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**Articles**



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# GAUSSIAN CONCENTRATION BOUNDS FOR STOCHASTIC CHAINS OF UNBOUNDED MEMORY

BY JEAN-RENÉ CHAZOTTES[1,a], SANDRO GALLO[2,b] AND DANIEL Y. TAKAHASHI[3,c]

[1]*Centre de Physique Théorique, CNRS, Institut Polytechnique de Paris,* [a]*jeanrene@cpht.polytechnique.fr*
[2]*Departamento de Estatística, Universidade Federal de São Carlos,* [b]*sandro.gallo@ufscar.br*
[3]*Instituto do Cérebro, Universidade Federal do Rio Grande do Norte,* [c]*takahashiyd@gmail.com*

Stochastic chains of unbounded memory (SCUMs) are generalization of Markov chains, also known in the literature as "chains with complete connections" or "$g$-measures". We obtain Gaussian concentration bounds (GCB) in this large class of models, for general alphabets, under two different conditions on the kernel: (1) when the sum of its oscillations is less than one, or (2) when the sum of its variations is finite, that is, belongs to $\ell^1(\mathbb{N})$. We also obtain explicit constants as functions of the parameters of the model. Our conditions are sharp in the sense that we exhibit examples of SCUMs that do not have GCB and for which the sum of oscillations is $1+\epsilon$, or the variation belongs to $\ell^{1+\epsilon}(\mathbb{N})$ for any $\epsilon > 0$. These examples are based on the existence of phase transitions.

We illustrate our results with four applications. First, we derive a Dvoretzky–Kiefer–Wolfowitz-type inequality which gives a uniform control on the fluctuations of the empirical measure. Second, in the finite-alphabet case, we obtain an upper bound on the $\bar{d}$-distance between two stationary SCUMs and, as a by-product, we obtain new explicit bounds on the speed of Markovian approximation in $\bar{d}$. Third, we derive new bounds on the fluctuations of the "plug-in" estimator for entropy. Fourth, we obtain new rate of convergence for the maximum likelihood estimator of conditional probability.

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# CONSTRUCTION OF BOLTZMANN AND MCKEAN–VLASOV TYPE FLOWS (THE SEWING LEMMA APPROACH)

BY AURÉLIEN ALFONSI[1,a] AND VLAD BALLY[2,b]

[1]*CERMICS, Ecole des Ponts, MathRisk, Inria,* [a]*aurelien.alfonsi@enpc.fr*
[2]*Université Gustave Eiffel, LAMA (UMR CNRS, UPEMLV, UPEC), MathRisk, Inria,* [b]*vlad.bally@univ-eiffel.fr*

We are concerned with a mixture of Boltzmann and McKean–Vlasov-type equations, this means (in probabilistic terms) equations with coefficients depending on the law of the solution itself, and driven by a Poisson point measure with the intensity depending also on the law of the solution. Both the analytical Boltzmann equation and the probabilistic interpretation initiated by Tanaka (*Z. Wahrsch. Verw. Gebiete* **46** (1978/79) 67–105; *J. Fac. Sci., Univ. Tokyo, Sect. IA, Math.* **34** (1987) 351–369) have intensively been discussed in the literature for specific models related to the behavior of gas molecules. In this paper, we consider general abstract coefficients that may include mean field effects and then we discuss the link with specific models as well. In contrast with the usual approach in which integral equations are used in order to state the problem, we employ here a new formulation of the problem in terms of flows of self-maps on the space of probability measure endowed with the Wasserstein distance. This point of view already appeared in the framework of rough differential equations. Our results concern existence and uniqueness of the solution, in the formulation of flows, but we also prove that the "flow solution" is a solution of the classical integral weak equation and admits a probabilistic interpretation. Moreover, we obtain stability results and regularity with respect to the time for such solutions. Finally we prove the convergence of empirical measures based on particle systems to the solution of our problem, and we obtain the rate of convergence. We discuss as examples the homogeneous and the inhomogeneous Boltzmann (Enskog) equation with hard potentials.

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# THE TRUNKS OF CLE(4) EXPLORATIONS

BY MATTHIS LEHMKUEHLER[a]

*D-MATH, ETH Zürich,* [a]*matthis.lehmkuehler@math.ethz.ch*

The family of $\mathrm{SLE}_4^{\langle\mu\rangle}(-2)$ exploration processes with parameter $\mu \in \mathbb{R}$ forms a natural class of conformally invariant ways for discovering the loops of a conformal loop ensemble $\mathrm{CLE}_4$. Such an exploration consists of one simple continuous path called the trunk of the exploration that discovers $\mathrm{CLE}_4$ loops along the way. The parameter $\mu$ appears in the Loewner chain description of the path that traces the trunk and all $\mathrm{CLE}_4$ loops encountered by the trunk in chronological order. These explorations can also be interpreted in terms of level lines of a Gaussian free field.

It has been shown by Miller, Sheffield and Werner that the trunk of such an exploration is an $\mathrm{SLE}_4(\rho, -2-\rho)$ process for some (unknown) value of $\rho \in (-2, 0)$. The main result of the present paper is to establish the relation between $\mu$ and $\rho$, more specifically to show that $\mu = -\pi \cot(\pi\rho/2)$.

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# SET-VALUED BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS

BY ÇAĞIN ARARAT[1,a], JIN MA[2,b] AND WENQIAN WU[3,c]

[1]*Department of Industrial Engineering, Bilkent University,* [a]*cararat@bilkent.edu.tr*
[2]*Department of Mathematics, University of Southern California,* [b]*jinma@usc.edu*
[3]*Fixed Income, Currency and Commodities, Guotai Junan Securities,* [c]*wuwenqian023709@gtjas.com*

In this paper, we establish an analytic framework for studying *set-valued backward stochastic differential equations (set-valued BSDE)*, motivated largely by the current studies of dynamic set-valued risk measures for multi-asset or network-based financial models. Our framework will make use of the notion of the *Hukuhara difference* between sets, in order to compensate the lack of "inverse" operation of the traditional Minkowski addition, whence the vector space structure in set-valued analysis. While proving the well-posedness of a class of set-valued BSDEs, we shall also address some fundamental issues regarding generalized Aumann–Itô integrals, especially when it is connected to the martingale representation theorem. In particular, we propose some necessary extensions of the integral that can be used to represent set-valued martingales with nonsingleton initial values. This extension turns out to be essential for the study of set-valued BSDEs.

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# VECTOR-VALUED STATISTICS OF BINOMIAL PROCESSES: BERRY–ESSEEN BOUNDS IN THE CONVEX DISTANCE

BY MIKOŁAJ J. KASPRZAK[a] AND GIOVANNI PECCATI[b]

*Department of Mathematics, University of Luxembourg,* [a]*mikolaj.kasprzak@uni.lu,* [b]*giovanni.peccati@uni.lu*

We study the discrepancy between the distribution of a vector-valued functional of i.i.d. random elements and that of a Gaussian vector. Our main contribution is an explicit bound on the convex distance between the two distributions, holding in every dimension. Such a finding constitutes a substantial extension of the one-dimensional bounds deduced in Chatterjee (*Ann. Probab.* **36** (2008) 1584–1610) and Lachièze-Rey and Peccati (*Ann. Appl. Probab.* **27** (2017) 1992–2031), as well as of the multidimensional bounds for smooth test functions and indicators of rectangles derived, respectively, in Dung (*Acta Math. Hungar.* **158** (2019) 173–201), and Fang and Koike (*Ann. Appl. Probab.* **31** (2021) 1660–1686). Our techniques involve the use of Stein's method, combined with a suitable adaptation of the recursive approach inaugurated by Schulte and Yukich (*Electron. J. Probab.* **24** (2019) 1–42): this yields rates of converge that have a presumably optimal dependence on the sample size. We develop several applications of a geometric nature, among which is a new collection of multidimensional quantitative limit theorems for the intrinsic volumes associated with coverage processes in Euclidean spaces.

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# HYDRODYNAMIC LIMIT FOR THE KOB–ANDERSEN MODEL

BY ASSAF SHAPIRA[a]

*MAP5, Université Paris Cité,* [a]*assaf.shapira@normalesup.org*

This paper concerns with the hydrodynamic limit of the Kob–Andersen model, an interacting particle system that has been introduced by physicists in order to explain glassy behavior, and widely studied since. We will see that the density profile evolves in the hydrodynamic limit according to a nondegenerate hydrodynamic equation, and understand how the diffusion coefficient decays as density grows.

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# GEOMETRY OF RANDOM CAYLEY GRAPHS OF ABELIAN GROUPS

BY JONATHAN HERMON[1,a] AND SAM OLESKER-TAYLOR[2,b]

[1]*Department of Mathematics, University of British Columbia,* [a]*jhermon@math.ubc.ca*
[2]*Department of Statistics, University of Warwick,* [b]*oleskertaylor.sam@gmail.com*

Consider the random Cayley graph of a finite Abelian group $G$ with respect to $k$ generators chosen uniformly at random, with $1 \ll \log k \ll \log |G|$. Draw a vertex $U \sim \text{Unif}(G)$.

We show that the graph distance $\text{dist}(\mathsf{id}, U)$ from the identity to $U$ concentrates at a particular value $M$, which is the minimal radius of a ball in $\mathbb{Z}^k$ of cardinality at least $|G|$, under mild conditions. In other words, the distance from the identity for all but $o(|G|)$ of the elements of $G$ lies in the interval $[M - o(M), M + o(M)]$. In the regime $k \gtrsim \log |G|$, we show that the diameter of the graph is also asymptotically $M$. In the spirit of a conjecture of Aldous and Diaconis (Technical Report 231 (1985)), this $M$ depends only on $k$ and $|G|$, not on the algebraic structure of $G$.

Write $d(G)$ for the minimal size of a generating subset of $G$. We prove that the order of the spectral gap is $|G|^{-2/k}$ when $k - d(G) \asymp k$ and $|G|$ lies in a density-1 subset of $\mathbb{N}$ or when $k - 2d(G) \asymp k$. This extends, for Abelian groups, a celebrated result of Alon and Roichman (*Random Structures Algorithms* **5** (1994) 271–284).

The aforementioned results all hold with high probability over the random Cayley graph.

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# STRONG ERROR BOUNDS FOR THE CONVERGENCE TO ITS MEAN FIELD LIMIT FOR SYSTEMS OF INTERACTING NEURONS IN A DIFFUSIVE SCALING

BY XAVIER ERNY[1,a], EVA LÖCHERBACH[2,b] AND DASHA LOUKIANOVA[3,c]

[1]*Ecole Polytechnique, Centre de mathématiques appliquées (CMAP),* [a]*xavier.erny@polytechnique.edu*

[2]*Statistique, Analyse et Modélisation Multidisciplinaire, Université Paris 1 Panthéon-Sorbonne, EA 4543,* [b]*eva.locherbach@univ-paris1.fr*

[3]*Laboratoire de Mathématiques et Modélisation d'Évry, Université d'Évry Val d'Essonne, UMR CNRS 8071,* [c]*dasha.loukianova@univ-evry.fr*

We consider the stochastic system of interacting neurons introduced in (*J. Stat. Phys.* **158** (2015) 866–902) and in (*Ann. Inst. Henri Poincaré Probab. Stat.* **52** (2016) 1844–1876) and then further studied in (*Electron. J. Probab.* **26** (2021) 20) in a diffusive scaling. The system consists of $N$ neurons, each spiking randomly with rate depending on its membrane potential. At its spiking time, the potential of the spiking neuron is reset to 0 and all other neurons receive an additional amount of potential which is a centred random variable of order $1/\sqrt{N}$. In between successive spikes, each neuron's potential follows a deterministic flow. In our previous article (*Electron. J. Probab.* **26** (2021) 20) we proved the convergence of the system, as $N \to \infty$, to a limit nonlinear jumping stochastic differential equation. In the present article we complete this study by establishing a strong convergence result, stated with respect to an appropriate distance, with an explicit rate of convergence. The main technical ingredient of our proof is the coupling introduced in (*Z. Wahrsch. Verw. Gebiete* **34** (1976) 33–58) of the point process representing the small jumps of the particle system with the limit Brownian motion.

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# GRAPHON MEAN FIELD SYSTEMS

BY ERHAN BAYRAKTAR[1,a], SUMAN CHAKRABORTY[2,b] AND RUOYU WU[3,c]

[1]*Department of Mathematics, University of Michigan,* [a]*erhan@umich.edu*
[2]*Department of Mathematics, Uppsala University,* [b]*suman.chakraborty@math.uu.se*
[3]*Department of Mathematics, Iowa State University,* [c]*ruoyu@iastate.edu*

We consider heterogeneously interacting diffusive particle systems and their large population limit. The interaction is of mean field type with weights characterized by an underlying graphon. A law of large numbers result is established as the system size increases and the underlying graphons converge. The limit is given by a graphon mean field system consisting of independent but heterogeneous nonlinear diffusions whose probability distributions are fully coupled. Well-posedness, continuity and stability of such systems are provided. We also consider a not-so-dense analogue of the finite particle system, obtained by percolation with vanishing rates and suitable scaling of interactions. A law of large numbers result is proved for the convergence of such systems to the corresponding graphon mean field system.

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# GENERAL DIFFUSION PROCESSES AS LIMIT OF TIME-SPACE MARKOV CHAINS

BY ALEXIS ANAGNOSTAKIS[a], ANTOINE LEJAY[b] AND DENIS VILLEMONAIS[c]

*Université de Lorraine, CNRS, Inria, IECL,* [a]*alexis.anagnostakis@univ-lorraine.fr,* [b]*antoine.lejay@univ-lorraine.fr,* [c]*denis.villemonais@univ-lorraine.fr*

We prove the convergence of the law of grid-valued random walks, which can be seen as time-space Markov chains, to the law of a general diffusion process. This includes processes with sticky features, reflecting or absorbing boundaries and skew behavior. We prove that the convergence occurs at any rate strictly inferior to $(1/4) \wedge (1/p)$ in terms of the maximum cell size of the grid, for any $p$-Wasserstein distance. We also show that it is possible to achieve any rate strictly inferior to $(1/2) \wedge (2/p)$ if the grid is adapted to the speed measure of the diffusion, which is optimal for $p \le 4$. This result allows us to set up asymptotically optimal approximation schemes for general diffusion processes. Last, we experiment numerically on diffusions that exhibit various features.

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# STOCHASTIC NONLINEAR SCHRÖDINGER EQUATIONS IN THE DEFOCUSING MASS AND ENERGY CRITICAL CASES

BY DENG ZHANG[a]

*School of Mathematical Sciences, Shanghai Jiao Tong University,* [a]*dzhang@sjtu.edu.cn*

We study the stochastic nonlinear Schrödinger equations with linear multiplicative noise, particularly in the defocusing mass-critical and energy-critical cases. For general initial data, we prove the global well-posedness of solutions in both mass-critical and energy-critical cases. We also prove the rescaled scattering behavior of global solutions in the spaces $L^2$, $H^1$ as well as the pseudo-conformal space for dimensions $d \geq 3$ in the case of finite global quadratic variation of noise. Furthermore, the Stroock–Varadhan type theorem is also obtained for the topological support of the probability distribution induced by global solutions in the Strichartz and local smoothing spaces. Our proof is based on the construction of a new family of rescaling transformations indexed by stopping times and on the stability analysis adapted to the multiplicative noise.

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# A SPATIAL MEASURE-VALUED MODEL FOR CHEMICAL REACTION NETWORKS IN HETEROGENEOUS SYSTEMS

BY LEA POPOVIC[1,a] AND AMANDINE VÉBER[2,b]

[1]*Department of Mathematics and Statistics, Concordia University,* [a]*lea.popovic@concordia.ca*
[2]*Université Paris Cité, CNRS, MAP5,* [b]*amandine.veber@parisdescartes.fr*

We propose a novel measure valued process which models the behaviour of chemical reaction networks in spatially heterogeneous systems. It models reaction dynamics between different molecular species and continuous movement of molecules in space. Reactions rates at a spatial location are proportional to the mass of different species present locally and to a location specific chemical rate, which may be a function of the local or global species mass as well. We obtain asymptotic limits for the process, with appropriate rescaling depending on the abundance of different molecular types. In particular, when the mass of some species in the scaling limit is discrete while the mass of the others is continuous, we obtain a new type of spatial random evolution process. This process can be shown, in some situations, to correspond to a measure-valued piecewise deterministic Markov process in which the discrete mass of the process evolves stochastically, and the continuous mass evolves in a deterministic way between consecutive jump times of the discrete part.

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# LARGE DEVIATION LOCAL LIMIT THEOREMS AND LIMITS OF BICONDITIONED PLANAR MAPS

BY IGOR KORTCHEMSKI[1,a] AND CYRIL MARZOUK[2,b]

[1] *CNRS & CMAP, École polytechnique,* [a]*igor.kortchemski@math.cnrs.fr*
[2] *CMAP, École polytechnique,* [b]*cyril.marzouk@polytechnique.edu*

We first establish new local limit estimates for the probability that a nondecreasing integer-valued random walk lies at time $n$ at an arbitrary value, encompassing in particular large deviation regimes on the boundary of the Cramér zone. This enables us to derive scaling limits of such random walks conditioned by their terminal value at time $n$ in various regimes. We believe both to be of independent interest. We then apply these results to obtain invariance principles for the Łukasiewicz path of Bienaymé–Galton–Watson trees conditioned on having a fixed number of leaves and of vertices at the same time, which constitutes a first step towards understanding their large scale geometry. We finally deduce from this scaling limit theorems for random bipartite planar maps under a new conditioning by fixing their number of vertices, edges, and faces at the same time. In the particular case of the uniform distribution, our results confirm a prediction of Fusy and Guitter on the growth of the typical distances and show furthermore that in all regimes, the scaling limit is the celebrated Brownian sphere.

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# A POTENTIAL-BASED CONSTRUCTION OF THE INCREASING SUPERMARTINGALE COUPLING

BY ERHAN BAYRAKTAR[1,a], SHUOQING DENG[2,c] AND DOMINYKAS NORGILAS[1,b]

[1]*Department of Mathematics, University of Michigan,* [a]*erhan@umich.edu,* [b]*dnorgila@umich.edu*
[2]*Department of Mathematics, Hong Kong University of Science and Technology,* [c]*masdeng@ust.hk*

The increasing supermartingale coupling, introduced by Nutz and Stebegg (*Ann. Probab.* **46** (2018) 3351–3398) is an extreme point of the set of "supermartingale" couplings between two real probability measures in convex-decreasing order. In the present paper we provide an explicit construction of a triple of functions, on the graph of which the increasing supermartingale coupling concentrates. In particular, we show that the increasing supermartingale coupling can be identified with the left-curtain martingale coupling and the antitone coupling to the left and to the right of a uniquely determined regime-switching point, respectively.

Our construction is based on the concept of the *shadow* measure. We show how to determine the potential of the shadow measure associated to a supermartingale, extending the recent results of Beiglböck et al. (*Electron. Commun. Probab.* **27** (2022) 1–12) obtained in the martingale setting.

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# STEIN'S METHOD, GAUSSIAN PROCESSES AND PALM MEASURES, WITH APPLICATIONS TO QUEUEING

BY A. D. BARBOUR[1,a], NATHAN ROSS[2,b] AND GUANGQU ZHENG[3,c]

[1]*Institut für Mathematik, Universität Zürich,* [a]*a.d.barbour@math.uzh.ch*
[2]*School of Mathematics and Statistics, University of Melbourne,* [b]*nathan.ross@unimelb.edu.au*
[3]*Department of Mathematical Sciences, University of Liverpool,* [c]*guangqu.zheng@liverpool.ac.uk*

We develop a general approach to Stein's method for approximating a random process in the path space $\mathbb{D}([0, T] \to \mathbb{R}^d)$ by a real continuous Gaussian process. We then use the approach in the context of processes that have a representation as integrals with respect to an underlying point process, deriving a general *quantitative* Gaussian approximation. The error bound is expressed in terms of couplings of the original process to processes generated from the reduced Palm measures associated with the point process. As applications, we study certain GI/GI/$\infty$ queues in the "heavy traffic" regime.

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# ON MEAN-FIELD SUPER-BROWNIAN MOTIONS

BY YAOZHONG HU[1,a], MICHAEL A. KOURITZIN[1,b], PANQIU XIA[2,c] AND JIAYU ZHENG[3,d]

[1]*Department of Mathematical and Statistical Sciences, University of Alberta at Edmonton,* [a]*yaozhong@ualberta.ca,* [b]*michaelk@ualberta.ca*

[2]*Department of Mathematical Science, University of Copenhagen,* [c]*px@math.ku.dk*

[3]*Faculty of Computational Mathematics and Cybernetics, Shenzhen MSU-BIT University,* [d]*jyzheng@smbu.edu.cn*

The mean-field stochastic partial differential equation (SPDE) corresponding to a mean-field super-Brownian motion (sBm) is obtained and studied. In this mean-field sBm, the branching-particle lifetime is allowed to depend upon the probability distribution of the sBm itself, producing an SPDE whose space-time white noise coefficient has, in addition to the typical sBm square root, an extra factor that is a function of the probability law of the density of the mean-field sBm. This novel mean-field SPDE is thus motivated by population models where things like overcrowding and isolation can affect growth. A two step approximation method is employed to show the existence for this SPDE under general conditions. Then, mild moment conditions are imposed to get uniqueness. Finally, smoothness of the SPDE solution is established under a further simplifying condition.

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# LOW-TEMPERATURE ISING DYNAMICS WITH RANDOM INITIALIZATIONS

BY REZA GHEISSARI[1,a] AND ALISTAIR SINCLAIR[2,b]

[1]*Departments of Statistics and EECS, University of California, Berkeley,* [a]*gheissari@berkeley.edu*
[2]*Computer Science Division, University of California, Berkeley,* [b]*sinclair@cs.berkeley.edu*

It is well known that Glauber dynamics on spin systems typically suffer exponential slowdowns at low temperatures. This is due to the emergence of multiple metastable phases in the state space, separated by narrow bottlenecks that are hard for the dynamics to cross. It is a folklore belief that if the dynamics is initialized from an appropriate random mixture of ground states, one for each phase, then convergence to the Gibbs distribution should be much faster. However, such phenomena have largely evaded rigorous analysis, as most tools in the study of Markov chain mixing times are tailored to worst-case initializations.

In this paper we develop a general framework towards establishing this conjectured behavior for the Ising model. In the classical setting of the Ising model on an $N$-vertex torus in $\mathbb{Z}^d$, our framework implies that the mixing time for the Glauber dynamics, initialized from a $\frac{1}{2}$-$\frac{1}{2}$ mixture of the all-plus and all-minus configurations, is $N^{1+o(1)}$ in dimension $d = 2$, and at most quasi-polynomial in all dimensions $d \geq 3$, at all temperatures below the critical one. The key innovation in our analysis is the introduction of the notion of "weak spatial mixing within a phase", a low-temperature adaptation of the classical concept of weak spatial mixing. We show both that this new notion is strong enough to control the mixing time from the above random initialization (by relating it to the mixing time with plus boundary condition at $O(\log N)$ scales), and that it holds at all low temperatures in all dimensions.

This framework naturally extends to more general families of graphs. To illustrate this, we use the same approach to establish optimal $O(N \log N)$ mixing for the Ising Glauber dynamics on random regular graphs at sufficiently low temperatures, when initialized from the same random mixture.

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# FUNCTIONAL CENTRAL LIMIT THEOREMS FOR LOCAL STATISTICS OF SPATIAL BIRTH–DEATH PROCESSES IN THE THERMODYNAMIC REGIME

BY EFE ONARAN[1,a], OMER BOBROWSKI[1,2,b] AND ROBERT J. ADLER[1,c]

[1]*Viterbi Faculty of Electrical and Computer Engineering, Technion–Israel Institute of Technology,* [a]*efeonaran@campus.technion.ac.il,* [b]*omer@ee.technion.ac.il,* [c]*radler@technion.ac.il*

[2]*School of Mathematical Sciences, Queen Mary University of London*

We present normal approximation results at the process level for local functionals defined on dynamic Poisson processes in $\mathbb{R}^d$. The dynamics we study here are those of a Markov birth–death process. We prove functional limit theorems in the so-called thermodynamic regime. Our results are applicable to several functionals of interest in the stochastic geometry literature, including subgraph and component counts in the random geometric graphs.

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# MULTIPARAMETER BERNOULLI FACTORIES

BY RENATO PAES LEME[a] AND JON SCHNEIDER[b]

*Google Research, NYC,* [a]*renatoppl@google.com,* [b]*jschnei@google.com*

We consider the problem of computing with many coins of unknown bias. We are given access to samples of $n$ coins with *unknown* biases $p_1, \dots, p_n$ and are asked to sample from a coin with bias $f(p_1, \dots, p_n)$ for a given function $f : [0, 1]^n \to [0, 1]$. We give a complete characterization of the functions $f$ for which this is possible. As a consequence, we show how to extend various combinatorial sampling procedures (most notably, the classic Sampford sampling for $k$-subsets) to the boundary of the hypercube.

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# LARGE DEVIATION PRINCIPLE FOR GEOMETRIC AND TOPOLOGICAL FUNCTIONALS AND ASSOCIATED POINT PROCESSES

BY CHRISTIAN HIRSCH[1,a] AND TAKASHI OWADA[2,b]

[1]*Department of Mathematics, Aarhus University,* [a]*hirsch@math.au.dk*
[2]*Department of Statistics, Purdue University,* [b]*owada@purdue.edu*

We prove a large deviation principle for the point process associated to $k$-element connected components in $\mathbb{R}^d$ with respect to the connectivity radii $r_n \to \infty$. The random points are generated from a homogeneous Poisson point process or the corresponding binomial point process, so that $(r_n)_{n\ge1}$ satisfies $n^k r_n^{d(k-1)} \to \infty$ and $nr_n^d \to 0$ as $n \to \infty$ (i.e., sparse regime). The rate function for the obtained large deviation principle can be represented as relative entropy. As an application, we deduce large deviation principles for various functionals and point processes appearing in stochastic geometry and topology. As concrete examples of topological invariants, we consider persistent Betti numbers of geometric complexes and the number of Morse critical points of the min-type distance function.

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# ANOMALOUS SCALING REGIME FOR ONE-DIMENSIONAL MOTT VARIABLE-RANGE HOPPING

BY DAVID A. CROYDON[1,a], RYOKI FUKUSHIMA[2,b] AND STEFAN JUNK[3,c]

[1]*Research Institute for Mathematical Sciences, Kyoto University,* [a]*croydon@kurims.kyoto-u.ac.jp*
[2]*Institute of Mathematics, University of Tsukuba,* [b]*ryoki@math.tsukuba.ac.jp*
[3]*Advanced Institute for Materials Research, Tohoku University,* [c]*sjunk@tohoku.ac.jp*

We derive an anomalous, sub-diffusive scaling limit for a one-dimensional version of the Mott random walk. The limiting process can be viewed heuristically as a one-dimensional diffusion with an absolutely continuous speed measure and a discontinuous scale function, as given by a two-sided stable subordinator. Corresponding to intervals of low conductance in the discrete model, the discontinuities in the scale function act as barriers off which the limiting process reflects for some time before crossing. We also discuss how, by incorporating a Bouchaud trap model element into the setting, it is possible to combine this "blocking" mechanism with one of "trapping". Our proof relies on a recently developed theory that relates the convergence of processes to that of associated resistance metric measure spaces.

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# DISAGREEMENT COUPLING OF GIBBS PROCESSES WITH AN APPLICATION TO POISSON APPROXIMATION

BY GÜNTER LAST[1,a] AND MORITZ OTTO[2,b]

[1]*Department of Mathematics, Karlsruhe Institute of Technology,* [a]*guenter.last@kit.edu*
[2]*Department of Mathematics, Aarhus University,* [b]*otto@math.au.dk*

We discuss a thinning and an embedding procedure to construct finite Gibbs processes with a given Papangelou intensity. Extending the approach of Hofer-Temmel (*Electron. J. Probab.* **24** (2019) 1–22) and Hofer-Temmel and Houdebert (*Stochastic Process. Appl.* **129** (2019) 3922–3940) we will use this to couple two finite Gibbs processes with different boundary conditions. As one application we will establish Poisson approximation of point processes derived from certain infinite volume Gibbs processes via dependent thinning. As another application we shall discuss empty space probabilities of certain Gibbs processes.

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# UTILITY MAXIMIZATION WITH RATCHET AND DRAWDOWN CONSTRAINTS ON CONSUMPTION IN INCOMPLETE SEMIMARTINGALE MARKETS

BY ANASTASIYA TANANA[a]

*Department of Mathematics, The University of Texas at Austin,* [a]*atanana@utexas.edu*

In this paper, we study expected utility maximization under ratchet and drawdown constraints on consumption in a general incomplete semimartingale market using duality methods. The optimization is considered with respect to two parameters: the initial wealth and the essential lower bound on consumption process. In order to state the problem and define the primal domains, we introduce a natural extension of the notion of running maximum to arbitrary nonnegative optional processes and study its properties. The dual domains for optimization are characterized in terms of solidity with respect to an ordering that is introduced on the set of nonnegative optional processes. The abstract duality result we obtain for the optimization problem is used in order to derive a more detailed characterization of solutions in the complete market case.

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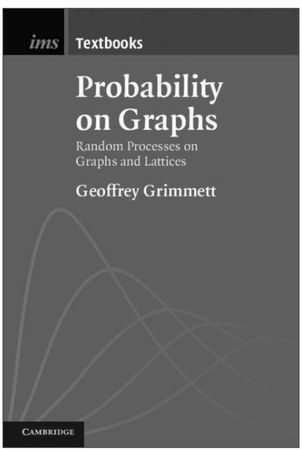


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