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DETECTING STRUCTURED SIGNALS IN ISING MODELS

BY NABARUN DEB[1,a], RAJARSHI MUKHERJEE[2,d], SUMIT MUKHERJEE[1,b] AND MING YUAN[1,c]

[1]*Department of Statistics, Columbia University,* [a]*nd2560@columbia.edu,* [b]*sm3949@columbia.edu,* [c]*ming.yuan@columbia.edu*

[2]*Department of Biostatistics, Harvard University,* [d]*ram521@mail.harvard.edu*

In this paper we study the effect of dependence on detecting a class of signals in Ising models, where the signals are present in a structured way. Examples include Ising models on lattices, and mean-field type Ising models (Erdős–Rényi, Random regular, and dense graphs). Our results rely on correlation decay and mixing type behavior for Ising models, and demonstrate the beneficial behavior of criticality in detection of strictly lower signals. As a by-product of our proof technique, we develop sharp control on mixing and spin-spin correlation for several mean-field type Ising models in all regimes of temperature–which might be of independent interest.

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THE DIRECTED SPANNING FOREST IN THE HYPERBOLIC SPACE

BY LUCAS FLAMMANT[a]

Université Polytechnique Hauts-de-France, CERAMATHS, CNRS, [a]*lucasflammant@aol.com*

The Euclidean directed spanning forest is a random forest in $\mathbb{R}^d$ introduced by Baccelli and Bordenave in 2007 and we introduce and study here the analogous tree in the hyperbolic space. The topological properties of the Euclidean DSF have been stated for $d = 2$ and conjectured for $d \geq 3$ (see further): it should be a tree for $d \in \{2, 3\}$ and a countable union of disjoint trees for $d \geq 4$. Moreover, it should not contain bi-infinite branches whatever the dimension d. In this paper, we construct the hyperbolic directed spanning forest (HDSF) and we give a complete description of its topological properties, which are radically different from the Euclidean case. Indeed, for any dimension, the hyperbolic DSF is a tree containing infinitely many bi-infinite branches, whose asymptotic directions are investigated. The strategy of our proofs consists in exploiting the mass transport principle, which is adapted to take advantage of the invariance by isometries. Using appropriate mass transports is the key to carry over the hyperbolic setting ideas developed in percolation and for spanning forests. This strategy provides an upper-bound for horizontal fluctuations of trajectories, which is the key point of the proofs. To obtain the latter, we exploit the representation of the forest in the hyperbolic half space.

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STRONG DIFFUSION APPROXIMATION IN AVERAGING AND VALUE COMPUTATION IN DYNKIN'S GAMES

BY YURI KIFER[a]

Institute of Mathematics, Hebrew University, [a]*kifer@math.huji.ac.il*

It is known since (*Theory Probab. Appl.* **11** (1966) 390–406) that the slow motion X^ε in the time-scaled multidimensional averaging setup

$$\frac{dX^\varepsilon(t)}{dt} = \frac{1}{\varepsilon}B\big(X^\varepsilon(t), \xi\big(t/\varepsilon^2\big)\big) + b\big(X^\varepsilon(t), \xi\big(t/\varepsilon^2\big)\big), \quad t \in [0, T]$$

converges weakly as $\varepsilon \to 0$ to a diffusion process provided $EB(x, \xi(s)) \equiv 0$ where ξ is a sufficiently fast mixing stochastic process. In this paper we show that both X^ε and a family of diffusions Ξ^ε can be redefined on a common sufficiently rich probability space so that $E\sup_{0\le t\le T}|X^\varepsilon(t) - \Xi^\varepsilon(t)|^{2M} \le C(M)\varepsilon^\delta$ for some $C(M), \delta > 0$ and all $M \ge 1$, $\varepsilon > 0$, where all $\Xi^\varepsilon, \varepsilon > 0$ have the same diffusion coefficients but underlying Brownian motions may change with ε. We obtain also a similar result for the corresponding discrete time averaging setup. As an application we consider Dynkin's games with path dependent payoffs involving a diffusion and obtain error estimates for computation of values of such games by means of such discrete time approximations which provides a more effective computational tool than the standard discretization of the diffusion itself.

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CAUCHY PROBLEM OF STOCHASTIC KINETIC EQUATIONS

BY XIAOLONG ZHANG[a] AND XICHENG ZHANG[b]

School of Mathematics and Statistics, Beijing Institute of Technology, [a]*zhangxl.math@bit.edu.cn,* [b]*XichengZhang@gmail.com*

In this paper we establish the optimal regularity estimates for the Cauchy problem of stochastic kinetic equations with random coefficients in anisotropic Besov spaces. As applications, we study the nonlinear filtering problem for a degenerate diffusion process, and obtain the existence and regularity of conditional probability densities under a few assumptions. Moreover, we also show the well-posedness for a class of super-linear growth stochastic kinetic equations driven by velocity-time white noises.

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WHEN RANDOM TENSORS MEET RANDOM MATRICES

BY MOHAMED EL AMINE SEDDIK[1,a], MAXIME GUILLAUD[2,b] AND ROMAIN COUILLET[3,c]

[1]*Technology Innovation Institute,* [a]*mohamed.seddik@tii.ae*
[2]*Inria / CITI Laboratory,* [b]*maxime.guillaud@inria.fr*
[3]*CNRS, GIPSA-lab, Université Grenoble Alpes,* [c]*romain.couillet@univ-grenoble-alpes.fr*

Relying on random matrix theory (RMT), this paper studies asymmetric order-d spiked tensor models with Gaussian noise. Using the variational definition of the singular vectors and values of Lim (In *Proc. IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing* (2005) 129–132), we show that the analysis of the considered model boils down to the analysis of an equivalent spiked symmetric *blockwise* random matrix, that is constructed from *contractions* of the studied tensor with the singular vectors associated to its best rank-1 approximation. Our approach allows the exact characterization of the almost sure asymptotic singular value and alignments of the corresponding singular vectors with the true spike components when $\frac{n_i}{\sum_{j=1}^d n_j} \to c_i \in (0,1)$ with n_i's the tensor dimensions. In contrast to other works that rely mostly on tools from statistical physics to study random tensors, our results rely solely on classical RMT tools such as Stein's lemma. Finally, classical RMT results concerning spiked random matrices are recovered as a particular case.

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IMPROVED ESTIMATION OF RELAXATION TIME IN NONREVERSIBLE MARKOV CHAINS

BY GEOFFREY WOLFER[1,a] AND ARYEH KONTOROVICH[2,b]

[1] *Center for AI Project, RIKEN,* [a]*geoffrey.wolfer@riken.jp*
[2] *Department of Computer Science, Ben–Gurion University of the Negev,* [b]*karyeh@cs.bgu.ac.il*

We show that the minimax sample complexity for estimating the pseudo-spectral gap γ_{ps} of an ergodic Markov chain in constant multiplicative error is of the order of

$$\tilde{\Theta}\left(\frac{1}{\gamma_{\mathsf{ps}}\pi_\star}\right),$$

where $\pi_\star$ is the minimum stationary probability, recovering the known bound in the reversible setting for estimating the absolute spectral gap (Hsu et al., *Ann. Appl. Probab.* **29** (2019) 2439–2480), and resolving an open problem of Wolfer and Kontorovich (In *Proceedings of the Thirty-Second Conference on Learning Theory* (2019) 3120–3159 PMLR). Furthermore, we strengthen the known empirical procedure by making it fully-adaptive to the data, thinning the confidence intervals and reducing the computational complexity. Along the way, we derive new properties of the pseudo-spectral gap and introduce the notion of a reversible dilation of a stochastic matrix.

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THE JOINT FLUCTUATIONS OF THE LENGTHS OF THE Beta($2-\alpha, \alpha$)-COALESCENTS

BY MATTHIAS BIRKNER[1,a], IULIA DAHMER[1,b], CHRISTINA S. DIEHL[2,c] AND GÖTZ KERSTING[2,d]

[1]*Institut für Mathematik, Johannes Gutenberg-Universität Mainz,* [a]*birkner@uni-mainz.de,* [b]*idahmer@uni-mainz.de*
[2]*Institut für Mathematik, Goethe-Universität Frankfurt,* [c]*ch.s.diehl@gmail.com,* [d]*kersting@math.uni-frankfurt.de*

We consider Beta($2-\alpha, \alpha$)-coalescents with parameter range $1 < \alpha < 2$ starting from n leaves. The length $\ell_r^{(n)}$ of order r in the n-Beta($2-\alpha, \alpha$)-coalescent tree is defined as the sum of the lengths of all branches that carry a subtree with r leaves. We show that for any $s \in \mathbb{N}$ the vector of suitably centered and rescaled lengths of orders $1 \leq r \leq s$ converges in distribution to a multivariate stable distribution as the number of leaves tends to infinity.

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VOLTERRA SQUARE-ROOT PROCESS: STATIONARITY AND REGULARITY OF THE LAW

BY MARTIN FRIESEN[1,a] AND PENG JIN[2,b]

[1]*School of Mathematical Sciences, Dublin City University,* [a]*martin.friesen@dcu.ie*
[2]*Guangdong Provincial Key Laboratory of IRADS, BNU-HKBU United International College,* [b]*pengjin@uic.edu.cn*

The Volterra square-root process on $\mathbb{R}_+^m$ is an affine Volterra process with continuous sample paths. Under a suitable integrability condition on the resolvent of the second kind associated with the Volterra convolution kernel, we establish the existence of limiting distributions. In contrast to the classical square-root diffusion process, here the limiting distributions may depend on the initial state of the process. Our result shows that the nonuniqueness of limiting distributions is closely related to the integrability of the Volterra convolution kernel. Using an extension of the exponential-affine transformation formula, we also give the construction of stationary processes associated with the limiting distributions. Finally, we prove that the time marginals as well as the limiting distributions, when restricted to the interior of the state space $\mathbb{R}_+^m$, are absolutely continuous with respect to the Lebesgue measure and their densities belong to some weighted Besov space of type $B_{1,\infty}^{\lambda}$.

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ON QUASILINEAR PARABOLIC SYSTEMS AND FBSDES OF QUADRATIC GROWTH

BY JOE JACKSON[a]

Department of Mathematics, University of Texas at Austin, [a]*jjackso1@utexas.edu*

Using probabilistic methods, we establish a priori estimates for two classes of quasilinear parabolic systems of partial differential equations (PDEs). We treat in particular the case of a nonlinearity, which has quadratic growth in the gradient of the unknown. As a result of our estimates, we obtain the existence of classical solutions of the PDE system. From this, we infer the existence of solutions to a corresponding class of forward–backward stochastic differential equations.

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MIXING TIME FOR THE ASYMMETRIC SIMPLE EXCLUSION PROCESS IN A RANDOM ENVIRONMENT

BY HUBERT LACOIN[1,a] AND SHANGJIE YANG[2,b]

[1]*Instituto de Matemática Pura e Aplicada,* [a]*lacoin@impa.br*
[2]*Department of Mathematics, Bar-Ilan University,* [b]*shangjie.yang@biu.ac.il*

We consider the simple exclusion process in the integer segment $[\![1, N]\!]$ with $k \le N/2$ particles and spatially inhomogenous jumping rates. A particle at site $x \in [\![1, N]\!]$ jumps to site $x-1$ (if $x \ge 2$) at rate $1-\omega_x$ and to site $x+1$ (if $x \le N-1$) at rate ω_x if the target site is not occupied. The sequence $\omega = (\omega_x)_{x\in\mathbb{Z}}$ is chosen by IID sampling from a probability law whose support is bounded away from zero and one (in other words the random environment satisfies the uniform ellipticity condition). We further assume $\mathbb{E}[\log \rho_1] < 0$ where $\rho_1 := (1-\omega_1)/\omega_1$, which implies that our particles have a tendency to move to the right. We prove that the mixing time of the exclusion process in this setup grows like a power of N. More precisely, for the exclusion process with $N^{\beta+o(1)}$ particles where $\beta \in [0, 1]$, we have in the large N asymptotic

$$N^{\max(1,\frac{1}{\lambda},\beta+\frac{1}{2\lambda})+o(1)} \le t_{\mathrm{mix}}^{N,k} \le N^{C+o(1)},$$

where $\lambda > 0$ is such that $\mathbb{E}[\rho_1^\lambda] = 1$ ($\lambda = \infty$ if the equation has no positive root) and C is a constant, which depends on the distribution of ω. We conjecture that our lower bound is sharp up to subpolynomial correction.

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TOWARD A MATHEMATICAL THEORY OF TRAJECTORY INFERENCE

BY HUGO LAVENANT[1,a], STEPHEN ZHANG[2,b], YOUNG-HEON KIM[2,c] AND GEOFFREY SCHIEBINGER[2,d]

[1]*Department of Decision Sciences and BIDSA, Bocconi University,* [a]*hugo.lavenant@unibocconi.it*
[2]*Department of Mathematics, University of British Columbia,* [b]*syz@math.ubc.ca,* [c]*yhkim@math.ubc.ca,* [d]*geoff@math.ubc.ca*

We devise a theoretical framework and a numerical method to infer trajectories of a stochastic process from samples of its temporal marginals. This problem arises in the analysis of single-cell RNA-sequencing data, which provide high-dimensional measurements of cell states but cannot track the trajectories of the cells over time. We prove that for a class of stochastic processes it is possible to recover the ground truth trajectories from limited samples of the temporal marginals at each time-point, and provide an efficient algorithm to do so in practice. The method we develop, Global Waddington-OT (gWOT), boils down to a smooth convex optimization problem posed globally over all time-points involving entropy-regularized optimal transport. We demonstrate that this problem can be solved efficiently in practice and yields good reconstructions, as we show on several synthetic and real data sets.

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QUANTITATIVE UNIFORM STABILITY OF THE ITERATIVE PROPORTIONAL FITTING PROCEDURE

BY GEORGE DELIGIANNIDIS[a], VALENTIN DE BORTOLI[b] AND ARNAUD DOUCET[c]

Department of Statistics, University of Oxford,
[a]*deligian@stats.ox.ac.uk,* [b]*valentin.debortoli@gmail.com,* [c]*doucet@stats.ox.ac.uk*

We establish that the iterates of the iterative proportional fitting procedure, also known as Sinkhorn's algorithm and commonly used to solve entropy-regularised optimal transport problems, are stable w.r.t. perturbations of the marginals, uniformly in time. Our result is quantitative and stated in terms of the 1-Wasserstein metric. As a corollary we establish a quantitative stability result for Schrödinger bridges.

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METASTABLE MIXING OF MARKOV CHAINS: EFFICIENTLY SAMPLING LOW TEMPERATURE EXPONENTIAL RANDOM GRAPHS

BY GUY BRESLER[1,a], DHEERAJ NAGARAJ[2,b] AND ESHAAN NICHANI[3,c]

[1]*Department of Electrical Engineering and Computer Science, Massachusetts Institute of Technology,* [a]*guy@mit.edu*
[2]*Google Research,* [b]*dheerajnagaraj@google.com*
[3]*Department of Electrical and Computer Engineering, Princeton University,* [c]*eshnich@princeton.edu*

In this paper, we consider the problem of sampling from the low-temperature exponential random graph model (ERGM). The usual approach is via Markov chain Monte Carlo, but Bhamidi et al. showed that any local Markov chain suffers from an exponentially large mixing time due to metastable states. We instead consider *metastable mixing*, a notion of approximate mixing relative to the stationary distribution, for which it turns out to suffice to mix only within a collection of metastable states. We show that the Glauber dynamics for the ERGM at any temperature—except at a lower-dimensional critical set of parameters—when initialized at $G(n, p)$ for the right choice of p has a metastable mixing time of $O(n^2 \log n)$ to within total variation distance $\exp(-\Omega(n))$.

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DIFFUSIVE LIMITS OF LIPSCHITZ FUNCTIONALS OF POISSON MEASURES

BY EUSTACHE BESANÇON[1,a], LAURE COUTIN[2,c], LAURENT DECREUSEFOND[1,b] AND PASCAL MOYAL[3,d]

[1]*LTCI, Telecom Paris,* [a]*eustache.besancon@telecom-paris.fr,* [b]*laurent.decreusefond@telecom-paris.fr*
[2]*Institut Mathématique de Toulouse, Université P. Sabatier,* [c]*laure.coutin@math.univ-toulouse.fr*
[3]*Institut Elie Cartan de Lorraine, Université de Lorraine,* [d]*pascal.moyal@univ-lorraine.fr*

Continuous time Markov Chains, Hawkes processes and many other interesting processes can be described as a solution of stochastic differential equations driven by Poisson measures. Previous works, using the Stein's method, give the convergence rate of a sequence of renormalized Poisson measures toward the Brownian motion in several distances, constructed on the model of the Kantorovitch–Rubinstein (or Wasserstein-1) distance. We show that many operations (like time change, convolution) on continuous functions are Lipschitz continuous to extend these quantified convergences to diffusive limits of Markov processes and long-time behavior of Hawkes processes.

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WEIGHTED SIGNATURE KERNELS

BY THOMAS CASS[1,a], TERRY LYONS[2,b] AND XINGCHENG XU[3,c]

[1]*Department of Mathematics, Imperial College London,* [a]*thomas.cass@imperial.ac.uk*
[2]*Matematical Institute, University of Oxford,* [b]*t.lyons@maths.ox.ac.uk*
[3]*Department of Mathematics, Imperial College London,* [c]*xingcheng.xu18@gmail.com*

Suppose that γ and σ are two continuous bounded variation paths, which take values in a finite-dimensional inner product space V. The recent papers (*J. Mach. Learn. Res.* **20** (2019) 1–45) and (*SIAM J. Math. Data Sci.* **3** (2021) 873–899), respectively, introduced the truncated and the untruncated signature kernel of γ and σ, and showed how these concepts can be used in classification and prediction tasks involving multivariate time series. In this paper, we introduce signature kernels $K_\phi^{\gamma,\sigma}$ indexed by a weight function ϕ, which generalise the ordinary signature kernel. We show how $K_\phi^{\gamma,\sigma}$ can be interpreted in many examples as an average of PDE solutions, and thus we show how it can be estimated computationally using suitable quadrature formulae. We extend this analysis to derive closed-form formulae for expressions involving the expected (Stratonovich) signature of Brownian motion. In doing so, we articulate a novel connection between signature kernels and the notion of the hyperbolic development of a path, which has been a broadly useful tool in the recent analysis of the signature; see, for example, (*Ann. of Math. (2)* **171** (2010) 109–167; *J. Funct. Anal.* **272** (2017) 2933–2955) and (*Trans. Amer. Math. Soc.* **372** (2019) 585–614). As applications, we evaluate the use of different general signature kernels as a basis for nonparametric goodness-of-fit tests to Wiener measure on path space.

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TIME-TIME COVARIANCE FOR LAST PASSAGE PERCOLATION IN HALF-SPACE

BY PATRIK FERRARI[1,a] AND ALESSANDRA OCCELLI[2,b]

[1]*Institute for Applied Mathematics, Bonn University,* [a]*ferrari@uni-bonn.de*
[2]*LAREMA, Université d'Angers,* [b]*alessandra.occelli@univ-angers.fr*

This article studies several properties of the half-space last passage percolation, in particular the two-time covariance. We show that, when the two end-points are at small macroscopic distance, then the first-order correction to the covariance for the point-to-point model is the same as the one of the stationary model. In order to obtain the result, we first derive comparison inequalities of the last passage increments for different models. This is used to prove tightness of the point-to-point process as well as localization of the geodesics. Unlike for the full-space case, for half-space we have to overcome the difficulty that the point-to-point model in half-space with generic start and end-points is not known.

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COMPUTATIONAL METHODS FOR ADAPTED OPTIMAL TRANSPORT

BY STEPHAN ECKSTEIN[a] AND GUDMUND PAMMER[b]

Department of Mathematics, ETH Zürich, [a]*stephan.eckstein@math.ethz.ch,* [b]*gudmund.pammer@math.ethz.ch*

Adapted optimal transport (AOT) problems are optimal transport problems for distributions of a time series where couplings are constrained to have a temporal causal structure. In this paper, we develop computational tools for solving AOT problems numerically. First, we show that AOT problems are stable with respect to perturbations in the marginals, and thus arbitrary AOT problems can be approximated by sequences of linear programs. We further study entropic methods to solve AOT problems. We show that any entropically regularized AOT problem converges to the corresponding unregularized problem if the regularization parameter goes to zero. The proof is based on a novel method—even in the nonadapted case—to easily obtain smooth approximations of a given coupling with fixed marginals. Finally, we show tractability of the adapted version of Sinkhorn's algorithm. We give explicit solutions for the occurring projections and prove that the procedure converges to the optimizer of the entropic AOT problem.

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A SHAPE THEOREM FOR EXPLODING SANDPILES

BY AHMED BOU-RABEE[a]

Department of Statistics, University of Chicago, [a]*ahmedb@uchicago.edu*

We study scaling limits of exploding Abelian sandpiles using ideas from percolation and front propagation in random media. We establish sufficient conditions under which a limit shape exists and show via a family of counterexamples that convergence may not occur in general. A corollary of our proof is a simple criterion for determining if a sandpile is explosive; this strengthens a result of Fey, Levine and Peres (*J. Stat. Phys.* (2010) **138** 143–159).

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PATHWISE LARGE DEVIATIONS FOR THE PURE JUMP k-NARY INTERACTING PARTICLE SYSTEMS

BY WEN SUN[a]

School of Mathematical Sciences, University of Science and Technology of China, [a]*wensun.ustc@gmail.com*

A pathwise large deviation result is proved for the pure jump models of the k-nary interacting particle system introduced by Kolokoltsov (*Markov Process. Related Fields* **12** (2006) 95–138; *Nonlinear Markov Processes and Kinetic Equations* (2010) Cambridge Univ. Press) that generalize classical Boltzmann's collision model, Smoluchovski's coagulation model and many others. The upper bound is obtained by following the standard methods (KOV (*Comm. Pure Appl. Math.* **42** (1989) 115–137)) of using a process "perturbed" by a regular function. To show the lower bound, we propose a family of orthogonal martingale measures and prove a coupling for the general perturbations. The rate function is studied based on the idea of Léonard (*Probab. Theory Related Fields* **101** (1995) 1–44) with a simplification by considering the conjugation of integral functionals on a subspace of L^{∞}. General "gelling" solutions in the domain of the rate function are also discussed.

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LONG-TERM BALANCED ALLOCATION VIA THINNING

BY OHAD NOY FELDHEIM[1,a], ORI GUREL-GUREVICH[1,b] AND JIANGE LI[2,c]

[1]*Einstein Institute of Mathematics, Hebrew University or Jerusalem,* [a]*ohad.feldheim@mail.huji.ac.il,* [b]*ori.gurel-gurevich@mail.huji.ac.il*

[2]*Institute for Advanced Study in Mathematics, Harbin Institute of Technology,* [c]*jiange.li@hit.edu.cn*

In the two-thinning balls-and-bins model, an overseer is provided with uniform random allocation of m balls into n bins in an on-line fashion. The overseer may reject the allocation of each ball, in which case it is placed into a new bin, drawn independently, uniformly at random. The purpose of the overseer is to reduce the *maximum load*, that is, the difference between the maximum number of balls in a single bin and the average number of balls among all bins.

We provide tight estimates for three quantities: the lowest achievable maximum load at a given time m, the lowest achievable maximum load uniformly over the entire time interval $[m] := \{1, 2, \dots, m\}$ and the lowest achievable *typical* maximum load over the interval $[m]$, that is, a load which upper-bounds $1 - o(1)$ portion of the times in $[m]$.

In particular, for m polynomial in n and sufficiently large, we provide an explicit strategy, which achieves a typical maximum load of $(\log n)^{1/2+o(1)}$, asymptotically the same as that can be achieved at a single time m. In contrast, we show that no strategy can achieve better than $\Theta(\frac{\log n}{\log\log n})$ maximum load for all times up to time m.

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