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ENERGY CORRELATIONS IN THE CRITICAL ISING MODEL ON A TORUS

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We compute rigorously the scaling limit of multipoint energy correlations in the critical Ising model on a torus. For the one-point function, averaged between horizontal and vertical edges of the square lattice, this result has been known since the 1969 work of Ferdinand and Fischer. We propose an alternative proof, in a slightly greater generality, via a new exact formula in terms of determinants of discrete Laplacians. We also compute the main term of the asymptotics of the difference $\mathbb{E}(\epsilon_V - \epsilon_H)$ of the energy density on a vertical and a horizontal edge, which is of order of δ^2, where δ is the mesh size. The observable $\epsilon_V - \epsilon_H$ has been identified by Kadanoff and Ceva as (a component of) the stress-energy tensor.

We then apply the discrete complex analysis methods of Smirnov and Hongler to compute the multipoint correlations. The fermionic observables are only periodic with doubled periods; by antisymmetrization, this leads to contributions from four "sectors." The main new challenge arises in the doubly periodic sector, due to the existence of nonzero constant (discrete) analytic functions. We show that some additional input, namely the scaling limit of the one-point function and of relative contribution of sectors to the partition function, is sufficient to overcome this difficulty and successfully compute all correlations.

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THE NEUMANN PROBLEM FOR FULLY NONLINEAR SPDE

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We generalize the notion of pathwise viscosity solutions, put forward by Lions and Souganidis to study fully nonlinear stochastic partial differential equations, to equations set on a sub-domain with Neumann boundary conditions. Under a convexity assumption on the domain, we obtain a comparison theorem which yields existence and uniqueness of solutions as well as continuity with respect to the driving noise. As an application, we study the long time behaviour of a stochastically perturbed mean-curvature flow in a cylinder-like domain with right angle contact boundary condition.

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SCALING LIMITS OF RANDOM WALKS, HARMONIC PROFILES, AND STATIONARY NONEQUILIBRIUM STATES IN LIPSCHITZ DOMAINS

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We consider the open symmetric exclusion (SEP) and inclusion (SIP) processes on a bounded Lipschitz domain Ω, with both fast and slow boundary. For the random walks on Ω dual to SEP/SIP we establish: a functional-CLT-type convergence to the Brownian motion on Ω with either Neumann (slow boundary), Dirichlet (fast boundary), or Robin (at criticality) boundary conditions; the discrete-to-continuum convergence of the corresponding harmonic profiles. As a consequence, we rigorously derive the hydrodynamic and hydrostatic limits for SEP/SIP on Ω, and analyze their stationary nonequilibrium fluctuations. All scaling limit results for SEP/SIP concern finite-dimensional distribution convergence only, as our duality techniques do not require to establish tightness for the fields associated to the particle systems.

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FIRST PASSAGE PERCOLATION WITH LONG-RANGE CORRELATIONS AND APPLICATIONS TO RANDOM SCHRÖDINGER OPERATORS

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We consider first passage percolation (FPP) with passage times generated by a general class of models with long-range correlations on $\mathbb{Z}^d$, $d \geq 2$, including discrete Gaussian free fields, Ginzburg–Landau $\nabla\phi$ interface models or random interlacements as prominent examples. We show that the associated time constant is positive, the FPP distance is comparable to the Euclidean distance, and we obtain a shape theorem. We also present two applications for random conductance models (RCM) with possibly unbounded and strongly correlated conductances. Namely, we obtain a Gaussian heat kernel upper bound for RCMs with a general class of speed measures, and an exponential decay estimate for the Green's function of RCMs with random killing measures.

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DEFORMED SEMICIRCLE LAW AND CONCENTRATION OF NONLINEAR RANDOM MATRICES FOR ULTRA-WIDE NEURAL NETWORKS

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In this paper, we investigate a two-layer fully connected neural network of the form $f(X)=\frac{1}{\sqrt{d_1}}\boldsymbol{a}^\top\sigma(WX)$, where $X\in {}^{d_0\times n}$ is a deterministic data matrix, $W\in\mathbb{R}^{d_1\times d_0}$ and $\boldsymbol{a}\in\mathbb{R}^{d_1}$ are random Gaussian weights, and σ is a nonlinear activation function. We study the limiting spectral distributions of two empirical kernel matrices associated with $f(X)$: the empirical conjugate kernel (CK) and neural tangent kernel (NTK), beyond the linear-width regime ($d_1\asymp n$). We focus on the *ultra-wide regime*, where the width d_1 of the first layer is much larger than the sample size n. Under appropriate assumptions on X and σ, a deformed semicircle law emerges as $d_1/n\to\infty$ and $n\to\infty$. We first prove this limiting law for generalized sample covariance matrices with some dependency. To specify it for our neural network model, we provide a nonlinear Hanson–Wright inequality suitable for neural networks with random weights and Lipschitz activation functions. We also demonstrate nonasymptotic concentrations of the empirical CK and NTK around their limiting kernels in the spectral norm, along with lower bounds on their smallest eigenvalues. As an application, we show that random feature regression induced by the empirical kernel achieves the same asymptotic performance as its limiting kernel regression under the ultra-wide regime. This allows us to calculate the asymptotic training and test errors for random feature regression using the corresponding kernel regression.

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MERTENS CONJECTURES IN ABSORBING GAMES WITH INCOMPLETE INFORMATION

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In a zero-sum stochastic game with signals (*Repeated Games* (2015) Cambridge Univ. Press, Chapter IV), at each stage, two adversary players make decisions and receive stage payoffs determined by these decisions and a variable called the *state*. The state follows a Markov chain controlled by both players. Actions and states are imperfectly observed by players, who receive private signals at each stage. Mertens (In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986)* (1987) 1528–1577 Amer. Math. Soc.) conjectured two properties regarding games with long durations: first, that a limit value always exists; second, that when Player 1 is more informed than Player 2, she can guarantee the limit value uniformly. These conjectures were recently disproved by the author (*Ann. Probab.* **44** (2016) 1107–1133) but remain widely open in many subclasses. A well-known particular subclass is *absorbing games with incomplete information on both sides*, in which the state can move at most once during the game, and players receive a private signal about it at the outset of the game. This paper proves Mertens' conjectures in this particular model by introducing a new approximation technique of belief dynamics likely to generalize to many other frameworks. In particular, this represents a significant step towards understanding the following broad question: in which games do Mertens' conjectures hold?

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CONTROLLED MEASURE-VALUED MARTINGALES: A VISCOSITY SOLUTION APPROACH

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We consider a class of stochastic control problems where the state process is a probability measure-valued process satisfying an additional martingale condition on its dynamics, called measure-valued martingales (MVMs). We establish the "classical" results of stochastic control for these problems: specifically, we prove that the value function for the problem can be characterised as the unique solution to the Hamilton–Jacobi–Bellman equation in the sense of viscosity solutions. In order to prove this result, we exploit structural properties of the MVM processes. Our results also include an appropriate version of Itô's formula for controlled MVMs.

We also show how problems of this type arise in a number of applications, including model-independent derivatives pricing, the optimal Skorokhod embedding problem, and two player games with asymmetric information.

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The Annals of Applied Probability
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LINEAR-TIME UNIFORM GENERATION OF RANDOM SPARSE CONTINGENCY TABLES WITH SPECIFIED MARGINALS

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We give an algorithm which generates a uniformly random contingency table with specified marginals, that is, a matrix with nonnegative integer values and specified row and column sums. Such algorithms are useful in statistics and combinatorics. When $\Delta^4 < M/5$, where Δ is the maximum of the row and column sums and M is the sum of all entries of the matrix, our algorithm runs in time linear in M in expectation. Most previously published algorithms for this problem are approximate samplers based on Markov chain Monte Carlo, whose provable bounds on the mixing time are typically polynomials with rather large degrees.

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LARGE-DIMENSIONAL CENTRAL LIMIT THEOREM WITH FOURTH-MOMENT ERROR BOUNDS ON CONVEX SETS AND BALLS

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We prove the large-dimensional Gaussian approximation of a sum of n independent random vectors in $\mathbb{R}^d$ together with fourth-moment error bounds on convex sets and Euclidean balls. Our bounds have near-optimal dependence on n and, compared with classical third-moment bounds, can achieve improved dependence on the dimension d. For centered balls, we obtain an additional error bound that has a sub-optimal dependence on n, but recovers the known result of the validity of the Gaussian approximation if and only if $d = o(n)$. We discuss an application to the bootstrap. We prove our main results using Stein's method.

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PROPAGATION OF CHAOS AND POISSON HYPOTHESIS FOR REPLICA MEAN-FIELD MODELS OF INTENSITY-BASED NEURAL NETWORKS

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Neural computations arising from myriads of interactions between spiking neurons can be modeled as network dynamics with punctuate interactions. However, most relevant dynamics do not allow for computational tractability. To circumvent this difficulty, the Poisson hypothesis regime replaces interaction times between neurons by Poisson processes. We prove that the Poisson hypothesis holds at the limit of an infinite number of replicas in the replica-mean-field model, which consists of randomly interacting copies of the network of interest. The proof is obtained through a novel application of the Chen–Stein method to the case of a random sum of Bernoulli random variables and a fixed point approach to prove a law of large numbers for exchangeable random variables.

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MCKEAN–VLASOV SDE AND SPDE WITH LOCALLY MONOTONE COEFFICIENTS

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In this paper we mainly investigate the strong and weak well-posedness of a class of McKean–Vlasov stochastic (partial) differential equations. The main existence and uniqueness results state that we only need to impose some local assumptions on the coefficients, that is, locally monotone condition both in state variable and distribution variable, which cause some essential difficulty since the coefficients of McKean–Vlasov stochastic equations typically are nonlocal. Furthermore, the large deviation principle is also derived for the McKean–Vlasov stochastic equations under those weak assumptions. The wide applications of main results are illustrated by various concrete examples such as the granular media equations, plasma-type models, kinetic equations, McKean–Vlasov-type porous media equations and Navier–Stokes equations. In particular, we could remove or relax some typical assumptions previously imposed on those models.

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ON A RANDOM MODEL OF FORGETTING

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Georgiou, Katkov and Tsodyks considered the following random process. Let $x_1, x_2, \dots$ be an infinite sequence of independent, identically distributed, uniform random points in $[0, 1]$. Starting with $S = \{0\}$, the elements x_k join S one by one, in order. When an entering element is larger than the current minimum element of S, this minimum leaves S. Let $S(1, n)$ denote the content of S after the first n elements x_k join. Simulations suggest that the size $|S(1, n)|$ of S at time n is typically close to n/e. Here we first give a rigorous proof that this is indeed the case, and that in fact the symmetric difference of $S(1, n)$ and the set $\{x_k \geq 1 - 1/e : 1 \leq k \leq n\}$ is of size at most $\tilde{O}(\sqrt{n})$ with high probability. Our main result is a more accurate description of the process implying, in particular, that as n tends to infinity $n^{-1/2}(|S(1, n)| - n/e)$ converges to a normal random variable with variance $3e^{-2} - e^{-1}$. We further show that the dynamics of the symmetric difference of $S(1, n)$ and the set $\{x_k \geq 1 - 1/e : 1 \leq k \leq n\}$ converges with proper scaling to a three-dimensional Bessel process.

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A FEASIBLE CENTRAL LIMIT THEOREM FOR REALISED COVARIATION OF SPDES IN THE CONTEXT OF FUNCTIONAL DATA

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This article establishes an asymptotic theory for volatility estimation in an infinite-dimensional setting. We consider mild solutions of semilinear stochastic partial differential equations and derive a stable central limit theorem for the *semigroup-adjusted realised covariation* (*SARCV*), which is a consistent estimator of the integrated volatility and a generalisation of the realised quadratic covariation to Hilbert spaces. Moreover, we introduce *semigroup-adjusted multipower variations* (*SAMPV*) and establish their weak law of large numbers; using *SAMPV*, we construct a consistent estimator of the asymptotic covariance of the mixed-Gaussian limiting process appearing in the central limit theorem for the SARCV, resulting in a feasible asymptotic theory. Finally, we outline how our results can be applied even if observations are only available on a discrete space-time grid.

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AN ENTROPIC APPROACH FOR HAMILTONIAN MONTE CARLO: THE IDEALIZED CASE

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Quantitative long-time entropic convergence and short-time regularization are established for an idealized Hamiltonian Monte Carlo chain which alternatively follows an Hamiltonian dynamics for a fixed time and then partially or totally refreshes its velocity with an auto-regressive Gaussian step. These results, in discrete time, are the analogues of similar results for the continuous-time kinetic Langevin diffusion, and the latter can be obtained from our bounds in a suitable limit regime. The dependency in the log-Sobolev constant of the target measure is sharp and is illustrated on a mean-field case and on a low-temperature regime, with an application to the simulated annealing algorithm. The practical unadjusted algorithm is briefly discussed.

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DISTRIBUTION OF THE NUMBER OF PIVOTS NEEDED USING GAUSSIAN ELIMINATION WITH PARTIAL PIVOTING ON RANDOM MATRICES

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Gaussian elimination with partial pivoting (GEPP) is a widely used method to solve dense linear systems. Each GEPP step uses a row transposition pivot movement if needed to ensure the leading pivot entry is maximal in magnitude for the leading column of the remaining untriangularized subsystem. We will use theoretical and numerical approaches to study how often this pivot movement is needed. We provide full distributional descriptions for the number of pivot movements needed using GEPP using particular Haar random ensembles as well as compare these models to other common transformations from randomized numerical linear algebra. Additionally, we introduce new random ensembles with fixed pivot movement counts and fixed sparsity, α. Experiments estimating the empirical spectral density (ESD) of these random ensembles leads to a new conjecture on a universality class of random matrices with fixed sparsity whose scaled ESD converges to a measure on the complex unit disk that depends on α and is an interpolation of the uniform measure on the unit disk and the Dirac measure at the origin.

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AN EXPLICIT MILSTEIN-TYPE SCHEME FOR INTERACTING PARTICLE SYSTEMS AND MCKEAN–VLASOV SDES WITH COMMON NOISE AND NON-DIFFERENTIABLE DRIFT COEFFICIENTS

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We propose an explicit drift-randomised Milstein scheme for both McKean–Vlasov stochastic differential equations and associated high-dimensional interacting particle systems with common noise. By using a drift randomisation step in space and measure, we establish the scheme's strong convergence rate of 1 under reduced regularity assumptions on the drift coefficient: no classical (Euclidean) derivatives in space or measure derivatives (e.g., Lions/Fréchet) are required. The main result is established by enriching the concepts of bistability and consistency of numerical schemes used previously for standard SDE. We introduce certain Spijker-type norms (and associated Banach spaces) to deal with the interaction of particles present in the stochastic systems being analysed. A discussion of the scheme's complexity is provided.

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THE STEFAN PROBLEM AND FREE TARGETS OF OPTIMAL BROWNIAN MARTINGALE TRANSPORT

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We formulate and solve a free target optimal Brownian stopping problem from a given distribution while the target distribution is free and is conditioned to satisfy a given density height constraint. The free target optimization problem exhibits monotonicity, from which a remarkable universality follows, in the sense that the optimal target is independent of its Lagrangian cost type. In particular, the solutions to this optimization problem generate solutions to both unstable and stable type of the Stefan problem, where the former stands for freezing of supercooled fluid (St_1) and the latter for ice melting (St_2). This unified approach to both types of the Stefan problem is new. In particular we obtain global-time existence and weak-strong uniqueness for the ill-posed freezing problem (St_1), for a given initial data and for a well-prepared class of initial domains generated from the initial data.

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THE WIRED MINIMAL SPANNING FOREST ON THE POISSON-WEIGHTED INFINITE TREE

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We study the spectral and diffusive properties of the wired minimal spanning forest (WMSF) on the Poisson-weighted infinite tree (PWIT). Let M be the tree containing the root in the WMSF on the PWIT and $(Y_n)_{n\geq 0}$ be a simple random walk on M starting from the root. We show that almost surely M has $\mathbb{P}[Y_{2n} = Y_0] = n^{-3/4+o(1)}$ and $\mathrm{dist}(Y_0, Y_n) = n^{1/4+o(1)}$ with high probability. That is, the spectral dimension of M is $3/2$ and its typical displacement exponent is $1/4$, almost surely. These confirm Addario–Berry's predictions (Addario-Berry (2013)).

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LIMIT DISTRIBUTION THEORY FOR SMOOTH *p*-WASSERSTEIN DISTANCES

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The Wasserstein distance is a metric on a space of probability measures that has seen a surge of applications in statistics, machine learning, and applied mathematics. However, statistical aspects of Wasserstein distances are bottlenecked by the curse of dimensionality, whereby the number of data points needed to accurately estimate them grows exponentially with dimension. Gaussian smoothing was recently introduced as a means to alleviate the curse of dimensionality, giving rise to a parametric convergence rate in any dimension, while preserving the Wasserstein metric and topological structure. To facilitate valid statistical inference, in this work, we develop a comprehensive limit distribution theory for the empirical smooth Wasserstein distance. The limit distribution results leverage the functional delta method after embedding the domain of the Wasserstein distance into a certain dual Sobolev space, characterizing its Hadamard directional derivative for the dual Sobolev norm, and establishing weak convergence of the smooth empirical process in the dual space. To estimate the distributional limits, we also establish consistency of the nonparametric bootstrap. Finally, we use the limit distribution theory to study applications to generative modeling via minimum distance estimation with the smooth Wasserstein distance, showing asymptotic normality of optimal solutions for the quadratic cost.

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CONVERGENCE OF SPACE-DISCRETISED GKPZ VIA REGULARITY STRUCTURES

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In this work, we show a convergence result for the discrete formulation of the generalised KPZ equation $\partial_t u = (\Delta u) + g(u)(\nabla u)^2 + k(\nabla u) + h(u) + f(u)\xi_t(x)$, where ξ is real-valued, Δ is the discrete Laplacian, and ∇ is a discrete gradient, without fixing the spatial dimension. Our convergence result is established within the discrete regularity structures introduced by Hairer and Erhard (*Ann. Inst. Henri Poincaré Probab. Stat.* **55** (2019) 2209–2248). We extend with new ideas the convergence result found in (*Comm. Pure Appl. Math.* **77** (2024) 1065–1125) that deals with a discrete form of the parabolic Anderson model driven by a (rescaled) symmetric simple exclusion process. This is the first time that a discrete generalised KPZ equation is treated and it is a major step toward a general convergence result that will cover a large family of discrete models.

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