean-field limit of systems of particles with singular interactions of the type $-\log|x|$ or $|x|^{-s}$, with $0 < s < d-2$, and with an additive noise in dimensions $d \geq 3$. We use a modulated-energy approach to prove a quantitative convergence rate to the solution of the corresponding limiting PDE. When $s > 0$, the convergence is global in time, and it is the first such result valid for both conservative and gradient flows in a singular setting on $\mathbb{R}^d$. The proof relies on an adaptation of an argument of Carlen–Loss (*Duke Math. J.* **81** (1995) 135–157) to show a decay rate of the solution to the limiting equation, and on an improvement of the modulated-energy method developed in (*SIAM J. Math. Anal.* **48** (2016) 2269–2300; *Duke Math. J.* **169** (2020) 2887–2935; Nguyen, Rosenzweig and Serfaty (2021)), making it so that all prefactors in the time derivative of the modulated energy are controlled by a decaying bound on the limiting solution.

REFERENCES

[1] BERMAN, R. J. and ÖNNHEIM, M. (2019). Propagation of chaos for a class of first order models with singular mean field interactions. *SIAM J. Math. Anal.* **51** 159–196. MR3904425 https://doi.org/10.1137/18M1196662

[2] BOLLEY, F., CAÑIZO, J. A. and CARRILLO, J. A. (2011). Stochastic mean-field limit: Non-Lipschitz forces and swarming. *Math. Models Methods Appl. Sci.* **21** 2179–2210. MR2860672 https://doi.org/10.1142/S0218202511005702

[3] BOLLEY, F., GUILLIN, A. and MALRIEU, F. (2010). Trend to equilibrium and particle approximation for a weakly selfconsistent Vlasov–Fokker–Planck equation. *ESAIM Math. Model. Numer. Anal.* **44** 867–884. MR2731396 https://doi.org/10.1051/m2an/2010045

[4] BRESCH, D., JABIN, P. and WANG, Z. (2019). Modulated free energy and mean field limit. In *Séminaire Laurent Schwartz–EDP et Applications* 1–22.

[5] BRESCH, D., JABIN, P.-E. and WANG, Z. (2019). On mean-field limits and quantitative estimates with a large class of singular kernels: Application to the Patlak–Keller–Segel model. *C. R. Math. Acad. Sci. Paris* **357** 708–720. MR4018082 https://doi.org/10.1016/j.crma.2019.09.007

[6] BRESCH, D., JABIN, P.-E. and WANG, Z. (2020). Mean-field limit and quantitative estimates with singular attractive kernels. Preprint. Available at arXiv:2011.08022.

[7] CARLEN, E. A. (1991). Some integral identities and inequalities for entire functions and their application to the coherent state transform. *J. Funct. Anal.* **97** 231–249. MR1105661 https://doi.org/10.1016/0022-1236(91)90022-W

[8] CARLEN, E. A. and LOSS, M. (1990). Extremals of functionals with competing symmetries. *J. Funct. Anal.* **88** 437–456. MR1038450 https://doi.org/10.1016/0022-1236(90)90114-Z

[9] CARLEN, E. A. and LOSS, M. (1995). Optimal smoothing and decay estimates for viscously damped conservation laws, with applications to the 2-D Navier–Stokes equation *Duke Math. J.* **81** 135–157. MR1381974 https://doi.org/10.1215/S0012-7094-95-08110-1

[10] CARRILLO, J. A., CHOI, Y.-P. and HAURAY, M. (2014). The derivation of swarming models: Mean-field limit and Wasserstein distances. In *Collective Dynamics from Bacteria to Crowds*. *CISM Courses and Lect.* **553** 1–46. Springer, Vienna. MR3331178 https://doi.org/10.1007/978-3-7091-1785-9_1

[11] CARRILLO, J. A., FERREIRA, L. C. F. and PRECIOSO, J. C. (2012). A mass-transportation approach to a one dimensional fluid mechanics model with nonlocal velocity. *Adv. Math.* **231** 306–327. MR2935390 https://doi.org/10.1016/j.aim.2012.03.036

MSC2020 subject classifications. 35Q70, 35Q35, 60H30.
Key words and phrases. Mean-field limit, McKean–Vlasov, uniform-in-time convergence, Riesz potentials.

[12] CATTIAUX, P., GUILLIN, A. and MALRIEU, F. (2008). Probabilistic approach for granular media equations in the non-uniformly convex case. *Probab. Theory Related Fields* **140** 19–40. MR2357669 https://doi.org/10.1007/s00440-007-0056-3

[13] CHIZAT, L. and BACH, F. (2018). On the global convergence of gradient descent for over-parameterized models using optimal transport. In *Proceedings of the* 32*nd International Conference on Neural Information Processing Systems*. *NIPS*18 3040–3050. Curran Associates Inc., Red Hook, NY, USA.

[14] CHORIN, A. J. (1973). Numerical study of slightly viscous flow. *J. Fluid Mech.* **57** 785–796. MR0395483 https://doi.org/10.1017/S0022112073002016

[15] DAVIES, E. B. (1987). Explicit constants for Gaussian upper bounds on heat kernels. *Amer. J. Math.* **109** 319–333. MR0882426 https://doi.org/10.2307/2374577

[16] DELARUE, F. and TSE, A. (2021). Uniform in time weak propagation of chaos on the torus. Preprint. Available at arXiv:2104.14973.

[17] DUERINCKX, M. (2016). Mean-field limits for some Riesz interaction gradient flows. *SIAM J. Math. Anal.* **48** 2269–2300. MR3514713 https://doi.org/10.1137/15M1042620

[18] DURMUS, A., EBERLE, A., GUILLIN, A. and ZIMMER, R. (2020). An elementary approach to uniform in time propagation of chaos. *Proc. Amer. Math. Soc.* **148** 5387–5398. MR4163850 https://doi.org/10.1090/proc/14612

[19] FOURNIER, N., HAURAY, M. and MISCHLER, S. (2014). Propagation of chaos for the 2D viscous vortex model. *J. Eur. Math. Soc.* (*JEMS*) **16** 1423–1466. MR3254330 https://doi.org/10.4171/JEMS/465

[20] FOURNIER, N. and JOURDAIN, B. (2017). Stochastic particle approximation of the Keller–Segel equation and two-dimensional generalization of Bessel processes. *Ann. Appl. Probab.* **27** 2807–2861. MR3719947 https://doi.org/10.1214/16-AAP1267

[21] GALLAY, T. and WAYNE, C. E. (2005). Global stability of vortex solutions of the two-dimensional Navier–Stokes equation. *Comm. Math. Phys.* **255** 97–129. MR2123378 https://doi.org/10.1007/s00220-004-1254-9

[22] GODINHO, D. and QUIÑINAO, C. (2015). Propagation of chaos for a subcritical Keller-Segel model. *Ann. Inst. Henri Poincaré Probab. Stat.* **51** 965–992. MR3365970 https://doi.org/10.1214/14-AIHP606

[23] GOLSE, F. (2016). On the dynamics of large particle systems in the mean field limit. In *Macroscopic and Large Scale Phenomena*: *Coarse Graining*, *Mean Field Limits and Ergodicity*. *Lect. Notes Appl. Math. Mech.* **3** 1–144. Springer, Cham. MR3468297 https://doi.org/10.1007/978-3-319-26883-5_1

[24] GRAFAKOS, L. (2014). *Classical Fourier Analysis*, 3rd ed. *Graduate Texts in Mathematics* **249**. Springer, New York. MR3243734 https://doi.org/10.1007/978-1-4939-1194-3

[25] GRAFAKOS, L. (2014). *Modern Fourier Analysis*, 3rd ed. *Graduate Texts in Mathematics* **250**. Springer, New York. MR3243741 https://doi.org/10.1007/978-1-4939-1230-8

[26] GROSS, L. (1975). Logarithmic Sobolev inequalities. *Amer. J. Math.* **97** 1061–1083. MR0420249 https://doi.org/10.2307/2373688

[27] GUILLIN, A., BRIS, P. L. and MONMARCHÉ, P. (2021). Uniform in time propagation of chaos for the 2D vortex model and other singular stochastic systems. Preprint. Available at arXiv:2108.08675.

[28] HAURAY, M. (2009). Wasserstein distances for vortices approximation of Euler-type equations. *Math. Models Methods Appl. Sci.* **19** 1357–1384. MR2555474 https://doi.org/10.1142/S0218202509003814

[29] HAURAY, M. and MISCHLER, S. (2014). On Kac's chaos and related problems. *J. Funct. Anal.* **266** 6055–6157. MR3188710 https://doi.org/10.1016/j.jfa.2014.02.030

[30] HEGSELMANN, R. and KRAUSE, U. (2002). Opinion dynamics and bounded confidence models, analysis and simulation. *Journal of Artificial Societies and Social Simulation* **5** 1–2.

[31] HOLDING, T. (2016). Propagation of chaos for Hölder continuous interaction kernels via Glivenko–Cantelli. Preprint. Available at arXiv:1608.02877.

[32] JABIN, P.-E. (2014). A review of the mean field limits for Vlasov equations. *Kinet. Relat. Models* **7** 661–711. MR3317577 https://doi.org/10.3934/krm.2014.7.661

[33] JABIN, P.-E. and WANG, Z. (2016). Mean field limit and propagation of chaos for Vlasov systems with bounded forces. *J. Funct. Anal.* **271** 3588–3627. MR3558251 https://doi.org/10.1016/j.jfa.2016.09.014

[34] JABIN, P.-E. and WANG, Z. (2017). Mean field limit for stochastic particle systems. In *Active Particles*. *Vol.* 1. *Advances in Theory*, *Models*, *and Applications*. *Model. Simul. Sci. Eng. Technol.* 379–402. Birkhäuser/Springer, Cham. MR3644596

[35] JABIN, P.-E. and WANG, Z. (2018). Quantitative estimates of propagation of chaos for stochastic systems with $W^{-1,\infty}$ kernels. *Invent. Math.* **214** 523–591. MR3858403 https://doi.org/10.1007/s00222-018-0808-y

[36] KARATZAS, I. and SHREVE, S. E. (1991). *Brownian Motion and Stochastic Calculus*, 2nd ed. *Graduate Texts in Mathematics* **113**. Springer, New York. MR1121940 https://doi.org/10.1007/978-1-4612-0949-2

[37] KRAUSE, U. (2000). A discrete nonlinear and non-autonomous model of consensus formation. In *Communications in Difference Equations* (*Poznan*, 1998) 227–236. Gordon & Breach, Amsterdam. MR1792007
[38] LACKER, D. (2021). Hierarchies, entropy, and quantitative propagation of chaos for mean field diffusions. Preprint. Available at arXiv:2105.02983.
[39] LI, L., LIU, J.-G. and YU, P. (2019). On the mean field limit for Brownian particles with Coulomb interaction in 3D. *J. Math. Phys.* **60** 111501, 34. MR4026330 https://doi.org/10.1063/1.5114854
[40] LIU, J.-G. and YANG, R. (2016). Propagation of chaos for large Brownian particle system with Coulomb interaction. *Res. Math. Sci.* **3** Paper No. 40, 33. MR3572548 https://doi.org/10.1186/s40687-016-0086-5
[41] MAEKAWA, Y. (2008). A lower bound for fundamental solutions of the heat convection equations. *Arch. Ration. Mech. Anal.* **189** 45–58. MR2403599 https://doi.org/10.1007/s00205-007-0088-y
[42] MALRIEU, F. (2003). Convergence to equilibrium for granular media equations and their Euler schemes. *Ann. Appl. Probab.* **13** 540–560. MR1970276 https://doi.org/10.1214/aoap/1050689593
[43] MARCHIORO, C. and PULVIRENTI, M. (1994). *Mathematical Theory of Incompressible Nonviscous Fluids. Applied Mathematical Sciences* **96**. Springer, New York. MR1245492 https://doi.org/10.1007/978-1-4612-4284-0
[44] MCKEAN, H. P. JR. (1967). Propagation of chaos for a class of non-linear parabolic equations. In *Stochastic Differential Equations* (*Lecture Series in Differential Equations*, *Session* 7, *Catholic Univ.*, 1967) 41–57. Air Force Office Sci. Res., Arlington, VA. MR0233437
[45] MEI, S., MONTANARI, A. and NGUYEN, P.-M. (2018). A mean field view of the landscape of two-layer neural networks. *Proc. Natl. Acad. Sci. USA* **115** E7665–E7671. MR3845070 https://doi.org/10.1073/pnas.1806579115
[46] MÉLÉARD, S. (1996). Asymptotic behaviour of some interacting particle systems; McKean–Vlasov and Boltzmann models. In *Probabilistic Models for Nonlinear Partial Differential Equations* (*Montecatini Terme*, 1995). *Lecture Notes in Math.* **1627** 42–95. Springer, Berlin. MR1431299 https://doi.org/10.1007/BFb0093177
[47] MISCHLER, S., MOUHOT, C. and WENNBERG, B. (2015). A new approach to quantitative propagation of chaos for drift, diffusion and jump processes. *Probab. Theory Related Fields* **161** 1–59. MR3304746 https://doi.org/10.1007/s00440-013-0542-8
[48] MOTSCH, S. and TADMOR, E. (2011). A new model for self-organized dynamics and its flocking behavior. *J. Stat. Phys.* **144** 923–947. MR2836613 https://doi.org/10.1007/s10955-011-0285-9
[49] NASH, J. (1958). Continuity of solutions of parabolic and elliptic equations. *Amer. J. Math.* **80** 931–954. MR0100158 https://doi.org/10.2307/2372841
[50] NGUYEN, Q. H., ROSENZWEIG, M. and SERFATY, S. (2021). Mean-field limits of Riesz-type singular flows with possible multiplicative transport noise. Preprint. Available at arXiv:2107.02592.
[51] ONSAGER, L. (1949). Statistical hydrodynamics. *Nuovo Cimento* (9) **6** 279–287. MR0036116
[52] OSADA, H. (1986). Propagation of chaos for the two-dimensional Navier–Stokes equation. *Proc. Japan Acad. Ser. A Math. Sci.* **62** 8–11. MR0839794
[53] OSADA, H. (1987). Propagation of chaos for the two-dimensional Navier–Stokes equation. In *Probabilistic Methods in Mathematical Physics* (*Katata/Kyoto*, 1985) 303–334. Academic Press, Boston, MA. MR0933829
[54] OSADA, H. (1987). Limit points of empirical distributions of vortices with small viscosity. In *Hydrodynamic Behavior and Interacting Particle Systems* (*Minneapolis*, *Minn.*, 1986). *IMA Vol. Math. Appl.* **9** 117–126. Springer, New York. MR0914990 https://doi.org/10.1007/978-1-4684-6347-7_10
[55] OSADA, H. (1987). Diffusion processes with generators of generalized divergence form. *J. Math. Kyoto Univ.* **27** 597–619. MR0916761 https://doi.org/10.1215/kjm/1250520601
[56] OTHMER, H. G. and STEVENS, A. (1997). Aggregation, blowup, and collapse: The ABCs of taxis in reinforced random walks. *SIAM J. Appl. Math.* **57** 1044–1081. MR1462051 https://doi.org/10.1137/S0036139995288976
[57] PERTHAME, B. (2007). *Transport Equations in Biology*. *Frontiers in Mathematics*. Birkhäuser, Basel. MR2270822
[58] ROSENZWEIG, M. (2020). The mean-field limit of stochastic point vortex systems with multiplicative noise. Preprint. Available at arXiv:2011.12180.
[59] ROSENZWEIG, M. (2022). Mean-field convergence of point vortices to the incompressible Euler equation with vorticity in L^∞. *Arch. Ration. Mech. Anal.* **243** 1361–1431. MR4381143 https://doi.org/10.1007/s00205-021-01735-3
[60] ROTSKOFF, G. M. and VANDEN-EIJNDEN, E. (2018). Trainability and accuracy of neural networks: An interacting particle system approach. Preprint. Available at arXiv:1805.00915.
[61] ROUGERIE, N. and SERFATY, S. (2016). Higher-dimensional Coulomb gases and renormalized energy functionals. *Comm. Pure Appl. Math.* **69** 519–605. MR3455593 https://doi.org/10.1002/cpa.21570

[62] SERFATY, S. (2017). Mean field limits of the Gross–Pitaevskii and parabolic Ginzburg–Landau equations. *J. Amer. Math. Soc.* **30** 713–768. MR3630086 https://doi.org/10.1090/jams/872

[63] SERFATY, S. (2020). Mean field limit for Coulomb-type flows. *Duke Math. J.* **169** 2887–2935. MR4158670 https://doi.org/10.1215/00127094-2020-0019

[64] SERFATY, S. and VÁZQUEZ, J. L. (2014). A mean field equation as limit of nonlinear diffusions with fractional Laplacian operators. *Calc. Var. Partial Differential Equations* **49** 1091–1120. MR3168624 https://doi.org/10.1007/s00526-013-0613-9

[65] STEIN, E. M. (1970). *Singular Integrals and Differentiability Properties of Functions*. *Princeton Mathematical Series*, *No*. 30. Princeton Univ. Press, Princeton, N.J. MR0290095

[66] STEIN, E. M. (1993). *Harmonic Analysis*: *Real-Variable Methods*, *Orthogonality*, *and Oscillatory Integrals*. *Princeton Mathematical Series* **43**. Princeton Univ. Press, Princeton, NJ. MR1232192

[67] SZNITMAN, A.-S. (1991). Topics in propagation of chaos. In *École D'Été de Probabilités de Saint-Flour XIX*—1989. *Lecture Notes in Math*. **1464** 165–251. Springer, Berlin. MR1108185 https://doi.org/10.1007/BFb0085169

[68] TOPAZ, C. M., BERTOZZI, A. L. and LEWIS, M. A. (2006). A nonlocal continuum model for biological aggregation. *Bull. Math. Biol.* **68** 1601–1623. MR2257718 https://doi.org/10.1007/s11538-006-9088-6

[69] WANG, Z., ZHAO, X. and ZHU, R. (2021). Gaussian fluctuations for interacting particle systems with singular kernels. Preprint. Available at arXiv:2105.13201.

[70] XIA, H., WANG, H. and XUAN, Z. (2011). Opinion dynamics: A multidisciplinary review and perspective on future research. *Int. J. Knowl. Syst. Sci.* **2** 72–91. https://doi.org/10.4018/jkss.2011100106

The Annals of Applied Probability
2023, Vol. 33, No. 2, 999–1033
https://doi.org/10.1214/22-AAP1834

SEQUENTIAL IMPORTANCE SAMPLING FOR ESTIMATING EXPECTATIONS OVER THE SPACE OF PERFECT MATCHINGS

BY YEGANEH ALIMOHAMMADI[1,a], PERSI DIACONIS[2,d], MOHAMMAD ROGHANI[1,b] AND AMIN SABERI[1,c]

[1]*Management Science and Engineering, Stanford University,* [a]*yeganeh@stanford.edu,* [b]*roghani@stanford.edu,* [c]*saberi@stanford.edu*

[2]*Department of Statistics, Stanford University,* [d]*diaconis@math.stanford.edu*

This paper makes three contributions to estimating the number of perfect matching in bipartite graphs. First, we prove that the popular sequential importance sampling algorithm works in polynomial time for dense bipartite graphs. More carefully, our algorithm gives a $(1 \pm \epsilon)$-approximation for the number of perfect matchings of a λ-dense bipartite graph, using $O(n^{\frac{1-2\lambda}{\lambda}}\epsilon^{-2})$ samples. With size n on each side and for $\frac{1}{2} > \lambda > 0$, a λ-dense bipartite graph has all degrees greater than $(\lambda + \frac{1}{2})n$.

Second, practical applications of the algorithm requires many calls to matching algorithms. A novel preprocessing step is provided which makes significant improvements.

Third, three applications are provided. The first is for counting Latin squares, the second is a practical way of computing the greedy algorithm for a card guessing game with feedback and the third is for stochastic block models. In all three examples, sequential importance sampling allows treating practical problems of reasonably large sizes.

REFERENCES

[1] ALIMOHAMMADI, Y., DIACONIS, P., ROGHANI, M. and SABERI, A. (2021). Sequential Importance Sampling of Perfect Matchings– Github repository. https://github.com/mohammadroghani/SIS.

[2] ALLEN-ZHU, Z., LI, Y., OLIVEIRA, R. and WIGDERSON, A. (2017). Much faster algorithms for matrix scaling. In 58*th Annual IEEE Symposium on Foundations of Computer Science—FOCS* 2017 890–901. IEEE Computer Soc., Los Alamitos, CA. MR3734290 https://doi.org/10.1109/FOCS.2017.87

[3] ANARI, N. and REZAEI, A. (2019). A tight analysis of Bethe approximation for permanent. In 2019 *IEEE* 60*th Annual Symposium on Foundations of Computer Science* 1434–1445. IEEE Comput. Soc. Press, Los Alamitos, CA. MR4228235 https://doi.org/10.1109/FOCS.2019.000-3

[4] BAPAT, R. B. (1990). Permanents in probability and statistics. *Linear Algebra Appl.* **127** 3–25. MR1048791 https://doi.org/10.1016/0024-3795(90)90332-7

[5] BARVINOK, A. (2016). *Combinatorics and Complexity of Partition Functions*. *Algorithms and Combinatorics* **30**. Springer, Cham. MR3558532 https://doi.org/10.1007/978-3-319-51829-9

[6] BAYATI, M., GAMARNIK, D., KATZ, D., NAIR, C. and TETALI, P. (2007). Simple deterministic approximation algorithms for counting matchings. In *STOC'*07*—Proceedings of the* 39*th Annual ACM Symposium on Theory of Computing* 122–127. ACM, New York. MR2402435 https://doi.org/10.1145/1250790.1250809

[7] BEICHL, I. and SULLIVAN, F. (1999). Approximating the permanent via importance sampling with application to the dimer covering problem. *J. Comput. Phys.* **149** 128–147. MR1669817 https://doi.org/10.1006/jcph.1998.6149

[8] BLACKWELL, D. and HODGES, J. L. JR. (1957). Design for the control of selection bias. *Ann. Math. Stat.* **28** 449–460. MR0088849 https://doi.org/10.1214/aoms/1177706973

[9] BLITZSTEIN, J. and DIACONIS, P. (2010). A sequential importance sampling algorithm for generating random graphs with prescribed degrees. *Internet Math.* **6** 489–522. MR2809836 https://doi.org/10.1080/15427951.2010.557277

MSC2020 subject classifications. Primary 65C05; secondary 05B15.

Key words and phrases. Perfect matching, sequential importance sampling, dense bipartite graph, KL-divergence, Latin squares, card guessing experiment, stochastic block model.

[10] BRÈGMAN, L. M. (1973). Certain properties of nonnegative matrices and their permanents. *Dokl. Akad. Nauk SSSR* **211** 27–30. MR0327788

[11] CAMERON, P. J. Random Latin Squares. School of Mathematical Sciences Queen Mary and Westfield College London E1 4NS UK.

[12] CHATTERJEE, S. and DIACONIS, P. (2018). The sample size required in importance sampling. *Ann. Appl. Probab.* **28** 1099–1135. MR3784496 https://doi.org/10.1214/17-AAP1326

[13] CHUNG, F. R. K., DIACONIS, P., GRAHAM, R. L. and MALLOWS, C. L. (1981). On the permanents of complements of the direct sum of identity matrices. *Adv. in Appl. Math.* **2** 121–137. MR0618698 https://doi.org/10.1016/0196-8858(81)90001-4

[14] DESALVO, S. (2017). Exact sampling algorithms for Latin squares and Sudoku matrices via probabilistic divide-and-conquer. *Algorithmica* **79** 742–762. MR3696094 https://doi.org/10.1007/s00453-016-0223-y

[15] DIACONIS, P. (1978). Statistical problems in ESP research. *Science* **201** 131–136.

[16] DIACONIS, P. (2019). Sequential importance sampling for estimating the number of perfect matchings in bipartite graphs: An ongoing conversation with Laci. In *Building Bridges II—Mathematics of László Lovász. Bolyai Soc. Math. Stud.* **28** 223–233. Springer, Berlin. MR4297782 https://doi.org/10.1007/978-3-662-59204-5_6

[17] DIACONIS, P. and GRAHAM, R. (1981). The analysis of sequential experiments with feedback to subjects. *Ann. Statist.* **9** 3–23. MR0600529

[18] DIACONIS, P., GRAHAM, R., HE, X. and SPIRO, S. (2022). Card guessing with partial feedback. *Combin. Probab. Comput.* **31** 1–20. MR4356453 https://doi.org/10.1017/s0963548321000134

[19] DIACONIS, P., GRAHAM, R. and HOLMES, S. P. (2001). Statistical problems involving permutations with restricted positions. In *State of the Art in Probability and Statistics* (*Leiden*, 1999). *Institute of Mathematical Statistics Lecture Notes—Monograph Series* **36** 195–222. IMS, Beachwood, OH. MR1836562 https://doi.org/10.1214/lnms/1215090070

[20] DIACONIS, P., GRAHAM, R. and SPIRO, S. (2021). Guessing about guessing: Practical strategies for card guessing with feedback. *Amer. Math. Monthly*.

[21] DIACONIS, P. and KOLESNIK, B. (2021). Randomized sequential importance sampling for estimating the number of perfect matchings in bipartite graphs. *Adv. in Appl. Math.* **131** Paper No. 102247, 41. MR4294202 https://doi.org/10.1016/j.aam.2021.102247

[22] DUBHASHI, D. and RANJAN, D. (1998). Balls and bins: A study in negative dependence. *Random Structures Algorithms* **13** 99–124. MR1642566 https://doi.org/10.1002/(SICI)1098-2418(199809)13:2<99::AID-RSA1>3.0.CO;2-M

[23] DUFOSSÉ, F., KAYA, K., PANAGIOTAS, I. and UÇAR, B. (2018). Scaling matrices and counting the perfect matchings in graphs Research Report No. RR-9161 Inria Grenoble Rhône-Alpes.

[24] EBERHARD, S., MANNERS, F. and MRAZOVIC, R. (2021). Oddly specific conjectures for counting latin squares. Personal Communication with Frederick Manners, April, 2021.

[25] EFRON, B. (1971). Forcing a sequential experiment to be balanced. *Biometrika* **58** 403–417. MR0312660 https://doi.org/10.1093/biomet/58.3.403

[26] ERDÖS, P. and KAPLANSKY, I. (1946). The asymptotic number of Latin rectangles. *Amer. J. Math.* **68** 230–236. MR0015356 https://doi.org/10.2307/2371834

[27] ETHIER, S. N. and LEVIN, D. A. (2005). On the fundamental theorem of card counting, with application to the game of trente et quarante. *Adv. in Appl. Probab.* **37** 90–107. MR2135155 https://doi.org/10.1239/aap/1113402401

[28] FISHER, R. A. (1935). *The Design of Experiments*. Oliver and Boyd, Edinburgh.

[29] GODSIL, C. D. and MCKAY, B. D. (1990). Asymptotic enumeration of Latin rectangles. *J. Combin. Theory Ser. B* **48** 19–44. MR1047551 https://doi.org/10.1016/0095-8956(90)90128-M

[30] HÄGGKVIST, R. and JANSSEN, J. C. (1996). All-even Latin squares. *Discrete Math.* **157** 199–206.

[31] HARRIS, D. G., SULLIVAN, F. and BEICHL, I. (2014). Fast sequential importance sampling to estimate the graph reliability polynomial. *Algorithmica* **68** 916–939. MR3165043 https://doi.org/10.1007/s00453-012-9703-x

[32] HUBER, M. and LAW, J. (2008). Fast approximation of the permanent for very dense problems. In *Proceedings of the Nineteenth Annual ACM-SIAM Symposium on Discrete Algorithms* 681–689. ACM, New York. MR2487637

[33] JACOBSON, M. T. and MATTHEWS, P. (1996). Generating uniformly distributed random Latin squares. *J. Combin. Des.* **4** 405–437. MR1410617 https://doi.org/10.1002/(SICI)1520-6610(1996)4:6<405::AID-JCD3>3.0.CO;2-J

[34] JENSEN, A. and BEICHL, I. (2020). A sequential importance sampling algorithm for counting linear extensions. *ACM J. Exp. Algorithmics* **25** Art. 1.7, 14. MR4102860 https://doi.org/10.1145/3385650

[35] JERRUM, M. and SINCLAIR, A. (1989). Approximating the permanent. *SIAM J. Comput.* **18** 1149–1178. MR1025467 https://doi.org/10.1137/0218077

[36] JERRUM, M., SINCLAIR, A. and VIGODA, E. (2004). A polynomial-time approximation algorithm for the permanent of a matrix with nonnegative entries. *J. ACM* **51** 671–697. MR2147852 https://doi.org/10.1145/1008731.1008738

[37] JERRUM, M. R., VALIANT, L. G. and VAZIRANI, V. V. (1986). Random generation of combinatorial structures from a uniform distribution. *Theoret. Comput. Sci.* **43** 169–188. MR0855970 https://doi.org/10.1016/0304-3975(86)90174-X

[38] JOAG-DEV, K. and PROSCHAN, F. (1983). Negative association of random variables, with applications. *Ann. Statist.* **11** 286–295. MR0684886 https://doi.org/10.1214/aos/1176346079

[39] KNUTH, D. E. (1981). A permanent inequality. *Amer. Math. Monthly* **88** 731–740, 798. MR0668399 https://doi.org/10.2307/2321472

[40] KOU, S. C. and MCCULLAGH, P. (2009). Approximating the α-permanent. *Biometrika* **96** 635–644. MR2538762 https://doi.org/10.1093/biomet/asp036

[41] KUZNETSOV, N. YU. (2009). Estimation of the number of Latin rectangles by an accelerated modeling method. *Kibernet. Sistem. Anal.* **45** 76–84, 190. MR2555849 https://doi.org/10.1007/s10559-009-9079-3

[42] LECKEY, K., LIEBENAU, A. and WORMALD, N. (2021). The asymptotic number of Latin rectangles. Personal Communication with Nick Wormald, April, 2021.

[43] LOVASZ, L. L. and PLUMMER, M. D. (1986). *Matching Theory*. Elsevier, Amsterdam.

[44] MCKAY, B. D. and ROGOYSKI, E. (1995). Latin squares of order 10. *Electron. J. Combin.* **2** Note 3, approx. 4. MR1346877

[45] MCKAY, B. D. and WANLESS, I. M. (2005). On the number of Latin squares. *Ann. Comb.* **9** 335–344. MR2176596 https://doi.org/10.1007/s00026-005-0261-7

[46] MICALI, S. and VAZIRANI, V. V. (1980). An $O(|V||E|)$ algorithm for finding maximum matching in general graphs. In *Proceedings of the* 21*st Annual Symposium on Foundations of Computer Science*. *SFCS'*80 17–27. IEEE Computer Society, USA. https://doi.org/10.1109/SFCS.1980.12

[47] MINC, H. (1963). Upper bounds for permanents of (0, 1)-matrices. *Bull. Amer. Math. Soc.* **69** 789–791. MR0155843 https://doi.org/10.1090/S0002-9904-1963-11031-9

[48] MULLEN, G. L. and PURDY, D. (1993). Some data concerning the number of Latin rectangles. *J. Combin. Math. Combin. Comput.* **13** 161–165. MR1220609

[49] NEWMAN, J. E. and VARDI, M. Y. (2020). FPRAS Approximation of the Matrix Permanent in Practice. E-prints arXiv:2012.03367.

[50] O'NEIL, P. E. (1970). Asymptotics in random $(O,1)$-matrices. *Proc. Amer. Math. Soc.* **25** 290–296. MR0255430 https://doi.org/10.2307/2037207

[51] OWEN, A. and ZHOU, Y. (2000). Safe and effective importance sampling. *J. Amer. Statist. Assoc.* **95** 135–143. MR1803146 https://doi.org/10.2307/2669533

[52] PITTENGER, A. O. (1997). Mappings of Latin squares. *Linear Algebra Appl.* **261** 251–268. MR1448875

[53] RASMUSSEN, L. E. (1994). Approximating the permanent: A simple approach. *Random Structures Algorithms* **5** 349–361. MR1262984 https://doi.org/10.1002/rsa.3240050208

[54] ROTHBLUM, U. G. and SCHNEIDER, H. (1989). Scalings of matrices which have prespecified row sums and column sums via optimization. *Linear Algebra Appl.* **114/115** 737–764. MR0986905 https://doi.org/10.1016/0024-3795(89)90491-6

[55] SANKOWSKI, P. (2003). Alternative algorithms for counting all matchings in graphs. In *STACS* 2003 (H. Alt and M. Habib, eds.) 427–438. Springer, Berlin, Heidelberg.

[56] SIEGMUND, D. (1976). Importance sampling in the Monte Carlo study of sequential tests. *Ann. Statist.* **4** 673–684. MR0418369

[57] STEIN, C. M. (1978). Asymptotic evaluation of the number of Latin rectangles. *J. Combin. Theory Ser. A* **25** 38–49. MR0499035 https://doi.org/10.1016/0097-3165(78)90029-8

[58] STONES, D. S. (2010). The many formulae for the number of Latin rectangles. *Electron. J. Combin.* **17** Article 1, 46. MR2661404

[59] SULLIVAN, F. and BEICHL, I. (2014). Permanents, α-permanents and Sinkhorn balancing. *Comput. Statist.* **29** 1793–1798. MR3279018 https://doi.org/10.1007/s00180-014-0506-1

[60] TARJAN, R. (1972). Depth-first search and linear graph algorithms. *SIAM J. Comput.* **1** 146–160. MR0304178 https://doi.org/10.1137/0201010

[61] TASSA, T. (2012). Finding all maximally-matchable edges in a bipartite graph. *Theoret. Comput. Sci.* **423** 50–58. MR2887980 https://doi.org/10.1016/j.tcs.2011.12.071

[62] TIMASHËV, A. N. (2002). On permanents of random doubly stochastic matrices and on asymptotic estimates for the number of Latin rectangles and Latin squares. *Diskret. Mat.* **14** 65–86. MR1964121 https://doi.org/10.1515/dma-2002-0502

[63] TSAO, A. (2020). Theoretical analysis of sequential importance sampling algorithms for a class of perfect matching problems. Preprint. Available at arXiv:2001.02273.

[64] VALIANT, L. G. (1979). The complexity of computing the permanent. *Theoret. Comput. Sci.* **8** 189–201. MR0526203 https://doi.org/10.1016/0304-3975(79)90044-6

[65] WANLESS, I. (2021). Research topics on Latin squares. https://users.monash.edu.au/~iwanless/resprojhome.html.

[66] WELLS, M. B. (1967). The number of Latin squares of order eight. *J. Combin. Theory* **3** 98–99. MR0214493

[67] YAMAMOTO, K. (1950). An asymptotic series for the number of three-line Latin rectangles. *J. Math. Soc. Japan* **1** 226–241. MR0039089 https://doi.org/10.2969/jmsj/00140226

[68] YAMAMOTO, K. (1951). On the asymptotic number of Latin rectangles. *Jpn. J. Math.* **21** 113–119. MR0051203 https://doi.org/10.4099/jjm1924.21.0_113

The Annals of Applied Probability
2023, Vol. 33, No. 2, 1034–1075
https://doi.org/10.1214/22-AAP1835

MEAN FIELD GAMES WITH BRANCHING

BY JULIEN CLAISSE[1,a], ZHENJIE REN[1,b] AND XIAOLU TAN[2,c]

[1]*CEREMADE, Université Paris-Dauphine, PSL University,* [a]*claisse@ceremade.dauphine.fr,* [b]*ren@ceremade.dauphine.fr*
[2]*Department of Mathematics, The Chinese University of Hong Kong,* [c]*xiaolu.tan@cuhk.edu.hk*

Mean field games are concerned with the limit of large-population stochastic differential games where the agents interact through their empirical distribution. In the classical setting, the number of players is large but fixed throughout the game. However, in various applications, such as population dynamics or economic growth, the number of players can vary across time and this may lead to different Nash equilibria. In order to account for this evolution, we introduce a branching mechanism in the population of agents and obtain a variant of the original mean field game problem. As a first step, we study a simple model using a PDE approach to illustrate the main differences with the classical setting. We prove existence of a solution and show that it provides an approximate Nash-equilibrium for large population games. We also present a numerical example for a linear–quadratic model. Then we study the problem in a general setting by a probabilistic approach. It is based upon the relaxed formulation of stochastic control problems which allows us to obtain a general existence result.

REFERENCES

[1] ALIPRANTIS, C. D. and BORDER, K. C. (2006). *Infinite Dimensional Analysis*: *A Hitchhiker's Guide*, 3rd ed. Springer, Berlin. MR2378491

[2] AMBROSIO, L., SAVARÉ, G. and ZAMBOTTI, L. (2009). Existence and stability for Fokker–Planck equations with log-concave reference measure. *Probab. Theory Related Fields* **145** 517–564. MR2529438 https://doi.org/10.1007/s00440-008-0177-3

[3] ATHREYA, K. B. and NEY, P. E. (2004). *Branching Processes*. Dover, Mineola, NY. MR2047480

[4] BARDI, M. (2012). Explicit solutions of some linear–quadratic mean field games. *Netw. Heterog. Media* **7** 243–261. MR2928378 https://doi.org/10.3934/nhm.2012.7.243

[5] BAYRAKTAR, E., CECCHIN, A., COHEN, A. and DELARUE, F. (2021). Finite state mean field games with Wright–Fisher common noise. *J. Math. Pures Appl.* (9) **147** 98–162. MR4213680 https://doi.org/10.1016/j.matpur.2021.01.003

[6] BENSOUSSAN, A., FREHSE, J. and GRÜN, C. (2014). Stochastic differential games with a varying number of players. *Commun. Pure Appl. Anal.* **13** 1719–1736. MR3230409 https://doi.org/10.3934/cpaa.2014.13.1719

[7] BERTUCCI, C. (2018). Optimal stopping in mean field games, an obstacle problem approach. *J. Math. Pures Appl.* (9) **120** 165–194. MR3906158 https://doi.org/10.1016/j.matpur.2017.09.016

[8] BERTUCCI, C., LASRY, J.-M. and LIONS, P.-L. (2019). Some remarks on mean field games. *Comm. Partial Differential Equations* **44** 205–227. MR3941633 https://doi.org/10.1080/03605302.2018.1542438

[9] BILLINGSLEY, P. (1999). *Convergence of Probability Measures*, 2nd ed. *Wiley Series in Probability and Statistics*: *Probability and Statistics*. Wiley, New York. MR1700749 https://doi.org/10.1002/9780470316962

[10] BONNANS, J. F., HADIKHANLOO, S. and PFEIFFER, L. (2021). Schauder estimates for a class of potential mean field games of controls. *Appl. Math. Optim.* **83** 1431–1464. MR4261263 https://doi.org/10.1007/s00245-019-09592-z

[11] BOUVERET, G., DUMITRESCU, R. and TANKOV, P. (2020). Mean-field games of optimal stopping: A relaxed solution approach. *SIAM J. Control Optim.* **58** 1795–1821. MR4119327 https://doi.org/10.1137/18M1233480

[12] CAMPI, L. and FISCHER, M. (2018). N-player games and mean-field games with absorption. *Ann. Appl. Probab.* **28** 2188–2242. MR3843827 https://doi.org/10.1214/17-AAP1354

MSC2020 subject classifications. Primary 60J80; secondary 91A13, 93E20.
Key words and phrases. Mean field games, branching diffusion process, relaxed control.

[13] CARDALIAGUET, P. (2010). Notes on mean field games. Technical report.
[14] CARMONA, R. and DELARUE, F. (2013). Probabilistic analysis of mean-field games. *SIAM J. Control Optim.* **51** 2705–2734. MR3072222 https://doi.org/10.1137/120883499
[15] CARMONA, R. and DELARUE, F. (2018). *Probabilistic Theory of Mean Field Games with Applications I–II*. Springer, Cham.
[16] CARMONA, R., DELARUE, F. and LACHAPELLE, A. (2013). Control of McKean–Vlasov dynamics versus mean field games. *Math. Financ. Econ.* **7** 131–166. MR3045029 https://doi.org/10.1007/s11579-012-0089-y
[17] CARMONA, R., DELARUE, F. and LACKER, D. (2017). Mean field games of timing and models for bank runs. *Appl. Math. Optim.* **76** 217–260. MR3679343 https://doi.org/10.1007/s00245-017-9435-z
[18] CARMONA, R. and LACKER, D. (2015). A probabilistic weak formulation of mean field games and applications. *Ann. Appl. Probab.* **25** 1189–1231. MR3325272 https://doi.org/10.1214/14-AAP1020
[19] CLAISSE, J. (2018). Optimal control of branching diffusion processes: A finite horizon problem. *Ann. Appl. Probab.* **28** 1–34. MR3770871 https://doi.org/10.1214/17-AAP1290
[20] DAWSON, D. A. (1993). Measure-valued Markov processes. In *École D'Été de Probabilités de Saint-Flour XXI*—1991. *Lecture Notes in Math.* **1541** 1–260. Springer, Berlin. MR1242575 https://doi.org/10.1007/BFb0084190
[21] EL KAROUI, N., NGUYEN, D. H. and JEANBLANC-PICQUÉ, M. (1987). Compactification methods in the control of degenerate diffusions: Existence of an optimal control. *Stochastics* **20** 169–219. MR0878312 https://doi.org/10.1080/17442508708833443
[22] EL KAROUI, N. and TAN, X. Capacities, measurable selection and dynamic programming, part I: Abstract framework. Preprint. Available at arXiv:1310.3363.
[23] FLEMING, W. H. (1976). Generalized solutions in optimal stochastic control. Technical report, Brown Univ. Providence RI Lefschetz Center for Dynamical Systems.
[24] GILES, M. B., JENTZEN, A. and WELTI, T. (2019). Generalised multilevel Picard approximations.
[25] GUÉANT, O., LASRY, J.-M. and LIONS, P.-L. (2011). Mean field games and applications. In *Paris–Princeton Lectures on Mathematical Finance* 2010. *Lecture Notes in Math.* **2003** 205–266. Springer, Berlin. MR2762362 https://doi.org/10.1007/978-3-642-14660-2_3
[26] HUANG, M., MALHAMÉ, R. P. and CAINES, P. E. (2006). Large population stochastic dynamic games: Closed-loop McKean–Vlasov systems and the Nash certainty equivalence principle. *Commun. Inf. Syst.* **6** 221–251. MR2346927
[27] IKEDA, N., NAGASAWA, M. and WATANABE, S. (1968). Branching Markov processes. I. *J. Math. Kyoto Univ.* **8** 233–278. MR0232439 https://doi.org/10.1215/kjm/1250524137
[28] IKEDA, N. and WATANABE, S. (1989). *Stochastic Differential Equations and Diffusion Processes*, 2nd ed. *North-Holland Mathematical Library* **24**. North-Holland, Amsterdam; Kodansha, Ltd., Tokyo. MR1011252
[29] JACOD, J., MÉMIN, J. and MÉTIVIER, M. (1983). On tightness and stopping times. *Stochastic Process. Appl.* **14** 109–146. MR0679668 https://doi.org/10.1016/0304-4149(83)90067-4
[30] JACOD, J. and SHIRYAEV, A. N. (2003). *Limit Theorems for Stochastic Processes*, 2nd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **288**. Springer, Berlin. MR1943877 https://doi.org/10.1007/978-3-662-05265-5
[31] KIMMEL, M. and AXELROD, D. E. (2015). *Branching Processes in Biology*, 2nd ed. *Interdisciplinary Applied Mathematics* **19**. Springer, New York. MR3310028 https://doi.org/10.1007/978-1-4939-1559-0
[32] LACKER, D. (2015). Mean field games via controlled martingale problems: Existence of Markovian equilibria. *Stochastic Process. Appl.* **125** 2856–2894. MR3332857 https://doi.org/10.1016/j.spa.2015.02.006
[33] LACKER, D. (2016). A general characterization of the mean field limit for stochastic differential games. *Probab. Theory Related Fields* **165** 581–648. MR3520014 https://doi.org/10.1007/s00440-015-0641-9
[34] LADYŽENSKAJA, O. A., SOLONNIKOV, V. A. and URAL'CEVA, N. N. (1968). *Linear and Quasilinear Equations of Parabolic Type*. *Translations of Mathematical Monographs* **23**. Amer. Math. Soc., Providence, RI. MR0241822
[35] LASRY, J.-M. and LIONS, P.-L. (2007). Mean field games. *Jpn. J. Math.* **2** 229–260. MR2295621 https://doi.org/10.1007/s11537-007-0657-8
[36] LE GALL, J.-F. (2005). Random trees and applications. *Probab. Surv.* **2** 245–311. MR2203728 https://doi.org/10.1214/154957805100000140
[37] NISIO, M. (1985). Stochastic control related to branching diffusion processes. *J. Math. Kyoto Univ.* **25** 549–575. MR0807497 https://doi.org/10.1215/kjm/1250521071
[38] NUTZ, M. (2018). A mean field game of optimal stopping. *SIAM J. Control Optim.* **56** 1206–1221. MR3780736 https://doi.org/10.1137/16M1078331
[39] PAO, C. V. (1992). *Nonlinear Parabolic and Elliptic Equations*. Plenum Press, New York. MR1212084

[40] ROELLY, S. and ROUAULT, A. (1990). Construction et propriétés de martingales des branchements spatiaux interactifs. *Int*. *Stat*. *Rev.* **58** 173–189.

[41] ROELLY-COPPOLETTA, S. (1986). A criterion of convergence of measure-valued processes: Application to measure branching processes. *Stochastics* **17** 43–65. MR0878553 https://doi.org/10.1080/17442508608833382

[42] SKOROHOD, A. V. (1964). Branching diffusion processes. *Teor. Veroyatn*. *Primen*. **9** 492–497. MR0168030

[43] STROOCK, D. W. and VARADHAN, S. R. S. (1979). *Multidimensional Diffusion Processes*. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **233**. Springer, Berlin. MR0532498

[44] USTUNEL, S. (1981). Construction of branching diffusion processes and their optimal stochastic control. *Appl*. *Math*. *Optim*. **7** 11–33. MR0600558 https://doi.org/10.1007/BF01442107

[45] VILLANI, C. (2009). *Optimal Transport*: *Old and New*. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **338**. Springer, Berlin. MR2459454 https://doi.org/10.1007/978-3-540-71050-9

The Annals of Applied Probability
2023, Vol. 33, No. 2, 1076–1101
https://doi.org/10.1214/22-AAP1836

PARKING ON THE INTEGERS

BY MICHAŁ PRZYKUCKI[1,a], ALEXANDER ROBERTS[2,b] AND ALEX SCOTT[2,c]

[1]*School of Mathematics, University of Birmingham,* [a]*michal.przykucki@gmail.com*
[2]*Mathematical Institute, University of Oxford,* [b]*alex.maths.roberts@yandex.com,* [c]*scott@maths.ox.ac.uk*

Models of parking in which cars are placed randomly and then move according to a deterministic rule have been studied since the work of Konheim and Weiss in the 1960s. Recently, Damron, Gravner, Junge, Lyu, and Sivakoff ((2019) *Ann. Appl. Probab.* **29** 2089–2113) introduced a model in which cars are both placed and move at random. Independently at each point of a Cayley graph G, we place a car with probability p, and otherwise an empty parking space. Each car independently executes a random walk until it finds an empty space in which to park. In this paper we introduce three new techniques for studying the model, namely the space-based parking model, and the strategies for parking and for car removal. These allow us to study the original model by coupling it with models where parking behaviour is easier to control. Applying our methods to the one-dimensional parking problem in $\mathbb{Z}$, we improve on previous work, showing that for $p < 1/2$ the expected journey length of a car is finite, and for $p = 1/2$ the expected journey length by time t grows like $t^{3/4}$ up to a polylogarithmic factor.

REFERENCES

[1] ARRATIA, R. (1983). Site recurrence for annihilating random walks on $\mathbf{Z}_d$. *Ann. Probab.* **11** 706–713. MR0704557

[2] BRAMSON, M. and LEBOWITZ, J. L. (1991). Asymptotic behavior of densities for two-particle annihilating random walks. *J. Stat. Phys.* **62** 297–372. MR1105266 https://doi.org/10.1007/BF01020872

[3] CABEZAS, M., ROLLA, L. T. and SIDORAVICIUS, V. (2014). Non-equilibrium phase transitions: Activated random walks at criticality. *J. Stat. Phys.* **155** 1112–1125. MR3207731 https://doi.org/10.1007/s10955-013-0909-3

[4] DAMRON, M., GRAVNER, J., JUNGE, M., LYU, H. and SIVAKOFF, D. (2019). Parking on transitive unimodular graphs. *Ann. Appl. Probab.* **29** 2089–2113. MR3983336 https://doi.org/10.1214/18-AAP1443

[5] ERDŐS, P. and NEY, P. (1974). Some problems on random intervals and annihilating particles. *Ann. Probab.* **2** 828–839. MR0373068 https://doi.org/10.1214/aop/1176996551

[6] FELLER, W. (1968). *An Introduction to Probability Theory and Its Applications. Vol. I*, 3rd ed. Wiley, New York. MR0228020

[7] GOLDSCHMIDT, C. and PRZYKUCKI, M. (2019). Parking on a random tree. *Combin. Probab. Comput.* **28** 23–45. MR3917904 https://doi.org/10.1017/S0963548318000457

[8] JOHNSON, T., JUNGE, M., LYU, H. and SIVAKOFF, D. (2020). Particle density in diffusion–limited annihilating systems. Available at arXiv:2005.06018.

[9] JUNGE, M. (2018). Parking. Seminar at the Informal Probability Workshop, Mathematical Institute, University of Oxford.

[10] KONHEIM, A. G. and WEISS, B. (1966). An occupancy discipline and applications. *SIAM J. Appl. Math.* **14** 1266–1274.

[11] LACKNER, M.-L. and PANHOLZER, A. (2016). Parking functions for mappings. *J. Combin. Theory Ser. A* **142** 1–28. MR3499489 https://doi.org/10.1016/j.jcta.2016.03.001

[12] LYONS, R. and PERES, Y. (2016). *Probability on Trees and Networks. Cambridge Series in Statistical and Probabilistic Mathematics* **42**. Cambridge Univ. Press, New York. MR3616205 https://doi.org/10.1017/9781316672815

[13] MITZENMACHER, M. and UPFAL, E. (2017). *Probability and Computing*, 2nd ed. Cambridge Univ. Press, Cambridge. MR3674428

MSC2020 subject classifications. Primary 60K35, 82C22; secondary 82B26.
Key words and phrases. Parking problems, Cayley graphs, random walks.

[14] ROLLA, L. T. (2020). Activated random walks on $\mathbb{Z}^d$. *Probab. Surv.* **17** 478–544. MR4152668 https://doi.org/10.1214/19-PS339

[15] ROLLA, L. T. and SIDORAVICIUS, V. (2012). Absorbing-state phase transition for driven-dissipative stochastic dynamics on $\mathbb{Z}$. *Invent. Math.* **188** 127–150. MR2897694 https://doi.org/10.1007/s00222-011-0344-5

The Annals of Applied Probability
2023, Vol. 33, No. 2, 1102–1135
https://doi.org/10.1214/22-AAP1837

SHARP LOWER ERROR BOUNDS FOR STRONG APPROXIMATION OF SDES WITH DISCONTINUOUS DRIFT COEFFICIENT BY COUPLING OF NOISE

BY THOMAS MÜLLER-GRONBACH[a] AND LARISA YAROSLAVTSEVA[b]

Faculty of Computer Science and Mathematics, University of Passau, [a]*thomas.mueller-gronbach@uni-passau.de,* [b]*larisa.yaroslavtseva@uni-passau.de*

In the past decade, an intensive study of strong approximation of stochastic differential equations (SDEs) with a drift coefficient that has discontinuities in space has begun. In the majority of these results it is assumed that the drift coefficient satisfies piecewise regularity conditions and that the diffusion coefficient is globally Lipschitz continuous and nondegenerate at the discontinuities of the drift coefficient. Under this type of assumptions the best L_p-error rate obtained so far for approximation of scalar SDEs at the final time is $3/4$ in terms of the number of evaluations of the driving Brownian motion. In the present article, we prove for the first time in the literature sharp lower error bounds for such SDEs. We show that for a huge class of additive noise driven SDEs of this type the L_p-error rate $3/4$ can not be improved.

For the proof of this result we employ a novel technique by studying equations with coupled noise: we reduce the analysis of the L_p-error of an arbitrary approximation based on evaluation of the driving Brownian motion at finitely many times to the analysis of the L_p-distance of two solutions of the same equation that are driven by Brownian motions that are coupled at the given time-points and independent, conditioned on their values at these points. To obtain lower bounds for the latter quantity, we prove a new quantitative version of positive association for bivariate normal random variables (Y, Z) by providing explicit lower bounds for the covariance $\mathrm{Cov}(f(Y), g(Z))$ in case of piecewise Lipschitz continuous functions f and g. In addition it turns out that our proof technique also leads to lower error bounds for estimating occupation time functionals $\int_0^1 f(W_t)\,dt$ of a Brownian motion W, which substantially extends known results for the case of f being an indicator function.

REFERENCES

[1] ALTMEYER, R. (2021). Approximation of occupation time functionals. *Bernoulli* **27** 2714–2739. MR4303901 https://doi.org/10.3150/21-BEJ1328

[2] BALAKRISHNAN, A. V. (1980). On stochastic bang-bang control. In *Stochastic Differential Systems* (*Proc. IFIP-WG* 7/1 *Working Conf.*, *Vilnius*, 1978). *Lecture Notes in Control and Information Sci.* **25** 221–238. Springer, Berlin. MR0609186

[3] CLARK, J. M. C. and CAMERON, R. J. (1980). The maximum rate of convergence of discrete approximations for stochastic differential equations. In *Stochastic Differential Systems* (*Proc. IFIP-WG* 7/1 *Working Conf.*, *Vilnius*, 1978). *Lecture Notes in Control and Information Sci.* **25** 162–171. Springer, Berlin. MR0609181

[4] DAREIOTIS, K. and GERENCSÉR, M. (2020). On the regularisation of the noise for the Euler–Maruyama scheme with irregular drift. *Electron. J. Probab.* **25** Paper No. 82, 18 pp. MR4125787 https://doi.org/10.1214/20-ejp479

[5] GÖTTLICH, S., LUX, K. and NEUENKIRCH, A. (2019). The Euler scheme for stochastic differential equations with discontinuous drift coefficient: A numerical study of the convergence rate. *Adv. Difference Equ.* Paper No. 429, 21 pp. MR4022682 https://doi.org/10.1186/s13662-019-2361-4

MSC2020 subject classifications. Primary 65C30, 60H35; secondary 60H10.

Key words and phrases. Stochastic differential equations, discontinuous drift coefficient, strong approximation, lower error bounds.

[6] GYÖNGY, I. (1998). A note on Euler's approximations. *Potential Anal.* **8** 205–216. MR1625576 https://doi.org/10.1023/A:1008605221617

[7] GYÖNGY, I. and KRYLOV, N. (1996). Existence of strong solutions for Itô's stochastic equations via approximations. *Probab. Theory Related Fields* **105** 143–158. MR1392450 https://doi.org/10.1007/BF01203833

[8] HALIDIAS, N. and KLOEDEN, P. E. (2008). A note on the Euler–Maruyama scheme for stochastic differential equations with a discontinuous monotone drift coefficient. *BIT* **48** 51–59. MR2386114 https://doi.org/10.1007/s10543-008-0164-1

[9] HEFTER, M., HERZWURM, A. and MÜLLER-GRONBACH, T. (2019). Lower error bounds for strong approximation of scalar SDEs with non-Lipschitzian coefficients. *Ann. Appl. Probab.* **29** 178–216. MR3910003 https://doi.org/10.1214/18-AAP1411

[10] HOFMANN, N., MÜLLER-GRONBACH, T. and RITTER, K. (2001). The optimal discretization of stochastic differential equations. *J. Complexity* **17** 117–153. MR1817611 https://doi.org/10.1006/jcom.2000.0570

[11] ICHIBA, T., PAPATHANAKOS, V., BANNER, A., KARATZAS, I. and FERNHOLZ, R. (2011). Hybrid atlas models. *Ann. Appl. Probab.* **21** 609–644. MR2807968 https://doi.org/10.1214/10-AAP706

[12] KALLENBERG, O. (1996). On the existence of universal functional solutions to classical SDE's. *Ann. Probab.* **24** 196–205. MR1387632 https://doi.org/10.1214/aop/1042644713

[13] KARATZAS, I. and SHREVE, S. E. (1991). *Brownian Motion and Stochastic Calculus*, 2nd ed. *Graduate Texts in Mathematics* **113**. Springer, New York. MR1121940 https://doi.org/10.1007/978-1-4612-0949-2

[14] KLENKE, A. (2008). *Probability Theory: A Comprehensive Course*. *Universitext*. Springer, London. MR2372119 https://doi.org/10.1007/978-1-84800-048-3

[15] LEOBACHER, G. and SZÖLGYENYI, M. (2016). A numerical method for SDEs with discontinuous drift. *BIT* **56** 151–162. MR3486457 https://doi.org/10.1007/s10543-015-0549-x

[16] LEOBACHER, G. and SZÖLGYENYI, M. (2017). A strong order 1/2 method for multidimensional SDEs with discontinuous drift. *Ann. Appl. Probab.* **27** 2383–2418. MR3693529 https://doi.org/10.1214/16-AAP1262

[17] LEOBACHER, G. and SZÖLGYENYI, M. (2018). Convergence of the Euler–Maruyama method for multidimensional SDEs with discontinuous drift and degenerate diffusion coefficient. *Numer. Math.* **138** 219–239. MR3745015 https://doi.org/10.1007/s00211-017-0903-9

[18] MÜLLER-GRONBACH, T. (2002). Strong approximation of systems of stochastic differential equations. Habilitation thesis, TU Darmstadt, iv+161.

[19] MÜLLER-GRONBACH, T. (2004). Optimal pointwise approximation of SDEs based on Brownian motion at discrete points. *Ann. Appl. Probab.* **14** 1605–1642. MR2099646 https://doi.org/10.1214/105051604000000954

[20] MÜLLER-GRONBACH, T. and YAROSLAVTSEVA, L. (2018). A note on strong approximation of SDEs with smooth coefficients that have at most linearly growing derivatives. *J. Math. Anal. Appl.* **467** 1013–1031. MR3842417 https://doi.org/10.1016/j.jmaa.2018.07.041

[21] MÜLLER-GRONBACH, T. and YAROSLAVTSEVA, L. (2020). On the performance of the Euler–Maruyama scheme for SDEs with discontinuous drift coefficient. *Ann. Inst. Henri Poincaré Probab. Stat.* **56** 1162–1178. MR4076779 https://doi.org/10.1214/19-AIHP997

[22] MÜLLER-GRONBACH, T. and YAROSLAVTSEVA, L. (2020). Sharp lower error bounds for strong approximation of SDEs with discontinuous drift coefficient by coupling of noise. Preprint. Available at arXiv:2010.00915v1.

[23] MÜLLER-GRONBACH, T. and YAROSLAVTSEVA, L. (2022). A strong order 3/4 method for SDEs with discontinuous drift coefficient. *IMA J. Numer. Anal.* **42** 229–259. MR4367654 https://doi.org/10.1093/imanum/draa078

[24] NEUENKIRCH, A. and SZÖLGYENYI, M. (2021). The Euler–Maruyama scheme for SDEs with irregular drift: Convergence rates via reduction to a quadrature problem. *IMA J. Numer. Anal.* **41** 1164–1196. MR4246871 https://doi.org/10.1093/imanum/draa007

[25] NEUENKIRCH, A., SZÖLGYENYI, M. and SZPRUCH, L. (2019). An adaptive Euler–Maruyama scheme for stochastic differential equations with discontinuous drift and its convergence analysis. *SIAM J. Numer. Anal.* **57** 378–403. MR3914573 https://doi.org/10.1137/18M1170017

[26] NGO, H.-L. and OGAWA, S. (2011). On the discrete approximation of occupation time of diffusion processes. *Electron. J. Stat.* **5** 1374–1393. MR2842909 https://doi.org/10.1214/11-EJS645

[27] NGO, H.-L. and TAGUCHI, D. (2016). Strong rate of convergence for the Euler–Maruyama approximation of stochastic differential equations with irregular coefficients. *Math. Comp.* **85** 1793–1819. MR3471108 https://doi.org/10.1090/mcom3042

[28] NGO, H.-L. and TAGUCHI, D. (2017). On the Euler–Maruyama approximation for one-dimensional stochastic differential equations with irregular coefficients. *IMA J. Numer. Anal.* **37** 1864–1883. MR3712177 https://doi.org/10.1093/imanum/drw058

[29] NGO, H.-L. and TAGUCHI, D. (2017). Strong convergence for the Euler–Maruyama approximation of stochastic differential equations with discontinuous coefficients. *Statist. Probab. Lett.* **125** 55–63. MR3626068 https://doi.org/10.1016/j.spl.2017.01.027

[30] PRZYBYŁOWICZ, P. and SZÖLGYENYI, M. (2021). Existence, uniqueness, and approximation of solutions of jump-diffusion SDEs with discontinuous drift. *Appl. Math. Comput.* **403** Paper No. 126191, 15 pp. MR4235987 https://doi.org/10.1016/j.amc.2021.126191

[31] QIAN, Z. and ZHENG, W. (2002). Sharp bounds for transition probability densities of a class of diffusions. *C. R. Math. Acad. Sci. Paris* **335** 953–957. MR1952556 https://doi.org/10.1016/S1631-073X(02)02579-7

[32] SHARDIN, A. A. and SZÖLGYENYI, M. (2016). Optimal control of an energy storage facility under a changing economic environment and partial information. *Int. J. Theor. Appl. Finance* **19** 1650026, 27 pp. MR3505500 https://doi.org/10.1142/S0219024916500266

[33] TONG, Y. L. (1990). *The Multivariate Normal Distribution*. *Springer Series in Statistics*. Springer, New York. MR1029032 https://doi.org/10.1007/978-1-4613-9655-0

[34] YAROSLAVTSEVA, L. (2022). An adaptive strong order 1 method for SDEs with discontinuous drift coefficient. *J. Math. Anal. Appl.* **513** Paper No. 126180, 29 pp. MR4404565 https://doi.org/10.1016/j.jmaa.2022.126180

The Annals of Applied Probability
2023, Vol. 33, No. 2, 1136–1171
https://doi.org/10.1214/22-AAP1838

MIXING OF THE AVERAGING PROCESS AND ITS DISCRETE DUAL ON FINITE-DIMENSIONAL GEOMETRIES

BY MATTEO QUATTROPANI[1,a] AND FEDERICO SAU[2,b]

[1]*Mathematical Institute, Leiden University,* [a]*m.quattropani@math.leidenuniv.nl*
[2]*Università degli Studi di Trieste,* [b]*federico.sau@units.it*

We analyze the L^1-mixing of a generalization of the averaging process introduced by Aldous (2011). The process takes place on a growing sequence of graphs which we assume to be finite-dimensional, in the sense that the random walk on those geometries satisfies a family of Nash inequalities. As a byproduct of our analysis, we provide a complete picture of the total variation mixing of a discrete dual of the averaging process, which we call binomial splitting process. A single particle of this process is essentially the random walk on the underlying graph. When several particles evolve together, they interact by synchronizing their jumps when placed on neighboring sites. We show that, given k the number of particles and n the (growing) size of the underlying graph, the system exhibits cutoff in total variation if $k \to \infty$ and $k = O(n^2)$. Finally, we exploit the duality between the two processes to show that the binomial splitting process satisfies a version of Aldous' spectral gap identity, namely, the relaxation time of the process is independent of the number of particles.

REFERENCES

[1] ALDOUS, D. (2011). Finite Markov information-exchange processes. Available at https://www.stat.berkeley.edu/~aldous/260-FMIE/index.html, Accessed 17 June 2021.

[2] ALDOUS, D. (2013). Interacting particle systems as stochastic social dynamics. *Bernoulli* **19** 1122–1149. MR3102546 https://doi.org/10.3150/12-BEJSP04

[3] ALDOUS, D. and DIACONIS, P. (1986). Shuffling cards and stopping times. *Amer. Math. Monthly* **93** 333–348. MR0841111 https://doi.org/10.2307/2323590

[4] ALDOUS, D. and LANOUE, D. (2012). A lecture on the averaging process. *Probab. Surv.* **9** 90–102. MR2908618 https://doi.org/10.1214/11-PS184

[5] ALDOUS, D. J. (1985). Exchangeability and related topics. In *École D'été de Probabilités de Saint-Flour, XIII*—1983. *Lecture Notes in Math.* **1117** 1–198. Springer, Berlin. MR0883646 https://doi.org/10.1007/BFb0099421

[6] BANERJEE, S. and BURDZY, K. (2021). Rates of convergence to equilibrium for potlatch and smoothing processes. *Ann. Probab.* **49** 1129–1163. MR4255140 https://doi.org/10.1214/20-aop1473

[7] BARLOW, M. T. (2017). *Random Walks and Heat Kernels on Graphs*. *London Mathematical Society Lecture Note Series* **438**. Cambridge Univ. Press, Cambridge. MR3616731 https://doi.org/10.1017/9781107415690

[8] BUFETOV, A. and NEJJAR, P. (2022). Cutoff profile of ASEP on a segment. *Probab. Theory Related Fields* **183** 229–253. MR4421174 https://doi.org/10.1007/s00440-021-01104-x

[9] CAPUTO, P. (2008). On the spectral gap of the Kac walk and other binary collision processes. *ALEA Lat. Am. J. Probab. Math. Stat.* **4** 205–222. MR2429910

[10] CAPUTO, P., LABBÉ, C. and LACOIN, H. (2020). Mixing time of the adjacent walk on the simplex. *Ann. Probab.* **48** 2449–2493. MR4152648 https://doi.org/10.1214/20-AOP1428

[11] CAPUTO, P., LABBÉ, C. and LACOIN, H. (2022). Spectral gap and cutoff phenomenon for the Gibbs sampler of $\nabla\varphi$ interfaces with convex potential. *Ann. Inst. Henri Poincaré Probab. Stat.* **58** 794–826. MR4421608 https://doi.org/10.1214/21-aihp1174

[12] CAPUTO, P., LIGGETT, T. M. and RICHTHAMMER, T. (2010). Proof of Aldous' spectral gap conjecture. *J. Amer. Math. Soc.* **23** 831–851. MR2629990 https://doi.org/10.1090/S0894-0347-10-00659-4

MSC2020 subject classifications. 60J25, 60J27, 37A25, 82C22.
Key words and phrases. Averaging process, mixing times, dualities and intertwinings, Nash inequality.

[13] CARLEN, E. A., KUSUOKA, S. and STROOCK, D. W. (1987). Upper bounds for symmetric Markov transition functions. *Ann. Inst. Henri Poincaré Probab. Stat.* **23** 245–287. MR0898496
[14] CHATTERJEE, S., DIACONIS, P., SLY, A. and ZHANG, L. (2022). A phase transition for repeated averages. *Ann. Probab.* **50** 1–17. MR4385355 https://doi.org/10.1214/21-AOP1526
[15] CHEN, M.-F. and WANG, F.-Y. (1997). Estimation of spectral gap for elliptic operators. *Trans. Amer. Math. Soc.* **349** 1239–1267. MR1401516 https://doi.org/10.1090/S0002-9947-97-01812-6
[16] CHEN, Z.-Q. and FAN, W.-T. (2017). Hydrodynamic limits and propagation of chaos for interacting random walks in domains. *Ann. Appl. Probab.* **27** 1299–1371. MR3678472 https://doi.org/10.1214/16-AAP1208
[17] CORUJO, J. (2020). On the spectrum of a neutral multi-allelic Moran model. arXiv:2010.08809.
[18] COULHON, T. (1996). Ultracontractivity and Nash type inequalities. *J. Funct. Anal.* **141** 510–539. MR1418518 https://doi.org/10.1006/jfan.1996.0140
[19] DE MASI, A. and PRESUTTI, E. (1991). *Mathematical Methods for Hydrodynamic Limits*. *Lecture Notes in Math.* **1501**. Springer, Berlin. MR1175626 https://doi.org/10.1007/BFb0086457
[20] DEMBO, A., KUMAGAI, T. and NAKAMURA, C. (2018). Cutoff for lamplighter chains on fractals. *Electron. J. Probab.* **23** Paper No. 73, 21. MR3835479 https://doi.org/10.1214/18-EJP196
[21] DIACONIS, P. and FREEDMAN, D. (1980). Finite exchangeable sequences. *Ann. Probab.* **8** 745–764. MR0577313
[22] DIACONIS, P. and SALOFF-COSTE, L. (1994). Moderate growth and random walk on finite groups. *Geom. Funct. Anal.* **4** 1–36. MR1254308 https://doi.org/10.1007/BF01898359
[23] DIACONIS, P. and SALOFF-COSTE, L. (1996). Nash inequalities for finite Markov chains. *J. Theoret. Probab.* **9** 459–510. MR1385408 https://doi.org/10.1007/BF02214660
[24] FABES, E. B. and STROOCK, D. W. (1986). A new proof of Moser's parabolic Harnack inequality using the old ideas of Nash. *Arch. Ration. Mech. Anal.* **96** 327–338. MR0855753 https://doi.org/10.1007/BF00251802
[25] GIARDINÀ, C., KURCHAN, J., REDIG, F. and VAFAYI, K. (2009). Duality and hidden symmetries in interacting particle systems. *J. Stat. Phys.* **135** 25–55. MR2505724 https://doi.org/10.1007/s10955-009-9716-2
[26] GRIFFITHS, R. C. (2014). The Λ–Fleming–Viot process and a connection with Wright–Fisher diffusion. *Adv. in Appl. Probab.* **46** 1009–1035. MR3290427 https://doi.org/10.1239/aap/1418396241
[27] HERMON, J. and SALEZ, J. (2019). A version of Aldous' spectral-gap conjecture for the zero range process. *Ann. Appl. Probab.* **29** 2217–2229. MR3984254 https://doi.org/10.1214/18-AAP1449
[28] HERMON, J. and SALEZ, J. (2020). Cutoff for the mean-field zero-range process with bounded monotone rates. *Ann. Probab.* **48** 742–759. MR4089493 https://doi.org/10.1214/19-AOP1373
[29] JONASSON, J. (2012). Mixing times for the interchange process. *ALEA Lat. Am. J. Probab. Math. Stat.* **9** 667–683. MR3069380
[30] JONES, O. D. (1996). Transition probabilities for the simple random walk on the Sierpiński graph. *Stochastic Process. Appl.* **61** 45–69. MR1378848 https://doi.org/10.1016/0304-4149(95)00074-7
[31] KIGAMI, J. (2001). *Analysis on Fractals*. *Cambridge Tracts in Mathematics* **143**. Cambridge Univ. Press, Cambridge. MR1840042 https://doi.org/10.1017/CBO9780511470943
[32] LABBÉ, C. and LACOIN, H. (2019). Cutoff phenomenon for the asymmetric simple exclusion process and the biased card shuffling. *Ann. Probab.* **47** 1541–1586. MR3945753 https://doi.org/10.1214/18-AOP1290
[33] LABBÉ, C. and LACOIN, H. (2020). Mixing time and cutoff for the weakly asymmetric simple exclusion process. *Ann. Appl. Probab.* **30** 1847–1883. MR4132639 https://doi.org/10.1214/19-AAP1545
[34] LACOIN, H. (2016). Mixing time and cutoff for the adjacent transposition shuffle and the simple exclusion. *Ann. Probab.* **44** 1426–1487. MR3474475 https://doi.org/10.1214/15-AOP1004
[35] LACOIN, H. and LEBLOND, R. (2011). Cutoff phenomenon for the simple exclusion process on the complete graph. *ALEA Lat. Am. J. Probab. Math. Stat.* **8** 285–301. MR2869447
[36] LEVIN, D. A., LUCZAK, M. J. and PERES, Y. (2010). Glauber dynamics for the mean-field Ising model: Cut-off, critical power law, and metastability. *Probab. Theory Related Fields* **146** 223–265. MR2550363 https://doi.org/10.1007/s00440-008-0189-z
[37] LEVIN, D. A. and PERES, Y. (2017). *Markov Chains and Mixing Times*. Amer. Math. Soc., Providence, RI. Second edition of [MR2466937], With contributions by Elizabeth L. Wilmer, With a chapter on "Coupling from the past" by James G. Propp and David B. Wilson. MR3726904 https://doi.org/10.1090/mbk/107
[38] LIGGETT, T. M. (2005). *Interacting Particle Systems*. *Classics in Mathematics*. Springer, Berlin. Reprint of the 1985 original. MR2108619 https://doi.org/10.1007/b138374
[39] LUBETZKY, E. and SLY, A. (2014). Cutoff for general spin systems with arbitrary boundary conditions. *Comm. Pure Appl. Math.* **67** 982–1027. MR3193965 https://doi.org/10.1002/cpa.21489

[40] MAAS, J. (2011). Gradient flows of the entropy for finite Markov chains. *J. Funct. Anal.* **261** 2250–2292. MR2824578 https://doi.org/10.1016/j.jfa.2011.06.009

[41] MATHIEU, P. and REMY, E. (2004). Isoperimetry and heat kernel decay on percolation clusters. *Ann. Probab.* **32** 100–128. MR2040777 https://doi.org/10.1214/aop/1078415830

[42] MERLE, M. and SALEZ, J. (2019). Cutoff for the mean-field zero-range process. *Ann. Probab.* **47** 3170–3201. MR4021248 https://doi.org/10.1214/19-AOP1336

[43] MICLO, L. (2018). On the Markovian similarity. In *Séminaire de Probabilités XLIX*. *Lecture Notes in Math.* **2215** 375–403. Springer, Cham. MR3838869

[44] MOSSEL, E. and TAMUZ, O. (2017). Opinion exchange dynamics. *Probab. Surv.* **14** 155–204. MR3677884 https://doi.org/10.1214/14-PS230

[45] NASH, J. (1958). Continuity of solutions of parabolic and elliptic equations. *Amer. J. Math.* **80** 931–954. MR0100158 https://doi.org/10.2307/2372841

[46] PILLAI, N. S. and SMITH, A. (2018). On the mixing time of Kac's walk and other high-dimensional Gibbs samplers with constraints. *Ann. Probab.* **46** 2345–2399. MR3813994 https://doi.org/10.1214/17-AOP1230

[47] REDIG, F. and SAU, F. (2017). Generalized immediate exchange models and their symmetries. *Stochastic Process. Appl.* **127** 3251–3267. MR3692314 https://doi.org/10.1016/j.spa.2017.02.005

[48] REDIG, F. and SAU, F. (2018). Factorized duality, stationary product measures and generating functions. *J. Stat. Phys.* **172** 980–1008. MR3830295 https://doi.org/10.1007/s10955-018-2090-1

[49] REN, P. and WANG, F.-Y. (2020). Spectral gap for measure-valued diffusion processes. *J. Math. Anal. Appl.* **483** 123624, 16. MR4024966 https://doi.org/10.1016/j.jmaa.2019.123624

[50] SALOFF-COSTE, L. (1997). Lectures on finite Markov chains. In *Lectures on Probability Theory and Statistics* (*Saint-Flour*, 1996). *Lecture Notes in Math.* **1665** 301–413. Springer, Berlin. MR1490046 https://doi.org/10.1007/BFb0092621

[51] SCHMID, D. (2019). Mixing times for the simple exclusion process in ballistic random environment. *Electron. J. Probab.* **24** Paper No. 22, 25. MR3933201 https://doi.org/10.1214/19-EJP286

[52] SCHMID, D. (2021). Mixing times for the TASEP in the maximal current phase. arXiv:2104.12745.

[53] SHIMAKURA, N. (1977). Équations différentielles provenant de la génétique des populations. *Tôhoku Math. J.* **29** 287–318. MR0504058 https://doi.org/10.2748/tmj/1178240658

[54] SMITH, A. (2013). Analysis of convergence rates of some Gibbs samplers on continuous state spaces. *Stochastic Process. Appl.* **123** 3861–3876. MR3084162 https://doi.org/10.1016/j.spa.2013.05.003

[55] SMITH, A. (2014). A Gibbs sampler on the n-simplex. *Ann. Appl. Probab.* **24** 114–130. MR3161643 https://doi.org/10.1214/12-AAP916

[56] WILSON, D. B. (2004). Mixing times of Lozenge tiling and card shuffling Markov chains. *Ann. Appl. Probab.* **14** 274–325. MR2023023 https://doi.org/10.1214/aoap/1075828054

The Annals of Applied Probability
2023, Vol. 33, No. 2, 1172–1212
https://doi.org/10.1214/22-AAP1839

MIXING TIMES FOR THE SIMPLE EXCLUSION PROCESS WITH OPEN BOUNDARIES

BY NINA GANTERT[1,a], EVITA NESTORIDI[2,c] AND DOMINIK SCHMID[1,b]

[1]*Department of Mathematics, Technical University of Munich,* [a]*nina.gantert@tum.de,* [b]*dominik.schmid@tum.de*
[2]*Department of Mathematics, Princeton University,* [c]*exn@princeton.edu*

We study mixing times of the symmetric and asymmetric simple exclusion process on the segment where particles are allowed to enter and exit at the endpoints. We consider different regimes depending on the entering and exiting rates as well as on the rates in the bulk, and show that the process exhibits pre-cutoff and in some cases cutoff. Our main contribution is to study mixing times for the asymmetric simple exclusion process with open boundaries. We show that the order of the mixing time can be linear or exponential in the size of the segment depending on the choice of the boundary parameters, proving a strikingly different (and richer) behavior for the simple exclusion process with open boundaries than for the process on the closed segment. Our arguments combine coupling, second class particle and censoring techniques with current estimates. A novel idea is the use of multi-species particle arguments, where the particles only obey a partial ordering.

REFERENCES

[1] BAIK, J. and LIU, Z. (2018). Fluctuations of TASEP on a ring in relaxation time scale. *Comm. Pure Appl. Math.* **71** 747–813. MR3772401 https://doi.org/10.1002/cpa.21702

[2] BALÁZS, M. and SEPPÄLÄINEN, T. (2007). Exact connections between current fluctuations and the second class particle in a class of deposition models. *J. Stat. Phys.* **127** 431–455. MR2314355 https://doi.org/10.1007/s10955-007-9291-3

[3] BALÁZS, M. and SEPPÄLÄINEN, T. (2010). Order of current variance and diffusivity in the asymmetric simple exclusion process. *Ann. of Math.* (2) **171** 1237–1265. MR2630064 https://doi.org/10.4007/annals.2010.171.1237

[4] BENJAMINI, I., BERGER, N., HOFFMAN, C. and MOSSEL, E. (2005). Mixing times of the biased card shuffling and the asymmetric exclusion process. *Trans. Amer. Math. Soc.* **357** 3013–3029. MR2135733 https://doi.org/10.1090/S0002-9947-05-03610-X

[5] BLYTHE, R. A. and EVANS, M. R. (2007). Nonequilibrium steady states of matrix-product form: A solver's guide. *J. Phys. A* **40** R333–R441. MR2437671 https://doi.org/10.1088/1751-8113/40/46/R01

[6] BLYTHE, R. A., EVANS, M. R., COLAIORI, F. and ESSLER, F. H. L. (2000). Exact solution of a partially asymmetric exclusion model using a deformed oscillator algebra. *J. Phys. A* **33** 2313–2332. MR1751977 https://doi.org/10.1088/0305-4470/33/12/301

[7] BRAK, R., CORTEEL, S., ESSAM, J., PARVIAINEN, R. and RECHNITZER, A. (2006). A combinatorial derivation of the PASEP stationary state. *Electron. J. Combin.* **13** Research Paper 108, 23. MR2274323

[8] CANTINI, L. (2017). Asymmetric simple exclusion process with open boundaries and Koornwinder polynomials. *Ann. Henri Poincaré* **18** 1121–1151. MR3626299 https://doi.org/10.1007/s00023-016-0540-3

[9] CANTINI, L., GARBALI, A., DE GIER, J. and WHEELER, M. (2016). Koornwinder polynomials and the stationary multi-species asymmetric exclusion process with open boundaries. *J. Phys. A* **49** 444002, 23. MR3562836 https://doi.org/10.1088/1751-8113/49/44/444002

[10] CORTEEL, S., MANDELSHTAM, O. and WILLIAMS, L. (2017). Combinatorics of the two-species ASEP and Koornwinder moments. *Adv. Math.* **321** 160–204. MR3715708 https://doi.org/10.1016/j.aim.2017.09.034

[11] CORTEEL, S. and WILLIAMS, L. K. (2011). Tableaux combinatorics for the asymmetric exclusion process and Askey–Wilson polynomials. *Duke Math. J.* **159** 385–415. MR2831874 https://doi.org/10.1215/00127094-1433385

MSC2020 subject classifications. Primary 60K35; secondary 60J27, 82C22.
Key words and phrases. Exclusion process, mixing times, coupling, second class particles.

[12] CORWIN, I. and DIMITROV, E. (2018). Transversal fluctuations of the ASEP, stochastic six vertex model, and Hall–Littlewood Gibbsian line ensembles. *Comm. Math. Phys.* **363** 435–501. MR3851820 https://doi.org/10.1007/s00220-018-3139-3
[13] CORWIN, I. and SHEN, H. (2018). Open ASEP in the weakly asymmetric regime. *Comm. Pure Appl. Math.* **71** 2065–2128. MR3861074 https://doi.org/10.1002/cpa.21744
[14] CRAMPE, N., MALLICK, K., RAGOUCY, E. and VANICAT, M. (2015). Open two-species exclusion processes with integrable boundaries. *J. Phys. A* **48** 175002, 18. MR3335684 https://doi.org/10.1088/1751-8113/48/17/175002
[15] DE GIER, J. and ESSLER, F. H. L. (2005). Bethe ansatz solution of the asymmetric exclusion process with open boundaries. *Phys. Rev. Lett.* **95** 240601, 4. MR2185698 https://doi.org/10.1103/PhysRevLett.95.240601
[16] DE GIER, J. and ESSLER, F. H. L. (2006). Exact spectral gaps of the asymmetric exclusion process with open boundaries. *J. Stat. Mech. Theory Exp.* **2006** P12011–P12011. https://doi.org/10.1088/1742-5468/2006/12/p12011
[17] DE GIER, J. and ESSLER, F. H. L. (2008). Slowest relaxation mode of the partially asymmetric exclusion process with open boundaries. *J. Phys. A* **41** 485002, 25. MR2515875 https://doi.org/10.1088/1751-8113/41/48/485002
[18] DERRIDA, B., EVANS, M. R., HAKIM, V. and PASQUIER, V. (1993). Exact solution of a 1D asymmetric exclusion model using a matrix formulation. *J. Phys. A* **26** 1493–1517. MR1219679
[19] DIACONIS, P. and STROOCK, D. (1991). Geometric bounds for eigenvalues of Markov chains. *Ann. Appl. Probab.* **1** 36–61. MR1097463
[20] FERRARI, P. A. (1992). Shock fluctuations in asymmetric simple exclusion. *Probab. Theory Related Fields* **91** 81–101. MR1142763 https://doi.org/10.1007/BF01194491
[21] FERRARI, P. A. and FONTES, L. R. G. (1994). Shock fluctuations in the asymmetric simple exclusion process. *Probab. Theory Related Fields* **99** 305–319. MR1278887 https://doi.org/10.1007/BF01199027
[22] FERRARI, P. A. and FONTES, L. R. G. (1994). Current fluctuations for the asymmetric simple exclusion process. *Ann. Probab.* **22** 820–832. MR1288133
[23] FERRARI, P. A., KIPNIS, C. and SAADA, E. (1991). Microscopic structure of travelling waves in the asymmetric simple exclusion process. *Ann. Probab.* **19** 226–244. MR1085334
[24] FILL, J. A. (1991). Eigenvalue bounds on convergence to stationarity for nonreversible Markov chains, with an application to the exclusion process. *Ann. Appl. Probab.* **1** 62–87. MR1097464
[25] FINN, C., RAGOUCY, E. and VANICAT, M. (2018). Matrix product solution to multi-species ASEP with open boundaries. *J. Stat. Mech. Theory Exp.* **4** 043201, 28. MR3801953 https://doi.org/10.1088/1742-5468/aab1b5
[26] GORISSEN, M., LAZARESCU, A., MALLICK, K. and VANDERZANDE, C. (2012). Exact current statistics of the asymmetric simple exclusion process with open boundaries. *Phys. Rev. Lett.* **109** 170601. https://doi.org/10.1103/PhysRevLett.109.170601
[27] HELBING, D. (2001). Traffic and related self-driven many-particle systems. *Rev. Modern Phys.* **73** 1067–1141. MR3875436 https://doi.org/10.1103/RevModPhys.73.1067
[28] HERMON, J. and PYMAR, R. (2020). The exclusion process mixes (almost) faster than independent particles. *Ann. Probab.* **48** 3077–3123. MR4164461 https://doi.org/10.1214/20-AOP1455
[29] JUNG, P. (2003). Extremal reversible measures for the exclusion process. *J. Stat. Phys.* **112** 165–191. MR1991035 https://doi.org/10.1023/A:1023679620839
[30] LABBÉ, C. and LACOIN, H. (2019). Cutoff phenomenon for the asymmetric simple exclusion process and the biased card shuffling. *Ann. Probab.* **47** 1541–1586. MR3945753 https://doi.org/10.1214/18-AOP1290
[31] LABBÉ, C. and LACOIN, H. (2020). Mixing time and cutoff for the weakly asymmetric simple exclusion process. *Ann. Appl. Probab.* **30** 1847–1883. MR4132639 https://doi.org/10.1214/19-AAP1545
[32] LACOIN, H. (2016). Mixing time and cutoff for the adjacent transposition shuffle and the simple exclusion. *Ann. Probab.* **44** 1426–1487. MR3474475 https://doi.org/10.1214/15-AOP1004
[33] LACOIN, H. (2016). The cutoff profile for the simple exclusion process on the circle. *Ann. Probab.* **44** 3399–3430. MR3551201 https://doi.org/10.1214/15-AOP1053
[34] LACOIN, H. (2017). The simple exclusion process on the circle has a diffusive cutoff window. *Ann. Inst. Henri Poincaré Probab. Stat.* **53** 1402–1437. MR3689972 https://doi.org/10.1214/16-AIHP759
[35] LANDIM, C., MILANÉS, A. and OLLA, S. (2008). Stationary and nonequilibrium fluctuations in boundary driven exclusion processes. *Markov Process. Related Fields* **14** 165–184. MR2437527
[36] LAZARESCU, A. (2013). Matrix ansatz for the fluctuations of the current in the ASEP with open boundaries. *J. Phys. A* **46** 145003, 21. MR3041516 https://doi.org/10.1088/1751-8113/46/14/145003
[37] LAZARESCU, A. (2015). The physicist's companion to current fluctuations: One-dimensional bulk-driven lattice gases. *J. Phys. A* **48** 503001, 80. MR3434835 https://doi.org/10.1088/1751-8113/48/50/503001

[38] LEVIN, D. A. and PERES, Y. (2016). Mixing of the exclusion process with small bias. *J. Stat. Phys.* **165** 1036–1050. MR3575636 https://doi.org/10.1007/s10955-016-1664-z

[39] LEVIN, D. A., PERES, Y. and WILMER, E. L. (2017). *Markov Chains and Mixing Times*. Amer. Math. Soc., Providence, RI. Second edition of [MR2466937], With a chapter on "Coupling from the past" by James G. Propp and David B. Wilson. MR3726904 https://doi.org/10.1090/mbk/107

[40] LIGGETT, T. M. (1975). Ergodic theorems for the asymmetric simple exclusion process. *Trans. Amer. Math. Soc.* **213** 237–261. MR0410986 https://doi.org/10.2307/1998046

[41] LIGGETT, T. M. (1999). *Stochastic Interacting Systems: Contact, Voter and Exclusion Processes. Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **324**. Springer, Berlin. MR1717346 https://doi.org/10.1007/978-3-662-03990-8

[42] LIGGETT, T. M. (2010). *Continuous Time Markov Processes: An Introduction. Graduate Studies in Mathematics* **113**. Amer. Math. Soc., Providence, RI. MR2574430 https://doi.org/10.1090/gsm/113

[43] MACDONALD, C. T., GIBBS, J. H. and PIPKIN, A. C. (1968). Kinetics of biopolymerization on nucleic acid templates. *Biopolymers: Original Research on Biomolecules* **6** 1–25.

[44] MANDELSHTAM, O. (2015). A determinantal formula for Catalan tableaux and TASEP probabilities. *J. Combin. Theory Ser. A* **132** 120–141. MR3311340 https://doi.org/10.1016/j.jcta.2014.12.005

[45] MANDELSHTAM, O. (2015). Matrix ansatz and combinatorics of the k-species PASEP. arXiv:1508.04115 [math.CO].

[46] NAM, D. and NESTORIDI, E. (2019). Cutoff for the cyclic adjacent transposition shuffle. *Ann. Appl. Probab.* **29** 3861–3892. MR4047994 https://doi.org/10.1214/19-AAP1495

[47] OLIVEIRA, R. I. (2013). Mixing of the symmetric exclusion processes in terms of the corresponding single-particle random walk. *Ann. Probab.* **41** 871–913. MR3077529 https://doi.org/10.1214/11-AOP714

[48] PAREKH, S. (2019). The KPZ limit of ASEP with boundary. *Comm. Math. Phys.* **365** 569–649. MR3907953 https://doi.org/10.1007/s00220-018-3258-x

[49] PERES, Y. and WINKLER, P. (2013). Can extra updates delay mixing? *Comm. Math. Phys.* **323** 1007–1016. MR3106501 https://doi.org/10.1007/s00220-013-1776-0

[50] PRÄHOFER, M. and SPOHN, H. (2002). Current fluctuations for the totally asymmetric simple exclusion process. In *In and Out of Equilibrium* (*Mambucaba*, 2000). *Progress in Probability* **51** 185–204. Birkhäuser, Boston, MA. MR1901953

[51] SASAMOTO, T. (1999). One-dimensional partially asymmetric simple exclusion process with open boundaries: Orthogonal polynomials approach. *J. Phys. A* **32** 7109–7131. MR1732546 https://doi.org/10.1088/0305-4470/32/41/306

[52] SASAMOTO, T. (2000). Density profile of the one-dimensional partially asymmetric simple exclusion process with open boundaries. *J. Phys. Soc. Jpn.* **69** 1055–1067. https://doi.org/10.1143/JPSJ.69.1055

[53] SCHMID, D. (2019). Mixing times for the simple exclusion process in ballistic random environment. *Electron. J. Probab.* **24** Paper No. 22, 25. MR3933201 https://doi.org/10.1214/19-EJP286

[54] SCHMID, D. (2021). Mixing times for the TASEP in the maximal current phase. Preprint, https://arxiv.org/abs/2104.12745.

[55] SPITZER, F. (1970). Interaction of Markov processes. *Adv. Math.* **5** 246–290. MR0268959 https://doi.org/10.1016/0001-8708(70)90034-4

[56] TESCHL, G. (2012). *Ordinary Differential Equations and Dynamical Systems. Graduate Studies in Mathematics* **140**. Amer. Math. Soc., Providence, RI. MR2961944 https://doi.org/10.1090/gsm/140

[57] UCHIYAMA, M. (2008). Two-species asymmetric simple exclusion process with open boundaries. *Chaos Solitons Fractals* **35** 398–407. MR2359487 https://doi.org/10.1016/j.chaos.2006.05.013

[58] UCHIYAMA, M., SASAMOTO, T. and WADATI, M. (2004). Asymmetric simple exclusion process with open boundaries and Askey–Wilson polynomials. *J. Phys. A* **37** 4985–5002. MR2065218 https://doi.org/10.1088/0305-4470/37/18/006

[59] UCHIYAMA, M. and WADATI, M. (2005). Correlation function of asymmetric simple exclusion process with open boundaries. *J. Nonlinear Math. Phys.* **12** 676–688. MR2118894 https://doi.org/10.2991/jnmp.2005.12.s1.52

[60] WILSON, D. B. (2004). Mixing times of Lozenge tiling and card shuffling Markov chains. *Ann. Appl. Probab.* **14** 274–325. MR2023023 https://doi.org/10.1214/aoap/1075828054

[61] WOOD, A. J., BLYTHE, R. A. and EVANS, M. R. (2020). Combinatorial mappings of exclusion processes. *J. Phys. A* **53** 123001, 51. MR4084240 https://doi.org/10.1088/1751-8121/ab73aa

The Annals of Applied Probability
2023, Vol. 33, No. 2, 1213–1251
https://doi.org/10.1214/22-AAP1840

A KESTEN–STIGUM TYPE THEOREM FOR A SUPERCRITICAL MULTITYPE BRANCHING PROCESS IN A RANDOM ENVIRONMENT

BY ION GRAMA[a], QUANSHENG LIU[b] AND ERWAN PIN[c]

Univ Bretagne Sud, CNRS UMR 6205, LMBA, [a]*ion.grama@univ-ubs.fr,* [b]*quansheng.liu@univ-ubs.fr,* [c]*erwan.pin@ac-rouen.fr*

Consider a multitype branching process in a random environment, whose reproduction law of generation n depends on the random environment at time n, unlike a constant distribution assumed in the Galton–Watson process. The famous Kesten–Stigum theorem for a supercritical multitype Galton–Watson process gives a precise description of the exponential increasing rate of the population size via a criterion for the nondegeneracy of the fundamental martingale. Finding the corresponding result in the random environment case is a longstanding problem. For the single-type case the problem has been solved by Athreya and Karlin for the sufficiency (*Ann. Math. Stat.* **42** (1971) 1499–1520) and Tanny for the necessity (*Stochastic Process. Appl.* **28** (1988) 123–139), but for the multitype case it has been open for 50 years. Here we solve this problem in the typical case, by constructing a suitable martingale which reduces to the fundamental one in the constat environment case, and by establishing a criterion for the nondegeneracy of its limit. The convergence in law of the direction of the branching process is also considered. Our results open ways in establishing other limit theorems, such as law of large numbers, central limit theorems, Berry–Essen bound, and large deviation results.

REFERENCES

[1] ATHREYA, K. B. and KARLIN, S. (1971). On branching processes with random environments. I. Extinction probabilities. *Ann. Math. Stat.* **42** 1499–1520. MR0298780 https://doi.org/10.1214/aoms/1177693150

[2] ATHREYA, K. B. and KARLIN, S. (1971). Branching processes with random environments. II. Limit theorems. *Ann. Math. Stat.* **42** 1843–1858. MR0298781 https://doi.org/10.1214/aoms/1177693051

[3] ATHREYA, K. B. and NEY, P. E. (1972). *Branching Processes. Die Grundlehren der Mathematischen Wissenschaften, Band* 196. Springer, New York. MR0373040

[4] BANSAYE, V. and BERESTYCKI, J. (2009). Large deviations for branching processes in random environment. *Markov Process. Related Fields* **15** 493–524. MR2598126

[5] BIGGINS, J. D., COHN, H. and NERMAN, O. (1999). Multi-type branching in varying environment. *Stochastic Process. Appl.* **83** 357–400. MR1708214 https://doi.org/10.1016/S0304-4149(99)00049-6

[6] CHOW, Y. S. and TEICHER, H. (2012). *Probability Theory: Independence, interchangeability, martingales*. Springer Science and Business Media, New York.

[7] COHN, H. (1989). On the growth of the multitype supercritical branching process in a random environment. *Ann. Probab.* **17** 1118–1123. MR1009447

[8] CORNFELD, I. P., FOMIN, S. V. and SINAĬ, Y. G. (1982). *Ergodic Theory. Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **245**. Springer, New York. MR0832433 https://doi.org/10.1007/978-1-4615-6927-5

[9] D'YAKONOVA, E. E. (2015). Limit theorem for a multitype critical branching process evolving in a random medium. *Discrete Math. Appl.* **25** 137–147.

[10] DOLGOPYAT, D., HEBBAR, P., KORALOV, L. and PERLMAN, M. (2018). Multi-type branching processes with time-dependent branching rates. *J. Appl. Probab.* **55** 701–727. MR3877306 https://doi.org/10.1017/jpr.2018.46

[11] DURRETT, R. (1991). *Probability: Theory and examples. The Wadsworth & Brooks/Cole Statistics/Probability Series*. Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, CA. MR1068527

MSC2020 subject classifications. Primary 60J80, 60K37; secondary 60J85.

Key words and phrases. Multitype branching processes, random environment, Kesten–Stigum theorem, products of random matrices, martingale.

[12] FURSTENBERG, H. and KESTEN, H. (1960). Products of random matrices. *Ann. Math. Stat.* **31** 457–469. MR0121828 https://doi.org/10.1214/aoms/1177705909

[13] GRAMA, I., LIU, Q. and MIQUEU, E. (2017). Berry–Esseen's bound and Cramér's large deviation expansion for a supercritical branching process in a random environment. *Stochastic Process. Appl.* **127** 1255–1281. MR3619270 https://doi.org/10.1016/j.spa.2016.07.014

[14] GRAMA, I., LIU, Q. and PIN, E. Berry–Esseen's bound and harmonic moments for supercritical multi-type branching processes in random environments. hal-02911865.

[15] GRAMA, I., LIU, Q. and PIN, E. Cramér type moderate deviation expansion for supercritical multi-type branching processes in random environments. hal-02934081.

[16] GRAMA, I., LIU, Q. and PIN, E. (2022). Convergence in L^p for a supercritical multi-type branching process in a random environment. *Proc. Steklov Inst. Math.* **316** 160–183; Trudy Mat. Inst. Steklova, 316, Steklov Math. Inst., Moscow, 2022, 169–194; hal-02934079.

[17] GUIVARC'H, Y. and LIU, Q. (2001). Propriétés asymptotiques des processus de branchement en environnement aléatoire. *C. R. Acad. Sci. Paris Sér. I Math.* **332** 339–344. MR1821473 https://doi.org/10.1016/S0764-4442(00)01783-3

[18] HARRIS, T. E. (1963). *The Theory of Branching Processes*. Springer, Berlin. MR0163361

[19] HENNION, H. (1997). Limit theorems for products of positive random matrices. *Ann. Probab.* **25** 1545–1587. MR1487428 https://doi.org/10.1214/aop/1023481103

[20] HESS, C., SERI, R. and CHOIRAT, C. (2010). Ergodic theorems for extended real-valued random variables. *Stochastic Process. Appl.* **120** 1908–1919. MR2673980 https://doi.org/10.1016/j.spa.2010.05.008

[21] HONG, W., LIU, M. and VATUTIN, V. (2019). Limit theorems for supercritical MBPRE with linear fractional offspring distributions. *Markov Process. Related Fields* **25** 1–31. MR3965092

[22] HUANG, C. and LIU, Q. (2014). Convergence in L^p and its exponential rate for a branching process in a random environment. *Electron. J. Probab.* **19** no. 104. MR3275856 https://doi.org/10.1214/EJP.v19-3388

[23] JONES, O. D. (1997). On the convergence of multitype branching processes with varying environments. *Ann. Appl. Probab.* **7** 772–801. MR1459270 https://doi.org/10.1214/aoap/1034801253

[24] KAPLAN, N. (1974). Some results about multidimensional branching processes with random environments. *Ann. Probab.* **2** 441–455. MR0378127 https://doi.org/10.1214/aop/1176996659

[25] KERSTING, G. and VATUTIN, V. (2017). *Discrete time branching processes in random environment*. Wiley-ISTE.

[26] KESTEN, H. and SPITZER, F. (1984). Convergence in Distribution of products of Random Matrices. *Z. Wahrsch. Verw. Gebiete* **67** 363–386.

[27] KESTEN, H. and STIGUM, B. P. (1966). A limit theorem for multidimensional Galton–Watson processes. *Ann. Math. Stat.* **37** 1211–1223. MR0198552 https://doi.org/10.1214/aoms/1177699266

[28] KINGMAN, J. F. C. (1973). Subadditive ergodic theory. *Ann. Probab.* **1** 883–909. MR0356192 https://doi.org/10.1214/aop/1176996798

[29] KURTZ, T., LYONS, R., PEMANTLE, R. and PERES, Y. (1997). A conceptual proof of the Kesten–Stigum theorem for multi-type branching processes. In *Classical and Modern Branching Processes* (*Minneapolis*, *MN*, 1994) (K. B. Athreya and P. Jagers, eds.). *IMA Vol. Math. Appl.* **84** 181–185. Springer, New York. MR1601737 https://doi.org/10.1007/978-1-4612-1862-3_14

[30] LE PAGE, E., PEIGNÉ, M. and PHAM, C. (2018). The survival probability of a critical multi-type branching process in i.i.d. random environment. *Ann. Probab.* **46** 2946–2972. MR3846842 https://doi.org/10.1214/17-AOP1243

[31] LYONS, R., PEMANTLE, R. and PERES, Y. (1995). Conceptual proofs of $L \log L$ criteria for mean behavior of branching processes. *Ann. Probab.* **23** 1125–1138. MR1349164

[32] SMITH, W. L. and WILKINSON, W. E. (1969). On branching processes in random environments. *Ann. Math. Stat.* **40** 814–827. MR0246380 https://doi.org/10.1214/aoms/1177697589

[33] TANNY, D. (1974). A zero-one law for stationary sequences. *Z. Wahrsch. Verw. Gebiete* **30** 139–148. MR0375448 https://doi.org/10.1007/BF00532266

[34] TANNY, D. (1981). On multitype branching processes in a random environment. *Adv. in Appl. Probab.* **13** 464–497. MR0615947 https://doi.org/10.2307/1426781

[35] TANNY, D. (1988). A necessary and sufficient condition for a branching process in a random environment to grow like the product of its means. *Stochastic Process. Appl.* **28** 123–139. MR0936379 https://doi.org/10.1016/0304-4149(88)90070-1

[36] VATUTIN, V. and WACHTEL, V. (2018). Multi-type subcritical branching processes in a random environment. *Adv. in Appl. Probab.* **50** 281–289. MR3905107 https://doi.org/10.1017/apr.2018.86

[37] VATUTIN, V. A. and D'YAKONOVA, E. E. (2017). Multitype branching processes in a random environment: Nonextinction probability in the critical case. *Theory Probab. Appl.* **62** 506–521.

The Annals of Applied Probability
2023, Vol. 33, No. 2, 1252–1294
https://doi.org/10.1214/22-AAP1841

COMPUTER-ASSISTED PROOF OF SHEAR-INDUCED CHAOS IN STOCHASTICALLY PERTURBED HOPF SYSTEMS

BY MAXIME BREDEN[1,a] AND MAXIMILIAN ENGEL[2,b]

[1]*CMAP, École Polytechnique,* [a]*maxime.breden@polytechnique.edu*
[2]*Department of Mathematics and Computer Science, Freie Universiät Berlin,* [b]*maximilian.engel@fu-berlin.de*

We confirm a long-standing conjecture concerning shear-induced chaos in stochastically perturbed systems exhibiting a Hopf bifurcation. The method of showing the main chaotic property, a positive *Lyapunov exponent*, is a *computer-assisted proof*. Using the recently developed theory of conditioned Lyapunov exponents on bounded domains and the modified Furstenberg–Khasminskii formula, the problem boils down to the rigorous computation of eigenfunctions of the Kolmogorov operators describing distributions of the underlying stochastic process.

REFERENCES

[1] ARIOLI, G. and KOCH, H. (2019). Non-radial solutions for some semilinear elliptic equations on the disk. *Nonlinear Anal*. **179** 294–308. MR3886634 https://doi.org/10.1016/j.na.2018.09.001
[2] ARIOLI, G., KOCH, H. and TERRACINI, S. (2005). Two novel methods and multi-mode periodic solutions for the Fermi-Pasta-Ulam model. *Comm. Math. Phys*. **255** 1–19. MR2123374 https://doi.org/10.1007/s00220-004-1251-z
[3] ARNOLD, L. (1998). *Random Dynamical Systems*. *Springer Monographs in Mathematics*. Springer, Berlin. MR1723992 https://doi.org/10.1007/978-3-662-12878-7
[4] BAXENDALE, P. H. (1991). Statistical equilibrium and two-point motion for a stochastic flow of diffeomorphisms. In *Spatial Stochastic Processes*. *Progress in Probability* **19** 189–218. Birkhäuser, Boston, MA. MR1144097
[5] BEDROSSIAN, J., BLUMENTHAL, A. and PUNSHON-SMITH, S. (2022). A regularity method for lower bounds on the Lyapunov exponent for stochastic differential equations. *Invent. Math*. **227** 429–516. MR4372219 https://doi.org/10.1007/s00222-021-01069-7
[6] BEHNKE, H. and GOERISCH, F. (1994). Inclusions for eigenvalues of selfadjoint problems. In *Topics in Validated Computations* (*Oldenburg*, 1993). *Stud. Comput. Math*. **5** 277–322. North-Holland, Amsterdam. MR1318957 https://doi.org/10.1016/0021-8502(94)90369-7
[7] BLACKBEARD, N., DUTTA, P. S., ERZGRABER, H. and WIECZOREK, S. (2014). From synchronisation to optical turbulence in laser arrays. *Phys. D* **286–287** 43–58.
[8] BLACKBEARD, N., ERZGRÄBER, H. and WIECZOREK, S. (2011). Shear-induced bifurcations and chaos in models of three coupled lasers. *SIAM J. Appl. Dyn. Syst*. **10** 469–509. MR2810625 https://doi.org/10.1137/100817383
[9] BLUMENTHAL, A., XUE, J. and YOUNG, L.-S. (2017). Lyapunov exponents for random perturbations of some area-preserving maps including the standard map. *Ann. of Math*. (2) **185** 285–310. MR3583355 https://doi.org/10.4007/annals.2017.185.1.5
[10] BREDEN, M. and ENGEL, M. (2021). Matlab code for "Computer-assisted proof of shear-induced chaos in stochastically perturbed Hopf systems." Available at https://github.com/MaximeBreden/Shear-inducedChaos.
[11] BREDEN, M. and KUEHN, C. (2019). Rigorous validation of stochastic transition paths. *J. Math. Pures Appl*. (9) **131** 88–129. MR4021171 https://doi.org/10.1016/j.matpur.2019.04.012
[12] BREYER, L. A. and ROBERTS, G. O. (1999). A quasi-ergodic theorem for evanescent processes. *Stochastic Process. Appl*. **84** 177–186. MR1719286 https://doi.org/10.1016/S0304-4149(99)00018-6
[13] CANCÈS, E., DUSSON, G., MADAY, Y., STAMM, B. and VOHRALÍK, M. (2020). Guaranteed a posteriori bounds for eigenvalues and eigenvectors: Multiplicities and clusters. *Math. Comp*. **89** 2563–2611. MR4136540 https://doi.org/10.1090/mcom/3549

MSC2020 subject classifications. Primary 35P99, 37-04, 37M25; secondary 37H15, 60J99.
Key words and phrases. Homotopy method, Kolmogorov operators, Lyapunov exponents, quasi-ergodic distribution.

[14] Champagnat, N., Coulibaly-Pasquier, K. A. and Villemonais, D. (2018). Criteria for exponential convergence to quasi-stationary distributions and applications to multi-dimensional diffusions. In *Séminaire de Probabilités XLIX*. *Lecture Notes in Math*. **2215** 165–182. Springer, Cham. MR3837104

[15] Champagnat, N. and Villemonais, D. (2016). Exponential convergence to quasi-stationary distribution and Q-process. *Probab. Theory Related Fields* **164** 243–283. MR3449390 https://doi.org/10.1007/s00440-014-0611-7

[16] Collet, P., Martínez, S. and San Martín, J. (2013). *Quasi-Stationary Distributions*: *Markov Chains*, *Diffusions and Dynamical Systems*. *Probability and Its Applications* (*New York*). Springer, Heidelberg. MR2986807 https://doi.org/10.1007/978-3-642-33131-2

[17] Crauel, H. (1991). Markov measures for random dynamical systems. *Stoch*. *Stoch*. *Rep*. **37** 153–173. MR1148346 https://doi.org/10.1080/17442509108833733

[18] Day, S., Lessard, J.-P. and Mischaikow, K. (2007). Validated continuation for equilibria of PDEs. *SIAM J. Numer. Anal*. **45** 1398–1424. MR2338393 https://doi.org/10.1137/050645968

[19] DeVille, R. E. L., Sri Namachchivaya, N. and Rapti, Z. (2011). Stability of a stochastic two-dimensional non-Hamiltonian system. *SIAM J. Appl. Math*. **71** 1458–1475. MR2835242 https://doi.org/10.1137/100782139

[20] Dijkstra, H. A., Frankcombe, L. M. and von der Heydt, A. S. (2008). A stochastic dynamical systems view of the Atlantic multidecadal oscillation. *Philos*. *Trans*. *R*. *Soc*. *Lond*. *Ser. A Math*. *Phys*. *Eng*. *Sci*. **366** 2545–2560. MR2415445 https://doi.org/10.1098/rsta.2008.0031

[21] Dimitroff, G. and Scheutzow, M. (2011). Attractors and expansion for Brownian flows. *Electron*. *J. Probab*. **16** 1193–1213. MR2827455 https://doi.org/10.1214/EJP.v16-894

[22] Doan, T. S., Engel, M., Lamb, J. S. W. and Rasmussen, M. (2018). Hopf bifurcation with additive noise. *Nonlinearity* **31** 4567–4601. MR3846439 https://doi.org/10.1088/1361-6544/aad208

[23] Duarte, P. (1994). Plenty of elliptic islands for the standard family of area preserving maps. *Ann*. *Inst*. *H*. *Poincaré C Anal*. *Non Linéaire* **11** 359–409. MR1287238 https://doi.org/10.1016/S0294-1449(16)30180-9

[24] Engel, M., Lamb, J. S. W. and Rasmussen, M. (2019). Bifurcation analysis of a stochastically driven limit cycle. *Comm*. *Math*. *Phys*. **365** 935–942. MR3916984 https://doi.org/10.1007/s00220-019-03298-7

[25] Engel, M., Lamb, J. S. W. and Rasmussen, M. (2019). Conditioned Lyapunov exponents for random dynamical systems. *Trans*. *Amer. Math*. *Soc*. **372** 6343–6370. MR4024524 https://doi.org/10.1090/tran/7803

[26] Flandoli, F., Gess, B. and Scheutzow, M. (2017). Synchronization by noise. *Probab. Theory Related Fields* **168** 511–556. MR3663624 https://doi.org/10.1007/s00440-016-0716-2

[27] Galatolo, S., Monge, M. and Nisoli, I. (2020). Existence of noise induced order, a computer aided proof. *Nonlinearity* **33** 4237–4276. MR4127778 https://doi.org/10.1088/1361-6544/ab86cd

[28] Galias, Z. and Zgliczyński, P. (1998). Computer assisted proof of chaos in the Lorenz equations. *Phys*. *D* **115** 165–188. MR1626596 https://doi.org/10.1016/S0167-2789(97)00233-9

[29] Goerisch, F. (1987). Ein Stufenverfahren zur Berechnung von Eigenwertschranken. In *Numerical Treatment of Eigenvalue Problems*, *Vol*. 4 (*Oberwolfach*, 1986). *Internat*. *Schriftenreihe Numer. Math*. **83** 104–114. Birkhäuser, Basel. MR1023876

[30] Gómez-Serrano, J. (2019). Computer-assisted proofs in PDE: A survey. *SeMA J*. **76** 459–484. MR3990999 https://doi.org/10.1007/s40324-019-00186-x

[31] Kapela, T., Mrozek, M., Wilczak, D. and Zgliczyński, P. (2021). CAPD::DynSys: A flexible C++ toolbox for rigorous numerical analysis of dynamical systems. *Commun*. *Nonlinear Sci*. *Numer. Simul*. **101** Paper No. 105578, 26 pp. MR4283203 https://doi.org/10.1016/j.cnsns.2020.105578

[32] Koch, H., Schenkel, A. and Wittwer, P. (1996). Computer-assisted proofs in analysis and programming in logic: A case study. *SIAM Rev*. **38** 565–604. MR1420838 https://doi.org/10.1137/S0036144595284180

[33] Lanford, O. E. III (1982). A computer-assisted proof of the Feigenbaum conjectures. *Bull*. *Amer. Math*. *Soc*. (*N.S.*) **6** 427–434. MR0648529 https://doi.org/10.1090/S0273-0979-1982-15008-X

[34] Lin, K. K. and Young, L.-S. (2008). Shear-induced chaos. *Nonlinearity* **21** 899–922. MR2412320 https://doi.org/10.1088/0951-7715/21/5/002

[35] Liu, X. (2015). A framework of verified eigenvalue bounds for self-adjoint differential operators. *Appl*. *Math*. *Comput*. **267** 341–355. MR3399052 https://doi.org/10.1016/j.amc.2015.03.048

[36] Marti, J. T. (1983). Evaluation of the least constant in Sobolev's inequality for $H^1(0,s)$. *SIAM J. Numer. Anal*. **20** 1239–1242. MR0723841 https://doi.org/10.1137/0720094

[37] Méléard, S. and Villemonais, D. (2012). Quasi-stationary distributions and population processes. *Probab. Surv*. **9** 340–410. MR2994898 https://doi.org/10.1214/11-PS191

[38] MISCHAIKOW, K. and MROZEK, M. (1995). Chaos in the Lorenz equations: A computer-assisted proof. *Bull. Amer. Math. Soc.* (*N.S.*) **32** 66–72. MR1276767 https://doi.org/10.1090/S0273-0979-1995-00558-6

[39] NAKAO, M. T., PLUM, M. and WATANABE, Y. (2019). *Numerical Verification Methods and Computer-Assisted Proofs for Partial Differential Equations*. *Springer Series in Computational Mathematics* **53**. Springer, Singapore. MR3971222 https://doi.org/10.1007/978-981-13-7669-6

[40] OISHI, S. (1995). Numerical verification of existence and inclusion of solutions for nonlinear operator equations. *J. Comput. Appl. Math.* **60** 171–185. MR1354654 https://doi.org/10.1016/0377-0427(94)00090-N

[41] PLUM, M. (1990). Eigenvalue inclusions for second-order ordinary differential operators by a numerical homotopy method. *Z. Angew. Math. Phys.* **41** 205–226. MR1045812 https://doi.org/10.1007/BF00945108

[42] PLUM, M. (1991). Computer-assisted existence proofs for two-point boundary value problems. *Computing* **46** 19–34. MR1100582 https://doi.org/10.1007/BF02239009

[43] RUMP, S. M. (1999). INTLAB—interval laboratory. In *Developments in Reliable Computing* 77–104. Springer, Berlin.

[44] RUMP, S. M. (2010). Verification methods: Rigorous results using floating-point arithmetic. *Acta Numer.* **19** 287–449. MR2652784 https://doi.org/10.1017/S096249291000005X

[45] SCHUSS, Z. (2010). *Theory and Applications of Stochastic Processes*: *An Analytical Approach*. *Applied Mathematical Sciences* **170**. Springer, New York. MR2583642 https://doi.org/10.1007/978-1-4419-1605-1

[46] SHEOMBARSING, R. (2018). Validated Chebyshev-based computations for ordinary and partial differential equations. Ph.D. thesis, VU Amsterdam.

[47] TAKAYASU, A., LIU, X. and OISHI, S. (2013). Verified computations to semilinear elliptic boundary value problems on arbitrary polygonal domains. *Nonlinear Theory Appl.*, *IEICE* **4** 34–61.

[48] TANAKA, K. (2020). Numerical verification method for positive solutions of elliptic problems. *J. Comput. Appl. Math.* **370** 112647, 10 pp. MR4042725 https://doi.org/10.1016/j.cam.2019.112647

[49] TREFETHEN, L. N. (2013). *Approximation Theory and Approximation Practice*. SIAM, Philadelphia, PA. MR3012510

[50] TUCKER, W. (2002). A rigorous ODE solver and Smale's 14th problem. *Found. Comput. Math.* **2** 53–117. MR1870856 https://doi.org/10.1007/s002080010018

[51] TUCKER, W. (2011). *Validated Numerics*: *A Short Introduction to Rigorous Computations*. Princeton Univ. Press, Princeton, NJ. MR2807595

[52] URABE, M. (1965). Galerkin's prodcedure for nonlinear periodic systems. *Arch. Ration. Mech. Anal.* **20** 120–152. MR0182771 https://doi.org/10.1007/BF00284614

[53] VAN DEN BERG, J. B. and LESSARD, J.-P. (2015). Rigorous numerics in dynamics. *Notices Amer. Math. Soc.* **62** 1057–1061. MR3444942 https://doi.org/10.1090/noti1276

[54] WATANABE, Y., KINOSHITA, T. and NAKAO, M. T. (2020). Some improvements of invertibility verifications for second-order linear elliptic operators. *Appl. Numer. Math.* **154** 36–46. MR4080596 https://doi.org/10.1016/j.apnum.2020.03.016

[55] WIECZOREK, S. (2009). Stochastic bifurcation in noise-driven lasers and Hopf oscillators. *Phys. Rev. E* (3) **79** 036209, 10 pp. MR2497189 https://doi.org/10.1103/PhysRevE.79.036209

[56] YAGLOM, A. M. (1947). Certain limit theorems of the theory of branching random processes. *Dokl. Akad. Nauk SSSR* (*N.S.*) **56** 795–798. MR0022045

[57] YAMAMOTO, N. (1998). A numerical verification method for solutions of boundary value problems with local uniqueness by Banach's fixed-point theorem. *SIAM J. Numer. Anal.* **35** 2004–2013. MR1639986 https://doi.org/10.1137/S0036142996304498

[58] YOUNG, L.-S. (2008). Chaotic phenomena in three settings: Large, noisy and out of equilibrium. *Nonlinearity* **21** T245–T252. MR2448225 https://doi.org/10.1088/0951-7715/21/11/T04

The Annals of Applied Probability
2023, Vol. 33, No. 2, 1295–1326
https://doi.org/10.1214/22-AAP1842

OPTIMAL CLEANING FOR SINGULAR VALUES OF CROSS-COVARIANCE MATRICES

BY FLORENT BENAYCH-GEORGES[a], JEAN-PHILIPPE BOUCHAUD[b] AND MARC POTTERS[c]

CFM, [a]*florent.benaych-georges@cfm.fr,* [b]*jean-philippe.bouchaud@cfm.fr,* [c]*marc.potters@cfm.fr*

We give a new algorithm for the estimation of the cross-covariance matrix $\mathbb{E}XY'$ of two large-dimensional signals $X \in \mathbb{R}^n$, $Y \in \mathbb{R}^p$ in the context where the number T of observations of the pair (X, Y) is large but n/T and p/T are not supposed to be small. In the asymptotic regime where n, p, T are large, with high probability, this algorithm is optimal for the Frobenius norm among *rotationally invariant estimators*, that is, estimators derived from the *empirical estimator* by *cleaning* the singular values, while letting singular vectors unchanged.

REFERENCES

[1] AMSALU, S., DUAN, J., MATZINGER, H. and POPESCU, I. (2018). Recovery of spectrum from estimated covariance matrices and statistical kernels for machine learning and big data. Available at arXiv:1804.09472.

[2] ANDERSON, G. W., GUIONNET, A. and ZEITOUNI, O. (2010). *An Introduction to Random Matrices*. *Cambridge Studies in Advanced Mathematics* **118**. Cambridge Univ. Press, Cambridge. MR2760897

[3] BENAYCH-GEORGES, F. (2022). A very short proof of Ledoit-Péché's RIE formula for covariance matrices. Available at arXiv:2201.05690.

[4] BENAYCH-GEORGES, F. and COUILLET, R. (2016). Spectral analysis of the Gram matrix of mixture models. *ESAIM Probab. Stat*. **20** 217–237. MR3528625 https://doi.org/10.1051/ps/2016007

[5] BENAYCH-GEORGES, F. and KNOWLES, A. (2017). Local semicircle law for Wigner matrices. In *Advanced Topics in Random Matrices*. *Panor. Synthèses* **53** 1–90. Soc. Math. France, Paris. MR3792624

[6] BOSE, A. and BHATTACHARJEE, M. (2019). *Large Covariance and Autocovariance Matrices*. *Monographs on Statistics and Applied Probability* **162**. CRC Press, Boca Raton, FL. MR3821539

[7] BOUCHAUD, J.-P., LALOUX, L., MICELI, M. A. and POTTERS, M. (2007). Large dimension forecasting models and random singular value spectra. *Eur. Phys. J. B* **55** 201–207. MR2293813 https://doi.org/10.1140/epjb/e2006-00204-0

[8] BUN, J., BOUCHAUD, J.-P. and POTTERS, M. (2016). Cleaning correlation matrices. Risk magazine.

[9] BUN, J., BOUCHAUD, J.-P. and POTTERS, M. (2017). Cleaning large correlation matrices: Tools from random matrix theory. *Phys. Rep*. **666** 1–109. MR3590056 https://doi.org/10.1016/j.physrep.2016.10.005

[10] CAPITAINE, M. (2013). Additive/multiplicative free subordination property and limiting eigenvectors of spiked additive deformations of Wigner matrices and spiked sample covariance matrices. *J. Theoret. Probab.* **26** 595–648. MR3090543 https://doi.org/10.1007/s10959-012-0416-5

[11] COUILLET, R., PASCAL, F. and SILVERSTEIN, J. W. (2014). Robust estimates of covariance matrices in the large dimensional regime. *IEEE Trans. Inf. Theory* **60** 7269–7278. MR3273053 https://doi.org/10.1109/TIT.2014.2354045

[12] COUILLET, R., PASCAL, F. and SILVERSTEIN, J. W. (2015). The random matrix regime of Maronna's M-estimator with elliptically distributed samples. *J. Multivariate Anal*. **139** 56–78. MR3349480 https://doi.org/10.1016/j.jmva.2015.02.020

[13] EL KAROUI, N. (2008). Spectrum estimation for large dimensional covariance matrices using random matrix theory. *Ann. Statist*. **36** 2757–2790. MR2485012 https://doi.org/10.1214/07-AOS581

[14] EL KAROUI, N. (2008). Operator norm consistent estimation of large-dimensional sparse covariance matrices. *Ann. Statist*. **36** 2717–2756. MR2485011 https://doi.org/10.1214/07-AOS559

MSC2020 subject classifications. Primary 60B20, 62G05; secondary 15B52.
Key words and phrases. Random matrices, cross-covariance matrices, rotationally invariant estimator.

[15] EL KAROUI, N. (2018). Random matrices and high-dimensional statistics: Beyond covariance matrices. In *Proceedings of the International Congress of Mathematicians—Rio de Janeiro* 2018. *Vol*. *IV*. *Invited Lectures* 2857–2876. World Sci. Publ., Hackensack, NJ. MR3966514

[16] ERDŐS, L., SCHLEIN, B. and YAU, H.-T. (2009). Semicircle law on short scales and delocalization of eigenvectors for Wigner random matrices. *Ann*. *Probab*. **37** 815–852. MR2537522 https://doi.org/10.1214/08-AOP421

[17] ERDŐS, L. and YAU, H.-T. (2017). *A Dynamical Approach to Random Matrix Theory*. *Courant Lecture Notes in Mathematics* **28**. Courant Institute of Mathematical Sciences, New York. MR3699468

[18] TIBSHIRANI, R., HASTIE, T. and FRIEDMAN, J. (2008). *Elements of Statistical Learning*, 2nd ed. *Springer Series in Statistics*. Springer, New York.

[19] KLOPP, O., LOUNICI, K. and TSYBAKOV, A. B. (2017). Robust matrix completion. *Probab*. *Theory Related Fields* **169** 523–564. MR3704775 https://doi.org/10.1007/s00440-016-0736-y

[20] KLOPP, O., LU, Y., TSYBAKOV, A. B. and ZHOU, H. H. (2019). Structured matrix estimation and completion. *Bernoulli* **25** 3883–3911. MR4010976 https://doi.org/10.3150/19-bej1114

[21] KOLTCHINSKII, V., LOUNICI, K. and TSYBAKOV, A. B. (2016). Estimation of low-rank covariance function. *Stochastic Process*. *Appl*. **126** 3952–3967. MR3565487 https://doi.org/10.1016/j.spa.2016.04.006

[22] LALOUX, L., CIZEAU, P., BOUCHAUD, J.-P. and POTTERS, M. (1999). Noise dressing of financial correlation matrices. *Phys*. *Rev*. *Lett*. **83** 1467.

[23] LEDOIT, O. and PÉCHÉ, S. (2011). Eigenvectors of some large sample covariance matrix ensembles. *Probab*. *Theory Related Fields* **151** 233–264. MR2834718 https://doi.org/10.1007/s00440-010-0298-3

[24] LEDOIT, O. and WOLF, M. (2004). A well-conditioned estimator for large-dimensional covariance matrices. *J*. *Multivariate Anal*. **88** 365–411. MR2026339 https://doi.org/10.1016/S0047-259X(03)00096-4

[25] LEDOIT, O. and WOLF, M. (2004). Honey, I shrunk the sample covariance matrix. *J*. *Portf*. *Manag*. **30** 110–119.

[26] LEDOIT, O. and WOLF, M. (2012). Nonlinear shrinkage estimation of large-dimensional covariance matrices. *Ann*. *Statist*. **40** 1024–1060. MR2985942 https://doi.org/10.1214/12-AOS989

[27] VERSHYNIN, R. (2018). *High-Dimensional Probability*. *Cambridge Series in Statistical and Probabilistic Mathematics*: *An Introduction with Applications in Data Science* **47**. Cambridge Univ. Press, Cambridge. MR3837109 https://doi.org/10.1017/9781108231596

The Annals of Applied Probability
2023, Vol. 33, No. 2, 1327–1372
https://doi.org/10.1214/22-AAP1843

A THEORETICAL ANALYSIS OF ONE-DIMENSIONAL DISCRETE GENERATION ENSEMBLE KALMAN PARTICLE FILTERS

BY PIERRE DEL MORAL[a] AND EMMA HORTON[b]

Centre de Recherche Inria Bordeaux Sud-Ouest, [a]*pierre.del-moral@inria.fr,* [b]*emma.horton@inria.fr*

Despite the widespread usage of discrete generation ensemble Kalman particle filtering methodology to solve nonlinear and high-dimensional filtering and inverse problems, little is known about their mathematical foundations. As genetic-type particle filters (a.k.a. sequential Monte Carlo), this ensemble-type methodology can also be interpreted as mean-field particle approximations of the Kalman–Bucy filtering equation. In contrast with conventional mean-field type interacting particle methods equipped with a globally Lipschitz interacting drift-type function, Ensemble Kalman filters depend on a nonlinear and quadratic-type interaction function defined in terms of the sample covariance of the particles.

Most of the literature in applied mathematics and computer science on these sophisticated interacting particle methods amounts to designing different classes of useable observer-type particle methods. These methods are based on a variety of inconsistent but judicious ensemble auxiliary transformations or include additional inflation/localisation-type algorithmic innovations, in order to avoid the inherent time-degeneracy of an insufficient particle ensemble size when solving a filtering problem with an unstable signal.

To the best of our knowledge, the first and the only rigorous mathematical analysis of these sophisticated discrete generation particle filters is developed in the pioneering articles by Le Gland–Monbet–Tran and by Mandel–Cobb–Beezley, which were published in the early 2010s. Nevertheless, besides the fact that these studies prove the asymptotic consistency of the ensemble Kalman filter, they provide exceedingly pessimistic mean-error estimates that grow exponentially fast with respect to the time horizon, even for linear Gaussian filtering problems with stable one-dimensional signals.

In the present article we develop a novel self-contained and complete stochastic perturbation analysis of the fluctuations, the stability, and the long-time performance of these discrete generation ensemble Kalman particle filters, including time-uniform and nonasymptotic mean-error estimates that apply to possibly unstable signals. To the best of our knowledge, these are the first results of this type in the literature on discrete generation particle filters, including the class of genetic-type particle filters and discrete generation ensemble Kalman filters. The stochastic Riccati difference equations considered in this work are also of interest in their own right, as a prototype of a new class of stochastic rational difference equation.

REFERENCES

[1] ALLEN, J. I., EKNES, M. and EVENSEN, G. (2002). An ensemble Kalman filter with a complex marine ecosystem model: Hindcasting phytoplankton in the Cretan Sea. *Ann. Geophys.* **20** 1–13.

[2] ANDERSON, B. D. O. (1971). Stability properties of Kalman–Bucy filters. *J. Franklin Inst.* **291** 137–144. MR0281515 https://doi.org/10.1016/0016-0032(71)90016-0

[3] ANDERSON, J. L. (2001). An ensemble adjustment Kalman filter for data assimilation. *Mon. Weather Rev.* **129** 2884–2903.

MSC2020 subject classifications. Primary 60G35, 60K35, 65C35, 60F05; secondary 60J10, 39A30, 65L07, 93D23.

Key words and phrases. Ensemble Kalman filter, stochastic Riccati difference equation, rational difference equations, noncentral χ-square, Feynman–Kac formulae, uniform estimates with respect to the time horizon, exponential stability, stochastic perturbation theorems.

[4] ANDERSON, J. L. (2003). A local least squares framework for ensemble filtering. *Mon. Weather Rev.* **131** 634–642.

[5] ANDERSON, J. L. (2007). An adaptive covariance inflation error correction algorithms for ensemble filters. *Tellus, Ser. A Dyn. Meteorol. Oceanogr.* **59** 210–224.

[6] ANDERSON, J. L. (2009). Spatially and temporally varying adaptive covariance inflation for ensemble filters. *Tellus, Ser. A Dyn. Meteorol. Oceanogr.* **61** 72–83.

[7] ARNAUDON, M. and DEL MORAL, P. (2019). A variational approach to nonlinear and interacting diffusions. *Stoch. Anal. Appl.* **37** 717–748. MR3991057 https://doi.org/10.1080/07362994.2019.1609985

[8] ARNAUDON, M. and DEL MORAL, P. (2020). A duality formula and a particle Gibbs sampler for continuous time Feynman–Kac measures on path spaces. *Electron. J. Probab.* **25** 157. MR4193898 https://doi.org/10.1214/20-ejp546

[9] ARNAUDON, M. and DEL MORAL, P. (2020). A second order analysis of McKean–Vlasov semigroups. *Ann. Appl. Probab.* **30** 2613–2664. MR4187123 https://doi.org/10.1214/20-AAP1568

[10] BASU, D. (1955). On statistics independent of a complete sufficient statistic. *Sankhyā* **15** 377–380. MR0074745 https://doi.org/10.1007/978-1-4419-5825-9_14

[11] BERGEMANN, K. and REICH, S. (2010). A localization technique for ensemble Kalman filters. *Q. J. R. Meteorol. Soc.* **136** 701–707.

[12] BEYOU, S., CORPETTI, T., GORTHI, S. and MÉMIN, E. (2013). Fluid flow estimation with multiscale ensemble filters based on motion measurements under location uncertainty. *Numer. Math. Theory Methods Appl.* **6** 21–46. MR3011630

[13] BEYOU, S., CUZOL, A., SUBRAHMANYAM GORTHI, S. A. I. and MÉMIN, E. (2013). Weighted ensemble transform Kalman filter for image assimilation. *Tellus, Ser. A Dyn. Meteorol. Oceanogr.* **65** 18803.

[14] BISHOP, A. N. and DEL MORAL, P. (2017). On the stability of Kalman–Bucy diffusion processes. *SIAM J. Control Optim.* **55** 4015–4047. ArXiv e-print. Available at arXiv:1610.04686.

[15] BISHOP, A. N. and DEL MORAL, P. (2019). On the stability of matrix-valued Riccati diffusions. *Electron. J. Probab.* **24** 84. MR4003137 https://doi.org/10.1214/19-ejp342

[16] BISHOP, A. N. and DEL MORAL, P. (2019). Stability properties of systems of linear stochastic differential equations with random coefficients. *SIAM J. Control Optim.* **57** 1023–1042. MR3924607 https://doi.org/10.1137/18M1182759

[17] BISHOP, A. N. and DEL MORAL, P. (2020). On the mathematical theory of ensemble (linear-Gaussian) Kalman–Bucy filtering. ArXiv preprint. Available at arXiv:2006.08843.

[18] BISHOP, A. N. and DEL MORAL, P. (2021). An explicit Floquet-type representation of Riccati aperiodic exponential semigroups. *Internat. J. Control* **94** 258–266. MR4194353 https://doi.org/10.1080/00207179.2019.1590647

[19] BISHOP, A. N., DEL MORAL, P., KAMATANI, K. and RÉMILLARD, B. (2019). On one-dimensional Riccati diffusions. *Ann. Appl. Probab.* **29** 1127–1187. MR3910025 https://doi.org/10.1214/18-AAP1431

[20] BISHOP, A. N., DEL MORAL, P. and NICLAS, A. (2020). A perturbation analysis of stochastic matrix Riccati diffusions. *Ann. Inst. Henri Poincaré Probab. Stat.* **56** 884–916. MR4076769 https://doi.org/10.1214/19-AIHP987

[21] BISHOP, A. N., DEL MORAL, P. and PATHIRAJA, S. D. (2018). Perturbations and projections of Kalman–Bucy semigroups. *Stochastic Process. Appl.* **128** 2857–2904. MR3834846 https://doi.org/10.1016/j.spa.2017.10.006

[22] BOCK, M. E., JUDGE, G. G. and YANCEY, T. A. (1984). A simple form for the inverse moments of non-central χ^2 and F random variables and certain confluent hypergeometric functions. *J. Econometrics* **25** 217–234. MR0748045 https://doi.org/10.1016/0304-4076(84)90048-4

[23] BRAND, L. (1955). Classroom Notes: A Sequence Defined by a Difference Equation. *Amer. Math. Monthly* **62** 489–492. MR1529078 https://doi.org/10.2307/2307362

[24] BUCY, R. S. (1967). Global theory of the Riccati equation. *J. Comput. System Sci.* **1** 349–361. MR0229916 https://doi.org/10.1016/S0022-0000(67)80025-4

[25] BUCY, R. S. (1972). The Riccati equation and its bounds. *J. Comput. System Sci.* **6** 343–353. MR0316122 https://doi.org/10.1016/S0022-0000(72)80026-6

[26] BURGERS, G., VAN LEEUWEN, P. J. and EVENSEN, G. (1998). Analysis scheme in the ensemble Kalman filter. *Mon. Weather Rev.* **126** 1719–1724.

[27] DEL MORAL, P. (2004). *Feynman-Kac Formulae. Genealogical and Interacting Particle Systems with Applications. Probability and Its Applications*. Springer, New York.

[28] DEL MORAL, P. (2013). *Mean Field Simulation for Monte Carlo Integration. Monographs on Statistics and Applied Probability* **126**. CRC Press, Boca Raton, FL. MR3060209

[29] DEL MORAL, P., KURTZMANN, A. and TUGAUT, J. (2017). On the stability and the uniform propagation of chaos of a class of extended ensemble Kalman–Bucy filters. *SIAM J. Control Optim.* **55** 119–155. MR3597159 https://doi.org/10.1137/16M1087497

[30] DEL MORAL, P. and MICLO, L. (2000). Branching and interacting particle systems approximations of Feynman–Kac formulae with applications to non-linear filtering. In *Séminaire de Probabilités, XXXIV*. *Lecture Notes in Math*. **1729** 1–145. Springer, Berlin. MR1768060 https://doi.org/10.1007/BFb0103798
[31] DEL MORAL, P. and PENEV, S. (2017). *Stochastic Processes*: *From Applications to Theory*. *Chapman & Hall/CRC Texts in Statistical Science Series*. CRC Press, Boca Raton, FL. MR3618157
[32] DEL MORAL, P. and TUGAUT, J. (2018). On the stability and the uniform propagation of chaos properties of ensemble Kalman–Bucy filters. *Ann*. *Appl*. *Probab*. **28** 790–850. MR3784489 https://doi.org/10.1214/17-AAP1317
[33] DELARUE, F. and TSE, A. (2021). Uniform in time weak propagation of chaos on the torus. ArXiv preprint. Available at arXiv:2104.14973.
[34] DELYON, B. (2001). A note on uniform observability. *IEEE Trans*. *Automat*. *Control* **46** 1326–1327. MR1847344 https://doi.org/10.1109/9.940944
[35] DEYST, J. J. JR. and PRICE, C. F. (1968). Conditions for asymptotic stability of the discrete minimum-variance linear estimator. *IEEE Trans*. *Automat*. *Control* **13** 702–704. MR0274103 https://doi.org/10.1109/tac.1968.1099024
[36] EKNES, M. and EVENSEN, G. (2002). An ensemble Kalman filter with a 1–D marine ecosystem model. *J. Mar. Syst*. **36** 75–100.
[37] EVENSEN, G. (1994). Sequential data assimilation with a non-linear quasi-geostrophic model using Monte Carlo methods to forecast error statistics. *J. Geophys*. *Res*. **99** 143–162.
[38] EVENSEN, G. (2003). The ensemble Kalman filter: Theoretical formulation and practical implementation. *Ocean Dyn*. **53** 343–367.
[39] EVENSEN, G. (2007). *Data Assimilation*: *The Ensemble Kalman Filter*. Springer, Berlin.
[40] EVENSEN, G., HOVE, J., MEISINGSET, H. C., REISO, E. and SEIM, K. S. (2007). Using the EnKF for assisted history matching of a North Sea reservoir model. *SPE J*. **106184**.
[41] HOUTEKAMER, P. and MITCHELL, H. L. (1998). Data assimilation using an ensemble Kalman filter technique. *Mon*. *Weather Rev*. **126** 796–811.
[42] JAZWINSKI, A. H. (1970). *Stochastic Processes and Filtering Theory*. Academic Press, San Diego.
[43] JOHNS, C. J. and MANDEL, J. (2008). A two-stage ensemble Kalman filter for smooth data assimilation. *Environ*. *Ecol*. *Stat*. **15** 101–110. MR2412683 https://doi.org/10.1007/s10651-007-0033-0
[44] KALMAN, R. E. (1960). Contributions to the theory of optimal control. *Bol*. *Soc*. *Mat*. *Mex*. **5** 102–119. MR0127472
[45] KALMAN, R. E. (1960). A new approach to linear filtering and prediction problems. *J. Basic Eng*. **82** 35–45. MR3931993
[46] KALMAN, R. E. and BERTRAM, J. E. (1960). Control system analysis and design via the "second method" of Lyapunov. I. Continuous-time systems. *J. Basic Eng*. **82** 371–393. MR0157810
[47] KALMAN, R. E. and BUCY, R. S. (1961). New results in linear filtering and prediction theory. *J. Basic Eng*. **83** 95–108. MR0234760
[48] KALNAY, E. (2003). *Atmospheric Modeling*, *Data Assimilation*, *and Predictability*. Cambridge University Press, Cambridge.
[49] KANTAS, N., BESKOS, A. and JASRA, A. (2014). Sequential Monte Carlo methods for high-dimensional inverse problems: A case study for the Navier–Stokes equations. *SIAM/ASA J*. *Uncertain*. *Quantificat*. **2** 464–489. MR3283917 https://doi.org/10.1137/130930364
[50] KELLY, D., MAJDA, A. J. and TONG, X. T. (2016). Concrete ensemble Kalman filters with rigorous catastrophic filter divergence. *Proc*. *Natl*. *Acad*. *Sci*. *USA* **112** 10589–10594.
[51] KELLY, D. T. B., LAW, K. J. H. and STUART, A. M. (2014). Well-posedness and accuracy of the ensemble Kalman filter in discrete and continuous time. *Nonlinearity* **27** 2579–2604. MR3265724 https://doi.org/10.1088/0951-7715/27/10/2579
[52] LANGE, T. (2022). Derivation of ensemble Kalman–Bucy filters with unbounded nonlinear coefficients. *Nonlinearity* **35** 1061–1092. MR4373995 https://doi.org/10.1088/1361-6544/ac4337
[53] LANGE, T. and STANNAT, W. (2020). Mean field limit of ensemble square root filters–discrete and continuous time. ArXiv preprint. Available at arXiv:2011.10516.
[54] LANGE, T. and STANNAT, W. (2021). On the continuous time limit of ensemble square root filters. *Commun*. *Math*. *Sci*. **19** 1855–1880. MR4312747 https://doi.org/10.4310/CMS.2021.v19.n7.a5
[55] LANGE, T. and STANNAT, W. (2021). On the continuous time limit of the ensemble Kalman filter. *Math*. *Comp*. **90** 233–265. MR4166460 https://doi.org/10.1090/mcom/3588
[56] LE GLAND, F., MONBET, V. and TRAN, V.-D. (2011). Large sample asymptotics for the ensemble Kalman filter. In *The Oxford Handbook of Nonlinear Filtering* 598–631. Oxford Univ. Press, Oxford. MR2884610

[57] LISAETER, K. A., ROSANOVA, J. and EVENSEN, G. (2003). Assimilation of ice concentration in a coupled ice-ocean model, using the ensemble Kalman filter. *Ocean Dyn*. **53** 368–388.

[58] MAJDA, A. J. and HARLIM, J. (2012). *Filtering Complex Turbulent Systems*. Cambridge Univ. Press, Cambridge. MR2934167 https://doi.org/10.1017/CBO9781139061308

[59] MAJDA, A. J. and TONG, X. T. (2018). Performance of ensemble Kalman filters in large dimensions. *Comm*. *Pure Appl*. *Math*. **71** 892–937. MR3794517 https://doi.org/10.1002/cpa.21722

[60] MANDEL, J., COBB, L. and BEEZLEY, J. D. (2011). On the convergence of the ensemble Kalman filter. *Appl*. *Math*. **56** 533–541. MR2886236 https://doi.org/10.1007/s10492-011-0031-2

[61] MAYBECK, P. S. (1979). *Stochastic Models*, *Estimation*, *and Control*. *Vol*. *I*. *Mathematics in Science and Engineering* **141**. Academic Press, New York. MR0539145

[62] NAEVDAL, G., JOHNSEN, L. M., AANONSEN, S. I. and VEFRING, E. H. (2005). Reservoir monitoring and continuous model updating using ensemble Kalman filter. *SPE J*. **10** 66–74.

[63] OTT, E., HUNT, B. R., SZUNYOGH, I., ZIMIN, A. V., KOSTELICH, E. J., CORAZZA, M., KALNAY, E., PATIL, D. and YORKE, J. A. (2004). A local ensemble Kalman filter for atmospheric data assimilation. *Tellus*, *Ser*. *A Dyn*. *Meteorol*. *Oceanogr*. **56** 415–428.

[64] PAPADAKIS, N., MÉMIN, E., CUZOL, A. and GENGEMBRE, N. (2010). Data assimilation with the weighted ensemble Kalman filter. *Tellus*, *Ser*. *A Dyn*. *Meteorol*. *Oceanogr*. **62** 673–697.

[65] PATHIRAJA, S., REICH, S. and STANNAT, W. (2021). McKean–Vlasov SDEs in nonlinear filtering. *SIAM J*. *Control Optim*. **59** 4188–4215. MR4334536 https://doi.org/10.1137/20M1355197

[66] REICH, S. and COTTER, C. (2015). *Probabilistic Forecasting and Bayesian Data Assimilation*. Cambridge Univ. Press, New York. MR3242790 https://doi.org/10.1017/CBO9781107706804

[67] REICH, S. and COTTER, C. J. (2013). Ensemble filter techniques for intermittent data assimilation. In *Large Scale Inverse Problems*. *Radon Ser*. *Comput*. *Appl*. *Math*. **13** 91–134. de Gruyter, Berlin. MR3185328

[68] RESSEGUIER, V., MÉMIN, E. and CHAPRON, B. (2017). Geophysical flows under location uncertainty, Part II Quasi-geostrophy and efficient ensemble spreading. *Geophys*. *Astrophys*. *Fluid Dyn*. **111** 177–208. MR3648563 https://doi.org/10.1080/03091929.2017.1312101

[69] RHUDY, M., GU, Y. and NAPOLITANO, M. (2012). Relaxation of initial error and noise bounds for stability of GPS/INS attitude estimation. In *AIAA Guidance*, *Navigation*, *and Control Conference* 13–16, Minneapolis, MN, USA.

[70] SAKOV, P. and OKE, P. R. (2008). A deterministic formulation of the ensemble Kalman filter: An alternative to ensemble square root filters. *Tellus*, *Ser*. *A Dyn*. *Meteorol*. *Oceanogr*. **60** 361–371.

[71] SAKOV, P. and OKE, P. R. (2008). Implications of the form of the ensemble transformation in the ensemble square root filters. *Mon*. *Weather Rev*. **136** 1042–1053.

[72] SEILER, A., EVENSEN, G., SKJERVHEIM, J.-A., HOVE, J. and VAB, J. G. (2010). Using the EnKF for history matching and uncertainty quantification of complex reservoir models. In *Computational Methods for Large Scale Inverse Problems and Quantification of Uncertainty* (L. Biegler, G. Biros, O. Ghattas, M. Heinkenschloss, D. Keyes, B. Mallick, L. Tenorio, B. van Bloemen Waanders and K. Willcox, eds.) Wiley, New York.

[73] SKJERVHEIM, J.-A., EVENSEN, G., AANONSEN, S. I., RUUD, B. O. and JOHANSEN, T. A. (2005). Incorporating 4D seismic data in reservoir simulation models using ensemble Kalman filter SPE 95789.

[74] TAGHVAEI, A. (2019). Design and analysis of particle-based algorithms for nonlinear filtering and sampling. Diss. University of Illinois at Urbana-Champaign.

[75] TAGHVAEI, A. and MEHTA, P. G. (2016). *An Optimal Transport Formulation of the Linear Feedback Particle Filter*. 2016 *American Control Conference* (*ACC*). IEEE.

[76] TAGHVAEI, A. and MEHTA, P. G. (2016). Gain function approximation in the feedback particle filter. In 2016 *IEEE* 55*th Conference on Decision and Control* (*CDC*) 5446–5452. IEEE.

[77] TAGHVAEI, A., MEHTA, P. G. and GEORGIOU, T. T. (2020). Optimality vs Stability Trade-off in Ensemble Kalman Filters. ArXiv preprint. Available at arXiv:2010.09920.

[78] TAO, Y., BLOM, H. A. P. and MEHTA, P. G. (2014). *The Continuous-Discrete Time Feedback Particle Filter*. *American Control Conference*. IEEE.

[79] TONG, X. T., MAJDA, A. J. and KELLY, D. (2016). Nonlinear stability and ergodicity of ensemble based Kalman filters. *Nonlinearity* **29** 657–691. MR3461611 https://doi.org/10.1088/0951-7715/29/2/657

[80] TONG, X. T., MAJDA, A. J. and KELLY, D. (2016). Nonlinear stability and ergodicity of ensemble based Kalman filters. *Nonlinearity* **29** 657–691. MR3461611 https://doi.org/10.1088/0951-7715/29/2/657

[81] VIT, K. (1972). Iterative solution of the Riccati equation. *IEEE Trans*. *Automat*. *Control* **17** 258–259. MR0371083 https://doi.org/10.1109/tac.1972.1099945

[82] VIT, K. (1972). Iterative solution of the Riccati equation. *IEEE Trans*. *Automat*. *Control* **17** 258–259. MR0371083 https://doi.org/10.1109/tac.1972.1099945

[83] VIT, K. (1972). Iterative solution of the Riccati equation. *IEEE Trans*. *Automat*. *Control* **17** 258–259. MR0371083 https://doi.org/10.1109/tac.1972.1099945

[84] WEN, X.-H. and CHEN, W. H. (2005). Real-time reservoir model updating using ensemble Kalman filter SPE-92991-MS. In *SPE Reservoir Simulation Symposium*, The Woodlands, TX.

[85] YANG, T., LAUGESEN, R. S., MEHTA, P. G. and MEYN, S. P. (2016). Multivariable feedback particle filter. *Automatica J. IFAC* **71** 10–23. MR3521947 https://doi.org/10.1016/j.automatica.2016.04.019

[86] YANG, T., MEHTA, P. G. and MEYN, S. P. (2013). Feedback particle filter. *IEEE Trans. Automat. Control* **58** 2465–2480. MR3106055 https://doi.org/10.1109/TAC.2013.2258825

The Annals of Applied Probability
2023, Vol. 33, No. 2, 1373–1395
https://doi.org/10.1214/22-AAP1844

CONVERGENCE TO THE THERMODYNAMIC LIMIT FOR RANDOM-FIELD RANDOM SURFACES

BY PAUL DARIO[a]

LAMA, Université Paris-Est Créteil , [a]*paul.dario@u-pec.fr*

We study random surfaces with a uniformly convex gradient interaction in the presence of quenched disorder taking the form of a random independent external field. Previous work on the model has focused on proving existence and uniqueness of infinite-volume gradient Gibbs measures with a given tilt and on studying the fluctuations of the surface and its discrete gradient.

In this work, we focus on the convergence of the thermodynamic limit, establishing convergence of the finite-volume distributions with Dirichlet boundary conditions to translation-covariant (gradient) Gibbs measures. Specifically, it is shown that, when the law of the random field has finite second moment and is symmetric, the distribution of the gradient of the surface converges in dimensions $d \geq 4$ while the distribution of the surface itself converges in dimensions $d \geq 5$. Moreover, a power-law upper bound on the rate of convergence in Wasserstein distance is obtained. The results partially answer a question discussed by Cotar and Külske (*Ann. Appl. Probab.* **22** (2012) 1650–1692).

REFERENCES

[1] ADAMS, S., KOTECKÝ, R. and MÜLLER, S. (2016). Strict convexity of the surface tension for non-convex potentials. Preprint. Available at arXiv:1606.09541.

[2] AIZENMAN, M., HAREL, M. and PELED, R. (2020). Exponential decay of correlations in the 2D random field Ising model. *J. Stat. Phys.* **180** 304–331. MR4130991 https://doi.org/10.1007/s10955-019-02401-5

[3] AIZENMAN, M. and PELED, R. (2019). A power-law upper bound on the correlations in the $2D$ random field Ising model. *Comm. Math. Phys.* **372** 865–892. MR4034778 https://doi.org/10.1007/s00220-019-03450-3

[4] AIZENMAN, M. and WEHR, J. (1989). Rounding of first-order phase transitions in systems with quenched disorder. *Phys. Rev. Lett.* **62** 2503–2506. MR0995009 https://doi.org/10.1103/PhysRevLett.62.2503

[5] AIZENMAN, M. and WEHR, J. (1990). Rounding effects of quenched randomness on first-order phase transitions. *Comm. Math. Phys.* **130** 489–528. MR1060388

[6] ARMSTRONG, S. and WU, W. (2022). C^2 regularity of the surface tension for the $\nabla\phi$ interface model. *Comm. Pure Appl. Math.* **75** 349–421. MR4373172 https://doi.org/10.1002/cpa.22031

[7] BISKUP, M. and KOTECKÝ, R. (2007). Phase coexistence of gradient Gibbs states. *Probab. Theory Related Fields* **139** 1–39. MR2322690 https://doi.org/10.1007/s00440-006-0013-6

[8] BISKUP, M. and SPOHN, H. (2011). Scaling limit for a class of gradient fields with nonconvex potentials. *Ann. Probab.* **39** 224–251. MR2778801 https://doi.org/10.1214/10-AOP548

[9] BOVIER, A. and KÜLSKE, C. (1994). A rigorous renormalization group method for interfaces in random media. *Rev. Math. Phys.* **6** 413–496. MR1305590 https://doi.org/10.1142/S0129055X94000171

[10] BOVIER, A. and KÜLSKE, C. (1996). There are no nice interfaces in $(2+1)$-dimensional SOS models in random media. *J. Stat. Phys.* **83** 751–759. MR1386357 https://doi.org/10.1007/BF02183747

[11] BRASCAMP, H. J. and LIEB, E. H. (1976). On extensions of the Brunn–Minkowski and Prékopa–Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation. *J. Funct. Anal.* **22** 366–389. MR0450480 https://doi.org/10.1016/0022-1236(76)90004-5

[12] BRASCAMP, H. J. and LIEB, E. H. (2002). Some inequalities for Gaussian measures and the long-range order of the one-dimensional plasma. In *Inequalities* 403–416. Springer, Berlin.

[13] BRASCAMP, H. J., LIEB, E. H. and LEBOWITZ, J. L. (1975). The statistical mechanics of anharmonic lattices. *Bull. Int. Stat. Inst.* **46** 393–404. MR0676341

MSC2020 subject classifications. 82B24, 82B44, 82C41.
Key words and phrases. Random surfaces, disordered systems.

[14] BRICMONT, J. and KUPIAINEN, A. (1988). Phase transition in the 3d random field Ising model. *Comm. Math. Phys.* **116** 539–572. MR0943702

[15] BRYDGES, D. and YAU, H.-T. (1990). Grad ϕ perturbations of massless Gaussian fields. *Comm. Math. Phys.* **129** 351–392. MR1048698

[16] CHATTERJEE, S. (2018). On the decay of correlations in the random field Ising model. *Comm. Math. Phys.* **362** 253–267. MR3833610 https://doi.org/10.1007/s00220-018-3085-0

[17] COTAR, C. and DEUSCHEL, J.-D. (2012). Decay of covariances, uniqueness of ergodic component and scaling limit for a class of $\nabla\phi$ systems with non-convex potential. *Ann. Inst. Henri Poincaré Probab. Stat.* **48** 819–853. MR2976565 https://doi.org/10.1214/11-AIHP437

[18] COTAR, C., DEUSCHEL, J.-D. and MÜLLER, S. (2009). Strict convexity of the free energy for a class of non-convex gradient models. *Comm. Math. Phys.* **286** 359–376. MR2470934 https://doi.org/10.1007/s00220-008-0659-2

[19] COTAR, C. and KÜLSKE, C. (2012). Existence of random gradient states. *Ann. Appl. Probab.* **22** 1650–1692. MR2985173 https://doi.org/10.1214/11-AAP808

[20] COTAR, C. and KÜLSKE, C. (2015). Uniqueness of gradient Gibbs measures with disorder. *Probab. Theory Related Fields* **162** 587–635. MR3383338 https://doi.org/10.1007/s00440-014-0580-x

[21] DARIO, P., HAREL, M. and PELED, R. (2021). Random-field random surfaces. Preprint. Available at arXiv:2101.01711.

[22] DE GIORGI, E. (1957). Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari. *Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat.* (3) **3** 25–43. MR0093649

[23] DELMOTTE, T. (1997). Inégalité de Harnack elliptique sur les graphes. *Colloq. Math.* **72** 19–37. MR1425544 https://doi.org/10.4064/cm-72-1-19-37

[24] DELMOTTE, T. and DEUSCHEL, J.-D. (2005). On estimating the derivatives of symmetric diffusions in stationary random environment, with applications to $\nabla\phi$ interface model. *Probab. Theory Related Fields* **133** 358–390. MR2198017 https://doi.org/10.1007/s00440-005-0430-y

[25] DEUSCHEL, J.-D., GIACOMIN, G. and IOFFE, D. (2000). Large deviations and concentration properties for $\nabla\phi$ interface models. *Probab. Theory Related Fields* **117** 49–111. MR1759509 https://doi.org/10.1007/s004400050266

[26] DING, J. and WIRTH, M. (2020). Correlation length of two-dimensional random field Ising model via greedy lattice animal. Preprint. Available at arXiv:2011.08768.

[27] DING, J. and XIA, J. (2021). Exponential decay of correlations in the two-dimensional random field Ising model. *Invent. Math.* **224** 999–1045. MR4258059 https://doi.org/10.1007/s00222-020-01024-y

[28] FUNAKI, T. (2005). Stochastic interface models. In *Lectures on Probability Theory and Statistics. Lecture Notes in Math.* **1869** 103–274. Springer, Berlin. MR2228384 https://doi.org/10.1007/11429579_2

[29] FUNAKI, T. and SPOHN, H. (1997). Motion by mean curvature from the Ginzburg-Landau $\nabla\phi$ interface model. *Comm. Math. Phys.* **185** 1–36. MR1463032 https://doi.org/10.1007/s002200050080

[30] GIACOMIN, G., OLLA, S. and SPOHN, H. (2001). Equilibrium fluctuations for $\nabla\phi$ interface model. *Ann. Probab.* **29** 1138–1172. MR1872740 https://doi.org/10.1214/aop/1015345600

[31] GILBARG, D. and TRUDINGER, N. S. (2001). *Elliptic Partial Differential Equations of Second Order. Classics in Mathematics*. Springer, Berlin. Reprint of the 1998 edition. MR1814364

[32] GLORIA, A. and OTTO, F. (2011). An optimal variance estimate in stochastic homogenization of discrete elliptic equations. *Ann. Probab.* **39** 779–856. MR2789576 https://doi.org/10.1214/10-AOP571

[33] GLORIA, A. and OTTO, F. (2012). An optimal error estimate in stochastic homogenization of discrete elliptic equations. *Ann. Appl. Probab.* **22** 1–28. MR2932541 https://doi.org/10.1214/10-AAP745

[34] HAN, Q. and LIN, F. (2011). *Elliptic Partial Differential Equations* **1**. Am. Math. Soc. , Providence.

[35] IMRY, Y. and MA, S.-K. (1975). Random-field instability of the ordered state of continuous symmetry. *Phys. Rev. Lett.* **35** 1399.

[36] KÜLSKE, C. and ORLANDI, E. (2006). A simple fluctuation lower bound for a disordered massless random continuous spin model in $D = 2$. *Electron. Commun. Probab.* **11** 200–205. MR2266710 https://doi.org/10.1214/ECP.v11-1218

[37] KÜLSKE, C. and ORLANDI, E. (2008). Continuous interfaces with disorder: Even strong pinning is too weak in two dimensions. *Stochastic Process. Appl.* **118** 1973–1981. MR2462282 https://doi.org/10.1016/j.spa.2007.11.005

[38] MILLER, J. (2011). Fluctuations for the Ginzburg–Landau $\nabla\phi$ interface model on a bounded domain. *Comm. Math. Phys.* **308** 591–639. MR2855536 https://doi.org/10.1007/s00220-011-1315-9

[39] MOSER, J. (1961). On Harnack's theorem for elliptic differential equations. *Comm. Pure Appl. Math.* **14** 577–591. MR0159138 https://doi.org/10.1002/cpa.3160140329

[40] MOSER, J. (1964). A Harnack inequality for parabolic differential equations. *Comm. Pure Appl. Math.* **17** 101–134. MR0159139 https://doi.org/10.1002/cpa.3160170106

[41] NADDAF, A. and SPENCER, T. (1997). On homogenization and scaling limit of some gradient perturbations of a massless free field. *Comm. Math. Phys.* **183** 55–84. MR1461951 https://doi.org/10.1007/BF02509796

[42] NADDAF, A. and SPENCER, T. (1998). Estimates on the variance of some homogenization problems. Unpublished preprint.

[43] NASH, J. (1958). Continuity of solutions of parabolic and elliptic equations. *Amer. J. Math.* **80** 931–954. MR0100158 https://doi.org/10.2307/2372841

[44] SHEFFIELD, S. (2005). Random surfaces. *Astérisque* **304** vi+175. MR2251117

[45] STROOCK, D. W. and ZHENG, W. (1997). Markov chain approximations to symmetric diffusions. *Ann. Inst. Henri Poincaré Probab. Stat.* **33** 619–649. MR1473568 https://doi.org/10.1016/S0246-0203(97)80107-0

[46] VAN ENTER, A. C. D. and KÜLSKE, C. (2008). Nonexistence of random gradient Gibbs measures in continuous interface models in $d = 2$. *Ann. Appl. Probab.* **18** 109–119. MR2380893 https://doi.org/10.1214/07-AAP446

[47] VAN ENTER, A. C. D. and SHLOSMAN, S. B. (2002). First-order transitions for n-vector models in two and more dimensions: Rigorous proof. *Phys. Rev. Lett.* **89** 285702. https://doi.org/10.1103/PhysRevLett.89.285702

[48] VAN ENTER, A. C. D. and SHLOSMAN, S. B. (2005). Provable first-order transitions for nonlinear vector and gauge models with continuous symmetries. *Comm. Math. Phys.* **255** 21–32. MR2123375 https://doi.org/10.1007/s00220-004-1286-1

[49] VELENIK, Y. (2006). Localization and delocalization of random interfaces. *Probab. Surv.* **3** 112–169. MR2216964 https://doi.org/10.1214/154957806000000050

[50] VILLAIN, J. (1975). Theory of one-and two-dimensional magnets with an easy magnetization plane. II. The planar, classical, two-dimensional magnet. *J. Phys.* **36** 581–590.

[51] VILLANI, C. (2009). *Optimal Transport: Old and New. Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **338**. Springer, Berlin. MR2459454 https://doi.org/10.1007/978-3-540-71050-9

The Annals of Applied Probability
2023, Vol. 33, No. 2, 1396–1458
https://doi.org/10.1214/22-AAP1845

ME, MYSELF AND I: A GENERAL THEORY OF NON-MARKOVIAN TIME-INCONSISTENT STOCHASTIC CONTROL FOR SOPHISTICATED AGENTS

BY CAMILO HERNÁNDEZ[1,a] AND DYLAN POSSAMAÏ[2,b]

[1]*Mathematics Department, Imperial College London,* [a]*c.hernandez@imperial.ac.uk*
[2]*Mathematics Department, ETH Zürich,* [b]*dylan.possamai@math.ethz.ch*

We develop a theory for continuous-time non-Markovian stochastic control problems which are inherently time-inconsistent. Their distinguishing feature is that the classical Bellman optimality principle no longer holds. Our formulation is cast within the framework of a controlled non-Markovian forward stochastic differential equation, and a general objective functional setting. We adopt a game-theoretic approach to study such problems, meaning that we seek for *subgame perfect Nash equilibrium* points. As a first novelty of this work, we introduce and motivate a refinement of the definition of equilibrium that allows us to establish a direct and rigorous proof of an *extended dynamic programming principle*, in the same spirit as in the classical theory. This in turn allows us to introduce a system consisting of an infinite family of backward stochastic differential equations analogous to the classical HJB equation. We prove that this system is fundamental, in the sense that its well-posedness is both necessary and sufficient to characterise the value function and equilibria. As a final step, we provide an existence and uniqueness result. Some examples and extensions of our results are also presented.

REFERENCES

[1] BARRO, R. J. (1999). Ramsey meets Laibson in the neoclassical growth model. *Q. J. Econ.* **114** 1125–1152.

[2] BASAK, S. and CHABAKAURI, G. (2010). Dynamic mean–variance asset allocation. *Rev. Financ. Stud.* **23** 2970–3016.

[3] BAYRAKTAR, E., ZHANG, J. and ZHOU, Z. (2019). On the notions of equilibria for time-inconsistent stopping problems in continuous time. ArXiv preprint arXiv:1909.01112.

[4] BERTSEKAS, D. P. and SHREVE, S. E. (1978). *Stochastic Optimal Control: The Discrete Time Case. Mathematics in Science and Engineering* **139**. Academic Press [Harcourt Brace Jovanovich, Publishers], New York–London. MR0511544

[5] BJÖRK, T., KHAPKO, M. and MURGOCI, A. (2016). Time inconsistent stochastic control in continuous time: theory and examples. ArXiv preprint arXiv:1612.03650.

[6] BJÖRK, T., KHAPKO, M. and MURGOCI, A. (2017). On time-inconsistent stochastic control in continuous time. *Finance Stoch.* **21** 331–360. MR3626618 https://doi.org/10.1007/s00780-017-0327-5

[7] BJÖRK, T. and MURGOCI, A. (2010). A general theory of Markovian time inconsistent stochastic control problems. SSRN preprint 1694759.

[8] BJÖRK, T. and MURGOCI, A. (2014). A theory of Markovian time-inconsistent stochastic control in discrete time. *Finance Stoch.* **18** 545–592. MR3232016 https://doi.org/10.1007/s00780-014-0234-y

[9] BJÖRK, T., MURGOCI, A. and ZHOU, X. Y. (2014). Mean-variance portfolio optimization with state-dependent risk aversion. *Math. Finance* **24** 1–24. MR3157686 https://doi.org/10.1111/j.1467-9965.2011.00515.x

[10] BOUCHARD, B., POSSAMAÏ, D., TAN, X. and ZHOU, C. (2018). A unified approach to *a priori* estimates for supersolutions of BSDEs in general filtrations. *Ann. Inst. Henri Poincaré Probab. Stat.* **54** 154–172. MR3765884 https://doi.org/10.1214/16-AIHP798

[11] BRUTSCHER, P. B. (2011). Payment matters? An exploratory study into the pre-payment electricity metering. Technical report, Univ. Cambridge.

MSC2020 subject classifications. Primary 60H10, 60H30; secondary 60G07.
Key words and phrases. Time inconsistency, consistent planning, nonexponential discounting, mean–variance, backward stochastic differential equations.

[12] CHRISTENSEN, S. and LINDENSJÖ, K. (2020). Time-inconsistent stopping, myopic adjustment and equilibrium stability: With a mean-variance application. In *Stochastic Modeling and Control*. *Banach Center Publ*. **122** 53–76. Polish Acad. Sci. Inst. Math., Warsaw. MR4276004 https://doi.org/10.4064/bc122-4

[13] CHRISTENSEN, S. and LINDENSJÖ, K. (2020). On time-inconsistent stopping problems and mixed strategy stopping times. *Stochastic Process*. *Appl*. **130** 2886–2917. MR4080735 https://doi.org/10.1016/j.spa.2019.08.010

[14] CHRISTENSEN, S. and LINDENSJÖ, K. (2022). Moment-constrained optimal dividends: Precommitment and consistent planning. *Adv. in Appl*. *Probab.* **54** 404–432. MR4434662 https://doi.org/10.1017/apr.2021.38

[15] CLAISSE, J., TALAY, D. and TAN, X. (2016). A pseudo-Markov property for controlled diffusion processes. *SIAM J. Control Optim*. **54** 1017–1029. MR3488161 https://doi.org/10.1137/151004252

[16] CONT, R. and FOURNIÉ, D.-A. (2010). Change of variable formulas for non-anticipative functionals on path space. *J. Funct*. *Anal*. **259** 1043–1072. MR2652181 https://doi.org/10.1016/j.jfa.2010.04.017

[17] CVITANIĆ, J., POSSAMAÏ, D. and TOUZI, N. (2018). Dynamic programming approach to principal-agent problems. *Finance Stoch*. **22** 1–37. MR3738664 https://doi.org/10.1007/s00780-017-0344-4

[18] CZICHOWSKY, C. (2013). Time-consistent mean-variance portfolio selection in discrete and continuous time. *Finance Stoch*. **17** 227–271. MR3038591 https://doi.org/10.1007/s00780-012-0189-9

[19] DA PRATO, G. and ZABCZYK, J. (2014). *Stochastic Equations in Infinite Dimensions*, 2nd ed. *Encyclopedia of Mathematics and Its Applications* **152**. Cambridge Univ. Press, Cambridge. MR3236753 https://doi.org/10.1017/CBO9781107295513

[20] DELLACHERIE, C. and MEYER, P.-A. (1978). *Probabilities and Potential*. *North-Holland Mathematics Studies* **29**. North-Holland, Amsterdam. MR0521810

[21] DENIS, L., HU, M. and PENG, S. (2011). Function spaces and capacity related to a sublinear expectation: Application to G-Brownian motion paths. *Potential Anal*. **34** 139–161. MR2754968 https://doi.org/10.1007/s11118-010-9185-x

[22] DJETE, M. F., POSSAMAÏ, D. and TAN, X. (2022). McKean–Vlasov optimal control: The dynamic programming principle. *Ann*. *Probab.* **50** 791–833. MR4399164 https://doi.org/10.1214/21-aop1548

[23] EKELAND, I. and LAZRAK, A. (2006). Being serious about non-commitment: Subgame perfect equilibrium in continuous time. ArXiv preprint arXiv:0604264.

[24] EKELAND, I. and LAZRAK, A. (2010). The golden rule when preferences are time inconsistent. *Math*. *Financ*. *Econ*. **4** 29–55. MR2746576 https://doi.org/10.1007/s11579-010-0034-x

[25] EKELAND, I. and PIRVU, T. A. (2008). Investment and consumption without commitment. *Math*. *Financ*. *Econ*. **2** 57–86. MR2461340 https://doi.org/10.1007/s11579-008-0014-6

[26] EL KAROUI, N. and TAN, X. (2013). Capacities, measurable selection and dynamic programming part I: Abstract framework. ArXiv preprint arXiv:1310.3363.

[27] EL KAROUI, N. and TAN, X. (2013). Capacities, measurable selection and dynamic programming part II: Application in stochastic control problems. ArXiv preprint arXiv:1310.3364.

[28] FLEMING, W. H. and SONER, H. M. (2006). *Controlled Markov Processes and Viscosity Solutions*, 2nd ed. *Stochastic Modelling and Applied Probability* **25**. Springer, New York. MR2179357

[29] FREI, C. and DOS REIS, G. (2011). A financial market with interacting investors: Does an equilibrium exist? *Math*. *Financ*. *Econ*. **4** 161–182. MR2796281 https://doi.org/10.1007/s11579-011-0039-0

[30] FUDENBERG, D. and LEVINE, D. K. (2006). A dual-self model of impulse control. *Am*. *Econ*. *Rev*. **96** 1449–1476.

[31] GOLDMAN, S. M. (1980). Consistent plans. *Rev*. *Econ*. *Stud*. **47** 533–537.

[32] GUL, F. and PESENDORFER, W. (2001). Temptation and self-control. *Econometrica* **69** 1403–1435. MR1865216 https://doi.org/10.1111/1468-0262.00252

[33] HAN, B. and WONG, H. Y. (2019). Time-consistent feedback strategies with Volterra processes. ArXiv preprint arXiv:1907.11378.

[34] HARTER, J. and RICHOU, A. (2019). A stability approach for solving multidimensional quadratic BSDEs. *Electron*. *J*. *Probab.* **24** Paper No. 4, 51. MR3916324 https://doi.org/10.1214/18-EJP260

[35] HE, X. D., HU, S., OBŁÓJ, J. and ZHOU, X. Y. (2019). Optimal exit time from casino gambling: Strategies of precommitted and naive gamblers. *SIAM J. Control Optim*. **57** 1845–1868. MR3955229 https://doi.org/10.1137/18M1174738

[36] HE, X. D. and JIANG, Z. L. (2021). On the equilibrium strategies for time-inconsistent problems in continuous time. *SIAM J. Control Optim*. **59** 3860–3886. MR4328502 https://doi.org/10.1137/20M1382106

[37] HE, X. D., STRUB, M. S. and ZARIPHOPOULOU, T. (2021). Forward rank-dependent performance criteria: Time-consistent investment under probability distortion. *Math*. *Finance* **31** 683–721. MR4237144 https://doi.org/10.1111/mafi.12298

[38] HE, X. D. and ZHOU, X. Y. (2022). Who are I: Time inconsistency and intrapersonal conflict and reconciliation. In *Stochastic Analysis, Filtering, and Stochastic Optimization* 177–208. Springer, Cham. MR4433814 https://doi.org/10.1007/978-3-030-98519-6_8

[39] HERNÁNDEZ, C. (2021). On quadratic multidimensional type-I BSVIEs, infinite families of BSDEs and their applications. ArXiv preprint arXiv:2111.11131.

[40] HERNÁNDEZ, C. and POSSAMAÏ, D. (2021). A unified approach to well-posedness of type-I backward stochastic Volterra integral equations. *Electron. J. Probab.* **26** Paper No. 89, 35. MR4278600 https://doi.org/10.1214/21-ejp653

[41] HERNANDEZ RAMIREZ, M. C. (2021). *Me, Myself and I: Time-Inconsistent Stochastic Control, Contract Theory and Backward Stochastic Volterra Integral Equations*. ProQuest LLC, Ann Arbor, MI. Thesis (Ph.D.)–Columbia Univ. MR4315347

[42] HU, Y., JIN, H. and ZHOU, X. Y. (2012). Time-inconsistent stochastic linear-quadratic control. *SIAM J. Control Optim.* **50** 1548–1572. MR2968066 https://doi.org/10.1137/110853960

[43] HU, Y., JIN, H. and ZHOU, X. Y. (2017). Time-inconsistent stochastic linear-quadratic control: Characterization and uniqueness of equilibrium. *SIAM J. Control Optim.* **55** 1261–1279. MR3639569 https://doi.org/10.1137/15M1019040

[44] HUANG, Y.-J., NGUYEN-HUU, A. and ZHOU, X. Y. (2020). General stopping behaviors of naïve and noncommitted sophisticated agents, with application to probability distortion. *Math. Finance* **30** 310–340. MR4067078 https://doi.org/10.1111/mafi.12224

[45] HUANG, Y.-J. and ZHOU, Z. (2020). Optimal equilibria for time-inconsistent stopping problems in continuous time. *Math. Finance* **30** 1103–1134. MR4116459 https://doi.org/10.1111/mafi.12251

[46] HUANG, Y.-J. and ZHOU, Z. (2021). Strong and weak equilibria for time-inconsistent stochastic control in continuous time. *Math. Oper. Res.* **46** 428–451. MR4288523 https://doi.org/10.1287/moor.2020.1066

[47] JACOD, J. and SHIRYAEV, A. N. (2003). *Limit Theorems for Stochastic Processes*, 2nd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **288**. Springer, Berlin. MR1943877 https://doi.org/10.1007/978-3-662-05265-5

[48] KARANDIKAR, R. L. (1995). On pathwise stochastic integration. *Stochastic Process. Appl.* **57** 11–18. MR1327950 https://doi.org/10.1016/0304-4149(95)00002-O

[49] KARATZAS, I. and SHREVE, S. E. (1991). *Brownian Motion and Stochastic Calculus*, 2nd ed. *Graduate Texts in Mathematics* **113**. Springer, New York. MR1121940 https://doi.org/10.1007/978-1-4612-0949-2

[50] KARNAM, C., MA, J. and ZHANG, J. (2017). Dynamic approaches for some time-inconsistent optimization problems. *Ann. Appl. Probab.* **27** 3435–3477. MR3737929 https://doi.org/10.1214/17-AAP1284

[51] LAIBSON, D. (1997). Golden eggs and hyperbolic discounting. *Q. J. Econ.* **112** 443–477.

[52] LANDRIAULT, D., LI, B., LI, D. and YOUNG, V. R. (2018). Equilibrium strategies for the mean-variance investment problem over a random horizon. *SIAM J. Financial Math.* **9** 1046–1073. MR3851043 https://doi.org/10.1137/17M1153479

[53] LINDENSJÖ, K. (2019). A regular equilibrium solves the extended HJB system. *Oper. Res. Lett.* **47** 427–432. MR3993968 https://doi.org/10.1016/j.orl.2019.07.011

[54] MA, J., WONG, T.-K. L. and ZHANG, J. (2021). Time-consistent conditional expectation under probability distortion. *Math. Oper. Res.* **46** 1149–1180. MR4312591 https://doi.org/10.1287/moor.2020.1101

[55] MCSHANE, E. J. (1983). *Unified Integration*. *Pure and Applied Mathematics* **107**. Academic Press [Harcourt Brace Jovanovich, Publishers], New York. MR0740710

[56] MEI, H. and ZHU, C. (2020). Closed-loop equilibrium for time-inconsistent McKean–Vlasov controlled problem. *SIAM J. Control Optim.* **58** 3842–3867. MR4188839 https://doi.org/10.1137/20M1319796

[57] NUTZ, M. and VAN HANDEL, R. (2013). Constructing sublinear expectations on path space. *Stochastic Process. Appl.* **123** 3100–3121. MR3062438 https://doi.org/10.1016/j.spa.2013.03.022

[58] NUTZ, M. and ZHANG, Y. (2020). Conditional optimal stopping: A time-inconsistent optimization. *Ann. Appl. Probab.* **30** 1669–1692. MR4133380 https://doi.org/10.1214/19-AAP1540

[59] O'DONOGHUE, T. and RABIN, M. (1999). Doing it now or later. *Am. Econ. Rev.* **89** 103–124.

[60] O'DONOGHUE, T. and RABIN, M. (1999). Incentives for procrastinators. *Q. J. Econ.* **114** 769–816.

[61] OSBORNE, M. J. and RUBINSTEIN, A. (1994). *A Course in Game Theory*. MIT Press, Cambridge, MA. MR1301776

[62] PAPAPANTOLEON, A., POSSAMAÏ, D. and SAPLAOURAS, A. (2018). Existence and uniqueness results for BSDE with jumps: The whole nine yards. *Electron. J. Probab.* **23** Paper No. 121, 69. MR3896858 https://doi.org/10.1214/18-EJP240

[63] PHELPS, E. S. and POLLAK, R. A. (1968). On second-best national saving and game-equilibrium growth. *Rev. Econ. Stud.* **35** 185–199.

[64] POLLAK, R. A. (1968). Consistent planning. *Rev. Econ. Stud.* **35** 201–208.

[65] POSSAMAÏ, D., TAN, X. and ZHOU, C. (2018). Stochastic control for a class of nonlinear kernels and applications. *Ann. Probab.* **46** 551–603. MR3758737 https://doi.org/10.1214/17-AOP1191

[66] PROTTER, P. E. (2005). *Stochastic Integration and Differential Equations*, 2nd ed. *Stochastic Modelling and Applied Probability* **21**. Springer, Berlin. Version 2.1, Corrected third printing. MR2273672 https://doi.org/10.1007/978-3-662-10061-5

[67] SAMUELSON, P. A. (1937). A note on measurement of utility. *Rev. Econ. Stud.* **4** 155–161.

[68] SCHÄL, M. (1974). A selection theorem for optimization problems. *Arch. Math.* (*Basel*) **25** 219–224. MR0346632 https://doi.org/10.1007/BF01238668

[69] SONER, H. M., TOUZI, N. and ZHANG, J. (2011). Martingale representation theorem for the G-expectation. *Stochastic Process. Appl.* **121** 265–287. MR2746175 https://doi.org/10.1016/j.spa.2010.10.006

[70] SONER, H. M., TOUZI, N. and ZHANG, J. (2012). Wellposedness of second order backward SDEs. *Probab. Theory Related Fields* **153** 149–190. MR2925572 https://doi.org/10.1007/s00440-011-0342-y

[71] STROOCK, D. W. and VARADHAN, S. R. S. (1979). *Multidimensional Diffusion Processes*. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **233**. Springer, Berlin. MR0532498

[72] STROTZ, R. H. (1955). Myopia and inconsistency in dynamic utility maximization. *Rev. Econ. Stud.* **23** 165–180.

[73] TAN, K. S., WEI, W. and ZHOU, X. Y. (2021). Failure of smooth pasting principle and nonexistence of equilibrium stopping rules under time-inconsistency. *SIAM J. Control Optim.* **59** 4136–4154. MR4332966 https://doi.org/10.1137/19M1291339

[74] VAN NEERVEN, J. M. A. M. (2002). Approximating Bochner integrals by Riemann sums. *Indag. Math.* (*N.S.*) **13** 197–208. MR2016338 https://doi.org/10.1016/S0019-3577(02)80005-2

[75] VIEILLE, N. and WEIBULL, J. W. (2009). Multiple solutions under quasi-exponential discounting. *Econom. Theory* **39** 513–526. MR2486772 https://doi.org/10.1007/s00199-008-0368-2

[76] WANG, H. and YONG, J. (2021). Time-inconsistent stochastic optimal control problems and backward stochastic Volterra integral equations. *ESAIM Control Optim. Calc. Var.* **27** Paper No. 22, 40. MR4238770 https://doi.org/10.1051/cocv/2021027

[77] WEI, Q., YONG, J. and YU, Z. (2017). Time-inconsistent recursive stochastic optimal control problems. *SIAM J. Control Optim.* **55** 4156–4201. MR3738841 https://doi.org/10.1137/16M1079415

[78] XING, H. and ŽITKOVIĆ, G. (2018). A class of globally solvable Markovian quadratic BSDE systems and applications. *Ann. Probab.* **46** 491–550. MR3758736 https://doi.org/10.1214/17-AOP1190

[79] YONG, J. (2012). Time-inconsistent optimal control problems and the equilibrium HJB equation. *Math. Control Relat. Fields* **2** 271–329. MR2991570 https://doi.org/10.3934/mcrf.2012.2.271

[80] ZHANG, J. (2017). *Backward Stochastic Differential Equations*: *From Linear to Fully Nonlinear Theory*. *Probability Theory and Stochastic Modelling* **86**. Springer, New York. MR3699487 https://doi.org/10.1007/978-1-4939-7256-2

The Annals of Applied Probability
2023, Vol. 33, No. 2, 1459–1500
https://doi.org/10.1214/22-AAP1846

COMPLEXITY RESULTS FOR MCMC DERIVED FROM QUANTITATIVE BOUNDS

BY JUN YANG[1,a] AND JEFFREY S. ROSENTHAL[2,b]

[1]*Department of Statistics, University of Oxford,* [a]*jun.yang@stats.ox.ac.uk*
[2]*Department of Statistical Sciences, University of Toronto,* [b]*jeff@math.toronto.edu*

This paper considers how to obtain MCMC quantitative convergence bounds which can be translated into tight complexity bounds in high-dimensional settings. We propose a modified drift-and-minorization approach, which establishes generalized drift conditions defined in subsets of the state space. The subsets are called the "large sets", and are chosen to rule out some "bad" states which have poor drift property when the dimension of the state space gets large. Using the "large sets" together with a "fitted family of drift functions", a quantitative bound can be obtained which can be translated into a tight complexity bound. As a demonstration, we analyze several Gibbs samplers and obtain complexity upper bounds for the mixing time. In particular, for one example of Gibbs sampler which is related to the James–Stein estimator, we show that the number of iterations required for the Gibbs sampler to converge is constant under certain conditions on the observed data and the initial state. It is our hope that this modified drift-and-minorization approach can be employed in many other specific examples to obtain complexity bounds for high-dimensional Markov chains.

REFERENCES

[1] ADAMCZAK, R. (2008). A tail inequality for suprema of unbounded empirical processes with applications to Markov chains. *Electron. J. Probab.* **13** 1000–1034. MR2424985 https://doi.org/10.1214/EJP.v13-521

[2] ADAMCZAK, R. and BEDNORZ, W. (2015). Exponential concentration inequalities for additive functionals of Markov chains. *ESAIM Probab. Stat.* **19** 440–481. MR3423302 https://doi.org/10.1051/ps/2014032

[3] BAXENDALE, P. H. (2005). Renewal theory and computable convergence rates for geometrically ergodic Markov chains. *Ann. Appl. Probab.* **15** 700–738. MR2114987 https://doi.org/10.1214/105051604000000710

[4] BOU-RABEE, N. and HAIRER, M. (2013). Nonasymptotic mixing of the MALA algorithm. *IMA J. Numer. Anal.* **33** 80–110. MR3020951 https://doi.org/10.1093/imanum/drs003

[5] BROOKS, S., GELMAN, A., JONES, G. L. and MENG, X.-L., eds. (2011). *Handbook of Markov Chain Monte Carlo. Chapman & Hall/CRC Handbooks of Modern Statistical Methods*. CRC Press, Boca Raton, FL. MR2742422 https://doi.org/10.1201/b10905

[6] CHOI, H. M. and HOBERT, J. P. (2013). The Polya-gamma Gibbs sampler for Bayesian logistic regression is uniformly ergodic. *Electron. J. Stat.* **7** 2054–2064. MR3091616 https://doi.org/10.1214/13-EJS837

[7] COBHAM, A. (1965). The intrinsic computational difficulty of functions. In *Logic, Methodology and Philos. Sci.* (*Proc.* 1964 *Internat. Congr.*) (Y. Bar-Hillel, ed.) 24–30. North-Holland, Amsterdam. MR0207561

[8] COOK, S. A. (1971). The complexity of theorem-proving procedures. In *Proceedings of the Third Annual ACM Symposium on Theory of Computing* 151–158. ACM, New York.

[9] DALALYAN, A. S. (2017). Theoretical guarantees for approximate sampling from smooth and log-concave densities. *J. R. Stat. Soc. Ser. B. Stat. Methodol.* **79** 651–676. MR3641401 https://doi.org/10.1111/rssb.12183

[10] DAVIS, B. and HOBERT, J. P. (2021). On the convergence complexity of Gibbs samplers for a family of simple Bayesian random effects models. *Methodol. Comput. Appl. Probab.* **23** 1323–1351. MR4335163 https://doi.org/10.1007/s11009-020-09808-8

[11] DURMUS, A. and MOULINES, É. (2017). Nonasymptotic convergence analysis for the unadjusted Langevin algorithm. *Ann. Appl. Probab.* **27** 1551–1587. MR3678479 https://doi.org/10.1214/16-AAP1238

MSC2020 subject classifications. Primary 60J20, 60J22; secondary 65C05.
Key words and phrases. Markov chain Monte Carlo, convergence complexity, mixing time, drift and minorization, high dimensions.

[12] Dwivedi, R., Chen, Y., Wainwright, M. J. and Yu, B. (2019). Log-concave sampling: Metropolis–Hastings algorithms are fast. *J. Mach. Learn. Res.* **20** Paper No. 183, 42 pp. MR4048994
[13] Dyer, M. and Frieze, A. (2003). Randomly coloring graphs with lower bounds on girth and maximum degree. *Random Structures Algorithms* **23** 167–179. MR1995689 https://doi.org/10.1002/rsa.10087
[14] Eberle, A. (2014). Error bounds for Metropolis–Hastings algorithms applied to perturbations of Gaussian measures in high dimensions. *Ann. Appl. Probab.* **24** 337–377. MR3161650 https://doi.org/10.1214/13-AAP926
[15] Efthymiou, C., Hayes, T. P., Štefankovič, D., Vigoda, E. and Yin, Y. (2016). Convergence of MCMC and loopy BP in the tree uniqueness region for the hard-core model. In 57*th Annual IEEE Symposium on Foundations of Computer Science—FOCS* 2016 704–713. IEEE Computer Soc., Los Alamitos, CA. MR3631033 https://doi.org/10.1109/FOCS.2016.80
[16] Flegal, J. M., Haran, M. and Jones, G. L. (2008). Markov chain Monte Carlo: Can we trust the third significant figure? *Statist. Sci.* **23** 250–260. MR2516823 https://doi.org/10.1214/08-STS257
[17] Gelfand, A. E. and Smith, A. F. M. (1990). Sampling-based approaches to calculating marginal densities. *J. Amer. Statist. Assoc.* **85** 398–409. MR1141740
[18] Gelman, A. and Rubin, D. B. (1992). Inference from iterative simulation using multiple sequences. *Statist. Sci.* **7** 457–472.
[19] Gilks, W. R., Richardson, S. and Spiegelhalter, D. J., eds. (1996). *Markov Chain Monte Carlo in Practice. Interdisciplinary Statistics*. CRC Press, London. MR1397966 https://doi.org/10.1007/978-1-4899-4485-6
[20] Hairer, M., Mattingly, J. C. and Scheutzow, M. (2011). Asymptotic coupling and a general form of Harris' theorem with applications to stochastic delay equations. *Probab. Theory Related Fields* **149** 223–259. MR2773030 https://doi.org/10.1007/s00440-009-0250-6
[21] Jerrum, M., Son, J.-B., Tetali, P. and Vigoda, E. (2004). Elementary bounds on Poincaré and log-Sobolev constants for decomposable Markov chains. *Ann. Appl. Probab.* **14** 1741–1765. MR2099650 https://doi.org/10.1214/105051604000000639
[22] Jin, Z. and Hobert, J. P. (2022). On the convergence rate of the "out-of-order" block Gibbs sampler. *Statist. Probab. Lett.* **188** 109538. MR4432414 https://doi.org/10.1016/j.spl.2022.109538
[23] Jones, G. L., Haran, M., Caffo, B. S. and Neath, R. (2006). Fixed-width output analysis for Markov chain Monte Carlo. *J. Amer. Statist. Assoc.* **101** 1537–1547. MR2279478 https://doi.org/10.1198/016214506000000492
[24] Jones, G. L. and Hobert, J. P. (2001). Honest exploration of intractable probability distributions via Markov chain Monte Carlo. *Statist. Sci.* **16** 312–334. MR1888447 https://doi.org/10.1214/ss/1015346317
[25] Jones, G. L. and Hobert, J. P. (2004). Sufficient burn-in for Gibbs samplers for a hierarchical random effects model. *Ann. Statist.* **32** 784–817. MR2060178 https://doi.org/10.1214/009053604000000184
[26] Khare, K. and Hobert, J. P. (2013). Geometric ergodicity of the Bayesian lasso. *Electron. J. Stat.* **7** 2150–2163. MR3104915 https://doi.org/10.1214/13-EJS841
[27] Łatuszyński, K., Miasojedow, B. and Niemiro, W. (2013). Nonasymptotic bounds on the estimation error of MCMC algorithms. *Bernoulli* **19** 2033–2066. MR3129043 https://doi.org/10.3150/12-BEJ442
[28] Łatuszyński, K. and Niemiro, W. (2011). Rigorous confidence bounds for MCMC under a geometric drift condition. *J. Complexity* **27** 23–38. MR2745298 https://doi.org/10.1016/j.jco.2010.07.003
[29] Lovász, L. and Vempala, S. (2004). Hit-and-run from a corner. In *Proceedings of the* 36*th Annual ACM Symposium on Theory of Computing* 310–314. ACM, New York. MR2121613 https://doi.org/10.1145/1007352.1007403
[30] Lovász, L. and Vempala, S. (2006). Hit-and-run from a corner. *SIAM J. Comput.* **35** 985–1005. MR2203735 https://doi.org/10.1137/S009753970544727X
[31] Mangoubi, O. and Smith, A. (2017). Rapid mixing of Hamiltonian Monte Carlo on strongly log-concave distributions. Preprint. Available at arXiv:1708.07114.
[32] Martin, R. A. and Randall, D. (2000). Sampling adsorbing staircase walks using a new Markov chain decomposition method. In 41*st Annual Symposium on Foundations of Computer Science* (*Redondo Beach*, *CA*, 2000) 492–502. IEEE Comput. Soc. Press, Los Alamitos, CA. MR1931846 https://doi.org/10.1109/SFCS.2000.892137
[33] Medina-Aguayo, F., Rudolf, D. and Schweizer, N. (2020). Perturbation bounds for Monte Carlo within Metropolis via restricted approximations. *Stochastic Process. Appl.* **130** 2200–2227. MR4074698 https://doi.org/10.1016/j.spa.2019.06.015
[34] Meyn, S. P. and Tweedie, R. L. (1993). *Markov Chains and Stochastic Stability. Communications and Control Engineering Series*. Springer London, Ltd., London. MR1287609 https://doi.org/10.1007/978-1-4471-3267-7

[35] MEYN, S. P. and TWEEDIE, R. L. (1994). Computable bounds for geometric convergence rates of Markov chains. *Ann. Appl. Probab.* **4** 981–1011. MR1304770
[36] PAULIN, D. (2015). Concentration inequalities for Markov chains by Marton couplings and spectral methods. *Electron. J. Probab.* **20** Paper No. 79, 32 pp. MR3383563 https://doi.org/10.1214/EJP.v20-4039
[37] QIN, Q. and HOBERT, J. P. (2019). Convergence complexity analysis of Albert and Chib's algorithm for Bayesian probit regression. *Ann. Statist.* **47** 2320–2347. MR3953453 https://doi.org/10.1214/18-AOS1749
[38] RAJARATNAM, B. and SPARKS, D. (2015). MCMC-based inference in the era of big data: A fundamental analysis of the convergence complexity of high-dimensional chains. Preprint. Available at arXiv:1508.00947.
[39] ROBERTS, G. O., GELMAN, A. and GILKS, W. R. (1997). Weak convergence and optimal scaling of random walk Metropolis algorithms. *Ann. Appl. Probab.* **7** 110–120. MR1428751 https://doi.org/10.1214/aoap/1034625254
[40] ROBERTS, G. O. and ROSENTHAL, J. S. (1998). Optimal scaling of discrete approximations to Langevin diffusions. *J. R. Stat. Soc. Ser. B. Stat. Methodol.* **60** 255–268. MR1625691 https://doi.org/10.1111/1467-9868.00123
[41] ROBERTS, G. O. and ROSENTHAL, J. S. (2016). Complexity bounds for Markov chain Monte Carlo algorithms via diffusion limits. *J. Appl. Probab.* **53** 410–420. MR3514287 https://doi.org/10.1017/jpr.2016.9
[42] ROBERTS, G. O. and TWEEDIE, R. L. (1999). Bounds on regeneration times and convergence rates for Markov chains. *Stochastic Process. Appl.* **80** 211–229. MR1682243 https://doi.org/10.1016/S0304-4149(98)00085-4
[43] ROBERTS, G. O. and TWEEDIE, R. L. (2000). Rates of convergence of stochastically monotone and continuous time Markov models. *J. Appl. Probab.* **37** 359–373. MR1780996 https://doi.org/10.1239/jap/1014842542
[44] ROSENTHAL, J. S. (1995). Minorization conditions and convergence rates for Markov chain Monte Carlo. *J. Amer. Statist. Assoc.* **90** 558–566. MR1340509
[45] ROSENTHAL, J. S. (1995). Rates of convergence for Gibbs sampling for variance component models. *Ann. Statist.* **23** 740–761. MR1345197 https://doi.org/10.1214/aos/1176324619
[46] ROSENTHAL, J. S. (1996). Analysis of the Gibbs sampler for a model related to James–Stein estimators. *Stat. Comput.* **6** 269–275.
[47] ROSENTHAL, J. S. (2002). Quantitative convergence rates of Markov chains: A simple account. *Electron. Commun. Probab.* **7** 123–128. MR1917546 https://doi.org/10.1214/ECP.v7-1054
[48] RUDOLF, D. (2009). Explicit error bounds for lazy reversible Markov chain Monte Carlo. *J. Complexity* **25** 11–24. MR2475305 https://doi.org/10.1016/j.jco.2008.05.005
[49] RUDOLF, D. (2010). Error bounds for computing the expectation by Markov chain Monte Carlo. *Monte Carlo Methods Appl.* **16** 323–342. MR2747819 https://doi.org/10.1515/MCMA.2010.012
[50] RUDOLF, D. (2012). Explicit error bounds for Markov chain Monte Carlo. *Dissertationes Math.* **485** 1–93. MR2977521 https://doi.org/10.4064/dm485-0-1
[51] RUDOLF, D. and SPRUNGK, B. (2018). On a generalization of the preconditioned Crank–Nicolson Metropolis algorithm. *Found. Comput. Math.* **18** 309–343. MR3777781 https://doi.org/10.1007/s10208-016-9340-x
[52] SINCLAIR, A. and JERRUM, M. (1989). Approximate counting, uniform generation and rapidly mixing Markov chains. *Inform. and Comput.* **82** 93–133. MR1003059 https://doi.org/10.1016/0890-5401(89)90067-9
[53] VEMPALA, S. (2005). Geometric random walks: A survey. In *Combinatorial and Computational Geometry. Math. Sci. Res. Inst. Publ.* **52** 577–616. Cambridge Univ. Press, Cambridge. MR2178341
[54] WOODARD, D. B., SCHMIDLER, S. C. and HUBER, M. (2009). Sufficient conditions for torpid mixing of parallel and simulated tempering. *Electron. J. Probab.* **14** 780–804. MR2495560 https://doi.org/10.1214/EJP.v14-638
[55] WOODARD, D. B., SCHMIDLER, S. C. and HUBER, M. (2009). Conditions for rapid mixing of parallel and simulated tempering on multimodal distributions. *Ann. Appl. Probab.* **19** 617–640. MR2521882 https://doi.org/10.1214/08-AAP555
[56] YANG, J., ROBERTS, G. O. and ROSENTHAL, J. S. (2020). Optimal scaling of random-walk Metropolis algorithms on general target distributions. *Stochastic Process. Appl.* **130** 6094–6132. MR4140028 https://doi.org/10.1016/j.spa.2020.05.004
[57] ZHOU, Q., YANG, J., VATS, D., ROBERTS, G. O. and ROSENTHAL, J. S. (2021). Dimension-free mixing for high-dimensional Bayesian variable selection. Preprint. Available at arXiv:2105.05719.
[58] ZORICH, V. A. and COOKE, R. (2004). *Mathematical Analysis II*. Springer, Berlin.

The Annals of Applied Probability
2023, Vol. 33, No. 2, 1501–1514
https://doi.org/10.1214/22-AAP1847

MODIFIED LOG-SOBOLEV INEQUALITIES FOR STRONG-RAYLEIGH MEASURES

BY JONATHAN HERMON[1,a] AND JUSTIN SALEZ[2,b]

[1]*Mathematics Department, University of British Columbia,* [a]*jhermon@math.ubc.ca*
[2]*CEREMADE, Université Paris-Dauphine and PSL,* [b]*justin.salez@dauphine.psl.eu*

We establish universal modified log-Sobolev inequalities for reversible Markov chains on the boolean lattice $\{0, 1\}^n$, when the invariant law π satisfies a form of negative dependence known as the *stochastic covering property*. This condition is strictly weaker than the strong Rayleigh property, and is satisfied in particular by all determinantal measures, as well as the uniform distribution over the set of bases of any balanced matroid. In the special case where π is k-homogeneous, our results imply the celebrated concentration inequality for Lipschitz functions due to Pemantle and Peres (*Combin. Probab. Comput.* **23** (2014) 140–160). As another application, we deduce that the natural Monte-Carlo Markov chain used to sample from π has mixing time at most $kn \log\log \frac{1}{\pi(x)}$ when initialized in state x. To the best of our knowledge, this is the first work relating negative dependence and modified log-Sobolev inequalities.

REFERENCES

[1] ADAMCZAK, R. and POLACZYK, B. (2021). Concentration inequalities for some negatively dependent binary random variables.

[2] ANARI, N., GHARAN, S. O. and REZAEI, A. (2016). Monte Carlo Markov chain algorithms for sampling strongly Rayleigh Distributions and Determinantal Point Processes.

[3] ANARI, N., GHARAN, S. O. and VINZANT, C. (2021). Log-concave polynomials, I: Entropy and a deterministic approximation algorithm for counting bases of matroids. *Duke Math. J.* **170** 3459–3504. MR4332671 https://doi.org/10.1215/00127094-2020-0091

[4] ANARI, N., LIU, K., GHARAN, S. O. and VINZANT, C. (2019). Log-concave polynomials II: High-dimensional walks and an FPRAS for counting bases of a matroid. In *STOC'19—Proceedings of the 51st Annual ACM SIGACT Symposium on Theory of Computing* 1–12. ACM, New York. MR4003314 https://doi.org/10.1145/3313276.3316385

[5] ANARI, N., LIU, K., OVEIS GHARAN, S. and VINZANT, C. (2018). Log-Concave Polynomials III: Mason's Ultra-Log-Concavity Conjecture for Independent Sets of Matroids. ArXiv e-prints. Available at arXiv:1811.01600.

[6] ANARI, N., VINZANT, C., LIU, K., OVEIS GHARAN, S. and VUONG, T.-D. (2021). Log-concave polynomials IV: Approximate exchange, tight mixing times, and near-optimal sampling of forests. In *STOC '21—Proceedings of the 53rd Annual ACM SIGACT Symposium on Theory of Computing* 408–420. ACM, New York. MR4398852 https://doi.org/10.1145/3406325.3451091

[7] AOUN, R., BANNA, M. and YOUSSEF, P. (2020). Matrix Poincaré inequalities and concentration. *Adv. Math.* **371** 107251. MR4108222 https://doi.org/10.1016/j.aim.2020.107251

[8] BOBKOV, S. G. and TETALI, P. (2006). Modified logarithmic Sobolev inequalities in discrete settings. *J. Theoret. Probab.* **19** 289–336. MR2283379 https://doi.org/10.1007/s10959-006-0016-3

[9] BORCEA, J., BRÄNDÉN, P. and LIGGETT, T. M. (2009). Negative dependence and the geometry of polynomials. *J. Amer. Math. Soc.* **22** 521–567. MR2476782 https://doi.org/10.1090/S0894-0347-08-00618-8

[10] BOUCHERON, S., LUGOSI, G. and MASSART, P. (2013). *Concentration Inequalities: A Nonasymptotic Theory of Independence*. Oxford Univ. Press, Oxford. MR3185193 https://doi.org/10.1093/acprof:oso/9780199535255.001.0001

[11] BRÄNDÉN, P. and HUH, J. (2018). Hodge–Riemann relations for Potts model partition functions.

[12] BRÄNDÉN, P. and HUH, J. (2020). Lorentzian polynomials. *Ann. of Math.* (2) **192** 821–891. MR4172622 https://doi.org/10.4007/annals.2020.192.3.4

MSC2020 subject classifications. 60J10, 60J45.
Key words and phrases. Modified log-Sobolev inequalities, negative dependence, stochastic covering.

[13] Brändén, P. and Jonasson, J. (2012). Negative dependence in sampling. *Scand. J. Stat.* **39** 830–838. MR3000852 https://doi.org/10.1111/j.1467-9469.2011.00766.x

[14] Caputo, P. and Posta, G. (2007). Entropy dissipation estimates in a zero-range dynamics. *Probab. Theory Related Fields* **139** 65–87. MR2322692 https://doi.org/10.1007/s00440-006-0039-9

[15] Choe, Y. and Wagner, D. G. (2006). Rayleigh matroids. *Combin. Probab. Comput.* **15** 765–781. MR2248326 https://doi.org/10.1017/S0963548306007541

[16] Cohen, E., Tetali, P. and Yeliussizov, D. (2015). Lattice path matroids: Negative correlation and fast mixing. ArXiv preprint. Available at arXiv:1505.06710.

[17] Cryan, M., Guo, H. and Mousa, G. (2021). Modified log-Sobolev inequalities for strongly log-concave distributions. *Ann. Probab.* **49** 506–525. MR4203344 https://doi.org/10.1214/20-AOP1453

[18] Diaconis, P. and Saloff-Coste, L. (1996). Logarithmic Sobolev inequalities for finite Markov chains. *Ann. Appl. Probab.* **6** 695–750. MR1410112 https://doi.org/10.1214/aoap/1034968224

[19] Diaconis, P. and Shahshahani, M. (1981). Generating a random permutation with random transpositions. *Z. Wahrsch. Verw. Gebiete* **57** 159–179. MR0626813 https://doi.org/10.1007/BF00535487

[20] Feder, T. and Mihail, M. (1992). Balanced matroids. In *Proceedings of the Twenty-Fourth Annual ACM Symposium on Theory of Computing*. *STOC'*92 26–38. ACM, New York.

[21] Hermon, J. and Salez, J. (2021). Entropy dissipation estimates for inhomogeneous zero-range processes. *Ann. Appl. Probab.* **31** 2275–2283. MR4332696 https://doi.org/10.1214/20-aap1646

[22] Jerrum, M. and Son, J.-B. (2002). Spectral gap and log-Sobolev constant for balanced matroids. In *The* 43*rd Annual IEEE Symposium on Foundations of Computer Science*, 2002. *Proceedings* 721–729.

[23] Jerrum, M., Son, J.-B., Tetali, P. and Vigoda, E. (2004). Elementary bounds on Poincaré and log-Sobolev constants for decomposable Markov chains. *Ann. Appl. Probab.* **14** 1741–1765. MR2099650 https://doi.org/10.1214/105051604000000639

[24] Kathuria, T. (2020). A Matrix Bernstein Inequality for Strong Rayleigh Distributions.

[25] Kirkpatrick, A., Patton, K., Tetali, P. and Mitchell, C. (2020). Markov chain-based sampling for exploring RNA secondary structure under the nearest neighbor thermodynamic model and extended applications. *Math. Comput. Appl.* **25** Paper No. 67. MR4219656 https://doi.org/10.3390/mca25040067

[26] Ledoux, M. (1999). Concentration of measure and logarithmic Sobolev inequalities. In *Séminaire de Probabilités*, *XXXIII*. *Lecture Notes in Math.* **1709** 120–216. Springer, Berlin. MR1767995 https://doi.org/10.1007/BFb0096511

[27] Levin, D. A. and Peres, Y. (2017). *Markov Chains and Mixing Times*. Amer. Math. Soc., Providence, RI. Second edition of [MR2466937]. MR3726904 https://doi.org/10.1090/mbk/107

[28] Lu, S. L. and Yau, H.-T. (1993). Spectral gap and logarithmic Sobolev inequality for Kawasaki and Glauber dynamics. *Comm. Math. Phys.* **156** 399–433. MR1233852

[29] Montenegro, R. and Tetali, P. (2006). Mathematical aspects of mixing times in Markov chains. *Found. Trends Theor. Comput. Sci.* **1** x+121. MR2341319 https://doi.org/10.1561/0400000003

[30] Oxley, J. (2003). What is a matroid? *Cubo Mat. Educ.* **5** 179–218. MR2065730

[31] Pemantle, R. (2000). Towards a theory of negative dependence. *J. Math. Phys.* **41** 1371–1390.

[32] Pemantle, R. and Peres, Y. (2014). Concentration of Lipschitz functionals of determinantal and other strong Rayleigh measures. *Combin. Probab. Comput.* **23** 140–160. MR3197973 https://doi.org/10.1017/S0963548313000345

[33] Sammer, M. (2005). Aspects of mass transportation in discrete concentration inequalities. Ph.D. thesis, Georgia Institute of Technology, Atlanta, GA.

[34] Sammer, M. and Tetali, P. (2009). Concentration on the discrete torus using transportation. *Combin. Probab. Comput.* **18** 835–860. MR2534272 https://doi.org/10.1017/S0963548309990319

The Annals of Applied Probability
2023, Vol. 33, No. 2, 1515–1568
https://doi.org/10.1214/22-AAP1848

THE MAXIMUM OF BRANCHING BROWNIAN MOTION IN $\mathbb{R}^d$

BY YUJIN H. KIM[1,a], EYAL LUBETZKY[1,b] AND OFER ZEITOUNI[2,c]

[1]*Courant Institute of Mathematical Sciences, New York University,* [a]*yujin.kim@courant.nyu.edu,* [b]*eyal@courant.nyu.edu*
[2]*Department of Mathematics, Weizmann Institute of Science,* [c]*ofer.zeitouni@weizmann.ac.il*

We show that in branching Brownian motion (BBM) in $\mathbb{R}^d$, $d \geq 2$, the law of R_t^*, the maximum distance of a particle from the origin at time t, converges as $t \to \infty$ to the law of a randomly shifted Gumbel random variable.

REFERENCES

[1] AÏDÉKON, E. (2013). Convergence in law of the minimum of a branching random walk. *Ann. Probab.* **41** 1362–1426. MR3098680 https://doi.org/10.1214/12-AOP750
[2] ARGUIN, L.-P., BOVIER, A. and KISTLER, N. (2013). The extremal process of branching Brownian motion. *Probab. Theory Related Fields* **157** 535–574. MR3129797 https://doi.org/10.1007/s00440-012-0464-x
[3] BELIUS, D., ROSEN, J. and ZEITOUNI, O. (2019). Barrier estimates for a critical Galton–Watson process and the cover time of the binary tree. *Ann. Inst. Henri Poincaré Probab. Stat.* **55** 127–154. MR3901643 https://doi.org/10.1214/17-aihp878
[4] BERESTYCKI, J., KIM, Y., LUBETZKY, E., MALLEIN, B. and ZEITOUNI, O. (2021). The extremal point process of branching Brownian motion in $\mathbb{R}^d$. Preprint. Available at arXiv:2112.08407.
[5] BRAMSON, M. (1983). Convergence of solutions of the Kolmogorov equation to travelling waves. *Mem. Amer. Math. Soc.* **44** iv+190. MR0705746 https://doi.org/10.1090/memo/0285
[6] BRAMSON, M., DING, J. and ZEITOUNI, O. (2016). Convergence in law of the maximum of nonlattice branching random walk. *Ann. Inst. Henri Poincaré Probab. Stat.* **52** 1897–1924. MR3573300 https://doi.org/10.1214/15-AIHP703
[7] BRAMSON, M. D. (1978). Maximal displacement of branching Brownian motion. *Comm. Pure Appl. Math.* **31** 531–581. MR0494541 https://doi.org/10.1002/cpa.3160310502
[8] GÄRTNER, J. (1982). Location of wave fronts for the multidimensional KPP equation and Brownian first exit densities. *Math. Nachr.* **105** 317–351. MR0670523 https://doi.org/10.1002/mana.19821050117
[9] HARRIS, S. C. and ROBERTS, M. I. (2017). The many-to-few lemma and multiple spines. *Ann. Inst. Henri Poincaré Probab. Stat.* **53** 226–242. MR3606740 https://doi.org/10.1214/15-AIHP714
[10] LALLEY, S. P. and SELLKE, T. (1987). A conditional limit theorem for the frontier of a branching Brownian motion. *Ann. Probab.* **15** 1052–1061. MR0893913
[11] MAILLARD, P. (2012). Branching Brownian motion with selection. Ph.D. thesis Univ. Paris.
[12] MALLEIN, B. (2015). Maximal displacement of d-dimensional branching Brownian motion. *Electron. Commun. Probab.* **20** 76. MR3417448 https://doi.org/10.1214/ECP.v20-4216
[13] MCKEAN, H. P. (1975). Application of Brownian motion to the equation of Kolmogorov–Petrovskii–Piskunov. *Comm. Pure Appl. Math.* **28** 323–331. MR0400428 https://doi.org/10.1002/cpa.3160280302
[14] REVUZ, D. and YOR, M. (1999). *Continuous Martingales and Brownian Motion*, 3rd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **293**. Springer, Berlin. MR1725357 https://doi.org/10.1007/978-3-662-06400-9
[15] STASIŃSKI, R., BERESTYCKI, J. and MALLEIN, B. (2021). Derivative martingale of the branching Brownian motion in dimension $d \geq 1$. *Ann. Inst. Henri Poincaré Probab. Stat.* **57** 1786–1810. MR4291461 https://doi.org/10.1214/20-aihp1131

MSC2020 subject classifications. 60J65, 60J70, 60J80.
Key words and phrases. Branching Brownian motion, Bessel process, extremal process, log-correlated field.

The Annals of Applied Probability
2023, Vol. 33, No. 2, 1569–1587
https://doi.org/10.1214/22-AAP1849

NONPARAMETRIC LEARNING FOR IMPULSE CONTROL PROBLEMS—EXPLORATION VS. EXPLOITATION

BY SÖREN CHRISTENSEN[1,a] AND CLAUDIA STRAUCH[2,b]

[1] *Department of Mathematics, Christian-Albrechts-Universität Kiel,* [a] *christensen@math.uni-kiel.de*
[2] *Department of Mathematics, Aarhus University,* [b] *strauch@math.au.dk*

One of the fundamental assumptions in stochastic control of continuous time processes is that the dynamics of the underlying (diffusion) process is known. This is, however, usually obviously not fulfilled in practice. On the other hand, over the last decades, a rich theory for nonparametric estimation of the drift (and volatility) for continuous time processes has been developed. The aim of this paper is bringing together techniques from stochastic control with methods from statistics for stochastic processes to find a way to both learn the dynamics of the underlying process and control in a reasonable way at the same time. More precisely, we study a long-term average impulse control problem, a stochastic version of the classical Faustmann timber harvesting problem. One of the problems that immediately arises is an exploration-exploitation dilemma as is well known for problems in machine learning. We propose a way to deal with this issue by combining exploration and exploitation periods in a suitable way. Our main finding is that this construction can be based on the rates of convergence of estimators for the invariant density. Using this, we obtain that the average cumulated regret is of uniform order $O(T^{-1/3})$.

REFERENCES

[1] AECKERLE-WILLEMS, C. and STRAUCH, C. (2021). Concentration of scalar ergodic diffusions and some statistical implications. *Ann. Inst. Henri Poincaré Probab. Stat.* **57** 1857–1887. MR4328556 https://doi.org/10.1214/20-aihp1144

[2] ALVAREZ, L. H. R. (2004). A class of solvable impulse control problems. *Appl. Math. Optim.* **49** 265–295. MR2041852 https://doi.org/10.1007/s00245-004-0792-z

[3] ALVAREZ, L. H. R. (2004). Stochastic forest stand value and optimal timber harvesting. *SIAM J. Control Optim.* **42** 1972–1993. MR2080924 https://doi.org/10.1137/S0363012901393456

[4] AUER, P., ORTNER, R. and SZEPESVÁRI, C. (2007). Improved rates for the stochastic continuum-armed bandit problem. In *Learning Theory. Lecture Notes in Computer Science* **4539** 454–468. Springer, Berlin. MR2397605 https://doi.org/10.1007/978-3-540-72927-3_33

[5] BAYRAKTAR, E. and ZHANG, Y. (2015). Minimizing the probability of lifetime ruin under ambiguity aversion. *SIAM J. Control Optim.* **53** 58–90. MR3295680 https://doi.org/10.1137/140955999

[6] BERTSEKAS, D. P. (2012). *Dynamic Programming and Optimal Control. Vol. II. Approximate Dynamic Programming*, 4th ed. Athena Scientific, Belmont, MA. MR3642732

[7] BERTSEKAS, D. P. (2019). *Reinforcement Learning and Optimal Control*. Athena Scientific, Belmont, MA.

[8] BIELECKI, T. R., CHEN, T., CIALENCO, I., COUSIN, A. and JEANBLANC, M. (2019). Adaptive robust control under model uncertainty. *SIAM J. Control Optim.* **57** 925–946. MR3922910 https://doi.org/10.1137/17M1137917

[9] BORODIN, A. N. and SALMINEN, P. (2015). *Handbook of Brownian Motion—Facts and Formulae*, 2nd ed. *Probability and Its Applications*. Birkhäuser, Basel. MR1477407 https://doi.org/10.1007/978-3-0348-7652-0

[10] BUBECK, S. and CESA-BIANCHI, N. (2012). Regret analysis of stochastic and nonstochastic multi-armed bandit problems. *Found. Trends Mach. Learn.* **5** 1–122.

[11] BUBECK, S., MUNOS, R., STOLTZ, G. and SZEPESVÁRI, C. (2011). $\mathscr{X}$-Armed bandits. *J. Mach. Learn. Res.* **12** 1655–1695. MR2813150

MSC2020 subject classifications. Primary 68T05, 93E20, 62M05; secondary 60J60.
Key words and phrases. Nonparametric statistics, stochastic impulse control, reinforcement learning, diffusion processes, optimal harvesting problem, exploration vs. exploitation, Faustmann problem.

[12] CHRISTENSEN, S. (2013). Optimal decision under ambiguity for diffusion processes. *Math. Methods Oper. Res.* **77** 207–226. MR3070026 https://doi.org/10.1007/s00186-012-0425-2

[13] CHRISTENSEN, S., NEUMANN, B. A. and SOHR, T. (2021). Competition versus cooperation: A class of solvable mean field impulse control problems. *SIAM J. Control Optim.* **59** 3946–3972. MR4328505 https://doi.org/10.1137/20M1373372

[14] CHRISTENSEN, S., STRAUCH, C. and TROTTNER, L. (2021). Learning to reflect: A unifying approach for data-driven stochastic control strategies. Preprint. Available at arXiv:2104.11496.

[15] GOBET, E., HOFFMANN, M. and REISS, M. (2004). Nonparametric estimation of scalar diffusions based on low frequency data. *Ann. Statist.* **32** 2223–2253. MR2102509 https://doi.org/10.1214/009053604000000797

[16] HELMES, K. L., STOCKBRIDGE, R. H. and ZHU, C. (2017). Continuous inventory models of diffusion type: Long-term average cost criterion. *Ann. Appl. Probab.* **27** 1831–1885. MR3678486 https://doi.org/10.1214/16-AAP1247

[17] HELMES, K. L., STOCKBRIDGE, R. H. and ZHU, C. (2018). A weak convergence approach to inventory control using a long-term average criterion. *Adv. in Appl. Probab.* **50** 1032–1074. MR3881109 https://doi.org/10.1017/apr.2018.50

[18] JOBJÖRNSSON, S. and CHRISTENSEN, S. (2018). Anscombe's model for sequential clinical trials revisited. *Sequential Anal.* **37** 115–144. MR3773160 https://doi.org/10.1080/07474946.2018.1427982

[19] KOHLER, M. and WALK, H. (2013). On data-based optimal stopping under stationarity and ergodicity. *Bernoulli* **19** 931–953. MR3079301 https://doi.org/10.3150/12-BEJ439

[20] KUTOYANTS, Y. A. (2004). *Statistical Inference for Ergodic Diffusion Processes*. *Springer Series in Statistics*. Springer, London. MR2144185 https://doi.org/10.1007/978-1-4471-3866-2

[21] KUTOYANTS, Y. A. AND NEGRI, I. (2001). On L_2 efficiency of empiric distribution for ergodic diffusion processes. *Teor. Veroyatn. Primen.* **46** 164–169. MR1968714 https://doi.org/10.1137/S0040585X97978816

[22] LATTIMORE, T. and SZEPESVÁRI, C. (2020). *Bandit Algorithms*. Cambridge Univ. Press, Cambridge.

[23] LE GALL, J.-F. (2016). *Brownian Motion, Martingales, and Stochastic Calculus*, 4th ed. *Graduate Texts in Mathematics* **274**. Springer, Cham. MR3497465 https://doi.org/10.1007/978-3-319-31089-3

[24] LOCATELLI, A. and CARPENTIER, A. (2018). Adaptivity to smoothness in X-armed bandits. In *Conference on Learning Theory* 1463–1492.

[25] PESKIR, G. and SHIRYAEV, A. (2006). *Optimal Stopping and Free-Boundary Problems*. *Lectures in Mathematics ETH Zürich*. Birkhäuser, Basel. MR2256030

[26] RIEDEL, F. (2009). Optimal stopping with multiple priors. *Econometrica* **77** 857–908. MR2531363 https://doi.org/10.3982/ECTA7594

[27] RIEDER, U. and BÄUERLE, N. (2005). Portfolio optimization with unobservable Markov-modulated drift process. *J. Appl. Probab.* **42** 362–378. MR2145482 https://doi.org/10.1239/jap/1118777176

[28] STETTNER, Ł. (1986). On continuous time adaptive impulsive control. In *System Modelling and Optimization* (*Budapest*, 1985). *Lect. Notes Control Inf. Sci.* **84** 913–922. Springer, Berlin. MR0903523 https://doi.org/10.1007/BFb0043919

The Annals of Applied Probability
2023, Vol. 33, No. 2, 1588–1618
https://doi.org/10.1214/22-AAP1850

PROPAGATION OF MINIMALITY IN THE SUPERCOOLED STEFAN PROBLEM

BY CHRISTA CUCHIERO[1,a], STEFAN RIGGER[2,b] AND SARA SVALUTO-FERRO[3,c]

[1]*Department of Statistics and Operations Research, Vienna University,* [a]*christa.cuchiero@univie.ac.at*
[2]*Department of Statistics and Operations Research, Vienna University,* [b]*stefan.rigger@univie.ac.at*
[3]*Department of Economics, University of Verona,* [c]*sara.svalutoferro@univr.it*

Supercooled Stefan problems describe the evolution of the boundary between the solid and liquid phases of a substance, where the liquid is assumed to be cooled below its freezing point. Following the methodology of Delarue, Nadtochiy and Shkolnikov, we construct solutions to the one-phase one-dimensional supercooled Stefan problem through a certain McKean–Vlasov equation, which allows to define global solutions even in the presence of blow-ups. Solutions to the McKean–Vlasov equation arise as mean-field limits of particle systems interacting through hitting times, which is important for systemic risk modeling. Our main contributions are: (i) A general tightness theorem for the Skorokhod M_1-topology which applies to processes that can be decomposed into a continuous and a monotone part. (ii) A propagation of chaos result for a perturbed version of the particle system for general initial conditions. (iii) The proof of a conjecture of Delarue, Nadtochiy and Shkolnikov, relating the solution concepts of so-called minimal and physical solutions, showing that minimal solutions of the McKean–Vlasov equation are physical whenever the initial condition is integrable.

REFERENCES

[1] BAKER, G. and SHKOLNIKOV, M. (2022). Zero kinetic undercooling limit in the supercooled Stefan problem. *Ann. Inst. Henri Poincaré Probab. Stat.* **58** 861–871. MR4421610 https://doi.org/10.1214/21-aihp1194

[2] CARMONA, R. and DELARUE, F. (2018). *Probabilistic Theory of Mean Field Games with Applications I*. Springer, Berlin.

[3] DELARUE, F., INGLIS, J., RUBENTHALER, S. and TANRÉ, E. (2015). Particle systems with a singular mean-field self-excitation. Application to neuronal networks. *Stochastic Process. Appl.* **125** 2451–2492. MR3322871 https://doi.org/10.1016/j.spa.2015.01.007

[4] DELARUE, F., INGLIS, J., RUBENTHALER, S. and TANRÉ, E. (2015). Global solvability of a networked integrate-and-fire model of McKean–Vlasov type. *Ann. Appl. Probab.* **25** 2096–2133. MR3349003 https://doi.org/10.1214/14-AAP1044

[5] DELARUE, F., NADTOCHIY, S. and SHKOLNIKOV, M. (2022). Global solutions to the supercooled Stefan problem with blow-ups: Regularity and uniqueness. *Probab. Math. Phys.* **3** 171–213.

[6] DEWYNNE, J. N. (1992). A survey of supercooled Stefan problems. In *Mini-Conference on Free and Moving Boundary and Diffusion Problems* (*Canberra*, 1990). *Proc. Centre Math. Appl. Austral. Nat. Univ.* **30** 42–56. Austral. Nat. Univ., Canberra. MR1210749

[7] DUDLEY, R. M. (2018). *Real Analysis and Probability*: 0. CRC Press/CRC, Boca Raton.

[8] ETHIER, S. N. and KURTZ, T. G. (2009). *Markov Processes*: *Characterization and Convergence* **282**. Wiley, New York.

[9] FEINSTEIN, Z. and SØJMARK, A. (2021). Short communication: Dynamic default contagion in heterogeneous interbank systems. *SIAM J. Financial Math.* **12** SC83–SC97. MR4323036 https://doi.org/10.1137/20M1376765

[10] HAMBLY, B., LEDGER, S. and SØJMARK, A. (2019). A McKean–Vlasov equation with positive feedback and blow-ups. *Ann. Appl. Probab.* **29** 2338–2373. MR3983340 https://doi.org/10.1214/18-AAP1455

MSC2020 subject classifications. 60H30, 60K35, 35Q84.

Key words and phrases. Supercooled Stefan problem, McKean–Vlasov equations, singular interactions, propagation of chaos, systemic risk.

[11] HAMBLY, B. and SØJMARK, A. (2019). An SPDE model for systemic risk with endogenous contagion. *Finance Stoch*. **23** 535–594. MR3968278 https://doi.org/10.1007/s00780-019-00396-1

[12] JACOD, J. and SHIRYAEV, A. (2013). *Limit Theorems for Stochastic Processes* **288**. Springer, Berlin.

[13] KAUSHANSKY, V. and REISINGER, C. (2019). Simulation of a simple particle system interacting through hitting times. *Discrete Contin. Dyn. Syst. Ser. B* **24** 5481–5502. MR4026935 https://doi.org/10.3934/dcdsb.2019067

[14] KAUSHANSKY, V., REISINGER, C., SHKOLNIKOV, M. and SONG, Z. Q. (2020). Convergence of a time-stepping scheme to the free boundary in the supercooled Stefan problem. ArXiv preprint. Available at arXiv:2010.05281.

[15] KLENKE, A. (2013). *Probability Theory: A Comprehensive Course*. Springer, Berlin.

[16] LAMÉ, G. and CLAPEYRON, B. P. E. (1831). Mémoire sur la solidification par refroidissement d'un globe liquide. In *Annales Chimie Physique* **47** 250–256.

[17] LEDGER, S. and SØJMARK, A. (2020). Uniqueness for contagious McKean–Vlasov systems in the weak feedback regime. *Bull. Lond. Math. Soc.* **52** 448–463. MR4171379 https://doi.org/10.1112/blms.12337

[18] LEDGER, S. and SØJMARK, A. (2021). At the mercy of the common noise: Blow-ups in a conditional McKean–Vlasov problem. *Electron. J. Probab.* **26** 35. MR4235486 https://doi.org/10.1214/21-EJP597

[19] LIPTON, A., KAUSHANSKY, V. and REISINGER, C. (2021). Semi-analytical solution of a McKean–Vlasov equation with feedback through hitting a boundary. *European J. Appl. Math.* **32** 1035–1068. MR4335143 https://doi.org/10.1017/S0956792519000342

[20] MASSART, P. (1990). The tight constant in the Dvoretzky–Kiefer–Wolfowitz inequality. *Ann. Probab.* **18** 1269–1283. MR1062069

[21] NADTOCHIY, S. and SHKOLNIKOV, M. (2019). Particle systems with singular interaction through hitting times: Application in systemic risk modeling. *Ann. Appl. Probab.* **29** 89–129. MR3910001 https://doi.org/10.1214/18-AAP1403

[22] NADTOCHIY, S. and SHKOLNIKOV, M. (2020). Mean field systems on networks, with singular interaction through hitting times. *Ann. Probab.* **48** 1520–1556. MR4112723 https://doi.org/10.1214/19-AOP1403

[23] PESKIR, G. and SHIRYAEV, A. (2006). *Optimal Stopping and Free-Boundary Problems*. *Lectures in Mathematics ETH Zürich*. Birkhäuser, Basel. MR2256030

[24] SCHWARTZ, L. (1973). *Radon Measures on Arbitrary Topological Spaces and Cylindrical Measures*. *Tata Institute of Fundamental Research Studies in Mathematics*, **6**. Published for the Tata Institute of Fundamental Research, Bombay by Oxford University Press, London. MR0426084

[25] SHERMAN, B. (1970). A general one-phase Stefan problem. *Quart. Appl. Math.* **28** 377–382. MR0282082 https://doi.org/10.1090/qam/282082

[26] STEFAN, J. (1891). Über die Theorie der Eisbildung, insbesondere über die Eisbildung im Polarmeere. *Ann. Phys. Chem.* **42** 269–286.

[27] SZNITMAN, A.-S. (1991). Topics in propagation of chaos. In *École D'Été de Probabilités de Saint-Flour XIX*—1989. *Lecture Notes in Math.* **1464** 165–251. Springer, Berlin. MR1108185 https://doi.org/10.1007/BFb0085169

[28] WHITT, W. (2002). *Stochastic-Process Limits: An Introduction to Stochastic-Process Limits and Their Application to Queues*. *Springer Series in Operations Research*. Springer, New York. MR1876437

The Annals of Applied Probability
2023, Vol. 33, No. 2, 1619–1659
https://doi.org/10.1214/22-AAP1853

A PROBABILITY APPROXIMATION FRAMEWORK: MARKOV PROCESS APPROACH

BY PENG CHEN[1,a], QI-MAN SHAO[2,b] AND LIHU XU[3,c]

[1]*College of Mathematics, Nanjing University of Aeronautics and Astronautics,* [a]*chenpengmath@nuaa.edu.cn*
[2]*Department of Statistics and Data Science, SICM, NCAMS, Southern University of Science and Technology,* [b]*shaoqm@sustech.edu.cn*
[3]*Department of Mathematics, Faculty of Science and Technology, University of Macau,* [c]*lihuxu@um.edu.mo*

We view the classical Lindeberg principle in a Markov process setting to establish a probability approximation framework by the associated Itô's formula and Markov operator. As applications, we study the error bounds of the following three approximations: approximating a family of online stochastic gradient descents (SGDs) by a stochastic differential equation (SDE) driven by multiplicative Brownian motion, Euler–Maruyama (EM) discretization for multi-dimensional Ornstein–Uhlenbeck stable process and multivariate normal approximation. All these error bounds are in Wasserstein-1 distance.

REFERENCES

[1] ARRAS, B. and HOUDRÉ, C. (2019). On Stein's method for multivariate self-decomposable laws. *Electron. J. Probab.* **24** Paper No. 128, 63. MR4029431 https://doi.org/10.1214/19-ejp378

[2] ARRATIA, R., GOLDSTEIN, L. and GORDON, L. (1990). Poisson approximation and the Chen–Stein method. *Statist. Sci.* **5** 403–434. With comments and a rejoinder by the authors. MR1092983

[3] AVELIN, B. and NYSTRÖM, K. (2021). Neural ODEs as the deep limit of ResNets with constant weights. *Anal. Appl.* (*Singap.*) **19** 397–437. MR4233706 https://doi.org/10.1142/S0219530520400023

[4] BAHLALI, K. (1999). Flows of homeomorphisms of stochastic differential equations with measurable drift. *Stoch. Stoch. Rep.* **67** 53–82. MR1717807 https://doi.org/10.1080/17442509908834203

[5] BALLY, V., CARAMELLINO, L. and POLY, G. (2018). Convergence in distribution norms in the CLT for non identical distributed random variables. *Electron. J. Probab.* **23** Paper No. 45, 51. MR3814239 https://doi.org/10.1214/18-ejp174

[6] BANNA, M., MERLEVÈDE, F. and PELIGRAD, M. (2015). On the limiting spectral distribution for a large class of symmetric random matrices with correlated entries. *Stochastic Process. Appl.* **125** 2700–2726. MR3332852 https://doi.org/10.1016/j.spa.2015.01.010

[7] BARBOUR, A. D. (1988). Stein's method and Poisson process convergence. *J. Appl. Probab.* **25A** 175–184. Special Vol., A celebration of applied probability. MR0974580 https://doi.org/10.1017/s0021900200040341

[8] BARBOUR, A. D. (1990). Stein's method for diffusion approximations. *Probab. Theory Related Fields* **84** 297–322. MR1035659 https://doi.org/10.1007/BF01197887

[9] BARBOUR, A. D. and BROWN, T. C. (1992). Stein's method and point process approximation. *Stochastic Process. Appl.* **43** 9–31. MR1190904 https://doi.org/10.1016/0304-4149(92)90073-Y

[10] BARBOUR, A. D., CHEN, L. H. Y. and LOH, W.-L. (1992). Compound Poisson approximation for nonnegative random variables via Stein's method. *Ann. Probab.* **20** 1843–1866. MR1188044

[11] BARBOUR, A. D. and HALL, P. (1984). On the rate of Poisson convergence. *Math. Proc. Cambridge Philos. Soc.* **95** 473–480. MR0755837 https://doi.org/10.1017/S0305004100061806

[12] BARBOUR, A. D., HOLST, L. and JANSON, S. (1992). *Poisson Approximation*. *Oxford Studies in Probability* **2**. The Clarendon Press, Oxford Univ. Press, New York. Oxford Science Publications. MR1163825

[13] BARMAN, K. and UPADHYE, N. S. (2020). Stein's method for tempered stable distributions. Available at arXiv:2008.05818.

MSC2020 subject classifications. Primary 60F05, 60H07; secondary 60J20.
Key words and phrases. Probability approximation, Markov process, Itô's formula, online stochastic gradient descent, stochastic differential equation, Euler–Maruyama (EM) discretization, stable process, normal approximation, Wasserstein-1 distance.

[14] BICHTELER, K. (2002). *Stochastic Integration with Jumps*. *Encyclopedia of Mathematics and Its Applications* **89**. Cambridge Univ. Press, Cambridge. MR1906715 https://doi.org/10.1017/CBO9780511549878

[15] BLUMENTHAL, R. M. and GETOOR, R. K. (1968). *Markov Processes and Potential Theory*. *Pure and Applied Mathematics* **29**. Academic Press, New York. MR0264757

[16] BOFFI, N. M. and SLOTINE, J.-J. E. (2020). A continuous-time analysis of distributed stochastic gradient. *Neural Comput*. **32** 36–96. MR4055382 https://doi.org/10.1162/neco_a_01248

[17] BOGACHEV, V. I. and RÖCKNER, M. (1995). Regularity of invariant measures on finite- and infinite-dimensional spaces and applications. *J. Funct. Anal*. **133** 168–223. MR1351647 https://doi.org/10.1006/jfan.1995.1123

[18] BOGDAN, K. and JAKUBOWSKI, T. (2007). Estimates of heat kernel of fractional Laplacian perturbed by gradient operators. *Comm. Math. Phys*. **271** 179–198. MR2283957 https://doi.org/10.1007/s00220-006-0178-y

[19] BRAVERMAN, A. and DAI, J. G. (2017). Stein's method for steady-state diffusion approximations of $M/Ph/n+M$ systems. *Ann. Appl. Probab.* **27** 550–581. MR3619795 https://doi.org/10.1214/16-AAP1211

[20] BRAVERMAN, A., DAI, J. G. and FENG, J. (2016). Stein's method for steady-state diffusion approximations: An introduction through the Erlang-A and Erlang-C models. *Stoch. Syst*. **6** 301–366. MR3633538 https://doi.org/10.1214/15-SSY212

[21] CARAVENNA, F., SUN, R. and ZYGOURAS, N. (2017). Polynomial chaos and scaling limits of disordered systems. *J. Eur. Math. Soc*. (*JEMS*) **19** 1–65. MR3584558 https://doi.org/10.4171/JEMS/660

[22] CERRAI, S. (1996). Elliptic and parabolic equations in $\mathbb{R}^n$ with coefficients having polynomial growth. *Comm. Partial Differential Equations* **21** 281–317. MR1373775 https://doi.org/10.1080/03605309608821185

[23] CHATTERJEE, S. (2006). A generalization of the Lindeberg principle. *Ann. Probab.* **34** 2061–2076. MR2294976 https://doi.org/10.1214/009117906000000575

[24] CHATTERJEE, S. and MECKES, E. (2008). Multivariate normal approximation using exchangeable pairs. *ALEA Lat. Am. J. Probab. Math. Stat*. **4** 257–283. MR2453473

[25] CHEN, L. H. Y. (1975). Poisson approximation for dependent trials. *Ann. Probab.* **3** 534–545. MR0428387 https://doi.org/10.1214/aop/1176996359

[26] CHEN, L. H. Y., GOLDSTEIN, L. and SHAO, Q.-M. (2011). *Normal Approximation by Stein's Method*. *Probability and Its Applications* (*New York*). Springer, Heidelberg. MR2732624 https://doi.org/10.1007/978-3-642-15007-4

[27] CHEN, P., NOURDIN, I. and XU, L. (2021). Stein's method for asymmetric α-stable distributions, with application to the stable CLT. *J. Theoret. Probab.* **34** 1382–1407. MR4289888 https://doi.org/10.1007/s10959-020-01004-1

[28] CHEN, P., NOURDIN, I., XU, L. and YANG, X. (2019). Multivariate stable approximation in Wasserstein distance by Stein's method. Available at arXiv:1911.12917.

[29] CHEN, P., NOURDIN, I., XU, L., YANG, X. and ZHANG, R. (2022). Non-integrable stable approximation by Stein's method. *J. Theoret. Probab.* **35** 1137–1186. MR4414414 https://doi.org/10.1007/s10959-021-01094-5

[30] CHEN, P. and XU, L. (2019). Approximation to stable law by the Lindeberg principle. *J. Math. Anal. Appl.* **480** 123338, 28. MR4000076 https://doi.org/10.1016/j.jmaa.2019.07.028

[31] CHEN, X., LEE, J. D., TONG, X. T. and ZHANG, Y. (2020). Statistical inference for model parameters in stochastic gradient descent. *Ann. Statist*. **48** 251–273. MR4065161 https://doi.org/10.1214/18-AOS1801

[32] CHEN, Z., LI, R. and WU, Y. (2012). Weighted quantile regression for AR model with infinite variance errors. *J. Nonparametr. Stat*. **24** 715–731. MR2968898 https://doi.org/10.1080/10485252.2012.698280

[33] CHERNOZHUKOV, V., CHETVERIKOV, D. and KATO, K. (2014). Gaussian approximation of suprema of empirical processes. *Ann. Statist*. **42** 1564–1597. MR3262461 https://doi.org/10.1214/14-AOS1230

[34] CHERNOZHUKOV, V., CHETVERIKOV, D. and KATO, K. (2017). Central limit theorems and bootstrap in high dimensions. *Ann. Probab.* **45** 2309–2352. MR3693963 https://doi.org/10.1214/16-AOP1113

[35] CHIARINI, A. and FISCHER, M. (2014). On large deviations for small noise Itô processes. *Adv. in Appl. Probab.* **46** 1126–1147. MR3290432 https://doi.org/10.1239/aap/1418396246

[36] CLINE, D. B. H. (1983). *Estimation and Linear Prediction for Regression*, *Autoregression and ARMA with Infinite Variance Data*. ProQuest LLC, Ann Arbor, MI. Thesis (Ph.D.)–Colorado State Univ. MR2633007

[37] DA PRATO, G. and GOLDYS, B. (2001). Elliptic operators on $\mathbb{R}^d$ with unbounded coefficients. *J. Differential Equations* **172** 333–358. MR1829633 https://doi.org/10.1006/jdeq.2000.3866

[38] DAVIS, R. A., KNIGHT, K. and LIU, J. (1992). M-estimation for autoregressions with infinite variance. *Stochastic Process. Appl.* **40** 145–180. MR1145464 https://doi.org/10.1016/0304-4149(92)90142-D

[39] ETHIER, S. N. and KURTZ, T. G. (2009). *Markov Processes: Characterization and Convergence* **282**. Wiley. MR0838085 https://doi.org/10.1002/9780470316658

[40] EVANS, L. C. (2013). *An Introduction to Stochastic Differential Equations*. Amer. Math. Soc., Providence, RI. MR3154922 https://doi.org/10.1090/mbk/082

[41] FANG, X., SHAO, Q.-M. and XU, L. (2019). Multivariate approximations in Wasserstein distance by Stein's method and Bismut's formula. *Probab. Theory Related Fields* **174** 945–979. MR3980309 https://doi.org/10.1007/s00440-018-0874-5

[42] FENG, Y., GAO, T., LI, L., LIU, J.-G. and LU, Y. (2020). Uniform-in-time weak error analysis for stochastic gradient descent algorithms via diffusion approximation. *Commun. Math. Sci.* **18** 163–188. MR4084147 https://doi.org/10.4310/CMS.2020.v18.n1.a7

[43] FONTAINE, X., DE BORTOLI, V. and DURMUS, A. (2020). Continuous and discrete-time analysis of stochastic gradient descent for convex and non-convex functions. Available at arXiv:2004.04193.

[44] GARDINER, C. W. (1985). *Handbook of Stochastic Methods*, 2nd ed. *Springer Series in Synergetics* **13**. Springer, Berlin. MR0858704

[45] GORHAM, J. and MACKEY, L. (2015). Measuring sample quality with Stein's method. In *Advances in Neural Information Processing Systems* 226–234.

[46] GURVICH, I. (2014). Diffusion models and steady-state approximations for exponentially ergodic Markovian queues. *Ann. Appl. Probab.* **24** 2527–2559. MR3262510 https://doi.org/10.1214/13-AAP984

[47] HAIRER, M. and MATTINGLY, J. C. (2006). Ergodicity of the 2D Navier–Stokes equations with degenerate stochastic forcing. *Ann. of Math.* (2) **164** 993–1032. MR2259251 https://doi.org/10.4007/annals.2006.164.993

[48] HU, W., LI, C. J., LI, L. and LIU, J.-G. (2019). On the diffusion approximation of nonconvex stochastic gradient descent. *Ann. Math. Sci. Appl.* **4** 3–32. MR3921998 https://doi.org/10.4310/AMSA.2019.v4.n1.a1

[49] JANICKI, A., MICHNA, Z. and WERON, A. (1996). Approximation of stochastic differential equations driven by α-stable Lévy motion. *Appl. Math.* (*Warsaw*) **24** 149–168. MR1424549 https://doi.org/10.4064/am-24-2-149-168

[50] KASPRZAK, M. J. (2017). Diffusion approximations via Stein's method and time changes. Available at arXiv:1701.07633.

[51] KLOEDEN, P. E. and PLATEN, E. (2013). *Numerical Solution of Stochastic Differential Equations*. *Applications of Mathematics* (*New York*) **23**. Springer, Berlin. MR1214374 https://doi.org/10.1007/978-3-662-12616-5

[52] KNOWLES, A. and YIN, J. (2017). Anisotropic local laws for random matrices. *Probab. Theory Related Fields* **169** 257–352. MR3704770 https://doi.org/10.1007/s00440-016-0730-4

[53] KORADA, S. B. and MONTANARI, A. (2011). Applications of the Lindeberg principle in communications and statistical learning. *IEEE Trans. Inf. Theory* **57** 2440–2450. MR2809100 https://doi.org/10.1109/TIT.2011.2112231

[54] KUNITA, H. (1984). Stochastic differential equations and stochastic flows of diffeomorphisms. In *École D'été de Probabilités de Saint-Flour, XII—1982*. *Lecture Notes in Math.* **1097** 143–303. Springer, Berlin. MR0876080 https://doi.org/10.1007/BFb0099433

[55] KUNITA, H. (1997). *Stochastic Flows and Stochastic Differential Equations*. *Cambridge Studies in Advanced Mathematics* **24**. Cambridge Univ. Press, Cambridge. Reprint of the 1990 original. MR1472487

[56] LI, Q., TAI, C. and E, W. (2017). Stochastic modified equations and adaptive stochastic gradient algorithms. In *International Conference on Machine Learning*. *PMLR* 2101–2110.

[57] LI, Q., TAI, C. and E, W. (2019). Stochastic modified equations and dynamics of stochastic gradient algorithms I: Mathematical foundations. *J. Mach. Learn. Res.* **20** Paper No. 40, 47. MR3948080

[58] LINDEBERG, J. W. (1922). Eine neue Herleitung des Exponentialgesetzes in der Wahrscheinlichkeitsrechnung. *Math. Z.* **15** 211–225. MR1544569 https://doi.org/10.1007/BF01494395

[59] LIU, W. (2019). Strong convergence rate of Euler–Maruyama method for stochastic differential equations with Hölder continuous drift coefficient driven by symmetric α-stable process. Available at arXiv:1901.08742.

[60] LOH, W.-L. (1992). Stein's method and multinomial approximation. *Ann. Appl. Probab.* **2** 536–554. MR1177898

[61] MERLEVÈDE, F. and PELIGRAD, M. (2016). On the empirical spectral distribution for matrices with long memory and independent rows. *Stochastic Process. Appl.* **126** 2734–2760. MR3522299 https://doi.org/10.1016/j.spa.2016.02.016

[62] MEYN, S. P. and TWEEDIE, R. L. (1993). Stability of Markovian processes. III. Foster–Lyapunov criteria for continuous-time processes. *Adv. in Appl. Probab.* **25** 518–548. MR1234295 https://doi.org/10.2307/1427522

[63] NORRIS, J. (1986). Simplified Malliavin calculus. In *Séminaire de Probabilités*, *XX*, 1984/85. *Lecture Notes in Math*. **1204** 101–130. Springer, Berlin. MR0942019 https://doi.org/10.1007/BFb0075716

[64] NOURDIN, I. and PECCATI, G. (2009). Stein's method on Wiener chaos. *Probab. Theory Related Fields* **145** 75–118. MR2520122 https://doi.org/10.1007/s00440-008-0162-x

[65] ØKSENDAL, B. (2013). *Stochastic Differential Equations*: *An Introduction with Applications*. *Universitext*. Springer, Berlin. MR2001996 https://doi.org/10.1007/978-3-642-14394-6

[66] POLYAK, B. T. and JUDITSKY, A. B. (1992). Acceleration of stochastic approximation by averaging. *SIAM J. Control Optim*. **30** 838–855. MR1167814 https://doi.org/10.1137/0330046

[67] REINERT, G. and RÖLLIN, A. (2009). Multivariate normal approximation with Stein's method of exchangeable pairs under a general linearity condition. *Ann*. *Probab.* **37** 2150–2173. MR2573554 https://doi.org/10.1214/09-AOP467

[68] REN, Y.-F. (2008). On the Burkholder–Davis–Gundy inequalities for continuous martingales. *Statist*. *Probab. Lett*. **78** 3034–3039. MR2474395 https://doi.org/10.1016/j.spl.2008.05.024

[69] RESNICK, S. I. (1997). Heavy tail modeling and teletraffic data. *Ann*. *Statist*. **25** 1805–1869. With discussion and a rejoinder by the author. MR1474072 https://doi.org/10.1214/aos/1069362376

[70] SATO, K. (1999). *Lévy Processes and Infinitely Divisible Distributions*. *Cambridge Studies in Advanced Mathematics* **68**. Cambridge Univ. Press, Cambridge. Translated from the 1990 Japanese original, Revised by the author. MR1739520

[71] SONG, Y. (2020). Normal approximation by Stein's method under sublinear expectations. *Stochastic Process*. *Appl*. **130** 2838–2850. MR4080731 https://doi.org/10.1016/j.spa.2019.08.005

[72] STEIN, C. (1972). A bound for the error in the normal approximation to the distribution of a sum of dependent random variables. In *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability* (*Univ. California*, *Berkeley*, *Calif.*, 1970/1971), *Vol. II*: *Probability Theory* 583–602. MR0402873

[73] TAO, T. and VU, V. (2011). Random matrices: Universality of local eigenvalue statistics. *Acta Math*. **206** 127–204. MR2784665 https://doi.org/10.1007/s11511-011-0061-3

[74] TARAMI, B. and AVAJI, M. (2018). Convergence of Euler–Maruyama method for stochastic differential equations driven by α-stable Lévy motion. *J. Math*. *Ext*. **12** 31–53.

[75] TEH, Y. W., THIERY, A. H. and VOLLMER, S. J. (2016). Consistency and fluctuations for stochastic gradient Langevin dynamics. *J. Mach*. *Learn*. *Res*. **17** Paper No. 7, 33. MR3482927

[76] VALIANT, G. and VALIANT, P. (2010). A CLT and tight lower bounds for estimating entropy. *Electron*. *Colloq*. *Comput*. *Complex*. **17** 179.

[77] WANG, J. (2013). Exponential ergodicity and strong ergodicity for SDEs driven by symmetric α-stable processes. *Appl*. *Math*. *Lett*. **26** 654–658. MR3028071 https://doi.org/10.1016/j.aml.2013.01.004

[78] WANG, L., AUE, A. and PAUL, D. (2017). Spectral analysis of sample autocovariance matrices of a class of linear time series in moderately high dimensions. *Bernoulli* **23** 2181–2209. MR3648029 https://doi.org/10.3150/16-BEJ807

[79] WANG, Y. and YUAN, C. (2007). Convergence of the Euler–Maruyama method for stochastic differential equations with respect to semimartingales. *Appl*. *Math*. *Sci*. (*Ruse*) **1** 2063–2077. MR2369925

[80] XI, F. B. (2001). Invariant measures for a random evolution equation with small perturbations. *Acta Math*. *Sin*. (*Engl*. *Ser.*) **17** 631–642. MR1891754 https://doi.org/10.1007/s101140100127

[81] XU, L. (2019). Approximation of stable law in Wasserstein-1 distance by Stein's method. *Ann*. *Appl*. *Probab*. **29** 458–504. MR3910009 https://doi.org/10.1214/18-AAP1424

[82] YOSIDA, K. (1980). *Functional Analysis*, 6th ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **123**. Springer, Berlin. MR0617913