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THE STOCHASTIC HEAT EQUATION AS THE LIMIT OF A STIRRING DYNAMICS PERTURBED BY A VOTER MODEL

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We prove that in dimension $d \leq 3$ a modified density field of a stirring dynamics perturbed by a voter model converges to the stochastic heat equation.

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MULTITYPE Λ-COALESCENTS

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Consider a multitype coalescent process in which each block has a colour in $\{1, \dots, d\}$. Individual blocks may change colour, and some number of blocks of various colours may merge to form a new block of some colour. We show that if the law of a multitype coalescent process is invariant under permutations of blocks of the same colour, has consistent Markovian projections, and has asynchronous mergers, then it is a multitype Λ-coalescent: a process in which single blocks may change colour, two blocks of like colour may merge to form a single block of that colour, or large mergers across various colours happen at rates governed by a d-tuple of measures on $[0, 1]^d$. We go on to identify when such processes come down from infinity. Our framework generalises Pitman's celebrated classification theorem for singletype coalescent processes, and provides a unifying setting for numerous examples that have appeared in the literature, including the seed-bank model, the island model, and the coalescent structure of continuous-state branching processes.

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KYLE–BACK MODELS WITH RISK AVERSION AND NON-GAUSSIAN BELIEFS

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We show that the problem of existence of equilibrium in Kyle's continuous time insider trading model can be tackled by considering a forward-backward system coupled via an optimal transport type constraint at maturity. The forward component is a stochastic differential equation representing an endogenously determined state variable and the backward component is a quasilinear parabolic equation representing the pricing function. By obtaining a stochastic representation for the solution of such a system, we show the well-posedness of solutions and study the properties of the equilibrium obtained for small enough risk aversion parameter. In our model, the insider has exponential type utility and the belief of the market maker on the distribution of the price at final time can be non-Gaussian.

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CONTIGUITY UNDER HIGH-DIMENSIONAL GAUSSIANITY WITH APPLICATIONS TO COVARIANCE TESTING

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Le Cam's third/contiguity lemma is a fundamental probabilistic tool to compute the limiting distribution of a given statistic T_n under a nonnull sequence of probability measures $\{Q_n\}$, provided its limiting distribution under a null sequence $\{P_n\}$ is available, and the log likelihood ratio $\{\log(\mathrm{d}Q_n/\mathrm{d}P_n)\}$ has a distributional limit. Despite its wide-spread applications to low-dimensional statistical problems, the stringent requirement of Le Cam's third/contiguity lemma on the distributional limit of the log likelihood ratio makes it challenging, or even impossible to use in many modern high-dimensional statistical problems.

This paper provides a nonasymptotic analogue of Le Cam's third/contiguity lemma under high-dimensional normal populations. Our contiguity method is particularly compatible with sufficiently regular statistics T_n: the regularity of T_n effectively reduces both the problems of (i) obtaining a null (Gaussian) limit distribution and of (ii) verifying our new quantitative contiguity condition, to those of derivative calculations and moment bounding exercises. More important, our method bypasses the need to understand the precise behavior of the log likelihood ratio, and therefore possibly works even when it necessarily fails to stabilize—a regime beyond the reach of classical contiguity methods.

As a demonstration of the scope of our new contiguity method, we obtain asymptotically exact power formulae for a number of widely used high-dimensional covariance tests, including the likelihood ratio tests and trace tests, that hold uniformly over all possible alternative covariance under mild growth conditions on the dimension-to-sample ratio. These new results go much beyond the scope of previous available case-specific techniques, and exhibit new phenomenon regarding the behavior of these important class of covariance tests.

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A DISCRETE COMPLEMENT OF LYAPUNOV'S INEQUALITY AND ITS INFORMATION THEORETIC CONSEQUENCES

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We establish a reversal of Lyapunov's inequality for monotone log-concave sequences, settling a conjecture of Havrilla–Tkocz and Melbourne–Tkocz. A strengthened version of the same conjecture is disproved through counter example. We also derive several information theoretic inequalities as consequences. In particular sharp bounds are derived for the varentropy, Rényi entropies, and the concentration of information of monotone log-concave random variables. Moreover, the majorization approach utilized in the proof of the main theorem, is applied to derive analogous information theoretic results in the symmetric setting, where the Lyapunov reversal is known to fail.

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ONE-DEPENDENT COLORINGS OF THE STAR GRAPH

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This paper is concerned with symmetric 1-dependent colorings of the d-ray star graph $\mathscr{S}^d$ for $d \ge 2$. We compute the critical point of the 1-dependent hard-core processes on $\mathscr{S}^d$, which gives a lower bound for the number of colors needed for a 1-dependent coloring of $\mathscr{S}^d$. We provide an explicit construction of a 1-dependent q-coloring for any $q \ge 5$ of the infinite subgraph $\mathscr{S}^3_{(1,1,\infty)}$, which is symmetric in the colors and whose restriction to any path is some symmetric 1-dependent q-coloring. We also prove that there is no such coloring of $\mathscr{S}^3_{(1,1,\infty)}$ with $q = 4$ colors. A list of open problems are presented.

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RANKING-BASED RICH-GET-RICHER PROCESSES

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We study a discrete-time Markov process $X_n \in \mathbb{R}^d$ for which the distribution of the future increments depends only on the relative ranking of its components (descending order by value). We endow the process with a rich-get-richer assumption and show that, together with a finite second moments assumption, it is enough to guarantee almost sure convergence of X_n/n. We characterize the possible limits if one is free to choose the initial state and we give a condition under which the initial state is irrelevant. Finally, we show how our framework can account for ranking-based Pólya urns and can be used to study ranking algorithms for web interfaces.

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GENERALIZED WASSERSTEIN BARYCENTERS BETWEEN PROBABILITY MEASURES LIVING ON DIFFERENT SUBSPACES

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In this paper, we introduce a generalization of the Wasserstein barycenter, to a case where the initial probability measures live on different subspaces of $\mathbb{R}^d$. We study the existence and uniqueness of this barycenter, we show how it is related to a larger multimarginal optimal transport problem, and we propose a dual formulation. Finally, we explain how to compute numerically this generalized barycenter on discrete distributions, and we propose an explicit solution for Gaussian distributions.

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ON THE STABILITY OF POSITIVE SEMIGROUPS

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The stability and contraction properties of positive integral semigroups on Polish spaces are investigated. Our novel analysis is based on the extension of V-norm contraction methods, associated to functionally weighted Banach spaces for Markov semigroups, to positive semigroups. This methodology is applied to a general class of positive and possibly time-inhomogeneous bounded integral semigroups and their normalised versions. The spectral theorems that we develop are an extension of Perron–Frobenius and Krein–Rutman theorems for positive operators to a class of time-varying positive semigroups. In the context of time-homogeneous models, the regularity conditions discussed in the present article appear to be necessary and sufficient condition for the existence of leading eigenvalues. We review and illustrate the impact of these results in the context of positive semigroups arising in transport theory, physics, mathematical biology and signal processing.

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FUNCTIONAL CONVEX ORDER FOR THE SCALED MCKEAN–VLASOV PROCESSES

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We establish the functional convex order results for two scaled McKean–Vlasov processes $X = (X_t)_{t\in[0,T]}$ and $Y = (Y_t)_{t\in[0,T]}$ defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ by

$$\begin{cases} dX_t = b(t, X_t, \mu_t)\,dt + \sigma(t, X_t, \mu_t)\,dB_t, & X_0 \in L^p(\mathbb{P}), \\ dY_t = b(t, Y_t, \nu_t)\,dt + \theta(t, Y_t, \nu_t)\,dB_t, & Y_0 \in L^p(\mathbb{P}), \end{cases}$$

where $p \geq 2$, for every $t \in [0, T]$, μ_t, ν_t denote the probability distribution of X_t, Y_t respectively and the drift coefficient $b(t, x, \mu)$ is affine in x (scaled). If we make the convexity and monotony assumption (only) on σ and if $\sigma \preceq \theta$ with respect to the partial matrix order, the convex order for the initial random variable $X_0 \preceq_{cv} Y_0$ can be propagated to the whole path of process X and Y. That is, if we consider a convex functional F defined on the path space with polynomial growth, we have $\mathbb{E}F(X) \leq \mathbb{E}F(Y)$; for a convex functional G defined on the product space involving the path space and its marginal distribution space, we have $\mathbb{E}G(X, (\mu_t)_{t\in[0,T]}) \leq \mathbb{E}G(Y, (\nu_t)_{t\in[0,T]})$ under appropriate conditions. The symmetric setting is also valid, that is, if $\theta \preceq \sigma$ and $Y_0 \leq X_0$ with respect to the convex order, then $\mathbb{E}F(Y) \leq \mathbb{E}F(X)$ and $\mathbb{E}G(Y, (\nu_t)_{t\in[0,T]}) \leq \mathbb{E}G(X, (\mu_t)_{t\in[0,T]})$. The proof is based on several forward and backward dynamic programming principles and the convergence of the Euler scheme of the McKean–Vlasov equation. Two applications of these results, to mean field control and mean field games, are proposed.

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EVOLVING GENEALOGIES FOR BRANCHING POPULATIONS UNDER SELECTION AND COMPETITION

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For a continuous state branching process with two types of individuals which are subject to selection and density dependent competition, we characterize the joint evolution of population size, type configurations and genealogies as the unique strong solution of a system of SDEs. Our construction is achieved in the lookdown framework and provides a synthesis as well as a generalization of cases considered separately in two seminal papers by Donnelly and Kurtz (*Ann. Appl. Probab.* **9** (1999) 1091–1148; *Ann. Probab.* **27** (1999) 166–205), namely fluctuating population sizes under neutrality, and selection with constant population size. As a conceptual core in our approach we introduce the *selective lookdown space* which is obtained from its neutral counterpart through a state-dependent thinning of "potential" selection/competition events whose rates interact with the evolution of the type densities. The updates of the genealogical distance matrix at the "active" selection/competition events are obtained through an appropriate sampling from the selective lookdown space. The solution of the above mentioned system of SDEs is then mapped into the joint evolution of population size and symmetrized type configurations and genealogies, that is, marked distance matrix distributions. By means of Kurtz' Markov mapping theorem, we characterize the latter process as the unique solution of a martingale problem. For the sake of transparency we restrict the main part of our presentation to a prototypical example with two types, which contains the essential features. In the final section we outline an extension to processes with multiple types including mutation.

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DYNAMICAL GIBBS VARIATIONAL PRINCIPLES FOR IRREVERSIBLE INTERACTING PARTICLE SYSTEMS WITH APPLICATIONS TO ATTRACTOR PROPERTIES

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We consider irreversible translation-invariant interacting particle systems on the d-dimensional cubic lattice with finite local state space, which admit at least one Gibbs measure as a time-stationary measure. Under some mild degeneracy conditions on the rates and the specification we prove that zero relative entropy loss of a translation-invariant measure implies that the measure is Gibbs w.r.t. the same specification as the time-stationary Gibbs measure. As an application, we obtain the attractor property for irreversible interacting particle systems, which says that any weak limit point of any trajectory of translation-invariant measures is a Gibbs measure w.r.t. the same specification as the time-stationary measure. This extends previously known results to fairly general irreversible interacting particle systems.

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COUPLING BY REFLECTION FOR CONTROLLED DIFFUSION PROCESSES: TURNPIKE PROPERTY AND LARGE TIME BEHAVIOR OF HAMILTON–JACOBI–BELLMAN EQUATIONS

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We investigate the long time behavior of weakly dissipative semilinear Hamilton–Jacobi–Bellman (HJB) equations and the turnpike property for the corresponding stochastic control problems. To this aim, we develop a probabilistic approach based on a variant of coupling by reflection adapted to the study of controlled diffusion processes. We prove existence and uniqueness of solutions for the ergodic Hamilton–Jacobi–Bellman equation and different kind of quantitative exponential convergence results at the level of the value function, of the optimal controls and of the optimal processes. Moreover, we provide uniform in time gradient and Hessian estimates for the solutions of the HJB equation that are of independent interest.

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CONTINUITY OF THE MARTINGALE OPTIMAL TRANSPORT PROBLEM ON THE REAL LINE

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We show continuity of the martingale optimal transport optimisation problem as a functional of its marginals. This is achieved via an estimate on the projection in the nested/causal Wasserstein distance of an arbitrary coupling on to the set of martingale couplings with the same marginals. As a corollary we obtain an independent proof of sufficiency of the monotonicity principle established in Beiglböck and Juillet (*Ann. Probab.* **44** (2016) 42–106) for cost functions of polynomial growth.

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SPEED UP ZIG-ZAG

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The Zig-Zag process is a piecewise deterministic Markov process, efficiently used for simulation in an MCMC setting. As we show in this article, it fails to be exponentially ergodic on heavy tailed target distributions. We introduce an extension of the Zig-Zag process by allowing the process to move with a nonconstant speed function s, depending on the current state of the process. We call this process Speed Up Zig-Zag (SUZZ). We provide conditions that guarantee stability properties for the SUZZ process, including nonexplosivity, exponential ergodicity in heavy tailed targets and central limit theorem. Interestingly, we find that using speed functions that induce explosive deterministic dynamics may lead to stable algorithms that can even mix faster. We further discuss the choice of an efficient speed function by providing an efficiency criterion for the one-dimensional process and we support our findings with simulation results.

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CRAMÉR-TYPE MODERATE DEVIATIONS UNDER LOCAL DEPENDENCE

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We establish Cramér-type moderate deviation theorems for the sums of locally dependent random variables and combinatorial central limit theorems. Optimal error bounds and convergence ranges are obtained under some mild exponential moment conditions. Our main results are more general or sharper than the results in the literature. The main results follow from a more general Cramér-type moderate deviation theorem for dependent random variables without any boundedness assumptions, which is of independent interest. The proofs couple Stein's method with a recursive argument.

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RANDOM NEURAL NETWORKS IN THE INFINITE WIDTH LIMIT AS GAUSSIAN PROCESSES

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This article gives a new proof that fully connected neural networks with random weights and biases converge to Gaussian processes in the regime where the input dimension, output dimension, and depth are kept fixed, while the hidden layer widths tend to infinity. Unlike prior work, convergence is shown assuming only moment conditions for the distribution of weights and for quite general nonlinearities.

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THE COALESCENT STRUCTURE OF UNIFORM AND POISSON SAMPLES FROM MULTITYPE BRANCHING PROCESSES

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We introduce a Poissonization method to study the coalescent structure of uniform samples from branching processes. This method relies on the simple observation that a uniform sample of size k taken from a random set with positive Lebesgue measure may be represented as a mixture of Poisson samples with rate λ and mixing measure $k\,d\lambda/\lambda$. We develop a multitype analogue of this mixture representation, and use it to characterise the coalescent structure of multitype continuous-state branching processes in terms of random multitype forests. Thereafter we study the small time asymptotics of these random forests, establishing a correspondence between multitype continuous-state branching processes and multitype Λ-coalescents.

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THE LANGEVIN MONTE CARLO ALGORITHM IN THE NON-SMOOTH LOG-CONCAVE CASE

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We prove nonasymptotic polynomial bounds on the convergence of the Langevin Monte Carlo algorithm in the case where the potential is a convex function which is globally Lipschitz on its domain, typically the maximum of a finite number of affine functions on an arbitrary convex set. In particular the potential is not assumed to be gradient Lipschitz, in contrast with most existing works on the topic.

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COMBINATORICS OF ANCESTRAL LINES FOR A WRIGHT–FISHER DIFFUSION WITH SELECTION IN A LÉVY ENVIRONMENT

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Wright–Fisher diffusions describe the evolution of the type composition of an infinite haploid population with two types (say type 0 and type 1) subject to neutral reproductions, and possibly selection and mutations. In the present paper we study a Wright–Fisher diffusion in a Lévy environment that gives a selective advantage to sometimes one type, sometimes the other. Classical methods using the ancestral selection graph (ASG) fail in the study of this model because of the complexity, resulting from the two-sided selection, of the structure of the information contained in the ASG. We propose a new method that consists in encoding the relevant combinatorics of the ASG into a function. We show that the expectations of the coefficients of this function form a (nonstochastic) semigroup and deduce that they satisfy a linear system of differential equations. As a result we obtain a series representation for the fixation probability $h(x)$ (where x is the initial proportion of individuals of type 0 in the population) as an infinite sum of polynomials whose coefficients satisfy explicit linear relations. Our approach then allows to derive Taylor expansions at every order for $h(x)$ near $x = 0$ and to obtain an explicit recursion formula for the coefficients.

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CONVERGENCE OF A STOCHASTIC COLLOCATION FINITE VOLUME METHOD FOR THE COMPRESSIBLE NAVIER–STOKES SYSTEM

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We propose a stochastic collocation method based on the piecewise constant interpolation on the probability space combined with a finite volume method to solve the compressible Navier–Stokes system at the nodal points. We show convergence of numerical solutions to a statistical solution of the Navier–Stokes system on condition that the numerical solutions are bounded in probability. The analysis uses the stochastic compactness method based on the Skorokhod/Jakubowski representation theorem and the criterion of convergence in probability due to Gyöngy and Krylov.

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CONVERGENCE IN LAW FOR THE CAPACITY OF THE RANGE OF A CRITICAL BRANCHING RANDOM WALK

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Let R_n be the range of a critical branching random walk with n particles on $\mathbb{Z}^d$, which is the set of sites visited by a random walk indexed by a critical Galton–Watson tree conditioned on having exactly n vertices. For $d \in \{3, 4, 5\}$, we prove that $n^{-\frac{d-2}{4}} \mathtt{cap}^{(d)}(R_n)$, the renormalized capacity of R_n, converges in law to the capacity of the support of the integrated super-Brownian excursion. The proof relies on a study of the intersection probabilities between the critical branching random walk and an independent simple random walk on $\mathbb{Z}^d$.

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ERRATUM TO "ON MANY-SERVER QUEUES IN HEAVY TRAFFIC"

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We correct the proof of the limit (4.38) in A.A. Puhalskii and J.E. Reed, on many-server queues in heavy traffic (*Ann. Appl. Probab.* **20** (2010) 129–195).

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