ISSN 1050-5164 (print)
ISSN 2168-8737 (online)

THE ANNALS *of* APPLIED PROBABILITY

AN OFFICIAL JOURNAL OF THE
INSTITUTE OF MATHEMATICAL STATISTICS

Articles

continued

Vol. 33, No. 3—June 2023

ISSN 1050-5164 (print)
ISSN 2168-8737 (online)

THE ANNALS *of* APPLIED PROBABILITY

AN OFFICIAL JOURNAL OF THE
INSTITUTE OF MATHEMATICAL STATISTICS

Articles—*Continued from front cover*

Vol. 33, No. 3—June 2023

THE ANNALS OF APPLIED PROBABILITY Vol. 33, No. 3, pp. 1661–2492 June 2023

INSTITUTE OF MATHEMATICAL STATISTICS

(Organized September 12, 1935)

The purpose of the Institute is to foster the development and dissemination of the theory and applications of statistics and probability.

The Annals of Applied Probability **[ISSN 1050-5164 (print); ISSN 2168-8737 (online)],** Volume 33, Number 3, June 2023. Published bimonthly by the Institute of Mathematical Statistics, 9760 Smith Road, Waite Hill, Ohio 44094, USA. Periodicals postage paid at Cleveland, Ohio, and at additional mailing offices.

POSTMASTER: Send address changes to *The Annals of Applied Probability*, Institute of Mathematical Statistics, Dues and Subscriptions Office, PO Box 729, Middletown, Maryland 21769, USA.

The Annals of Applied Probability
2023, Vol. 33, No. 3, 1661–1700
https://doi.org/10.1214/22-AAP1794

A DYNAMIC ANALYTIC METHOD FOR RISK-AWARE CONTROLLED MARTINGALE PROBLEMS

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We present a new, tractable method for solving risk-aware problems over finite and infinite, discounted time-horizons where the dynamics of the controlled process are described using the martingale method. Supposing general Polish state and action spaces, and using the martingale characterization, we state a risk-aware dynamic optimal control problem of minimizing risk of costs described by a generic risk function. From this, we construct an alternative formulation of the optimization problem that takes the form of a nonlinear programming problem, constrained by the dynamic, that is, time-dependent and linear Kolmogorov forward equation describing the time-dependent distribution of the state and running costs. This formulation is similar to the convex analytic method, in that the control problem is recast into a form where the objective is optimized over distributions representing the state space visitation frequencies. However, in our approach, the distributions are dynamic and also encode the cost distribution. As our main results, we prove the equivalence of the original martingale and dynamic analytic problems, in the sense that both have the same optimal values, and that the solution of either problem yields a solution of the other. Moreover, we find an optimal control process can be taken to be Markov in the controlled process state, running costs, and time. We further show that under additional assumptions the optimal value is attained. An example numeric problem is presented and solved.

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MSC2020 subject classifications. Primary 93E20, 60J25; secondary 60J35, 90C30.
Key words and phrases. Optimal control, stochastic processes, martingale problems, forward equation, nonlinear programming.

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The Annals of Applied Probability
2023, Vol. 33, No. 3, 1701–1731
https://doi.org/10.1214/22-AAP1808

UNIVERSALITY OF THE TIME CONSTANT FOR 2*D* CRITICAL FIRST-PASSAGE PERCOLATION

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We consider first-passage percolation (FPP) on the triangular lattice with vertex weights (t_v) whose common distribution function F satisfies $F(0) = 1/2$. This is known as the critical case of FPP because large (critical) zero-weight clusters allow travel between distant points in time which is sublinear in the distance. Denoting by $T(0, \partial B(n))$ the first-passage time from 0 to $\{x : \|x\|_\infty = n\}$, we show existence of a "time constant" and find its exact value to be

$$\lim_{n\to\infty} \frac{T(0, \partial B(n))}{\log n} = \frac{I}{2\sqrt{3}\pi} \quad \text{almost surely,}$$

where $I = \inf\{x > 0 : F(x) > 1/2\}$ and F is any critical distribution for t_v. This result shows that this time constant is universal and depends only on the value of I. Furthermore, we find the exact value of the limiting normalized variance, which is also only a function of I, under the optimal moment condition on F. The proof method also shows an analogous universality on other two-dimensional lattices, assuming the time constant exists.

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MSC2020 subject classifications. Primary 60K35; secondary 82B43.
Key words and phrases. First-passage percolation, universality, time constant.

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The Annals of Applied Probability
2023, Vol. 33, No. 3, 1732–1757
https://doi.org/10.1214/22-AAP1851

THE MEAN-FIELD ZERO-RANGE PROCESS WITH UNBOUNDED MONOTONE RATES: MIXING TIME, CUTOFF, AND POINCARÉ CONSTANT

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We consider the mean-field zero-range process in the regime where the potential function r is increasing to infinity at sublinear speed, and the density of particles is bounded. We determine the mixing time of the system, and establish cutoff. We also prove that the Poincaré constant is bounded away from zero and infinity. This mean-field estimate extends to arbitrary geometries via a comparison argument. Our proof uses the path-coupling method of Bubley and Dyer and stochastic calculus.

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MSC2020 subject classifications. 37A25, 60J27, 60K35, 82C22.
Key words and phrases. Zero-range process, mixing time, Poincaré constant, cutoff phenomenon.

The Annals of Applied Probability
2023, Vol. 33, No. 3, 1758–1826
https://doi.org/10.1214/22-AAP1852

LARGE DEVIATIONS OF KAC'S CONSERVATIVE PARTICLE SYSTEM AND ENERGY NONCONSERVING SOLUTIONS TO THE BOLTZMANN EQUATION: A COUNTEREXAMPLE TO THE PREDICTED RATE FUNCTION

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We consider the dynamic large deviation behaviour of Kac's collisional process for a range of initial conditions including equilibrium. We prove an upper bound with a rate function of the type which has previously been found for kinetic large deviation problems, and a matching lower bound restricted to a class of sufficiently good paths. However, we are able to show by an explicit counterexample that the predicted rate function does not extend to a global lower bound: even though the particle system almost surely conserves energy, large deviation behaviour includes solutions to the Boltzmann equation which do not conserve energy, as found by Lu and Wennberg, and these occur strictly more rarely than predicted by the proposed rate function. At the level of the particle system, this occurs because a macroscopic proportion of energy can concentrate in $\mathfrak{o}(N)$ particles with probability $e^{-\mathcal{O}(N)}$.

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MSC2020 subject classifications. Primary 35Q20, 60F10, 00X00; secondary 82C22, 82C26.
Key words and phrases. Boltzmann equation, large deviations, Kac process.

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The Annals of Applied Probability
2023, Vol. 33, No. 3, 1827–1873
https://doi.org/10.1214/22-AAP1854

SPECTRAL GAPS AND ERROR ESTIMATES FOR INFINITE-DIMENSIONAL METROPOLIS–HASTINGS WITH NON-GAUSSIAN PRIORS

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We study a class of Metropolis–Hastings algorithms for target measures that are absolutely continuous with respect to a large class of non-Gaussian prior measures on Banach spaces. The algorithm is shown to have a spectral gap in a Wasserstein-like semimetric weighted by a Lyapunov function. A number of error bounds are given for computationally tractable approximations of the algorithm including bounds on the closeness of Cesáro averages and other pathwise quantities via perturbation theory. Several applications illustrate the breadth of problems to which the results apply such as various likelihood approximations and perturbations of prior measures.

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MSC2020 subject classifications. 65C05, 60J05, 62G99, 62M40.
Key words and phrases. Markov chain Monte Carlo, spectral gap, non-Gaussian, Bayesian inverse problem, weak Harris' theorem.

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The Annals of Applied Probability
2023, Vol. 33, No. 3, 1874–1903
https://doi.org/10.1214/22-AAP1855

ROBUST ONE-BIT COMPRESSED SENSING WITH PARTIAL CIRCULANT MATRICES

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We present optimal sample complexity estimates for one-bit compressed sensing problems in a realistic scenario: the procedure uses a structured matrix (a randomly subsampled circulant matrix) and is robust to analog pre-quantization noise as well as to adversarial bit corruptions in the quantization process. Our results imply that quantization is not a statistically expensive procedure in the presence of nontrivial analog noise: recovery requires the same sample size one would have needed had the measurement matrix been Gaussian and the noisy analog measurements been given as data.

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MSC2020 subject classifications. 60B20, 94A12.
Key words and phrases. Compressed sensing, quantization, random circulant matrices, empirical processes, generic chaining.

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The Annals of Applied Probability
2023, Vol. 33, No. 3, 1904–1960
https://doi.org/10.1214/22-AAP1856
This research was funded, in whole or in part, by [EPSRC, EP/P006175/1, EPSRC EP/N510129/1].

SIMPLEST RANDOM WALK FOR APPROXIMATING ROBIN BOUNDARY VALUE PROBLEMS AND ERGODIC LIMITS OF REFLECTED DIFFUSIONS

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A simple-to-implement weak-sense numerical method to approximate reflected stochastic differential equations (RSDEs) is proposed and analysed. It is proved that the method has the first order of weak convergence. Together with the Monte Carlo technique, it can be used to numerically solve linear parabolic and elliptic PDEs with Robin boundary condition. One of the key results of this paper is the use of the proposed method for computing ergodic limits, that is, expectations with respect to the invariant law of RSDEs, both inside a domain in $\mathbb{R}^d$ and on its boundary. This allows to efficiently sample from distributions with compact support. Both time-averaging and ensemble-averaging estimators are considered and analysed. A number of extensions are considered including a second-order weak approximation, the case of arbitrary oblique direction of reflection, and a new adaptive weak scheme to solve a Poisson PDE with Neumann boundary condition. The presented theoretical results are supported by several numerical experiments.

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MSC2020 subject classifications. Primary 60H35; secondary 65C30, 60H10, 37H10.

Key words and phrases. Reflected stochastic differential equations, weak approximation, ergodic limits, reflected Brownian dynamics, stochastic gradient system in bounded domains, Neumann boundary value problem, sampling from distributions with compact support, sampling on manifold.

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The Annals of Applied Probability
2023, Vol. 33, No. 3, 1961–2003
https://doi.org/10.1214/22-AAP1857

FLUCTUATIONS IN MEAN-FIELD ISING MODELS

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In this paper, we study the fluctuations of the average magnetization in an Ising model on an approximately d_N regular graph G_N on N vertices. In particular, if G_N satisfies a "spectral gap" condition, we show that whenever $d_N \gg \sqrt{N}$, the fluctuations are universal and the same as that of the Curie–Weiss model in the entire ferromagnetic parameter regime. We give a counterexample to demonstrate that the condition $d_N \gg \sqrt{N}$ is tight, in the sense that the limiting distribution changes if $d_N \sim \sqrt{N}$ except in the high temperature regime. By refining our argument, we extend universality in the high temperature regime up to $d_N \gg N^{1/3}$. Our results include universal fluctuations of the average magnetization in Ising models on regular graphs, Erdős–Rényi graphs (directed and undirected), stochastic block models, and sparse regular graphons. In fact, our results apply to general matrices with nonnegative entries, including Ising models on a Wigner matrix, and the block spin Ising model. As a by-product of our proof technique, we obtain Berry–Esseen bounds for these fluctuations, exponential concentration for the average of spins, tight error bounds for the mean-field approximation of the partition function, and tail bounds for various statistics of interest.

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MSC2020 subject classifications. Primary 82B20; secondary 82B26.
Key words and phrases. Berry–Esseen bound, Ising model, regular graphs, mean-field, partition function.

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The Annals of Applied Probability
2023, Vol. 33, No. 3, 2004–2023
https://doi.org/10.1214/22-AAP1858

A GENERAL CONDITIONAL MCKEAN–VLASOV STOCHASTIC DIFFERENTIAL EQUATION

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In this paper we consider a class of *conditional McKean–Vlasov SDEs* (CMVSDE for short). Such an SDE can be considered as an extended version of McKean–Vlasov SDEs with common noises, as well as the general version of the so-called *conditional mean-field SDEs* (CMFSDE) studied previously by the authors (*Ann. Appl. Probab.* **27** (2017) 3201–3245; *SIAM J. Control Optim.* **56** (2018) 1154–1180), but with some fundamental differences. In particular, due to the lack of compactness of the iterated conditional laws, the existing arguments of Schauder's fixed point theorem do not seem to apply in this situation, and the heavy nonlinearity on the conditional laws caused by change of probability measure adds more technical subtleties. Under some structural assumptions on the coefficients of the observation equation, we prove the well-posedness of the solutions in a weak sense along a more direct approach. Our result is the first that deals with McKean–Vlasov type SDEs involving state-dependent conditional laws.

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MSC2020 subject classifications. 60H07, 60H15, 60H30, 35R60, 34F05.
Key words and phrases. Conditional McKean–Vlasov SDEs, Kantorovich–Rubinstein's duality, weak solution.

The Annals of Applied Probability
2023, Vol. 33, No. 3, 2024–2065
https://doi.org/10.1214/22-AAP1859

THE SCALING LIMIT OF A CRITICAL RANDOM DIRECTED GRAPH

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We consider the random directed graph $\vec{G}(n, p)$ with vertex set $\{1, 2, \dots, n\}$ in which each of the $n(n-1)$ possible directed edges is present independently with probability p. We are interested in the strongly connected components of this directed graph. A phase transition for the emergence of a giant strongly connected component is known to occur at $p = 1/n$, with critical window $p = 1/n + \lambda n^{-4/3}$ for $\lambda \in \mathbb{R}$. We show that, within this critical window, the strongly connected components of $\vec{G}(n, p)$, ranked in decreasing order of size and rescaled by $n^{-1/3}$, converge in distribution to a sequence $(\mathcal{C}_1, \mathcal{C}_2, \dots)$ of finite strongly connected directed multigraphs with edge lengths which are either 3-regular or loops. The convergence occurs in the sense of an ℓ^1 sequence metric for which two directed multigraphs are close if there are compatible isomorphisms between their vertex and edge sets which roughly preserve the edge lengths. Our proofs rely on a depth-first exploration of the graph which enables us to relate the strongly connected components to a particular spanning forest of the undirected Erdős–Rényi random graph $G(n, p)$, whose scaling limit is well understood. We show that the limiting sequence $(\mathcal{C}_1, \mathcal{C}_2, \dots)$ contains only finitely many components which are not loops. If we ignore the edge lengths, any fixed finite sequence of 3-regular strongly connected directed multigraphs occurs with positive probability.

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MSC2020 subject classifications. Primary 60C05, 05C80; secondary 05C20, 60F05.
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The Annals of Applied Probability
2023, Vol. 33, No. 3, 2066–2102
https://doi.org/10.1214/22-AAP1860

SCALING LIMIT OF MODERATELY INTERACTING PARTICLE SYSTEMS WITH SINGULAR INTERACTION AND ENVIRONMENTAL NOISE

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We consider moderately interacting particle systems with singular interaction kernel and environmental noise. It is shown that the mollified empirical measures converge in strong norms to the unique (local) solutions of nonlinear Fokker–Planck equations. The approach works for the Biot–Savart and repulsive Poisson kernels.

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MSC2020 subject classifications. 60K35, 60H30, 35Q84.
Key words and phrases. Interacting particle system, moderate interaction, environmental noise, entropy, semigroup method.

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The Annals of Applied Probability
2023, Vol. 33, No. 3, 2103–2135
https://doi.org/10.1214/22-AAP1861

A VARIATIONAL FORMULA FOR LARGE DEVIATIONS IN FIRST-PASSAGE PERCOLATION UNDER TAIL ESTIMATES

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Consider the first passage percolation on the d-dimensional lattice $\mathbb{Z}^d$ with identical and independent weight distributions and the first passage time T. In this paper, we study the upper tail large deviations $\mathbb{P}(\mathrm{T}(0, nx) > n(\mu + \xi))$, for $\xi > 0$ and $x \neq 0$ with a time constant μ, for weights that satisfy a tail assumption $\mathbb{P}(\tau_e > t) \asymp \beta \exp(-\alpha t^r)$. When $r \leq 1$ (this includes the well-known Eden growth model), we show that the upper tail large deviation decays as $\exp(-(2d\alpha\xi^r + o(1))n)$. When $1 < r \leq d$, we find that the rate function can be naturally described by a variational formula, called the discrete p-Capacity, and we study its asymptotics. The case $r = d$ is critical and logarithmic corrections appear. For $r \in (1, d)$, we show that the large deviation event $\{\mathrm{T}(0, nx) > n(\mu + \xi)\}$ is described by a localization of high weights around the endpoints. The picture changes for $r \geq d$ where the configuration is not anymore localized.

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MSC2020 subject classifications. Primary 60K37; secondary 60K35, 82A51, 82D30.
Key words and phrases. Eden growth model, first-passage percolation, large deviations.

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The Annals of Applied Probability
2023, Vol. 33, No. 3, 2136–2175
https://doi.org/10.1214/22-AAP1862
This research was funded, in whole or in part, by [Swiss National Science Foundation, P2EZP2-188068]. A CC BY 4.0 license is applied to this article arising from this submission, in accordance with the grant's open access conditions.

ADAPTED TOPOLOGIES AND HIGHER RANK SIGNATURES

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Two adapted stochastic processes can have similar laws but give different results in applications such as optimal stopping, queuing theory, or stochastic programming. The reason is that the topology of weak convergence does not account for the growth of information over time that is captured in the filtration of an adapted stochastic process. To address such discontinuities, Aldous introduced the extended weak topology, and subsequently, Hoover and Keisler showed that both, weak topology and extended weak topology, are just the first two topologies in a sequence of topologies that get increasingly finer. We introduce higher rank expected signatures to embed adapted processes into graded linear spaces and show that these embeddings induce the adapted topologies of Hoover–Keisler.

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MSC2020 subject classifications. Primary 60G07; secondary 60L10.
Key words and phrases. Adapted topologies, signatures, extended weak convergence.

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The Annals of Applied Probability
2023, Vol. 33, No. 3, 2176–2209
https://doi.org/10.1214/22-AAP1863

HELLINGER AND TOTAL VARIATION DISTANCE IN APPROXIMATING LÉVY DRIVEN SDES

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In this paper, we get some convergence rates in total variation distance in approximating discretized paths of Lévy driven stochastic differential equations, assuming that the driving process is locally stable. The particular case of the Euler approximation is studied. Our results are based on sharp local estimates in Hellinger distance obtained using Malliavin calculus for jump processes.

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MSC2020 subject classifications. 60H10, 60G51, 60B10, 60H07.
Key words and phrases. Lévy process, stable process, stochastic differential equation, total variation, Hellinger distance.

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The Annals of Applied Probability
2023, Vol. 33, No. 3, 2210–2246
https://doi.org/10.1214/22-AAP1864

QUENCHED AND AVERAGED LARGE DEVIATIONS FOR RANDOM WALKS IN RANDOM ENVIRONMENTS: THE IMPACT OF DISORDER

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In 2003, Varadhan (*Comm. Pure Appl. Math.* **56** (2003) 1222–1245) developed a robust method for proving quenched and averaged large deviations for random walks in a uniformly elliptic and i.i.d. environment (RWRE) on $\mathbb{Z}^d$. One fundamental question which remained open was to determine when the quenched and averaged large deviation rate functions agree, and when they do not. In this article we show that for RWRE in uniformly elliptic and i.i.d. environment in $d \geq 4$, the two rate functions agree on any compact set contained in the interior of their domain which does not contain the origin, provided that the disorder of the environment is sufficiently low. Our result provides a new formulation which encompasses a set of sufficient conditions under which these rate functions agree without assuming that the RWRE is ballistic (see (*Probab. Theory Related Fields* **149** (2011) 463–491)), satisfies a CLT or even a law of large numbers (*Electron. Commun. Probab.* **7** (2002)191–197; *Ann. Probab.* **36** (2008) 728–738). Also, the equality of rate functions is not restricted to neighborhoods around given points, as long as the disorder of the environment is kept low. One of the novelties of our approach is the introduction of an auxiliary random walk in a deterministic environment which is itself ballistic (regardless of the actual RWRE behavior) and whose large deviation properties approximate those of the original RWRE in a robust manner, even if the original RWRE is not ballistic itself.

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MSC2020 subject classifications. 60K37, 60F10, 82C41.

Key words and phrases. Random walks in random environment, large deviations, disorder, quenched and averaged rate functions, random walks in random scenery, disordered media.

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The Annals of Applied Probability
2023, Vol. 33, No. 3, 2247–2290
https://doi.org/10.1214/22-AAP1865

THE BI-DIMENSIONAL DIRECTED IDLA FOREST

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We investigate three types of internal diffusion limited aggregation (IDLA) models. These models are based on simple random walks on $\mathbb{Z}^2$ with infinitely many sources that are the points of the vertical axis $I(\infty) = \{0\} \times \mathbb{Z}$. Various properties are provided, such as stationarity, mixing, stabilization and shape theorems. Our results allow us to define a new directed (w.r.t. the horizontal direction) random forest spanning $\mathbb{Z}^2$, based on an IDLA protocol, which is invariant in distribution w.r.t. vertical translations.

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MSC2020 subject classifications. Primary 60K35, 05C80, 82C24; secondary 82B41, 60G50.
Key words and phrases. Internal diffusion limited aggregation, cluster growth, random trees and forests, shape theorems, random walks.

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The Annals of Applied Probability
2023, Vol. 33, No. 3, 2291–2323
https://doi.org/10.1214/22-AAP1867

QUANTIFYING A CONVERGENCE THEOREM OF GYÖNGY AND KRYLOV

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We derive sharp strong convergence rates for the Euler–Maruyama scheme approximating multidimensional SDEs with multiplicative noise without imposing *any* regularity condition on the drift coefficient. In case the noise is additive, we show that Sobolev regularity can be leveraged to obtain improved rate: drifts with regularity of order $\alpha \in (0, 1)$ lead to rate $(1+\alpha)/2$.

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MSC2020 subject classifications. 60H10, 60H35, 60H50.
Key words and phrases. Euler–Maruyama scheme, regularisation by noise, stochastic differential equations, rate of convergence, strong approximation.

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The Annals of Applied Probability
2023, Vol. 33, No. 3, 2324–2341
https://doi.org/10.1214/22-AAP1868

MINIMAX FORMULA FOR THE REPLICA SYMMETRIC FREE ENERGY OF DEEP RESTRICTED BOLTZMANN MACHINES

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We study the free energy of a most used deep architecture for restricted Boltzmann machines, where the layers are disposed in series. Assuming independent Gaussian distributed random weights, we show that the error term in the so-called replica symmetric sum rule can be optimised as a saddle point. This leads us to conjecture that in the replica symmetric approximation the free energy is given by a min max formula, which parallels the one achieved for two-layer case.

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MSC2020 subject classifications. 82B44, 60G15.
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The Annals of Applied Probability
2023, Vol. 33, No. 3, 2342–2373
https://doi.org/10.1214/22-AAP1869

(UN-)BOUNDED TRANSITION FRONTS FOR THE PARABOLIC ANDERSON MODEL AND THE RANDOMIZED F-KPP EQUATION

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We investigate the uniform boundedness of the fronts of the solutions to the randomized Fisher-KPP equation and to its linearization, the parabolic Anderson model. It has been known that for the standard (i.e., deterministic) Fisher-KPP equation, as well as for the special case of a randomized Fisher-KPP equation with so-called ignition type nonlinearity, the transition front is uniformly bounded (in time). Here, we show that this property of having a uniformly bounded transition front fails to hold for the general randomized Fisher-KPP equation. In contrast, for the parabolic Anderson model we do establish this property under some assumptions.

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MSC2020 subject classifications. Primary 35A18, 35B40, 60K37; secondary 60J80, 60H15.
Key words and phrases. Parabolic Anderson model, F-KPP equation, tightness, transition front, traveling waves.

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The Annals of Applied Probability
2023, Vol. 33, No. 3, 2374–2425
https://doi.org/10.1214/22-AAP1870

NEARLY OPTIMAL CENTRAL LIMIT THEOREM AND BOOTSTRAP APPROXIMATIONS IN HIGH DIMENSIONS

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In this paper, we derive new, nearly optimal bounds for the Gaussian approximation to scaled averages of n independent high-dimensional centered random vectors $X_1, \dots, X_n$ over the class of rectangles in the case when the covariance matrix of the scaled average is nondegenerate. In the case of bounded X_i's, the implied bound for the Kolmogorov distance between the distribution of the scaled average and the Gaussian vector takes the form

$$C(B_n^2 \log^3 d/n)^{1/2} \log n,$$

where d is the dimension of the vectors and B_n is a uniform envelope constant on components of X_i's. This bound is sharp in terms of d and B_n, and is nearly (up to $\log n$) sharp in terms of the sample size n. In addition, we show that similar bounds hold for the multiplier and empirical bootstrap approximations. Moreover, we establish bounds that allow for unbounded X_i's, formulated solely in terms of moments of X_i's. Finally, we demonstrate that the bounds can be further improved in some special smooth and moment-constrained cases.

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MSC2020 subject classifications. 60F05, 62E17.
Key words and phrases. Berry–Esseen bound, bootstrap limit theorems, central limit theorem, high dimensions, smoothing inequalities.

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The Annals of Applied Probability
2023, Vol. 33, No. 3, 2426–2492
https://doi.org/10.1214/22-AAP1871

THE SPATIAL Λ-FLEMING–VIOT PROCESS IN A RANDOM ENVIRONMENT

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We study the large scale behaviour of a population consisting of two types which evolve in dimension $d = 1, 2$ according to a spatial Lambda-Fleming–Viot process subject to random time-independent selection. If one of the two types is rare compared to the other, we prove that its evolution can be approximated by a super-Brownian motion in a random (and singular) environment. Without the sparsity assumption, a diffusion approximation leads to a Fisher–KPP equation in a random potential. The proofs build on two-scale Schauder estimates and semidiscrete approximations of the Anderson Hamiltonian.

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MSC2020 subject classifications. Primary 35R60, 60F05, 60J68; secondary 60G51, 60J70, 92D15.
Key words and phrases. Spatial Lambda-Fleming–Viot model, super-processes, super-Brownian motion, Anderson Hamiltonian, scaling limits, Schauder estimates.

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