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SAMPLING FROM POTTS ON RANDOM GRAPHS OF UNBOUNDED DEGREE VIA RANDOM-CLUSTER DYNAMICS

BY ANTONIO BLANCA[1,a] AND REZA GHEISSARI[2,b]

[1]*Department of CSE, Pennsylvania State University,* [a]*ablanca@cse.psu.edu*
[2]*Departments of Statistics and EECS, University of California, Berkeley,* [b]*gheissari@berkeley.edu*

We consider the problem of sampling from the ferromagnetic Potts and random-cluster models on a general family of random graphs via the Glauber dynamics for the random-cluster model. The random-cluster model is parametrized by an edge probability $p \in (0,1)$ and a cluster weight $q > 0$. We establish that for every $q \geq 1$, the random-cluster Glauber dynamics mixes in optimal $\Theta(n \log n)$ steps on n-vertex random graphs having a prescribed degree sequence with bounded average branching γ throughout the full high-temperature uniqueness regime $p < p_u(q, \gamma)$.

The family of random graph models we consider includes the Erdős–Rényi random graph $G(n, \gamma/n)$, and so we provide the first polynomial-time sampling algorithm for the ferromagnetic Potts model on Erdős–Rényi random graphs for the full tree uniqueness regime. We accompany our results with mixing time lower bounds (exponential in the largest degree) for the Potts Glauber dynamics, in the same settings where our $\Theta(n \log n)$ bounds for the random-cluster Glauber dynamics apply. This reveals a novel and significant computational advantage of random-cluster based algorithms for sampling from the Potts model at high temperatures.

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THE BERRY–ESSEEN THEOREM FOR CIRCULAR β-ENSEMBLE

BY RENJIE FENG[1,a], GANG TIAN[2,b] AND DONGYI WEI[2,c]

[1]*Max Planck Institute for Mathematics,* [a]*feng@mpim-bonn.mpg.de*
[2]*Beijing International Center for Mathematical Research, Peking University,* [b]*gtian@math.pku.edu.cn,* [c]*jnwdyi@pku.edu.cn*

We will prove the Berry–Esseen theorem for the number counting function of the circular β-ensemble (CβE), which will imply the central limit theorem for the number of points in arcs of the unit circle in mesoscopic and macroscopic scales. We will prove the main result by estimating the characteristic functions of the Prüfer phases and the number counting function, which will imply the uniform upper and lower bounds of their variance. We also show that the similar results hold for the Sine$_\beta$ process. As a direct application of the uniform variance bound, we can prove the normality of the linear statistics when the test function $f(\theta) \in W^{1,p}(S^1)$ for some $p \in (1, +\infty)$.

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LIMIT DISTRIBUTIONS FOR THE DISCRETIZATION ERROR OF STOCHASTIC VOLTERRA EQUATIONS WITH FRACTIONAL KERNEL

BY MASAAKI FUKASAWA[a] AND TAKUTO UGAI[b]

Graduate School of Engineering Science, Osaka University, [a]*fukasawa@sigmath.es.osaka-u.ac.jp,* [b]*ugataku@gmail.com*

Our study aims to specify the asymptotic error distribution in the discretization of a stochastic Volterra equation with a fractional kernel. It is well known that for a standard stochastic differential equation, the discretization error, normalized with its rate of convergence $1/\sqrt{n}$, converges in law to the solution of a certain linear equation. Similar to this, we show that a suitably normalized discretization error of the Volterra equation converges in law to the solution of a certain linear Volterra equation with the same fractional kernel.

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RIGIDITY OF EIGENVALUES FOR β ENSEMBLE IN MULTI-CUT REGIME

BY YITING LI[a]

Department of Mathematical Sciences, Korea Advanced Institute of Science and Technology, [a]*yitingli@kaist.ac.kr*

For a β ensemble on $\Sigma^{(N)} = \{(x_1, \dots, x_N) \in \mathbb{R}^N | x_1 \leq \cdots \leq x_N\}$ with real analytic potential and general $\beta > 0$, under the assumption that its equilibrium measure is supported on q intervals where $q > 1$, we prove the following rigidity property for its particles.

1. In the bulk of the spectrum, with overwhelming probability, the distance between a particle and its classical position is of order $O(N^{-1+\epsilon})$.

2. If k is close to 1 or close to N, that is, near the extreme edges of the spectrum, then with overwhelming probability, the distance between the kth largest particle and its classical position is of order $O(N^{-\frac{2}{3}+\epsilon} \min(k, N+1-k)^{-\frac{1}{3}})$.

Here $\epsilon > 0$ is an arbitrarily small constant. Our main idea is to decompose the multi-cut β ensemble as a product of probability measures on spaces with lower dimensions and show that each of these measures is very close to a β ensemble in one-cut regime for which the rigidity of particles is known.

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STATIONARITY AND ERGODIC PROPERTIES FOR SOME OBSERVATION-DRIVEN MODELS IN RANDOM ENVIRONMENTS

BY PAUL DOUKHAN[1,a], MICHAEL H. NEUMANN[2,b] AND LIONEL TRUQUET[3,c]

[1]*UMR 8088 Analyse, Géométrie et Modélisation, Université Cergy-Pontoise,* [a]*paul.doukhan@gmail.com*
[2]*Institut für Mathematik, Friedrich-Schiller-Universität Jena,* [b]*michael.neumann@uni-jena.de*
[3]*Campus de Ker-Lann, ENSAI-CREST UMR 9194,* [c]*lionel.truquet@ensai.fr*

The first motivation of this paper is to study stationarity and ergodic properties for a general class of time series models defined conditional on an exogenous covariates process. The dynamic of these models is given by an autoregressive latent process which forms a Markov chain in random environments. Contrarily to existing contributions in the field of Markov chains in random environments, the state space is not discrete and we do not use small set type assumptions or uniform contraction conditions for the random Markov kernels. Our assumptions are quite general and allow us to deal with models that are not fully contractive, such as threshold autoregressive processes. Using a coupling approach, we study the existence of a limit, in Wasserstein metric, for the backward iterations of the chain. We also derive ergodic properties for the corresponding skew-product Markov chain. Our results are illustrated with many examples of autoregressive processes widely used in statistics or in econometrics, including GARCH type processes, count autoregressions and categorical time series.

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QUANTITATIVE CLT FOR LINEAR EIGENVALUE STATISTICS OF WIGNER MATRICES

BY ZHIGANG BAO[1,a] AND YUKUN HE[2,b]

[1]*Department of Mathematics, Hong Kong University of Science and Technology,* [a]*mazgbao@ust.hk*
[2]*Department of Mathematics, City University of Hong Kong,* [b]*yukunhe@cityu.edu.hk*

In this article, we establish a near-optimal convergence rate for the CLT of linear eigenvalue statistics of $N \times N$ Wigner matrices, in Kolmogorov–Smirnov distance. For all test functions $f \in C^5(\mathbb{R})$, we show that the convergence rate is either $N^{-1/2+\varepsilon}$ or $N^{-1+\varepsilon}$, depending on the first Chebyshev coefficient of f and the third moment of the diagonal matrix entries. The condition that distinguishes these two rates is necessary and sufficient. For a general class of test functions, we further identify matching lower bounds for the convergence rates. In addition, we identify an explicit, nonuniversal contribution in the linear eigenvalue statistics, which is responsible for the slow rate $N^{-1/2+\varepsilon}$ for non-Gaussian ensembles. By removing this nonuniversal part, we show that the shifted linear eigenvalue statistics have the unified convergence rate $N^{-1+\varepsilon}$ for all test functions.

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CARD GUESSING AND THE BIRTHDAY PROBLEM FOR SAMPLING WITHOUT REPLACEMENT

BY JIMMY HE[1,a] AND ANDREA OTTOLINI[2,b]

[1] *Department of Mathematics, Massachusetts Institute of Technology,* [a]*jimmyhe@mit.edu*
[2] *Department of Mathematics, University of Washington,* [b]*ottolini@uw.edu*

Consider a uniformly random deck consisting of cards labelled by numbers from 1 through n, possibly with repeats. A guesser guesses the top card, after which it is revealed and removed and the game continues. What is the expected number of correct guesses under the best and worst strategies? We establish sharp asymptotics for both strategies. For the worst case, this answers a recent question of Diaconis, Graham, He and Spiro, who found the correct order. As part of the proof, we study the birthday problem for sampling without replacement using Stein's method.

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A GROWTH-FRAGMENTATION-ISOLATION PROCESS ON RANDOM RECURSIVE TREES AND CONTACT TRACING

BY VINCENT BANSAYE[1,a], CHENLIN GU[2,b] AND LINGLONG YUAN[3,c]

[1]*CMAP, Ecole Polytechnique, IP Paris,* [a]*vincent.bansaye@polytechnique.edu*
[2]*Yau Mathematical Sciences Center, Tsinghua University,* [b]*guchenlin@hotmail.com*
[3]*Department of Mathematical Sciences, University of Liverpool,* [c]*linglong.yuan@liverpool.ac.uk*

We consider a random process on recursive trees, with three types of events. Vertices give birth at a constant rate (growth), each edge may be removed independently (fragmentation of the tree) and clusters (or trees) are frozen with a rate proportional to their sizes (isolation of connected component). A phase transition occurs when the isolation is able to stop the growth-fragmentation process and cause extinction. When the process survives, the number of clusters increases exponentially and we prove that the normalised empirical measure of clusters a.s. converges to a limit law on recursive trees. We exploit the branching structure associated to the size of clusters, which is inherited from the splitting property of random recursive trees. This work is motivated by the control of epidemics and contact tracing where clusters correspond to trees of infected individuals that can be identified and isolated. We complement this work by providing results on the Malthusian exponent to describe the effect of control policies on epidemics.

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ON THE ACCEPT–REJECT MECHANISM FOR METROPOLIS–HASTINGS ALGORITHMS

BY NATHAN GLATT-HOLTZ[1,a], JUSTIN KROMETIS[2,b] AND CECILIA MONDAINI[3,c]

[1]*Department of Mathematics, Tulane University,* [a]*negh@tulane.edu*
[2]*National Security Institute and Department of Mathematics, Virginia Tech,* [b]*jkrometi@vt.edu*
[3]*Department of Mathematics, Drexel University,* [c]*cf823@drexel.edu*

This work develops a powerful and versatile framework for determining acceptance ratios in Metropolis–Hastings-type Markov kernels widely used in statistical sampling problems. Our approach allows us to derive new classes of kernels which unify random walk or diffusion-type sampling methods with more complicated "extended phase space" algorithms based around ideas from Hamiltonian dynamics. Our starting point is an abstract result developed in the generality of measurable state spaces that addresses proposal kernels that possess a certain involution structure. Note that, while this underlying proposal structure suggests a scope which includes Hamiltonian-type kernels, we demonstrate that our abstract result is, in an appropriate sense, equivalent to an earlier general state space setting developed in (*Ann. Appl. Probab.* **8** (1998) 1–9) where the connection to Hamiltonian methods was more obscure.

On the basis of our abstract results we develop several new classes of extended phase space, HMC-like algorithms. First we tackle the classical finite-dimensional setting of a continuously distributed target measure. We then consider an infinite-dimensional framework for targets which are absolutely continuous with respect to a Gaussian measure with a trace-class covariance. Each of these algorithm classes can be viewed as "surrogate-trajectory" methods, providing a versatile methodology to bypass expensive gradient computations through skillful reduced order modeling and/or data driven approaches as we begin to explore in a forthcoming companion work (Glatt-Holtz et al. (2023)). On the other hand, along with the connection of our main abstract result to the framework in (*Ann. Appl. Probab.* **8** (1998) 1–9), these algorithm classes provide a unifying picture connecting together a number of popular existing algorithms which arise as special cases of our general frameworks under suitable parameter choices. In particular we show that, in the finite-dimensional setting, we can produce an algorithm class which includes the Metropolis adjusted Langevin algorithm (MALA) and random walk Metropolis method (RWMC) alongside a number of variants of the HMC algorithm including the geometric approach introduced in (*J. R. Stat. Soc. Ser. B. Stat. Methodol.* **73** (2011) 123–214). In the infinite-dimensional situation, we show that the algorithm class we derive includes the preconditioned Crank–Nicolson (pCN), ∞MALA and ∞HMC methods considered in (*Stoch. Dyn.* **8** (2008) 319–350; *Stochastic Process. Appl.* **121** (2011) 2201–2230; *Statist. Sci.* **28** (2013) 424–446) as special cases.

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MODEL-FREE MEAN-FIELD REINFORCEMENT LEARNING: MEAN-FIELD MDP AND MEAN-FIELD Q-LEARNING

BY RENÉ CARMONA[1,a], MATHIEU LAURIÈRE[2,c] AND ZONGJUN TAN[1,b]

[1]*Department of Operations Research and Financial Engineering, Princeton University,* [a]*rcarmona@princeton.edu,* [b]*zongjun.tan@princeton.edu*

[2]*NYU-ECNU Institute of Mathematical Sciences and Center for Data Science and Artificial Intelligence, New York University Shanghai,* [c]*mathieu.lauriere@nyu.edu*

We study infinite horizon discounted mean field control (MFC) problems with common noise through the lens of mean field Markov decision processes (MFMDP). We allow the agents to use actions that are randomized not only at the individual level but also at the level of the population. This common randomization is introduced for the purpose of exploration from a reinforcement learning (RL) paradigm. It also allows us to establish connections between both closed-loop and open-loop policies for MFC and Markov policies for the MFMDP. In particular, we show that there exists an optimal closed-loop policy for the original MFC and we prove dynamic programming principles for the state and state-action value functions. Building on this framework and the notion of state-action value function, we then propose RL methods for such problems, by adapting existing tabular and deep RL methods to the mean-field setting. The main difficulty is the treatment of the population state, which is an input of the policy and the value function. We provide convergence guarantees for the tabular Q-learning algorithm based on discretizations of the simplex. We also show that neural network based deep RL algorithms are more suitable for continuous spaces as they allow us to avoid discretizing the mean field state space. Numerical examples are provided.

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STABILITY OF THE WEAK MARTINGALE OPTIMAL TRANSPORT PROBLEM

BY MATHIAS BEIGLBÖCK[1,a] BENJAMIN JOURDAIN[2,b], WILLIAM MARGHERITI[2,c] AND GUDMUND PAMMER[3,d]

[1]*Department of Mathematics, University of Vienna,* [a]*mathias.beiglboeck@univie.ac.at*
[2]*CERMICS, École Nationale des Ponts et Chaussées,* [b]*benjamin.jourdain@enpc.fr,* [c]*william.margheriti@gmail.com*
[3]*Department of Mathematics, ETH Zürich,* [d]*gudmund.pammer@math.ethz.ch*

While many questions in (robust) finance can be posed in the martingale optimal transport (MOT) framework, others require to consider also nonlinear cost functionals. Following the terminology of Gozlan, Roberto, Samson and Tetali (*J. Funct. Anal.* **273** (2017) 3327–3405) for classical optimal transport, this corresponds to *weak* martingale optimal transport (WMOT).

In this article we establish stability of WMOT which is important since financial data can give only imprecise information on the underlying marginals. As application, we deduce the stability of the superreplication bound for VIX futures as well as the stability of the stretched Brownian motion and we derive a monotonicity principle for WMOT.

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STABILITY OF OVERSHOOTS OF MARKOV ADDITIVE PROCESSES

BY LEIF DÖRING[a] AND LUKAS TROTTNER[b]

Institute of Mathematics, University of Mannheim, [a]*doering@uni-mannheim.de,* [b]*trottner@math.au.dk*

We prove precise stability results for overshoots of Markov additive processes (MAPs) with finite modulating space. Our approach is based on the Markovian nature of overshoots of MAPs whose mixing and ergodic properties are investigated in terms of the characteristics of the MAP. On our way we extend fluctuation theory of MAPs, contributing among others to the understanding of the Wiener–Hopf factorization for MAPs by generalizing Vigon's équations amicales inversés known for Lévy processes. Using the Lamperti transformation the results can be applied to self-similar Markov processes. Among many possible applications, we study the mixing behavior of stable processes sampled at symmetric first hitting times as a concrete example.

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STOCHASTIC BILLIARDS WITH MARKOVIAN REFLECTIONS IN GENERALIZED PARABOLIC DOMAINS

BY CONRADO DA COSTA[a], MIKHAIL V. MENSHIKOV[b] AND ANDREW R. WADE[c]

Department of Mathematical Sciences, Durham University, [a]*conrado.da-costa@durham.ac.uk,* [b]*mikhail.menshikov@durham.ac.uk,* [c]*andrew.wade@durham.ac.uk*

We study recurrence and transience for a particle that moves at constant velocity in the interior of an unbounded planar domain, with random reflections at the boundary governed by a Markov kernel producing outgoing angles from incoming angles. Our domains have a single unbounded direction and sub-linear growth. We characterize recurrence in terms of the reflection kernel and growth rate of the domain. The results are obtained by transforming the stochastic billiards model to a Markov chain on a half-strip $\mathbb{R}_+ \times S$ where S is a compact set. We develop the recurrence classification for such processes in the near-critical regime in which drifts of the $\mathbb{R}_+$ component are of generalized Lamperti type, and the S component is asymptotically Markov; this extends earlier work that dealt with finite S.

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ALGORITHMIC OBSTRUCTIONS IN THE RANDOM NUMBER PARTITIONING PROBLEM

BY DAVID GAMARNIK[1,a] AND EREN C. KIZILDAĞ[2,b]

[1]*Sloan School of Management, Massachusetts Institute of Technology,* [a]*gamarnik@mit.edu*
[2]*Department of Statistics, Columbia University,* [b]*eck2170@columbia.edu*

We consider the algorithmic problem of finding a near-optimal solution for the number partitioning problem (NPP). This problem appears in many practical applications, including the design of randomized controlled trials, multiprocessor scheduling, and cryptography. It is also of theoretical significance. The NPP possesses a so-called *statistical-to-computational gap*: when its input X has distribution $\mathcal{N}(0, I_n)$, the optimal value of the NPP is $\Theta(\sqrt{n}2^{-n})$ w.h.p., whereas the best-known polynomial-time algorithm achieves an objective value of only $2^{-\Theta(\log^2 n)}$ w.h.p.

In this paper we initiate the study of the nature of this gap. Inspired by insights from statistical physics, we study the landscape of the NPP and establish the presence of the overlap gap property (OGP), an intricate geometrical property which is known to be a rigorous evidence of an algorithmic hardness for large classes of algorithms. By leveraging the OGP, we establish that: (a) any sufficiently stable algorithm, appropriately defined, fails to find a near-optimal solution with energy below $2^{-\omega(n \log^{-1/5} n)}$, and (b) a very natural Markov chain Monte Carlo dynamic fails to find near-optimal solutions. Our simulation results suggest that the state-of-the-art algorithm achieving the value of $2^{-\Theta(\log^2 n)}$ is indeed stable, but formally verifying this is left as an open problem.

OGP regards the overlap structure of m-tuples of solutions achieving a certain objective value. When m is constant, we prove the presence of OGP for the objective values of order $2^{-\Theta(n)}$ and the absence of it in the regime $2^{-o(n)}$. Interestingly though, by considering overlaps with growing values of m, we prove the presence of the OGP up to the level $2^{-\omega(\sqrt{n \log n})}$. Our proof of the failure of stable algorithms at values $2^{-\omega(n \log^{-1/5} n)}$ employs methods from Ramsey theory from the extremal combinatorics and is of independent interest.

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VISCOSITY SOLUTIONS TO SECOND ORDER PATH-DEPENDENT HAMILTON–JACOBI–BELLMAN EQUATIONS AND APPLICATIONS

BY JIANJUN ZHOU[a]

College of Science, Northwest A&F University, [a]*zhoujianjun@nwsuaf.edu.cn*

In this article a notion of viscosity solutions is introduced for second-order path-dependent Hamilton–Jacobi–Bellman (PHJB) equations associated with optimal control problems for path-dependent stochastic differential equations. We identify the value functional of optimal control problems as unique viscosity solution to the associated PHJB equations. We also show that our notion of viscosity solutions is consistent with the corresponding notion of classical solutions and satisfies a stability property. Applications to backward stochastic Hamilton–Jacobi–Bellman equations are also given.

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RANKINGS IN DIRECTED CONFIGURATION MODELS WITH HEAVY TAILED IN-DEGREES

BY XING SHI CAI[1,a], PIETRO CAPUTO[2,b], GUILLEM PERARNAU[3,c] AND MATTEO QUATTROPANI[4,d]

[1]*Mathematics Department, Duke Kunshan University,* [a]*xingshi.cai@dukekunshan.edu.cn*
[2]*Dipartimento di Matematica e Fisica, Università di Roma Tre,* [b]*pietro.caputo@uniroma3.it*
[3]*Departament de Matemàtiques (MAT), Universitat Politècnica de Catalunya (UPC)*
Centre de Recerca Matemàtica, Barcelona, Spain, [c]*guillem.perarnau@upc.edu*
[4]*Dipartimento di Matematica "Guido Castelnuovo", Sapienza Università di Roma,* [d]*matteo.quattropani@uniroma1.it*

We consider the extremal values of the stationary distribution of sparse directed random graphs with given degree sequences and their relation to the extremal values of the in-degree sequence. The graphs are generated by the directed configuration model. Under the assumption of bounded $(2+\eta)$-moments on the in-degrees and of bounded out-degrees, we obtain tight comparisons between the maximum value of the stationary distribution and the maximum in-degree. Under the further assumption that the order statistics of the in-degrees have a power-law behavior, we show that the extremal values of the stationary distribution also have a power-law behavior with the same index. In the same setting, we prove that these results extend to the PageRank scores of the random digraph, thus confirming a version of the so-called *power-law hypothesis*. Along the way we establish several facts about the model, including the mixing time cutoff and the characterization of the typical values of the stationary distribution, which were previously obtained under the assumption of bounded in-degrees.

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THE EFFECTIVE RADIUS OF SELF REPELLING ELASTIC MANIFOLDS

BY CARL MUELLER[1,a] AND EYAL NEUMAN[2,b]

[1]*Department of Mathematics, University of Rochester,* [a]*carl.e.mueller@rochester.edu*
[2]*Department of Mathematics, Imperial College London,* [b]*e.neumann@imperial.ac.uk*

We study elastic manifolds with self-repelling terms and estimate their effective radius. This class of manifolds is modelled by a self-repelling vector-valued Gaussian free field with Neumann boundary conditions over the domain $[-N, N]^d \cap \mathbb{Z}^d$, that takes values in $\mathbb{R}^d$. Our main result states that in two dimensions ($d = 2$), the effective radius R_N of the manifold is approximately N. This verifies the conjecture of Kantor, Kardar and Nelson (*Phys. Rev. Lett.* **58** (1987) 1289–1292) up to a logarithmic correction. Our results in $d \geq 3$ give a similar lower bound on R_N and an upper of order $N^{d/2}$. This result implies that self-repelling elastic manifolds undergo a substantial stretching at any dimension.

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ROUGH MCKEAN–VLASOV DYNAMICS FOR ROBUST ENSEMBLE KALMAN FILTERING

BY MICHELE COGHI[1,a], TORSTEIN NILSSEN[2,b], NIKOLAS NÜSKEN[3,c] AND SEBASTIAN REICH[4,d]

[1]*Department of Mathematics, Università di Trento,* [a]*michele.coghi@unitn.it*
[2]*Institute of Mathematics, University of Agder,* [b]*torstein.nilssen@uia.no*
[3]*Department of Mathematics, King's College London,* [c]*nikolas.nusken@kcl.ac.uk*
[4]*Institute of Mathematics, Universität Potsdam,* [d]*sebastian.reich@uni-potsdam.de*

Motivated by the challenge of incorporating data into misspecified and multiscale dynamical models, we study a McKean–Vlasov equation that contains the data stream as a common driving rough path. This setting allows us to prove well-posedness as well as continuity with respect to the driver in an appropriate rough-path topology. The latter property is key in our subsequent development of a robust data assimilation methodology: We establish propagation of chaos for the associated interacting particle system, which in turn is suggestive of a numerical scheme that can be viewed as an extension of the ensemble Kalman filter to a rough-path framework. Finally, we discuss a data-driven method based on subsampling to construct suitable rough path lifts and demonstrate the robustness of our scheme in a number of numerical experiments related to parameter estimation problems in multiscale contexts.

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LONG RANDOM MATRICES AND TENSOR UNFOLDING

BY GÉRARD BEN AROUS[1,a], DANIEL ZHENGYU HUANG[2,b] AND JIAOYANG HUANG[3,c]

[1]*Courant Institute, New York University,* [a]*benarous@cims.nyu.edu*

[2] *Beijing International Center for Mathematical Research, Peking University,* [b]*huangdz@bicmr.pku.edu.cn*

[3]*Department of Statistics and Data Science, University of Pennsylvania,* [c]*huangjy@wharton.upenn.edu*

In this paper, we consider the singular values and singular vectors of low rank perturbations of large rectangular random matrices, in the regime the matrix is "long": we allow the number of rows (columns) to grow polynomially in the number of columns (rows). We prove there exists a critical signal-to-noise ratio (depending on the dimensions of the matrix), and the extreme singular values and singular vectors exhibit a BBP-type phase transition. As a main application, we investigate the tensor unfolding algorithm for the asymmetric rank-one spiked tensor model, and obtain an exact threshold, which is independent of the procedure of tensor unfolding. If the signal-to-noise ratio is above the threshold, tensor unfolding detects the signals; otherwise, it fails to capture the signals.

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APPROXIMATE VISCOSITY SOLUTIONS OF PATH-DEPENDENT PDES AND DUPIRE'S VERTICAL DIFFERENTIABILITY

BY BRUNO BOUCHARD[1,a], GRÉGOIRE LOEPER[2,b] AND XIAOLU TAN[3,c]

[1]*CEREMADE, Université Paris-Dauphine, PSL, CNRS,* [a]*bouchard@ceremade.dauphine.fr*

[2]*BNP Paribas Global Markets, School of Mathematical Sciences & Centre for Quantitative Finance and Investment Strategies (CQFIS) Monash University,* [b]*gregoire.loeper@monash.edu*

[3]*Department of Mathematics, Chinese University of Hong Kong,* [c]*xiaolu.tan@cuhk.edu.hk*

We introduce a notion of approximate viscosity solutions for a class of nonlinear path-dependent PDEs (PPDEs), including the Hamilton–Jacobi–Bellman-type equations. Existence, comparaison and stability results have been established under fairly general conditions. It is also consistent with the notion of smooth solution when the dimension is less or equal to two, or the nonlinearity is concave in the second order space derivative. We finally investigate the regularity (in the sense of Dupire) of the solution to the PPDE.

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ERRATUM: DIFFUSION MODELS AND STEADY-STATE APPROXIMATIONS FOR EXPONENTIALLY ERGODIC MARKOVIAN QUEUES

BY ITAI GURVICH[a]

Kellogg School of Management, Northwestern University
[a]*i-gurvich@kellogg.northwestern.edu*

This is a correction to (*Ann. Appl. Probab.* **24** (2014) 2527–2559). The corrected result is that the gap between the steady-state moments of the diffusion and those of the properly centered and scaled CTMCs shrinks at a rate of $n^{\frac{1}{2}-\epsilon}$ for *any* $\epsilon > 0$.

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VOLUME 33

2023

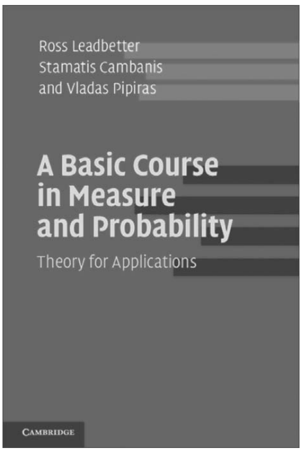
CONTENTS OF VOLUME 33

Articles

Erratum

A Basic Course in Measure and Probability: *Theory for Applications*

Ross Leadbetter, Stamatis Cambanis, and Vladas Pipiras



Originating from the authors' own graduate course at the University of North Carolina, this material has been thoroughly tried and tested over many years, making the book perfect for a two-term course or for self-study. It provides a concise introduction that covers all of the measure theory and probability most useful for statisticians, including Lebesgue integration, limit theorems in probability, martingales, and some theory of stochastic processes. Readers can test their understanding of the material through the 300 exercises provided.

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