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MEAN-FIELD REFLECTED BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS

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In this paper, we study a class of reflected backward stochastic differential equations (BSDEs) of mean-field type, where the mean-field interaction in terms of the distribution of the Y-component of the solution enters in both the driver and the lower obstacle. We consider in details the case where the lower obstacle is a deterministic function of $(Y, \mathbb{E}[Y])$ and discuss the more general dependence on the distribution of Y. Under mild Lipschitz and integrability conditions on the coefficients, we obtain the well-posedness of such a class of equations. Under further monotonicity conditions, we show convergence of the standard penalization scheme to the solution of the equation, which hence satisfies a minimality property. This class of equations is motivated by applications in pricing life insurance contracts with surrender options.

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TESTING CORRELATION OF UNLABELED RANDOM GRAPHS

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We study the problem of detecting the edge correlation between two random graphs with n unlabeled nodes. This is formalized as a hypothesis testing problem, where under the null hypothesis, the two graphs are independently generated; under the alternative, the two graphs are edge-correlated under some latent node correspondence, but have the same marginal distributions as the null. For both Gaussian-weighted complete graphs and dense Erdős–Rényi graphs (with edge probability $n^{-o(1)}$), we determine the sharp threshold at which the optimal testing error probability exhibits a phase transition from zero to one as $n \to \infty$. For sparse Erdős–Rényi graphs with edge probability $n^{-\Omega(1)}$, we determine the threshold within a constant factor.

The proof of the impossibility results is an application of the conditional second-moment method, where we bound the truncated second moment of the likelihood ratio by carefully conditioning on the typical behavior of the intersection graph (consisting of edges in both observed graphs) and taking into account the cycle structure of the induced random permutation on the edges. Notably, in the sparse regime, this is accomplished by leveraging the pseudoforest structure of subcritical Erdős–Rényi graphs and a careful enumeration of subpseudoforests that can be assembled from short orbits of the edge permutation.

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LARGE-SCALE REGULARITY IN STOCHASTIC HOMOGENIZATION WITH DIVERGENCE-FREE DRIFT

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We provide a proof of stochastic homogenization for random environments with a mean zero, divergence-free drift. We prove that the environment homogenizes weakly in H^1 if the drift admits a stationary L^2-integrable stream matrix, and we prove that the two-scale expansion converges strongly in H^1 if the drift admits a stationary $L^{d\vee(2+\delta)}$-integrable stream matrix. Additionally, under this stronger integrability assumption, we show that the environment almost surely satisfies a large-scale Hölder regularity estimate and first-order Liouville principle.

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AN SPDE APPROACH TO PERTURBATION THEORY OF Φ_2^4: ASYMPTOTICITY AND SHORT DISTANCE BEHAVIOR

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In this paper we study the perturbation theory of Φ_2^4 model on the whole plane via stochastic quantization. We use integration by parts formula (i.e., Dyson–Schwinger equations) to generate the perturbative expansion for the k-point correlation functions, and prove bounds on the remainder of the truncated expansion using PDE estimates; this in particular proves that the expansion is asymptotic. Furthermore, we derive short distance behaviors of the 2-point function and the connected 4-point function, also via suitable Dyson–Schwinger equations combined with PDE arguments.

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THE TAP FREE ENERGY FOR HIGH-DIMENSIONAL LINEAR REGRESSION

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We derive a variational representation for the log-normalizing constant of the posterior distribution in Bayesian linear regression with a uniform spherical prior and an i.i.d. Gaussian design. We work under the "proportional" asymptotic regime, where the number of observations and the number of features grow at a proportional rate. Our representation holds when the variance of the additive noise is sufficiently large, which corresponds to a high-temperature condition in statistical physics. This rigorously establishes the Thouless–Anderson–Palmer (TAP) approximation arising from spin glass theory, and proves a conjecture of (In *2014 IEEE International Symposium on Information Theory* (2014) 1499–1503 IEEE) in the special case of the spherical prior (at sufficiently high temperature).

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CLOSED-LOOP CONVERGENCE FOR MEAN FIELD GAMES WITH COMMON NOISE

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This paper studies the convergence problem for mean field games with common noise. We define a suitable notion of weak mean field equilibria, which we prove captures all subsequential limit points, as $n \to \infty$, of closed-loop approximate equilibria from the corresponding n-player games. This extends to the common noise setting a recent result of the first author, while also simplifying a key step in the proof and allowing unbounded coefficients and non-i.i.d. initial conditions. Conversely, we show that every weak mean field equilibrium arises as the limit of some sequence of approximate equilibria for the n-player games, as long as the latter are formulated over a broader class of closed-loop strategies which may depend on an additional common signal.

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WELL-POSEDNESS AND WAVE-BREAKING FOR THE STOCHASTIC ROTATION-TWO-COMPONENT CAMASSA–HOLM SYSTEM

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We study the global well-posedness and wave-breaking phenomenon for the stochastic rotation-two-component Camassa–Holm (R2CH) system. First, we find a Hamiltonian structure of the R2CH system and use the stochastic Hamiltonian to derive the stochastic R2CH system. Then, we establish the local well-posedness of the stochastic R2CH system using a dispersion-dissipation approximation system and the regularization method. We also show a precise blow-up criterion for the stochastic R2CH system. Moreover, we prove that the global existence of the stochastic R2CH system occurs with high probability. At the end, we consider the transport noise case and establish the local well-posedness and another blow-up criterion.

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A UNIFIED APPROACH TO LINEAR-QUADRATIC-GAUSSIAN MEAN-FIELD TEAM: HOMOGENEITY, HETEROGENEITY AND QUASI-EXCHANGEABILITY

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This paper aims to systematically solve stochastic team optimization of a large-scale system, in a linear-quadratic-Gaussian framework. Concretely, the underlying large-scale system involves considerable weakly coupled cooperative agents for which the individual admissible controls: (i) enter the diffusion terms, (ii) are constrained in some closed-convex subsets and (iii) subject to a general *partial decentralized information* structure. A more important but serious feature: (iv) all agents are heterogenous with *continuum* instead of *finite* diversity. Combination of (i)–(iv) yields a quite general modeling of stochastic team-optimization, but on the other hand, also fails current existing techniques of team analysis. In particular, classical team consistency with continuum heterogeneity collapses because of (i). As the resolution, a novel *unified approach* is proposed under which the intractable *continuum heterogeneity* can be converted to a more tractable *homogeneity*. As a trade-off, the underlying randomness is augmented, and all agents become (quasi) weakly exchangeable. Such an approach essentially involves a subtle balance between homogeneity v.s. heterogeneity, and left (prior-sampling)- v.s. right (posterior-sampling) information filtration. Subsequently, the consistency condition (CC) system takes a new type of forward-backward stochastic system with *double-projections* (due to (ii), (iii)), along with *spatial mean* on continuum heterogenous index (due to (iv)). Such a system is new in team literature and its well-posedness is also challenging. We address this issue under mild conditions. Related asymptotic optimality is also established.

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MEAN FIELD GAMES OF CONTROLS: ON THE CONVERGENCE OF NASH EQUILIBRIA

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In this paper, we provide convergence and existence results for mean field games of controls. Mean field games of controls are a class of mean field games where the mean field interactions are achieved through the joint (conditional) distribution of the controlled state and the control process. The framework we are considering allows to control the diffusion coefficient σ, and the controls/strategies are supposed to be of *open loop* type. Using (controlled) Fokker–Planck equations, we introduce a notion of measure-valued solution of mean field game of controls and prove a relation between these solutions on the one hand, and the approximate Nash equilibria on the other hand. First of all, in the N-player game associated to the mean field game of controls, given a sequence of approximate Nash equilibria, it is shown that, this sequence admits limits as N tends to infinity, and each limit is a measure-valued solution of the corresponding mean field game of controls. Conversely, any measure-valued solution can be obtained as the limit of a sequence of approximate Nash equilibria of the N-player game. In other words, the measure-valued solutions are the accumulation points of the approximate Nash equilibria. Then, by considering an approximate strong solution of mean field game of controls which is the classical strong solution where the optimality is obtained by admitting a small error ε, we prove that the measure-valued solutions are the accumulation points of this type of solutions when ε goes to zero. Finally, the existence of a measure-valued solution of mean field game of controls is proved in the case without common noise.

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OPTIMAL CONTROL OF PATH-DEPENDENT MCKEAN–VLASOV SDES IN INFINITE-DIMENSION

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We study the optimal control of path-dependent McKean–Vlasov equations valued in Hilbert spaces motivated by non-Markovian mean-field models driven by stochastic PDEs. We first establish the well-posedness of the state equation, and then we prove the dynamic programming principle (DPP) in such a general framework. The crucial law invariance property of the value function V is rigorously obtained, which means that V can be viewed as a function on the Wasserstein space of probability measures on the set of continuous functions valued in Hilbert space. We then define a notion of pathwise measure derivative, which extends the Wasserstein derivative due to Lions (Lions (Audio Conference, 2006–2012)), and prove a related functional Itô formula in the spirit of Dupire ((2009), Functional Itô Calculus, Bloomberg Portfolio Research Paper No. 2009-04-FRONTIERS) and Wu and Zhang (*Ann. Appl. Probab.* **30** (2020) 936–986). The Master Bellman equation is derived from the DPP by means of a suitable notion of viscosity solution. We provide different formulations and simplifications of such a Bellman equation notably in the special case when there is no dependence on the law of the control.

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FLUCTUATION BOUNDS FOR CONTINUOUS TIME BRANCHING PROCESSES AND EVOLUTION OF GROWING TREES WITH A CHANGE POINT

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We consider dynamic random trees constructed using an attachment function $f : \mathbb{N} \to \mathbb{R}_+$ where, at each step of the evolution, a new vertex attaches to an existing vertex v in the current tree with probability proportional to $f(\text{degree}(v))$. We explore the effect of a change point in the system; the dynamics are initially driven by a function f until the tree reaches size $\tau(n) \in (0, n)$, at which point the attachment function switches to another function, g, until the tree reaches size n. Two change point time scales are considered, namely the *standard model* where $\tau(n) = \gamma n$, and the *quick big bang model* where $\tau(n) = n^\gamma$, for some $0 < \gamma < 1$. In the former case, we obtain deterministic approximations for the evolution of the empirical degree distribution (EDF) in sup-norm and use these to devise a provably consistent nonparametric estimator for the change point γ. In the latter case, we show that the effect of pre-change point dynamics asymptotically vanishes in the EDF, although this effect persists in functionals such as the maximal degree. Our proofs rely on embedding the discrete time tree dynamics in an associated (time) inhomogeneous continuous time branching process (CTBP). In the course of proving the above results, we develop novel mathematical techniques to analyze both homogeneous and inhomogeneous CTBPs and obtain rates of convergence for functionals of such processes, which are of independent interest.

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LOCAL LAWS FOR MULTIPLICATION OF RANDOM MATRICES

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Consider the random matrix model $A^{1/2}UBU^*A^{1/2}$, where A and B are two $N \times N$ deterministic matrices and U is either an $N \times N$ Haar unitary or orthogonal random matrix. It is well known that on the macroscopic scale (*Invent. Math.* **104** (1991) 201–220), the limiting empirical spectral distribution (ESD) of the above model is given by the free multiplicative convolution of the limiting ESDs of A and B, denoted as $\mu_\alpha \boxtimes \mu_\beta$, where μ_α and μ_β are the limiting ESDs of A and B, respectively. In this paper, we study the asymptotic microscopic behavior of the edge eigenvalues and eigenvectors statistics. We prove that both the density of $\mu_A \boxtimes \mu_B$, where μ_A and μ_B are the ESDs of A and B, respectively and the associated subordination functions have a regular behavior near the edges. Moreover, we establish the local laws near the edges on the optimal scale. In particular, we prove that the entries of the resolvent are close to some functionals depending only on the eigenvalues of A, B and the subordination functions with optimal convergence rates. Our proofs and calculations are based on the techniques developed for the additive model $A + UBU^*$ in (*J. Funct. Anal.* **271** (2016) 672–719; *Comm. Math. Phys.* **349** (2017) 947–990; *Adv. Math.* **319** (2017) 251–291; *J. Funct. Anal.* **279** (2020) 108639), and our results can be regarded as the counterparts of (*J. Funct. Anal.* **279** (2020) 108639) for the multiplicative model.

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EXISTENCE OF GRADIENT GIBBS MEASURES ON REGULAR TREES WHICH ARE NOT TRANSLATION INVARIANT

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We provide an existence theory for gradient Gibbs measures for $\mathbb{Z}$-valued spin models on regular trees which are not invariant under translations of the tree, assuming only summability of the transfer operator. The gradient states we obtain are delocalized. The construction we provide for them starts from a two-layer hidden Markov model representation in a setup which is not invariant under tree-automorphisms, involving internal q-spin models. The proofs of existence and lack of translation invariance of infinite-volume gradient states are based on properties of the local pseudo-unstable manifold of the corresponding discrete dynamical systems of these internal models, around the free state, at large q.

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NEURAL NETWORK APPROXIMATION AND ESTIMATION OF CLASSIFIERS WITH CLASSIFICATION BOUNDARY IN A BARRON CLASS

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We prove bounds for the approximation and estimation of certain binary classification functions using ReLU neural networks. Our estimation bounds provide a priori performance guarantees for empirical risk minimization using networks of a suitable size, depending on the number of training samples available. The obtained approximation and estimation *rates* are independent of the dimension of the input, showing that the curse of dimensionality can be overcome in this setting; in fact, the input dimension only enters in the form of a polynomial factor. Regarding the regularity of the target classification function, we assume the interfaces between the different classes to be locally of Barron-type. We complement our results by studying the relations between various Barron-type spaces that have been proposed in the literature. These spaces differ substantially more from each other than the current literature suggests.

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CYCLIC CELLULAR AUTOMATA AND GREENBERG–HASTINGS MODELS ON REGULAR TREES

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We study the cyclic cellular automaton (CCA) and the Greenberg–Hastings model (GHM) with $\kappa \geq 3$ colors and contact threshold $\theta \geq 2$ on the infinite $(d+1)$-regular tree, T_d. When the initial state has the uniform product distribution, we show that these dynamical systems exhibit at least two distinct phases. For sufficiently large d, we show that if $\kappa(\theta - 1) \leq d - O(\sqrt{d\kappa \ln(d)})$, then every vertex almost surely changes its color infinitely often, while if $\kappa\theta \geq d + O(\kappa\sqrt{d \ln(d)})$, then every vertex almost surely changes its color only finitely many times. Roughly, this implies that as $d \to \infty$, there is a phase transition where $\kappa\theta/d = 1$. For the GHM dynamics, in the scenario where every vertex changes color finitely many times, we moreover give an exponential tail bound for the distribution of the time of the last color change at a given vertex.

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RANDOMLY COUPLED DIFFERENTIAL EQUATIONS WITH ELLIPTIC CORRELATIONS

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We consider the long time asymptotic behavior of a large system of N linear differential equations with random coefficients. We allow for general elliptic correlation structures among the coefficients, thus we substantially generalize our previous work (*SIAM J. Math. Anal.* **50** (2018) 3271–3290) that was restricted to the independent case. In particular, we analyze a recent model in the theory of neural networks (*Phys. Rev. E* **97** (2018) 062314) that specifically focused on the effect of the distributional asymmetry in the random connectivity matrix X. We rigorously prove and slightly correct the explicit formula from (*J. Math. Phys.* **41** (2000) 3233–3256) on the time decay as a function of the asymmetry parameter. Our main tool is an asymptotically precise formula for the normalized trace of $f(X)g(X^*)$, in the large N limit, where f and g are analytic functions.

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PHASE TRANSITION FOR PERCOLATION ON A RANDOMLY STRETCHED SQUARE LATTICE

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Let $\{\xi_i\}_{i\geq 1}$ be a sequence of i.i.d. positive random variables. Starting from the usual square lattice replace each horizontal edge that links a site in the ith vertical column to another in the $(i+1)$th vertical column by an edge having length ξ_i. Then declare independently each edge e in the resulting lattice open with probability $p_e = p^{|e|}$ where $p \in [0, 1]$ and $|e|$ is the length of e. We relate the occurrence of a nontrivial phase transition for this model to moment properties of ξ_1. More precisely, we prove that the model undergoes a nontrivial phase transition when $\mathbb{E}(\xi_1^\eta) < \infty$, for some $\eta > 1$. On the other hand, when $\mathbb{E}(\xi_1) = \infty$, percolation never occurs for $p < 1$. We also show that the probability of the one-arm event decays no faster than a polynomial in an open interval of parameters p close to the critical point.

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CROSSING PROBABILITIES OF MULTIPLE ISING INTERFACES

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We prove that in the scaling limit, the crossing probabilities of multiple interfaces in the critical planar Ising model with alternating boundary conditions are conformally invariant expressions given by the pure partition functions of multiple SLE_κ with $\kappa = 3$. In particular, this identifies the scaling limits with ratios of specific correlation functions of conformal field theory.

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DENSE MULTIGRAPHON-VALUED STOCHASTIC PROCESSES AND EDGE-CHANGING DYNAMICS IN THE CONFIGURATION MODEL

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Time-evolving random graph models have appeared and have been studied in various fields of research over the past decades. However, the rigorous mathematical treatment of large graphs and their limits at the process-level is still in its infancy. In this article, we adapt the approach of Athreya, den Hollander and Röllin (*Ann. Appl. Probab.* **31** (2021) 1724–1745) to the setting of multigraphs and multigraphons, introduced by Kolossváry and Ráth (*Acta Math. Hungar.* **130** (2011) 1–34). We then generalise the work of Ráth (*Random Structures Algorithms* **41** (2012) 365–390) and Ráth and Szakács (*Acta Math. Hungar.* **136** (2012) 196–221), who analysed edge-flipping dynamics on the configuration model—in contrast to their work, we establish weak convergence at the process-level, and by allowing removal and addition of edges, these limits are nondeterministic.

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ON THE GENERATING FUNCTION OF THE PEARCEY PROCESS

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The Pearcey process is a universal point process in random matrix theory. In this paper, we study the generating function of the Pearcey process on any number m of intervals. We derive an integral representation for it in terms of a Hamiltonian that is related to a system of $6m+2$ coupled nonlinear equations. We also obtain asymptotics for the generating function as the size of the intervals get large, up to and including the constant term. This work generalizes some results of Dai, Xu, and Zhang, which correspond to $m=1$.

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A SAMPLE-PATH LARGE DEVIATION PRINCIPLE FOR DYNAMIC ERDŐS–RÉNYI RANDOM GRAPHS

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We consider a dynamic Erdős–Rényi random graph on n vertices in which each edge switches on at rate λ and switches off at rate μ, independently of other edges. The focus is on the analysis of the evolution of the associated empirical graphon in the limit as $n \to \infty$. Our main result is a large deviation principle (LDP) for the sample path of the empirical graphon observed until a fixed time horizon. The rate is $\binom{n}{2}$, the rate function is a specific action integral on the space of graphon trajectories. We apply the LDP to identify (i) the most likely path that starting from a constant graphon creates a graphon with an atypically large density of d-regular subgraphs, and (ii) the mostly likely path between two given graphons. It turns out that bifurcations may occur in the solutions of associated variational problems.

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