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SIMPLE FORM CONTROL POLICIES FOR RESOURCE SHARING NETWORKS WITH HGI PERFORMANCE

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We consider a family of resource sharing networks, known as bandwidth sharing models, in heavy traffic with general service and interarrival times. These networks, introduced in Massoulié and Roberts (*Telecommun. Syst.* **15** (2000) 185–201) as models for internet flows, have the feature that a typical job may require simultaneous processing by multiple resources in the network. We construct simple form threshold policies that asymptotically achieve the Hierarchical Greedy Ideal (HGI) performance. This performance benchmark, which was introduced in Harrison et al. (*Stoch. Syst.* **4** (2014) 524–555), is characterized by the following two features: every resource works at full capacity whenever there is work for that resource in the system; total holding cost of jobs of each type at any instant is the minimum cost possible for the associated vector of workloads. The control policy we provide is explicit in terms of a finite collection of vectors, which can be computed offline by solving a system of linear inequalities. Proof of convergence is based on path large deviation estimates for renewal processes, Lyapunov function constructions and analyses of suitable sample path excursions.

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Key words and phrases. Stochastic processing networks, dynamic control, asymptotic optimality, heavy traffic, Brownian control problems, diffusion approximations, resource sharing networks, reflected Brownian motions, internet flows.

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OPTIMAL STOPPING WITH EXPECTATION CONSTRAINTS

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We analyze an optimal stopping problem with a series of inequality-type and equality-type expectation constraints in a general non-Markovian framework. We show that the optimal stopping problem with expectation constraints (OSEC) in an arbitrary probability setting is equivalent to the constrained problem in weak formulation (an optimization over joint laws of stopping rules with Brownian motion and state dynamics on an enlarged canonical space), and thus the OSEC value is independent of a specific probabilistic setup. Using a martingale-problem formulation, we make an equivalent characterization of the probability classes in weak formulation, which implies that the OSEC value function is upper semianalytic. Then we exploit a measurable selection argument to establish a dynamic programming principle in weak formulation for the OSEC value function, in which the conditional expected costs act as additional states for constraint levels at the intermediate horizon.

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SHADOWS AND BARRIERS

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In this article, we show an intimate connection between two objects in probability theory, which received some attention in the last years: shadows of measures and barrier solutions to the Skorokhod embedding problem (SEP). The shadow of a measure μ in the measure ν is the key object in the construction of the left-curtain coupling and its siblings in martingale optimal transport by Beiglböck and Juillet (*Ann. Probab.* **44** (2016) 42–106; *Trans. Amer. Math. Soc.* **374** (2021) 4973–5002). Many prominent solutions to the SEP are first hitting times of barriers in certain phase spaces, that is, they are of the form $\inf\{t \geq 0 : (X_t, B_t) \in \mathcal{R}\}$ for some closed set $\mathcal{R}$, an increasing processes X and Brownian motion B.

We show that the property that a solution to the SEP is of barrier type can be characterized in terms of the shadow. This characterization allows us to construct new families of barrier solutions that naturally interpolate between two given barrier solutions. We exemplify this by an interpolation between the Root embedding and the left-monotone embedding.

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THE CRITICAL TWO-POINT FUNCTION FOR LONG-RANGE PERCOLATION ON THE HIERARCHICAL LATTICE

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We prove up-to-constants bounds on the two-point function (i.e., point-to-point connection probabilities) for critical long-range percolation on the d-dimensional hierarchical lattice. More precisely, we prove that if we connect each pair of points x and y by an edge with probability $1 - \exp(-\beta\|x - y\|^{-d-\alpha})$, where $0 < \alpha < d$ is fixed and $\beta \geq 0$ is a parameter, then the critical two-point function satisfies

$$\mathbb{P}_{\beta_c}(x \leftrightarrow y) \asymp \|x - y\|^{-d+\alpha}$$

for every pair of distinct points x and y. We deduce in particular that the model has mean-field critical behaviour when $\alpha < d/3$ and does *not* have mean-field critical behaviour when $\alpha > d/3$.

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ON THE DIFFERENCE BETWEEN ENTROPIC COST AND THE OPTIMAL TRANSPORT COST

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Consider the Monge–Kantorovich problem of transporting densities ρ_0 to ρ_1 on $\mathbb{R}^d$ with a strictly convex cost function. A popular regularization of the problem is the one-parameter family called the entropic cost problem. The entropic cost K_h, $h > 0$, is significantly faster to compute and hK_h is known to converge to the optimal transport cost as h goes to zero. We are interested in the rate of convergence. We show that the difference between K_h and $1/h$ times the optimal cost of transport has a pointwise limit when transporting a compactly supported density to another that satisfies a few other technical restrictions. This limit is the relative entropy of ρ_1 with respect to a weighted Riemannian volume measure on $\mathbb{R}^d$ that measures the local sensitivity of the transport map. For the quadratic Wasserstein transport, this relative entropy is exactly one half of the difference of entropies of ρ_1 and ρ_0. More surprisingly, we demonstrate that this difference of two entropies (plus the cost) is also the limit for the Dirichlet transport introduced by Pal and Wong (*Probab. Theory Related Fields* **178** (2020) 613–654) in the context of stochastic portfolio theory. The latter can be thought of as a multiplicative analog of the Wasserstein transport and corresponds to a nonlocal operator. The proofs are based on Gaussian approximations to Schrödinger bridges as h approaches zero.

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PARALLEL SERVER SYSTEMS UNDER AN EXTENDED HEAVY TRAFFIC CONDITION: A LOWER BOUND

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The standard setting for studying parallel server systems (PSS) at the diffusion scale is based on the heavy traffic condition (HTC), which assumes that the underlying static allocation linear program (LP) is critical and has a unique solution. This solution determines the graph of basic activities, which identifies the set of activities (i.e., class-server pairs) that are operational. In this paper we explore the extended HTC, where the LP is merely assumed to be critical. Because multiple solutions are allowed, multiple sets of operational activities, referred to as modes, are available. Formally, the scaling limit for the control problem associated with the model is given by a so-called workload control problem (WCP) in which a cost associated with a diffusion process is to be minimized by dynamically switching between these modes. Our main result is that the WCP's value constitutes an asymptotic lower bound on the cost associated with the PSS model.

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ANALYSIS OF THE ENSEMBLE KALMAN–BUCY FILTER FOR CORRELATED OBSERVATION NOISE

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Ensemble Kalman–Bucy filters (EnKBFs) are an important tool in data assimilation that aim to approximate the posterior distribution for continuous time filtering problems using an ensemble of interacting particles. In this work we extend a previously derived unifying framework for consistent representations of the posterior distribution to correlated observation noise and use these representations to derive an EnKBF suitable for this setting as a constant gain approximation of these optimal filters. Existence and uniqueness results for both the EnKBF and its mean field limit are provided. The existence and uniqueness of solutions to its limiting McKean-Vlasov equation does not seem to be covered by the existing literature. In the correlated noise case the evolution of the ensemble depends also on the pseudoinverse of its empirical covariance matrix, which has to be controlled for global well-posedness. These bounds may also be of independent interest. Finally the convergence to the mean field limit is proven. The results can also be extended to other versions of EnKBFs.

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SHARP CONVERGENCE RATES FOR EMPIRICAL OPTIMAL TRANSPORT WITH SMOOTH COSTS

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We revisit the question of characterizing the convergence rate of plug-in estimators of optimal transport costs. It is well known that an empirical measure comprising independent samples from an absolutely continuous distribution on $\mathbb{R}^d$ converges to that distribution at the rate $n^{-1/d}$ in Wasserstein distance, which can be used to prove that plug-in estimators of many optimal transport costs converge at this same rate. However, we show that when the cost is smooth, this analysis is loose: plug-in estimators based on empirical measures converge quadratically faster, at the rate $n^{-2/d}$. As a corollary, we show that the Wasserstein distance between two distributions is significantly easier to estimate when the measures are well-separated. We also prove lower bounds, showing not only that our analysis of the plug-in estimator is tight, but also that no other estimator can enjoy significantly faster rates of convergence uniformly over all pairs of measures. Our proofs rely on empirical process theory arguments based on tight control of L^2 covering numbers for locally Lipschitz and semiconcave functions. As a byproduct of our proofs, we derive L^∞ estimates on the displacement induced by the optimal coupling between any two measures satisfying suitable concentration and anticoncentration conditions, for a wide range of cost functions.

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ONE-POINT ASYMPTOTICS FOR HALF-FLAT ASEP

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We consider the asymmetric simple exclusion process (ASEP) with half-flat initial condition. We show that the one-point marginals of the ASEP height function are described by those of the $\text{Airy}_{2\to 1}$ process, introduced by Borodin–Ferrari–Sasamoto in (*Comm. Pure Appl. Math.* **61** (2008) 1603–1629). This result was conjectured by Ortmann–Quastel–Remenik (*Ann. Appl. Probab.* **26** (2016) 507–548), based on an informal asymptotic analysis of exact formulas for generating functions of the half-flat ASEP height function at one spatial point. Our present work provides a fully rigorous derivation and asymptotic analysis of the same generating functions, under certain parameter restrictions of the model.

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ON THE VALLEYS OF THE STOCHASTIC HEAT EQUATION

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We consider a generalization of the parabolic Anderson model driven by space-time white noise, also called the stochastic heat equation, on the real line:

$$\partial_t u(t,x) = \frac{1}{2}\partial_x^2 u(t,x) + \sigma\big(u(t,x)\big)\xi(t,x) \quad \text{for } t > 0 \text{ and } x \in \mathbb{R}.$$

High peaks of solutions have been extensively studied under the name of intermittency, but less is known about spatial regions between peaks, which we may loosely refer to as valleys. We present two results about the valleys of the solution.

Our first theorem provides information about the size of valleys and the supremum of the solution $u(t,x)$ over a valley. More precisely, when the initial function $u_0(x) = 1$ for all $x \in \mathbb{R}$, we show that the supremum of the solution over a valley vanishes as $t \to \infty$, and we establish an upper bound of $\exp\{-\text{const}\cdot t^{1/3}\}$ for $u(t,x)$ when x lies in a valley. We demonstrate also that the length of a valley grows at least as $\exp\{+\text{const}\cdot t^{1/3}\}$ as $t \to \infty$.

Our second theorem asserts that the length of the valleys are eventually infinite when the initial function $u(0,x)$ has subgaussian tails.

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FINITE SAMPLE COMPLEXITY OF SEQUENTIAL MONTE CARLO ESTIMATORS ON MULTIMODAL TARGET DISTRIBUTIONS

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We prove finite sample complexities for sequential Monte Carlo (SMC) algorithms which require only *local* mixing times of the associated Markov kernels. Our bounds are particularly useful when the target distribution is multimodal and global mixing of the Markov kernel is slow; in such cases our approach establishes the benefits of SMC over the corresponding Markov chain Monte Carlo (MCMC) estimator. The lack of global mixing is addressed by sequentially controlling the bias introduced by SMC resampling procedures. We apply these results to obtain complexity bounds for approximating expectations under mixtures of log-concave distributions and show that SMC provides a fully polynomial time randomized approximation scheme for some difficult multimodal problems where the corresponding Markov chain sampler is exponentially slow. Finally, we compare the bounds obtained by our approach to existing bounds for tempered Markov chains on the same problems.

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SMOLUCHOWSKI PROCESSES AND NONPARAMETRIC ESTIMATION OF FUNCTIONALS OF PARTICLE DISPLACEMENT DISTRIBUTIONS FROM COUNT DATA

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Suppose that particles are randomly distributed in $\mathbb{R}^d$, and they are subject to identical stochastic motion independently of each other. The Smoluchowski process describes fluctuations of the number of particles in an observation region over time. This paper studies properties of the Smoluchowski processes and considers related statistical problems. In the first part of the paper we revisit probabilistic properties of the Smoluchowski process in a unified and principled way: explicit formulas for generating functionals and moments are derived, conditions for stationarity and Gaussian approximation are discussed, and relations to other stochastic models are highlighted. The second part deals with statistics of the Smoluchowski processes. We consider two different models of the particle displacement process: the undeviated uniform motion (when a particle moves with random constant velocity along a straight line) and the Brownian motion displacement. In the setting of the undeviated uniform motion we study the problems of estimating the mean speed and the speed distribution, while for the Brownian displacement model the problem of estimating the diffusion coefficient is considered. In all these settings we develop estimators with provable accuracy guarantees.

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THE RELATIVE FREQUENCY BETWEEN TWO CONTINUOUS-STATE BRANCHING PROCESSES WITH IMMIGRATION AND THEIR GENEALOGY

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When two (possibly different in distribution) continuous-state branching processes with immigration are present, we study the relative frequency of one of them when the total mass is forced to be constant at a dense set of times. This leads to a SDE whose unique strong solution will be the definition of a Λ-asymmetric frequency process (Λ-AFP). We prove that it is a Feller process and we calculate a large population limit when the total mass tends to infinity. This allows us to study the fluctuations of the process around its deterministic limit. Furthermore, we find conditions for the Λ-AFP to have a moment dual. The dual can be interpreted in terms of selection, (coordinated) mutation, pairwise branching (efficiency), coalescence, and a novel component that comes from the asymmetry between the reproduction mechanisms. In the particular case of a pair of equally distributed continuous-state branching processes the associated Λ-AFP will be the dual of a Λ-coalescent. The map that sends each continuous-state branching process to its associated Λ-coalescent (according to the former procedure) is a homeomorphism between metric spaces.

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SPECTRAL TELESCOPE: CONVERGENCE RATE BOUNDS FOR RANDOM-SCAN GIBBS SAMPLERS BASED ON A HIERARCHICAL STRUCTURE

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Random-scan Gibbs samplers possess a natural hierarchical structure. The structure connects Gibbs samplers targeting higher-dimensional distributions to those targeting lower-dimensional ones. This leads to a quasi-telescoping property of their spectral gaps. Based on this property, we derive three new bounds on the spectral gaps and convergence rates of Gibbs samplers on general domains. The three bounds relate a chain's spectral gap to, respectively, the correlation structure of the target distribution, a class of random walk chains, and a collection of influence matrices. Notably, one of our results generalizes the technique of spectral independence, which has received considerable attention for its success on finite domains, to general state spaces. We illustrate our methods through a sampler targeting the uniform distribution on a corner of an n-cube.

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L^p OPTIMAL PREDICTION OF THE LAST ZERO OF A SPECTRALLY NEGATIVE LÉVY PROCESS

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Given a spectrally negative Lévy process X drifting to infinity, (inspired on the early ideas of Shiryaev (2002)) we are interested in finding a stopping time that minimises the L^p distance ($p > 1$) with g, the last time X is negative. The solution is substantially more difficult compared to the case $p = 1$, for which it was shown by Baurdoux and Pedraza (2020) that it is optimal to stop as soon as X exceeds a constant barrier. In the case of $p > 1$ treated here, we prove that solving this optimal prediction problem is equivalent to solving an optimal stopping problem in terms of a two-dimensional strong Markov process that incorporates the length of the current positive excursion away from 0. We show that an optimal stopping time is now given by the first time that X exceeds a nonincreasing and nonnegative curve depending on the length of the current positive excursion away from 0. We further characterise the optimal boundary and the value function as the unique solution of a nonlinear system of integral equations within a subclass of functions. As examples, the case of a Brownian motion with drift and a Brownian motion with drift perturbed by a Poisson process with exponential jumps are considered.

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LIMIT DISTRIBUTIONS AND SENSITIVITY ANALYSIS FOR EMPIRICAL ENTROPIC OPTIMAL TRANSPORT ON COUNTABLE SPACES

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For probability measures on countable spaces we derive distributional limits for empirical entropic optimal transport quantities. More precisely, we show that the empirical optimal transport plan weakly converges to a centered Gaussian process and that the empirical entropic optimal transport value is asymptotically normal. The results are valid for a large class of cost functions and generalize distributional limits for empirical entropic optimal transport quantities on finite spaces. Our proofs are based on a sensitivity analysis with respect to norms induced by suitable function classes, which arise from novel quantitative bounds for primal and dual optimizers, that are related to the exponential penalty term in the dual formulation. The distributional limits then follow from the functional delta method together with weak convergence of the empirical process in that respective norm, for which we provide sharp conditions on the underlying measures. As a byproduct of our proof technique, consistency of the bootstrap for statistical applications is shown.

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THE DIVIDE-AND-CONQUER SEQUENTIAL MONTE CARLO ALGORITHM: THEORETICAL PROPERTIES AND LIMIT THEOREMS

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We provide a comprehensive characterisation of the theoretical properties of the divide-and-conquer sequential Monte Carlo (DaC-SMC) algorithm. We firmly establish it as a well-founded method by showing that it possesses the same basic properties as conventional sequential Monte Carlo (SMC) algorithms do. In particular, we derive pertinent laws of large numbers, L^p inequalities, and central limit theorems; and we characterize the bias in the normalized estimates produced by the algorithm and argue the absence thereof in the unnormalized ones. We further consider its practical implementation and several interesting variants; obtain expressions for its globally and locally optimal intermediate targets, auxiliary measures, and proposal kernels; and show that, in comparable conditions, DaC-SMC proves more statistically efficient than its direct SMC analogue. We close the paper with a discussion of our results, open questions, and future research directions.

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MAPPING HYDRODYNAMICS FOR THE FACILITATED EXCLUSION AND ZERO-RANGE PROCESSES

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We derive the hydrodynamic limit for two degenerate lattice gases, the *facilitated exclusion process* (FEP) and the *facilitated zero-range process* (FZRP), both in the symmetric and the asymmetric case. For both processes, the hydrodynamic limit in the symmetric case takes the form of a diffusive Stefan problem, whereas the asymmetric case is characterized by a hyperbolic Stefan problem. Although the FZRP is attractive, a property that we extensively use to derive its hydrodynamic limits in both cases, the FEP is not. To derive the hydrodynamic limit for the latter, we exploit that of the zero-range process, together with a classical mapping between exclusion and zero-range processes, both at the microscopic and macroscopic level. Due to the degeneracy of both processes, the asymmetric case is a new result, but our work also provides a simpler proof than the one that was previously proposed for the FEP in the symmetric case in (*Probab. Math. Phys.* **2** (2021) 127–178).

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BACKWARD MARTINGALE TRANSPORT AND FITZPATRICK FUNCTIONS IN PSEUDO-EUCLIDEAN SPACES

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We study an optimal transport problem with a backward martingale constraint in a pseudo-Euclidean space S. We show that the dual problem consists in the minimization of the expected values of the Fitzpatrick functions associated with maximal S-monotone sets. An optimal plan γ and an optimal maximal S-monotone set G are characterized by the condition that the support of γ is contained in the graph of the S-projection on G. For a Gaussian random variable Y, we get a unique decomposition: $Y = X + Z$, where X and Z are independent Gaussian random variables taking values, respectively, in complementary positive and negative linear subspaces of the S-space.

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MCKEAN–VLASOV EQUATIONS INVOLVING HITTING TIMES: BLOW-UPS AND GLOBAL SOLVABILITY

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This paper is concerned with the analysis of blow-ups for two McKean–Vlasov equations involving hitting times. Let $(B(t); t \geq 0)$ be standard Brownian motion, and $\tau := \inf\{t \geq 0 : X(t) \leq 0\}$ be the hitting time to zero of a given process X. The first equation is $X(t) = X(0-) + B(t) - \alpha \mathbb{P}(\tau \leq t)$. We provide a simple condition on α and the distribution of $X(0-)$ such that the corresponding Fokker–Planck equation has no blow-up, and thus the McKean–Vlasov dynamics is well defined for all time $t \geq 0$. Our approach relies on a connection between the McKean–Vlasov equation and the supercooled Stefan problem, as well as several comparison principles. The second equation is $X(t) = X(0-) + \beta t + B(t) + \alpha \ln \mathbb{P}(\tau > t)$, $t \geq 0$, whose Fokker–Planck equation is nonlocal. We prove that for $\beta > 0$ sufficiently large and α no greater than a sufficiently small positive constant, there is no blow-up and the McKean–Vlasov dynamics is well defined for all time $t \geq 0$. The argument is based on a new transform, which removes the nonlocal term, followed by a relative entropy analysis.

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EXTREMAL STATISTICS OF QUADRATIC FORMS OF GOE/GUE EIGENVECTORS

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We consider quadratic forms of deterministic matrices A evaluated at the random eigenvectors of a large $N \times N$ GOE or GUE matrix, or equivalently evaluated at the columns of a Haar-orthogonal or Haar-unitary random matrix. We prove that, as long as the deterministic matrix has rank much smaller than $\sqrt{N}$, the distributions of the extrema of these quadratic forms are asymptotically the same as if the eigenvectors were independent Gaussians. This reduces the problem to Gaussian computations, which we carry out in several cases to illustrate our result, finding Gumbel or Weibull limiting distributions depending on the signature of A. Our result also naturally applies to the eigenvectors of any invariant ensemble.

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CONVERGENCE RATE OF THE EULER–MARUYAMA SCHEME APPLIED TO DIFFUSION PROCESSES WITH $L^q - L^\rho$ DRIFT COEFFICIENT AND ADDITIVE NOISE

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We are interested in the time discretization of stochastic differential equations with additive d-dimensional Brownian noise and $L^q - L^\rho$ drift coefficient when the condition $\frac{d}{\rho} + \frac{2}{q} < 1$, under which Krylov and Röckner (*Probab. Theory Related Fields* **131** (2005) 154–196) proved existence of a unique strong solution, is met. We show weak convergence with order $\frac{1}{2}(1 - (\frac{d}{\rho} + \frac{2}{q}))$ which corresponds to half the distance to the threshold for the Euler scheme with randomized time variable and cutoffed drift coefficient so that its contribution on each time-step does not dominate the Brownian contribution. More precisely, we prove that both the diffusion and this Euler scheme admit transition densities and that the difference between these densities is bounded from above by the time-step to this order multiplied by some centered Gaussian density.

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