ISSN 1050-5164 (print)
ISSN 2168-8737 (online)

THE ANNALS *of* APPLIED PROBABILITY

AN OFFICIAL JOURNAL OF THE
INSTITUTE OF MATHEMATICAL STATISTICS

Articles

Vol. 34, No. 5—October 2024

THE ANNALS OF APPLIED PROBABILITY Vol. 34, No. 5, pp. 4193–4997 October 2024

INSTITUTE OF MATHEMATICAL STATISTICS

(Organized September 12, 1935)

The purpose of the Institute is to foster the development and dissemination of the theory and applications of statistics and probability.

***The Annals of Applied Probability* [ISSN 1050-5164 (print); ISSN 2168-8737 (online)],** Volume 34, Number 5, October 2024. Published bimonthly by the Institute of Mathematical Statistics, 9760 Smith Road, Waite Hill, Ohio 44094, USA. Periodicals postage paid at Cleveland, Ohio, and at additional mailing offices.

POSTMASTER: Send address changes to *The Annals of Applied Probability*, Institute of Mathematical Statistics, Dues and Subscriptions Office, PO Box 729, Middletown, Maryland 21769, USA.

The Annals of Applied Probability
2024, Vol. 34, No. 5, 4193–4236
https://doi.org/10.1214/24-AAP2063

ON A STIFF PROBLEM IN TWO-DIMENSIONAL SPACE

BY LIPING LI[1,a] AND WENJIE SUN[2,b]

[1]*School of Mathematical Sciences, Fudan University,* [a]*liliping@fudan.edu.cn*
[2]*School of Mathematical Sciences, Key Laboratory of Intelligent Computing and Applications (Ministry of Education), Tongji University,* [b]*wjsun@tongji.edu.cn*

In this paper we study a stiff problem in two-dimensional space and especially characterize its probabilistic counterpart. More specifically, consider the heat equation with a parameter $\varepsilon > 0$:

$$\partial_t u^\varepsilon(t,x) = \frac{1}{2}\nabla \cdot \big(\mathbf{A}_\varepsilon(x)\nabla u^\varepsilon(t,x)\big), \quad t \geq 0, x \in \mathbb{R}^2,$$

where $\mathbf{A}_\varepsilon(x) := \mathrm{Id}_2$, the identity matrix, for $x \notin \Omega_\varepsilon := \{x = (x_1, x_2) \in \mathbb{R}^2 : |x_2| < \varepsilon\}$ and

$$\mathbf{A}_\varepsilon(x) := \begin{pmatrix} a_\varepsilon^{-} & 0 \\ 0 & a_\varepsilon^{|} \end{pmatrix}, \quad x \in \Omega_\varepsilon$$

with two constants $a_\varepsilon^{-}, a_\varepsilon^{|} > 0$. The solution u^ε is usually called a flux. Then the stiff problem is concerned with the existence and characterization of the limit u, called the limiting flux, of u^ε as $\varepsilon \downarrow 0$ in a certain sense. Note that there exists a diffusion process X^ε on $\mathbb{R}^2$ associated to this heat equation in the sense that $u^\varepsilon(t,x) := \mathbf{E}^x u^\varepsilon(0, X_t^\varepsilon)$ is its unique solution. The main result of this paper figures out the limiting process of X^ε as $\varepsilon \downarrow 0$ for all possible cases. As a byproduct, the limiting flux u in an L^2-sense and several boundary conditions on the x_1-axis satisfied by u regarding various cases will be further obtained.

REFERENCES

[1] ADAMS, R. A. and FOURNIER, J. J. F. (2003). *Sobolev Spaces*, 2nd ed. *Pure and Applied Mathematics* (*Amsterdam*) **140**. Elsevier/Academic Press, Amsterdam. MR2424078

[2] BAHOURI, H., CHEMIN, J.-Y. and DANCHIN, R. (2011). *Fourier Analysis and Nonlinear Partial Differential Equations*. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **343**. Springer, Heidelberg. MR2768550 https://doi.org/10.1007/978-3-642-16830-7

[3] CHEN, X., POND, C. and WANG, X. (2012). Effective boundary conditions resulting from anisotropic and optimally aligned coatings: The two dimensional case. *Arch. Ration. Mech. Anal.* **206** 911–951. MR2989447 https://doi.org/10.1007/s00205-012-0547-y

[4] CHEN, Z.-Q. and FUKUSHIMA, M. (2012). *Symmetric Markov Processes, Time Change, and Boundary Theory*. *London Mathematical Society Monographs Series* **35**. Princeton Univ. Press, Princeton, NJ. MR2849840

[5] CHEN, Z.-Q., FUKUSHIMA, M. and YING, J. (2006). Traces of symmetric Markov processes and their characterizations. *Ann. Probab.* **34** 1052–1102. MR2243879 https://doi.org/10.1214/009117905000000657

[6] FUKUSHIMA, M. (1980). *Dirichlet Forms and Markov Processes*. *North-Holland Mathematical Library* **23**. North-Holland, Amsterdam–New York. MR0569058

[7] FUKUSHIMA, M., OSHIMA, Y. and TAKEDA, M. (2011). *Dirichlet Forms and Symmetric Markov Processes*, extended ed. *De Gruyter Studies in Mathematics* **19**. de Gruyter, Berlin. MR2778606

[8] JONSSON, A. and WALLIN, H. (1978). A Whitney extension theorem in L_p and Besov spaces. *Ann. Inst. Fourier* (*Grenoble*) **28** 139–192. MR0500920

[9] LEJAY, A. (2016). The snapping out Brownian motion. *Ann. Appl. Probab.* **26** 1727–1742. MR3513604 https://doi.org/10.1214/15-AAP1131

[10] LI, H. and WANG, X. (2017). Using effective boundary conditions to model fast diffusion on a road in a large field. *Nonlinearity* **30** 3853–3894. MR3705138 https://doi.org/10.1088/1361-6544/aa82ee

MSC2020 subject classifications. Primary 60J46, 31C25; secondary 60J60.
Key words and phrases. Stiff problems, boundary conditions, Dirichlet forms, Mosco convergence.

[11] LI, H. and WANG, X. (2020). Effective boundary conditions for the heat equation with interior inclusion. *Commun. Math. Res.* **36** 272–295. MR4199956 https://doi.org/10.4208/cmr.2020-0012

[12] LI, L. and SUN, W. (2020). On stiff problems via Dirichlet forms. *Ann. Inst. Henri Poincaré Probab. Stat.* **56** 2051–2080. MR4116717 https://doi.org/10.1214/19-AIHP1028

[13] LI, L. and SUN, W. (2022). On trace of Brownian motion on the boundary of a strip. *Illinois J. Math.* **66** 271–287. MR4484221 https://doi.org/10.1215/00192082-9853356

[14] MAZ'JA, V. G. (1985). *Sobolev Spaces*. *Springer Series in Soviet Mathematics*. Springer, Berlin. MR0817985 https://doi.org/10.1007/978-3-662-09922-3

[15] MCLEAN, W. (2000). *Strongly Elliptic Systems and Boundary Integral Equations*. Cambridge Univ. Press, Cambridge. MR1742312

[16] MOSCO, U. (1994). Composite media and asymptotic Dirichlet forms. *J. Funct. Anal.* **123** 368–421. MR1283033 https://doi.org/10.1006/jfan.1994.1093

[17] ÔKURA, H. (1989). Recurrence criteria for skew products of symmetric Markov processes. *Forum Math.* **1** 331–357. MR1016677 https://doi.org/10.1515/form.1989.1.331

[18] REVUZ, D. and YOR, M. (1999). *Continuous Martingales and Brownian Motion*, 3rd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **293**. Springer, Berlin. MR1725357 https://doi.org/10.1007/978-3-662-06400-9

[19] SÁNCHEZ-PALENCIA, E. (1980). *Nonhomogeneous Media and Vibration Theory*. *Lecture Notes in Physics* **127**. Springer, Berlin. MR0578345

[20] WIDDER, D. V. (1961). Functions harmonic in a strip. *Proc. Amer. Math. Soc.* **12** 67–72. MR0132838 https://doi.org/10.2307/2034126

The Annals of Applied Probability
2024, Vol. 34, No. 5, 4237–4267
https://doi.org/10.1214/24-AAP2064

FROM FINITE POPULATION OPTIMAL STOPPING TO MEAN FIELD OPTIMAL STOPPING

BY MEHDI TALBI[1,a], NIZAR TOUZI[2,b] AND JIANFENG ZHANG[3,c]

[1]*Laboratoire de Probabilités, Statistiques et Modélisation, Université Paris-Cité,* [a]*talbi@lpsm.paris*
[2]*Tandon School of Engineering, New York University,* [b]*nt2635@nyu.edu*
[3]*Department of Mathematics, University of Southern California,* [c]*jianfenz@usc.edu*

This paper analyzes the convergence of the finite population optimal stopping problem towards the corresponding mean field limit. Building on the viscosity solution characterization of the mean field optimal stopping problem of our previous papers (*SIAM J. Control Optim.* **61** (2023) 1712–1736, 2140–2164), we prove the convergence of the value functions by adapting the Barles–Souganidis (*Asymptot. Anal.* **4** (1991) 271–283) monotone scheme method to our context. We next characterize the optimal stopping policies of the mean field problem by the accumulation points of the finite population optimal stopping strategies. In particular, if the limiting problem has a unique optimal stopping policy, then the finite population optimal stopping strategies do converge towards this solution. As a by-product of our analysis, we provide an extension of the standard propagation of chaos to the context of stopped McKean–Vlasov diffusions.

REFERENCES

[1] BARLES, G. and SOUGANIDIS, P. E. (1991). Convergence of approximation schemes for fully nonlinear second order equations. *Asymptot. Anal.* **4** 271–283. MR1115933

[2] BAYRAKTAR, E., CECCHIN, A. and CHAKRABORTY, P. (2023). Mean field control and finite agent approximation for regime-switching jump diffusions. *Appl. Math. Optim.* **88** Paper No. 36, 35. MR4595996 https://doi.org/10.1007/s00245-023-10015-3

[3] BAYRAKTAR, E., CECCHIN, A., COHEN, A. and DELARUE, F. (2022). Finite state mean field games with Wright–Fisher common noise as limits of N-player weighted games. *Math. Oper. Res.* **47** 2840–2890. MR4515487 https://doi.org/10.1287/moor.2021.1230

[4] BAYRAKTAR, E. and COHEN, A. (2018). Analysis of a finite state many player game using its master equation. *SIAM J. Control Optim.* **56** 3538–3568. MR3860894 https://doi.org/10.1137/17M113887X

[5] BILLINGSLEY, P. (1968). *Convergence of Probability Measures*. Wiley, New York. MR0233396

[6] CARDALIAGUET, P. (2017). The convergence problem in mean field games with local coupling. *Appl. Math. Optim.* **76** 177–215. MR3679342 https://doi.org/10.1007/s00245-017-9434-0

[7] CARDALIAGUET, P., DAUDIN, S., JACKSON, J. and SOUGANIDIS, P. E. (2023). An algebraic convergence rate for the optimal control of McKean–Vlasov dynamics. *SIAM J. Control Optim.* **61** 3341–3369. MR4665660 https://doi.org/10.1137/22M1486789

[8] CARDALIAGUET, P., DELARUE, F., LASRY, J.-M. and LIONS, P.-L. (2019). *The Master Equation and the Convergence Problem in Mean Field Games*. *Annals of Mathematics Studies* **201**. Princeton Univ. Press, Princeton, NJ. MR3967062 https://doi.org/10.2307/j.ctvckq7qf

[9] CARDALIAGUET, P. and SOUGANIDIS, P. E. (2023). Regularity of the value function and quantitative propagation of chaos for mean field control problems. *NoDEA Nonlinear Differential Equations Appl.* **30** Paper No. 25, 37. MR4530369 https://doi.org/10.1007/s00030-022-00823-x

[10] CARMONA, R. and DELARUE, F. (2013). Probabilistic analysis of mean-field games. *SIAM J. Control Optim.* **51** 2705–2734. MR3072222 https://doi.org/10.1137/120883499

[11] CARMONA, R. and DELARUE, F. (2018). *Probabilistic Theory of Mean Field Games with Applications II: Mean Field Games with Common Noise and Master Equations*. *Probability Theory and Stochastic Modelling* **84**. Springer, Cham. MR3753660

MSC2020 subject classifications. Primary 60G40, 35Q89; secondary 49N80, 49L25, 65K15.
Key words and phrases. Mean field optimal stopping, obstacle problems, viscosity solutions, propagation of chaos.

[12] CARMONA, R., DELARUE, F. and LACKER, D. (2017). Mean field games of timing and models for bank runs. *Appl. Math. Optim.* **76** 217–260. MR3679343 https://doi.org/10.1007/s00245-017-9435-z
[13] CAVAGNARI, G., LISINI, S., ORRIERI, C. and SAVARÉ, G. (2022). Lagrangian, Eulerian and Kantorovich formulations of multi-agent optimal control problems: Equivalence and gamma-convergence. *J. Differ. Equ.* **322** 268–364. MR4401975 https://doi.org/10.1016/j.jde.2022.03.019
[14] CECCHIN, A. (2021). Finite state N-agent and mean field control problems. *ESAIM Control Optim. Calc. Var.* **27** Paper No. 31, 33. MR4241175 https://doi.org/10.1051/cocv/2021032
[15] CECCHIN, A. and PELINO, G. (2019). Convergence, fluctuations and large deviations for finite state mean field games via the master equation. *Stochastic Process. Appl.* **129** 4510–4555. MR4013871 https://doi.org/10.1016/j.spa.2018.12.002
[16] CECCHIN, A., PRA, P. D., FISCHER, M. and PELINO, G. (2019). On the convergence problem in mean field games: A two state model without uniqueness. *SIAM J. Control Optim.* **57** 2443–2466. MR3981375 https://doi.org/10.1137/18M1222454
[17] DAUDIN, S., DELARUE, F. and JACKSON, J. (2023). On the optimal rate for the convergence problem in mean field control. Available at arXiv:2305.08423.
[18] DELARUE, F., LACKER, D. and RAMANAN, K. (2019). From the master equation to mean field game limit theory: A central limit theorem. *Electron. J. Probab.* **24** Paper No. 51, 54. MR3954791 https://doi.org/10.1214/19-EJP298
[19] DELARUE, F., LACKER, D. and RAMANAN, K. (2020). From the master equation to mean field game limit theory: Large deviations and concentration of measure. *Ann. Probab.* **48** 211–263. MR4079435 https://doi.org/10.1214/19-AOP1359
[20] DJETE, M. F. (2023). Mean field games of controls: On the convergence of Nash equilibria. *Ann. Appl. Probab.* **33** 2824–2862. MR4612656 https://doi.org/10.1214/22-aap1879
[21] DJETE, M. F. (2023). Large population games with interactions through controls and common noise: Convergence results and equivalence between open-loop and closed-loop controls. *ESAIM Control Optim. Calc. Var.* **29** Paper No. 39, 42. MR4598796 https://doi.org/10.1051/cocv/2023005
[22] DJETE, M. F., POSSAMAÏ, D. and TAN, X. (2022). McKean–Vlasov optimal control: Limit theory and equivalence between different formulations. *Math. Oper. Res.* **47** 2891–2930. MR4515488 https://doi.org/10.1287/moor.2021.1232
[23] DONCEL, J., GAST, N. and GAUJAL, B. (2019). Discrete mean field games: Existence of equilibria and convergence. *J. Dyn. Games* **6** 221–239. MR4026158 https://doi.org/10.3934/jdg.2019016
[24] EKREN, I. (2017). Viscosity solutions of obstacle problems for fully nonlinear path-dependent PDEs. *Stochastic Process. Appl.* **127** 3966–3996. MR3718103 https://doi.org/10.1016/j.spa.2017.03.016
[25] EKREN, I., KELLER, C., TOUZI, N. and ZHANG, J. (2014). On viscosity solutions of path dependent PDEs. *Ann. Probab.* **42** 204–236. MR3161485 https://doi.org/10.1214/12-AOP788
[26] EKREN, I., TOUZI, N. and ZHANG, J. (2016). Viscosity solutions of fully nonlinear parabolic path dependent PDEs: Part I. *Ann. Probab.* **44** 1212–1253. MR3474470 https://doi.org/10.1214/14-AOP999
[27] EKREN, I., TOUZI, N. and ZHANG, J. (2016). Viscosity solutions of fully nonlinear parabolic path dependent PDEs: Part II. *Ann. Probab.* **44** 2507–2553. MR3531674 https://doi.org/10.1214/15-AOP1027
[28] FELEQI, E. (2013). The derivation of ergodic mean field game equations for several populations of players. *Dyn. Games Appl.* **3** 523–536. MR3127148 https://doi.org/10.1007/s13235-013-0088-5
[29] FISCHER, M. (2017). On the connection between symmetric N-player games and mean field games. *Ann. Appl. Probab.* **27** 757–810. MR3655853 https://doi.org/10.1214/16-AAP1215
[30] FISCHER, M. and LIVIERI, G. (2016). Continuous time mean-variance portfolio optimization through the mean field approach. *ESAIM Probab. Stat.* **20** 30–44. MR3528616 https://doi.org/10.1051/ps/2016001
[31] FISCHER, M. and SILVA, F. J. (2021). On the asymptotic nature of first order mean field games. *Appl. Math. Optim.* **84** 2327–2357. MR4304905 https://doi.org/10.1007/s00245-020-09711-1
[32] FORNASIER, M., LISINI, S., ORRIERI, C. and SAVARÉ, G. (2019). Mean-field optimal control as gamma-limit of finite agent controls. *European J. Appl. Math.* **30** 1153–1186. MR4028474 https://doi.org/10.1017/s0956792519000044
[33] FOURNIER, N. and GUILLIN, A. (2015). On the rate of convergence in Wasserstein distance of the empirical measure. *Probab. Theory Related Fields* **162** 707–738. MR3383341 https://doi.org/10.1007/s00440-014-0583-7
[34] GANGBO, W., MAYORGA, S. and SWIECH, A. (2021). Finite dimensional approximations of Hamilton–Jacobi–Bellman equations in spaces of probability measures. *SIAM J. Math. Anal.* **53** 1320–1356. MR4226237 https://doi.org/10.1137/20M1331135
[35] GANGBO, W. and MÉSZÁROS, A. R. (2022). Global well-posedness of master equations for deterministic displacement convex potential mean field games. *Comm. Pure Appl. Math.* **75** 2685–2801. MR4509653
[36] GERMAIN, M., PHAM, H. and WARIN, X. (2022). Rate of convergence for particle approximation of PDEs in Wasserstein space. *J. Appl. Probab.* **59** 992–1008. MR4507678 https://doi.org/10.1017/jpr.2021.102

[37] HUANG, M., MALHAMÉ, R. P. and CAINES, P. E. (2006). Large population stochastic dynamic games: Closed-loop McKean–Vlasov systems and the Nash certainty equivalence principle. *Commun. Inf. Syst.* **6** 221–251. MR2346927

[38] ISERI, M. and ZHANG, J. (2021). Set values for mean field games. Available at arXiv:2107.01661.

[39] KOBYLANSKI, M., QUENEZ, M.-C. and ROUY-MIRONESCU, E. (2011). Optimal multiple stopping time problem. *Ann. Appl. Probab.* **21** 1365–1399. MR2857451 https://doi.org/10.1214/10-AAP727

[40] LACKER, D. (2015). Mean field games via controlled martingale problems: Existence of Markovian equilibria. *Stochastic Process. Appl.* **125** 2856–2894. MR3332857 https://doi.org/10.1016/j.spa.2015.02.006

[41] LACKER, D. (2016). A general characterization of the mean field limit for stochastic differential games. *Probab. Theory Related Fields* **165** 581–648. MR3520014 https://doi.org/10.1007/s00440-015-0641-9

[42] LACKER, D. (2017). Limit theory for controlled McKean–Vlasov dynamics. *SIAM J. Control Optim.* **55** 1641–1672. MR3654119 https://doi.org/10.1137/16M1095895

[43] LACKER, D. (2020). On the convergence of closed-loop Nash equilibria to the mean field game limit. *Ann. Appl. Probab.* **30** 1693–1761. MR4133381 https://doi.org/10.1214/19-AAP1541

[44] LACKER, D. and LE FLEM, L. (2023). Closed-loop convergence for mean field games with common noise. *Ann. Appl. Probab.* **33** 2681–2733. MR4612653 https://doi.org/10.1214/22-aap1876

[45] LASRY, J.-M. and LIONS, P.-L. (2007). Mean field games. *Jpn. J. Math.* **2** 229–260. MR2295621 https://doi.org/10.1007/s11537-007-0657-8

[46] LAURIÈRE, M. and TANGPI, L. (2022). Convergence of large population games to mean field games with interaction through the controls. *SIAM J. Math. Anal.* **54** 3535–3574. MR4438040 https://doi.org/10.1137/22M1469328

[47] MOU, C. and ZHANG, J. (2021). Wellposedness of second order master equations for mean field games with nonsmooth data. *Memoirs of the AMS*.

[48] NUTZ, M., SAN MARTIN, J. and TAN, X. (2020). Convergence to the mean field game limit: A case study. *Ann. Appl. Probab.* **30** 259–286. MR4068311 https://doi.org/10.1214/19-AAP1501

[49] OELSCHLÄGER, K. (1984). A martingale approach to the law of large numbers for weakly interacting stochastic processes. *Ann. Probab.* **12** 458–479. MR0735849

[50] POSSAMAÏ, D. (2021). Non-asymptotic convergence rates for mean-field games: Weak formulation and McKean–Vlasov BSDEs. Available at arXiv:2105.00484.

[51] REN, Z., TOUZI, N. and ZHANG, J. (2014). An overview of viscosity solutions of path-dependent PDEs. In *Stochastic Analysis and Applications* 2014. *Springer Proc. Math. Stat.* **100** 397–453. Springer, Cham. MR3332721 https://doi.org/10.1007/978-3-319-11292-3_15

[52] SZNITMAN, A.-S. (1991). Topics in propagation of chaos. In *École D'Été de Probabilités de Saint-Flour XIX*—1989. *Lecture Notes in Math.* **1464** 165–251. Springer, Berlin. MR1108185 https://doi.org/10.1007/BFb0085169

[53] TALBI, M. (2022). A finite-dimensional approximation for partial differential equations on Wasserstein space. Available at arXiv:2211.00719.

[54] TALBI, M., TOUZI, N. and ZHANG, J. (2023). Dynamic programming equation for the mean field optimal stopping problem. *SIAM J. Control Optim.* **61** 2140–2164. MR4613226 https://doi.org/10.1137/21M1404259

[55] TALBI, M., TOUZI, N. and ZHANG, J. (2023). Viscosity solutions for obstacle problems on Wasserstein space. *SIAM J. Control Optim.* **61** 1712–1736. MR4604196 https://doi.org/10.1137/22M1488119

[56] VILLANI, C. (2003). *Topics in Optimal Transportation*. *Graduate Studies in Mathematics* **58**. Amer. Math. Soc., Providence, RI. MR1964483 https://doi.org/10.1090/gsm/058

The Annals of Applied Probability
2024, Vol. 34, No. 5, 4268–4309
https://doi.org/10.1214/24-AAP2065

QUASILINEAR ROUGH EVOLUTION EQUATIONS

BY ANTOINE HOCQUET[1,a] AND ALEXANDRA NEAMŢU[2,b]

[1]*Institute of Mathematics, Technische Universität Berlin,* [a]*antoine.hocquet86@gmail.com*
[2]*Department of Mathematics and Statistics, Universität Konstanz,* [b]*alexandra.neamtu@uni-konstanz.de*

We investigate the abstract Cauchy problem for a quasilinear parabolic equation in a Banach space of the form $du_t - L_t(u_t)u_t\,dt = N_t(u_t)\,dt + F(u_t)\cdot d\mathbf{X}_t$, where $\mathbf{X}$ is a γ-Hölder rough path for $\gamma \in (1/3, 1/2)$. We explore the mild formulation that combines functional analysis techniques and controlled rough paths theory which entail the local well-posedness of such equations. We apply our results to the stochastic Landau–Lifshitz–Gilbert and Shigesada–Kawasaki–Teramoto equation.

REFERENCES

[1] AGRESTI, A. and VERAAR, M. (2020). Stability properties of stochastic maximal L^p-regularity. *J. Math. Anal. Appl.* **482** 123553, 35. MR4016507 https://doi.org/10.1016/j.jmaa.2019.123553

[2] AGRESTI, A. and VERAAR, M. (2022). Nonlinear parabolic stochastic evolution equations in critical spaces part I. Stochastic maximal regularity and local existence. *Nonlinearity* **35** 4100–4210. MR4459102

[3] ALÒS, E., LEÓN, J. A. and NUALART, D. (1999). Stochastic heat equation with random coefficients. *Probab. Theory Related Fields* **115** 41–94. MR1715545 https://doi.org/10.1007/s004400050236

[4] AMANN, H. (1985). Global existence for semilinear parabolic systems. *J. Reine Angew. Math.* **360** 47–83. MR0799657 https://doi.org/10.1515/crll.1985.360.47

[5] AMANN, H. (1986). Quasilinear evolution equations and parabolic systems. *Trans. Amer. Math. Soc.* **293** 191–227. MR0814920 https://doi.org/10.2307/2000279

[6] AMANN, H. (1995). *Linear and Quasilinear Parabolic Problems. Vol. I: Abstract Linear Theory. Monographs in Mathematics* **89**. Birkhäuser, Boston, MA. MR1345385 https://doi.org/10.1007/978-3-0348-9221-6

[7] ARNOLD, L. (2003). *Random Dynamical Systems*. Springer, Berlin.

[8] BAILLEUL, I., DEBUSSCHE, A. and HOFMANOVÁ, M. (2019). Quasilinear generalized parabolic Anderson model equation. *Stoch. Partial Differ. Equ. Anal. Comput.* **7** 40–63. MR3916262 https://doi.org/10.1007/s40072-018-0121-1

[9] BAILLEUL, I., RIEDEL, S. and SCHEUTZOW, M. (2017). Random dynamical systems, rough paths and rough flows. *J. Differ. Equ.* **262** 5792–5823. MR3624539 https://doi.org/10.1016/j.jde.2017.02.014

[10] BONNEFOI, T., CHANDRA, A., MOINAT, A. and WEBER, H. (2022). A priori bounds for rough differential equations with a non-linear damping term. *J. Differ. Equ.* **318** 58–93. MR4385876 https://doi.org/10.1016/j.jde.2022.02.006

[11] BRZEŹNIAK, Z., GOLDYS, B. and JEGARAJ, T. (2013). Weak solutions of a stochastic Landau–Lifshitz–Gilbert equation. *Appl. Math. Res. Express. AMRX* **2013** 1–33. MR3040886 https://doi.org/10.1093/amrx/abs009

[12] BRZEŹNIAK, Z., GOLDYS, B. and JEGARAJ, T. (2017). Large deviations and transitions between equilibria for stochastic Landau–Lifshitz–Gilbert equation. *Arch. Ration. Mech. Anal.* **226** 497–558. MR3687875 https://doi.org/10.1007/s00205-017-1117-0

[13] CARUANA, M., FRIZ, P. K. and OBERHAUSER, H. (2011). A (rough) pathwise approach to a class of non-linear stochastic partial differential equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **28** 27–46. MR2765508 https://doi.org/10.1016/j.anihpc.2010.11.002

[14] DENK, R., HIEBER, M. and PRÜSS, J. (2003). $\mathcal{R}$-boundedness, Fourier multipliers and problems of elliptic and parabolic type. *Mem. Amer. Math. Soc.* **166** viii+114. MR2006641 https://doi.org/10.1090/memo/0788

MSC2020 subject classifications. 60XXX, 60L50, 60L20, 60H15.
Key words and phrases. Quasilinear evolution equations, controlled rough paths, Landau–Lifshitz–Gilbert equation.

[15] DEYA, A., GUBINELLI, M., HOFMANOVÁ, M. and TINDEL, S. (2019). A priori estimates for rough PDEs with application to rough conservation laws. *J. Funct. Anal.* **276** 3577–3645. MR3957994 https://doi.org/10.1016/j.jfa.2019.03.008

[16] DEYA, A., GUBINELLI, M. and TINDEL, S. (2012). Non-linear rough heat equations. *Probab. Theory Related Fields* **153** 97–147. MR2925571 https://doi.org/10.1007/s00440-011-0341-z

[17] DHARIWAL, G., HUBER, F. and NEAMŢU, A. (2021). On the equivalence of pathwise mild and weak solutions for quasilinear SPDEs. *Stoch. Anal. Appl.* **39** 898–925. MR4310351 https://doi.org/10.1080/07362994.2020.1857268

[18] FAHIM, K., HAUSENBLAS, E. and MUKHERJEE, D. (2021). Wong–Zakai approximation for Landau–Lifshitz–Gilbert equation driven by geometric rough paths. *Appl. Math. Optim.* **84** S1685–S1730. MR4356908 https://doi.org/10.1007/s00245-021-09808-1

[19] FRIZ, P. K. and HAIRER, M. (2020). *A Course on Rough Paths: With an Introduction to Regularity Structures*. *Universitext*. Springer, Cham. MR4174393 https://doi.org/10.1007/978-3-030-41556-3

[20] FURLAN, M. and GUBINELLI, M. (2019). Paracontrolled quasilinear SPDEs. *Ann. Probab.* **47** 1096–1135. MR3916943 https://doi.org/10.1214/18-AOP1280

[21] GERASIMOVIČS, A. and HAIRER, M. (2019). Hörmander's theorem for semilinear SPDEs. *Electron. J. Probab.* **24** 1–56. MR4040992 https://doi.org/10.1214/19-ejp387

[22] GERASIMOVIČS, A., HOCQUET, A. and NILSSEN, T. (2021). Non-autonomous rough semilinear PDEs and the multiplicative sewing lemma. *J. Funct. Anal.* **281** Paper No. 109200, 65. MR4299812 https://doi.org/10.1016/j.jfa.2021.109200

[23] GERENCSÉR, M. and HAIRER, M. (2019). A solution theory for quasilinear singular SPDEs. *Comm. Pure Appl. Math.* **72** 1983–2005. MR3987723 https://doi.org/10.1002/cpa.21816

[24] GUBINELLI, M. (2004). Controlling rough paths. *J. Funct. Anal.* **216** 86–140. MR2091358 https://doi.org/10.1016/j.jfa.2004.01.002

[25] GUBINELLI, M. and TINDEL, S. (2010). Rough evolution equations. *Ann. Probab.* **38** 1–75. MR2599193 https://doi.org/10.1214/08-AOP437

[26] GUSSETTI, E. and HOCQUET, A. (2023). A pathwise stochastic Landau–Lifshitz–Gilbert equation with application to large deviations. *J. Funct. Anal.* **285** Paper No. 110094, 79. MR4620723 https://doi.org/10.1016/j.jfa.2023.110094

[27] HESSE, R. and NEAMŢU, A. (2019). Local mild solutions for rough stochastic partial differential equations. *J. Differ. Equ.* **267** 6480–6538. MR4001062 https://doi.org/10.1016/j.jde.2019.06.026

[28] HESSE, R. and NEAMŢU, A. (2020). Global solutions and random dynamical systems for rough evolution equations. *Discrete Contin. Dyn. Syst. Ser. B* **25** 2723–2748. MR4097587 https://doi.org/10.3934/dcdsb.2020029

[29] HESSE, R. and NEAMŢU, A. (2022). Global solutions for semilinear rough partial differential equations. *Stoch. Dyn.* **22** Paper No. 2240011, 18. MR4431448 https://doi.org/10.1142/S0219493722400111

[30] HOCQUET, A. (2021). Quasilinear rough partial differential equations with transport noise. *J. Differ. Equ.* **276** 43–95. MR4191349 https://doi.org/10.1016/j.jde.2020.12.011

[31] HOCQUET, A. and HOFMANOVÁ, M. (2018). An energy method for rough partial differential equations. *J. Differ. Equ.* **265** 1407–1466. MR3797622 https://doi.org/10.1016/j.jde.2018.04.006

[32] HOCQUET, A. and NILSSEN, T. (2021). An Itô formula for rough partial differential equations and some applications. *Potential Anal.* **54** 331–386. MR4202743 https://doi.org/10.1007/s11118-020-09830-y

[33] HOFMANOVÁ, M., LEAHY, J.-M. and NILSSEN, T. (2019). On the Navier–Stokes equation perturbed by rough transport noise. *J. Evol. Equ.* **19** 203–247. MR3918521 https://doi.org/10.1007/s00028-018-0473-z

[34] HOFMANOVÁ, M. and ZHANG, T. (2017). Quasilinear parabolic stochastic partial differential equations: Existence, uniqueness. *Stochastic Process. Appl.* **127** 3354–3371. MR3692318 https://doi.org/10.1016/j.spa.2017.01.010

[35] HORNUNG, L. (2019). Quasilinear parabolic stochastic evolution equations via maximal L^p-regularity. *Potential Anal.* **50** 279–326. MR3905531 https://doi.org/10.1007/s11118-018-9683-9

[36] KUEHN, C. and NEAMŢU, A. (2020). Pathwise mild solutions for quasilinear stochastic partial differential equations. *J. Differ. Equ.* **269** 2185–2227. MR4093727 https://doi.org/10.1016/j.jde.2020.01.032

[37] LEÓN, J. A. and NUALART, D. (1998). Stochastic evolution equations with random generators. *Ann. Probab.* **26** 149–186. MR1617045 https://doi.org/10.1214/aop/1022855415

[38] LIONS, P.-L., PERTHAME, B. and SOUGANIDIS, P. E. (2013). Scalar conservation laws with rough (stochastic) fluxes. *Stoch. Partial Differ. Equ. Anal. Comput.* **1** 664–686. MR3327520 https://doi.org/10.1007/s40072-013-0021-3

[39] LUNARDI, A. (2009). *Interpolation Theory*, 2nd ed. *Appunti. Scuola Normale Superiore di Pisa (Nuova Serie)*. [*Lecture Notes. Scuola Normale Superiore di Pisa (New Series)*]. Edizioni della Normale, Pisa. MR2523200

[40] OTTO, F., SAUER, J. and WEBER, H. (2021). A priori bounds for quasi-linear SPDEs in the full sub-critical regime. Available at: arXiv:2103.11039.

[41] OTTO, F. and WEBER, H. (2019). Quasilinear SPDEs via rough paths. *Arch. Ration. Mech. Anal.* **232** 873–950. MR3925533 https://doi.org/10.1007/s00205-018-01335-8

[42] PRONK, M. and VERAAR, M. (2014). A new approach to stochastic evolution equations with adapted drift. *J. Differ. Equ.* **256** 3634–3683. MR3186842 https://doi.org/10.1016/j.jde.2014.02.014

[43] RUSSO, F. and VALLOIS, P. (1993). Forward, backward and symmetric stochastic integration. *Probab. Theory Related Fields* **97** 403–421. MR1245252 https://doi.org/10.1007/BF01195073

[44] TANABE, H. (1960). On the equations of evolution in a Banach space. *Osaka Math. J.* **12** 363–376. MR0125455

[45] YAGI, A. (2010). *Abstract Parabolic Evolution Equations and Their Applications*. *Springer Monographs in Mathematics*. Springer, Berlin. MR2573296 https://doi.org/10.1007/978-3-642-04631-5

The Annals of Applied Probability
2024, Vol. 34, No. 5, 4310–4341
https://doi.org/10.1214/24-AAP2066

ANALYSIS OF TWO-COMPONENT GIBBS SAMPLERS USING THE THEORY OF TWO PROJECTIONS

BY QIAN QIN[a]

School of Statistics, University of Minnesota, [a]*qqin@umn.edu*

The theory of two projections is utilized to study two-component Gibbs samplers. Through this theory, previously intractable problems regarding the asymptotic variances of two-component Gibbs samplers are reduced to elementary matrix algebra exercises. It is found that in terms of asymptotic variance, the two-component random-scan Gibbs sampler is never much worse, and could be considerably better than its deterministic-scan counterpart, provided that the selection probability is appropriately chosen. This is especially the case when there is a large discrepancy in computation cost between the two components. The result contrasts with the known fact that the deterministic-scan version has a faster convergence rate, which can also be derived from the method herein. On the other hand, a modified version of the deterministic-scan sampler that accounts for computation cost can outperform the random-scan version.

REFERENCES

ALBERT, J. H. and CHIB, S. (1993). Bayesian analysis of binary and polychotomous response data. *J. Amer. Statist. Assoc.* **88** 669–679. MR1224394

AMIT, Y. (1991). On rates of convergence of stochastic relaxation for Gaussian and non-Gaussian distributions. *J. Multivariate Anal.* **38** 82–99. MR1128938 https://doi.org/10.1016/0047-259X(91)90033-X

ANDRIEU, C. (2016). On random- and systematic-scan samplers. *Biometrika* **103** 719–726. MR3551794 https://doi.org/10.1093/biomet/asw019

BADEA, C., GRIVAUX, S. and MÜLLER, V. (2012). The rate of convergence in the method of alternating projections. *St. Petersburg Math. J.* **23** 413–434. MR2896163 https://doi.org/10.1090/S1061-0022-2012-01202-1

BÖTTCHER, A. and SPITKOVSKY, I. M. (2010). A gentle guide to the basics of two projections theory. *Linear Algebra Appl.* **432** 1412–1459. MR2580440 https://doi.org/10.1016/j.laa.2009.11.002

BÖTTCHER, A. and SPITKOVSKY, I. M. (2018). Robert-Sheckley's answerer for two orthogonal projections. In *The Diversity and Beauty of Applied Operator Theory. Oper. Theory Adv. Appl.* **268** 125–138. Birkhäuser, Cham. MR3793301

CASELLA, G. and GEORGE, E. I. (1992). Explaining the Gibbs sampler. *Amer. Statist.* **46** 167–174. MR1183069 https://doi.org/10.2307/2685208

CHLEBICKA, I., ŁATUSZYŃSKI, K. and MIASOJEDOW, B. (2023). Solidarity of Gibbs samplers: The spectral gap. arXiv preprint.

DIACONIS, P., KHARE, K. and SALOFF-COSTE, L. (2010). Stochastic alternating projections. *Illinois J. Math.* **54** 963–979. MR2928343

DOBRUSHIN, R. (1956). Central limit theorem for nonstationary Markov chains. I. *Theory Probab. Appl.* **1** 65–80.

FILL, J. A. (1991). Eigenvalue bounds on convergence to stationarity for nonreversible Markov chains, with an application to the exclusion process. *Ann. Appl. Probab.* **1** 62–87. MR1097464

GELFAND, A. E. and SMITH, A. F. M. (1990). Sampling-based approaches to calculating marginal densities. *J. Amer. Statist. Assoc.* **85** 398–409. MR1141740

GREENWOOD, P. E., MCKEAGUE, I. W. and WEFELMEYER, W. (1998). Information bounds for Gibbs samplers. *Ann. Statist.* **26** 2128–2156. MR1700224 https://doi.org/10.1214/aos/1024691464

HALMOS, P. R. (1969). Two subspaces. *Trans. Amer. Math. Soc.* **144** 381–389. MR0251519 https://doi.org/10.2307/1995288

HELMBERG, G. (2014). *Introduction to Spectral Theory in Hilbert Space. North-Holland Series in Applied Mathematics and Mechanics, Vol.* 6. North-Holland, Amsterdam-London. MR0243367

MSC2020 subject classifications. 60J05.

Key words and phrases. Asymptotic variance, convergence rate, Markov operator, matrix representation, MCMC.

JONES, G. L. (2004). On the Markov chain central limit theorem. *Probab. Surv.* **1** 299–320. MR2068475 https://doi.org/10.1214/154957804100000051

JONES, G. L. and HOBERT, J. P. (2001). Honest exploration of intractable probability distributions via Markov chain Monte Carlo. *Statist. Sci.* **16** 312–334. MR1888447 https://doi.org/10.1214/ss/1015346317

KAYALAR, S. and WEINERT, H. L. (1988). Error bounds for the method of alternating projections. *Math. Control Signals Systems* **1** 43–59. MR0923275 https://doi.org/10.1007/BF02551235

KONTOROVICH, L. and RAMANAN, K. (2008). Concentration inequalities for dependent random variables via the martingale method. *Ann. Probab.* **36** 2126–2158. MR2478678 https://doi.org/10.1214/07-AOP384

KUBRUSLY, C. S. (2012). *Spectral Theory of Operators on Hilbert Spaces*. Birkhäuser, New York. MR2951608 https://doi.org/10.1007/978-0-8176-8328-3

LINK, W. A. and EATON, M. J. (2012). On thinning of chains in MCMC. *Methods Ecol. Evol.* **3** 112–115.

LIU, J. S., WONG, W. H. and KONG, A. (1994). Covariance structure of the Gibbs sampler with applications to the comparisons of estimators and augmentation schemes. *Biometrika* **81** 27–40. MR1279653 https://doi.org/10.1093/biomet/81.1.27

MACEACHERN, S. N. and BERLINER, L. M. (1994). Subsampling the Gibbs sampler. *Amer. Statist.* **48** 188–190.

MAIRE, F., DOUC, R. and OLSSON, J. (2014). Comparison of asymptotic variances of inhomogeneous Markov chains with application to Markov chain Monte Carlo methods. *Ann. Statist.* **42** 1483–1510. MR3262458 https://doi.org/10.1214/14-AOS1209

NISHIO, K. (1985). The structure of a real linear combination of two projections. *Linear Algebra Appl.* **66** 169–176. MR0781299 https://doi.org/10.1016/0024-3795(85)90130-2

PAULIN, D. (2015). Concentration inequalities for Markov chains by Marton couplings and spectral methods. *Electron. J. Probab.* **20** 1–32. MR3383563 https://doi.org/10.1214/EJP.v20-4039

POLSON, N. G., SCOTT, J. G. and WINDLE, J. (2013). Bayesian inference for logistic models using Pólya-Gamma latent variables. *J. Amer. Statist. Assoc.* **108** 1339–1349. MR3174712 https://doi.org/10.1080/01621459.2013.829001

QIN, Q. and HOBERT, J. P. (2021). On the limitations of single-step drift and minorization in Markov chain convergence analysis. *Ann. Appl. Probab.* **31** 1633–1659. MR4312841 https://doi.org/10.1214/20-aap1628

QIN, Q. and JONES, G. L. (2022). Convergence rates of two-component MCMC samplers. *Bernoulli* **28** 859–885. MR4388922 https://doi.org/10.3150/21-bej1369

ROBERTS, G. O. and ROSENTHAL, J. S. (1997). Geometric ergodicity and hybrid Markov chains. *Electron. Commun. Probab.* **2** 13–25. MR1448322 https://doi.org/10.1214/ECP.v2-981

ROBERTS, G. O. and ROSENTHAL, J. S. (2006). Harris recurrence of Metropolis-within-Gibbs and trans-dimensional Markov chains. *Ann. Appl. Probab.* **16** 2123–2139. MR2288716 https://doi.org/10.1214/105051606000000510

ROBERTS, G. O. and ROSENTHAL, J. S. (2016). Surprising convergence properties of some simple Gibbs samplers under various scans. *Int. J. Stat. Probab.* **5** 51–60.

ROBERTS, G. O. and TWEEDIE, R. L. (2001). Geometric L^2 and L^1 convergence are equivalent for reversible Markov chains. *J. Appl. Probab.* **38A** 37–41. MR1915532 https://doi.org/10.1239/jap/1085496589

TANNER, M. A. and WONG, W. H. (1987). The calculation of posterior distributions by data augmentation. *J. Amer. Statist. Assoc.* **82** 528–540. MR0898357

TIERNEY, L. (1994). Markov chains for exploring posterior distributions. *Ann. Statist.* **22** 1701–1728. MR1329166 https://doi.org/10.1214/aos/1176325750

VAN DYK, D. A. and MENG, X.-L. (2001). The art of data augmentation. *J. Comput. Graph. Statist.* **10** 1–50. MR1936358 https://doi.org/10.1198/10618600152418584

VATS, D., FLEGAL, J. M. and JONES, G. L. (2019). Multivariate output analysis for Markov chain Monte Carlo. *Biometrika* **106** 321–337. MR3949306 https://doi.org/10.1093/biomet/asz002

The Annals of Applied Probability
2024, Vol. 34, No. 5, 4342–4386
https://doi.org/10.1214/24-AAP2068

SYSTEMATIC JUMP RISK

BY JEAN JACOD[1,a], HUIDI LIN[2,b] AND VIKTOR TODOROV[2,c]

[1]*Institut de Mathématiques de Jussieu, Sorbonne Université,* [a]*jean.jacod@gmail.com*
[2]*Department of Finance, Northwestern University,* [b]*huidi.lin@northwestern.edu,* [c]*v-todorov@northwestern.edu*

In this paper we develop tests for detecting systematic jumps in asset prices of general form, including ones not explained by observable factors, and we further propose nonparametric estimates for them. The inference is based on a panel of high-frequency asset returns, with both the sampling frequency and the size of the cross-section increasing asymptotically. The feasible limit theory developed in the paper utilizes the different asymptotic roles played by diffusive versus jump risks and systematic versus idiosyncratic risks in statistics that involve cross-sectional averages of suitably chosen transforms of the high-frequency price increments. The rate of convergence of the statistics is determined by the two asymptotically increasing dimensions of the panel, without imposing restrictions on their relative size. In an empirical application, using the developed tools, we document the existence of systematic jump risk, that is not spanned by standard (observable) risk factors, and we further show that this risk commands a nontrivial risk premium.

REFERENCES

[1] AÏT-SAHALIA, Y. and JACOD, J. (2009). Testing for jumps in a discretely observed process. *Ann. Statist.* **37** 184–222. MR2488349 https://doi.org/10.1214/07-AOS568

[2] AÏT-SAHALIA, Y. and JACOD, J. (2014). High-frequency financial econometrics. In *High-Frequency Financial Econometrics*. Princeton Univ. Press, Princeton, NJ.

[3] AÏT-SAHALIA, Y., JACOD, J. and XIU, D. (2020). Inference on risk premia in continuous-time asset pricing models. Technical report, National Bureau of Economic Research.

[4] ANDERSEN, T. G., SU, T., TODOROV, V. and ZHANG, Z. (2023). Intraday periodic volatility curves. *J. Amer. Statist. Assoc.* 1–11.

[5] BARNDORFF-NIELSEN, O. E. and SHEPHARD, N. (2004). Power and bipower variation with stochastic volatility and jumps. *J. Financ. Econom.* **2** 1–37.

[6] BARNDORFF-NIELSEN, O. E. and SHEPHARD, N. (2004). Econometrics of testing for jumps in financial economics using bipower variation. *J. Financ. Econom.* **4** 1–30. MR2051439 https://doi.org/10.1111/j.1468-0262.2004.00515.x

[7] BIBINGER, M. and WINKELMANN, L. (2015). Econometrics of co-jumps in high-frequency data with noise. *J. Econometrics* **184** 361–378. MR3291008 https://doi.org/10.1016/j.jeconom.2014.10.004

[8] BOLLERSLEV, T., LAW, T. H. and TAUCHEN, G. (2008). Risk, jumps, and diversification. *J. Econometrics* **144** 234–256. MR2439931 https://doi.org/10.1016/j.jeconom.2008.01.006

[9] BOLLERSLEV, T., LI, S. Z. and TODOROV, V. (2016). Roughing up beta: Continuous versus discontinuous betas and the cross section of expected stock returns. *J. Financ. Econ.* **120** 464–490.

[10] CAPORIN, M., KOLOKOLOV, A. and RENÒ, R. (2017). Systemic co-jumps. *J. Financ. Econ.* **126** 563–591.

[11] CORRADI, V., DISTASO, W. and FERNANDES, M. (2020). Testing for jump spillovers without testing for jumps. *J. Amer. Statist. Assoc.* **115** 1214–1226. MR4143460 https://doi.org/10.1080/01621459.2019.1609971

[12] DING, Y., LI, Y., LIU, G. and ZHENG, X. (2024). Stock co-jump networks. *J. Econometrics* **239** Paper No. 105420, 12. MR4708617 https://doi.org/10.1016/j.jeconom.2023.01.026

[13] DUNGEY, M., ERDEMLIOGLU, D., MATEI, M. and YANG, X. (2018). Testing for mutually exciting jumps and financial flights in high frequency data. *J. Econometrics* **202** 18–44. MR3724963 https://doi.org/10.1016/j.jeconom.2017.09.002

MSC2020 subject classifications. Primary 60F05, 60J76; secondary 62G20, 62M09.

Key words and phrases. Dynamic factor models, high-dimensional analysis, high-frequency data, Itô semimartingale, nonparametric inference, stable convergence, systematic risk.

[14] GATHERAL, J., JAISSON, T. and ROSENBAUM, M. (2018). Volatility is rough. *Quant. Finance* **18** 933–949. MR3805308 https://doi.org/10.1080/14697688.2017.1393551

[15] GILDER, D., SHACKLETON, M. B. and TAYLOR, S. J. (2014). Cojumps in stock prices: Empirical evidence. *J. Bank. Financ.* **40** 443–459.

[16] JACOD, J. and PROTTER, P. (2012). *Discretization of Processes. Stochastic Modelling and Applied Probability* **67**. Springer, Heidelberg. MR2859096 https://doi.org/10.1007/978-3-642-24127-7

[17] JACOD, J. and TODOROV, V. (2009). Testing for common arrivals of jumps for discretely observed multidimensional processes. *Ann. Statist.* **37** 1792–1838. MR2533472 https://doi.org/10.1214/08-AOS624

[18] LAHAYE, J., LAURENT, S. and NEELY, C. J. (2011). Jumps, cojumps and macro announcements. *J. Appl. Econometrics* **26** 893–921. MR2843111 https://doi.org/10.1002/jae.1149

[19] LEE, S. S. and MYKLAND, P. A. (2008). Jumps in financial markets: A new nonparametric test and jump dynamics. *Rev. Financ. Stud.* **21** 2535–2563.

[20] MANCINI, C. and GOBBI, F. (2012). Identifying the Brownian covariation from the co-jumps given discrete observations. *Econometric Theory* **28** 249–273. MR2913631 https://doi.org/10.1017/S0266466611000326

[21] MERTON, R. (1976). Option pricing when underlying asset returns are discontinuous. *J. Financ. Econ.* **3** 125–144.

[22] NOVOTNÝ, J. and URGA, G. (2018). Testing for co-jumps in financial markets. *J. Financ. Econom.* **16** 118–128.

[23] PELGER, M. (2019). Large-dimensional factor modeling based on high-frequency observations. *J. Econometrics* **208** 23–42. MR3906960 https://doi.org/10.1016/j.jeconom.2018.09.004

[24] PELGER, M. (2020). Understanding systematic risk: A high-frequency approach. *J. Finance* **75** 2179–2220.

[25] TODOROV, V. and BOLLERSLEV, T. (2010). Jumps and betas: A new framework for disentangling and estimating systematic risks. *J. Econometrics* **157** 220–235. MR2661596 https://doi.org/10.1016/j.jeconom.2009.11.010

The Annals of Applied Probability
2024, Vol. 34, No. 5, 4387–4407
https://doi.org/10.1214/24-AAP2069

HARMONIC MEASURE IN A MULTIDIMENSIONAL GAMBLER'S PROBLEM

BY DENIS DENISOV[1,a] AND VITALI WACHTEL[2,b]

[1]*Department of Mathematics, University of Manchester,* [a]*denis.denisov@manchester.ac.uk*
[2]*Faculty of Mathematics, Bielefeld University,* [b]*wachtel@math.uni-bielefeld.de*

We consider a random walk in a truncated cone K_N, which is obtained by slicing cone K by a hyperplane at a growing level of order N. We study the behaviour of the Green function in this truncated cone as N increases. Using these results we also obtain the asymptotic behaviour of the harmonic measure.

The obtained results are applied to a multidimensional gambler's problem studied by Diaconis and Ethier (*Staist. Sci.* **37** (2022) 289–305). In particular we confirm their conjecture that the probability of eliminating players in a particular order has the same exact asymptotic behaviour as for the Brownian motion approximation. We also provide a rate of convergence of this probability towards this approximation.

REFERENCES

[1] ARMITAGE, D. H. and GARDINER, S. J. (2001). *Classical Potential Theory*. *Springer Monographs in Mathematics*. Springer, London. MR1801253 https://doi.org/10.1007/978-1-4471-0233-5

[2] COVER, T. M. and GOPINATH, B., eds. (1987). Gambler's ruin: A random walk on the simplex. In *Open Problems in Communication and Computation* 204–207. Springer, New York. MR0922073 https://doi.org/10.1007/978-1-4612-4808-8

[3] DENISOV, D. and WACHTEL, V. (2010). Conditional limit theorems for ordered random walks. *Electron. J. Probab.* **15** 292–322. MR2609589 https://doi.org/10.1214/EJP.v15-752

[4] DENISOV, D. and WACHTEL, V. (2015). Random walks in cones. *Ann. Probab.* **43** 992–1044. MR3342657 https://doi.org/10.1214/13-AOP867

[5] DENISOV, D. and WACHTEL, V. (2019). Alternative constructions of a harmonic function for a random walk in a cone. *Electron. J. Probab.* **24** Paper No. 92, 26. MR4003145 https://doi.org/10.1214/19-ejp349

[6] DENISOV, D. and WACHTEL, V. (2024). Random walks in cones revisited. *Ann. Inst. Henri Poincaré Probab. Stat.* **60** 126–166. MR4718377 https://doi.org/10.1214/22-aihp1331

[7] DIACONIS, P. and ETHIER, S. N. (2022). Gambler's ruin and the ICM. *Statist. Sci.* **37** 289–305. MR4444368 https://doi.org/10.1214/21-sts826

[8] DIACONIS, P., HOUSTON-EDWARDS, K. and SALOFF-COSTE, L. (2021). Gambler's ruin estimates on finite inner uniform domains. *Ann. Appl. Probab.* **31** 865–895. MR4254498 https://doi.org/10.1214/20-aap1607

[9] DUBÉDAT, J. (2004). Reflected planar Brownian motions, intertwining relations and crossing probabilities. *Ann. Inst. Henri Poincaré Probab. Stat.* **40** 539–552. MR2086013 https://doi.org/10.1016/j.anihpb.2003.11.005

[10] DURAJ, J., RASCHEL, K., TARRAGO, P. and WACHTEL, V. (2022). Martin boundary of random walks in convex cones. *Ann. Henri Lebesgue* **5** 559–609. MR4443298 https://doi.org/10.5802/ahl.130

[11] DURAJ, J. and WACHTEL, V. (2020). Invariance principles for random walks in cones. *Stochastic Process. Appl.* **130** 3920–3942. MR4102254 https://doi.org/10.1016/j.spa.2019.11.004

[12] HOBSON, D. G. and WERNER, W. (1996). Non-colliding Brownian motions on the circle. *Bull. Lond. Math. Soc.* **28** 643–650. MR1405497 https://doi.org/10.1112/blms/28.6.643

[13] O'CONNOR, K. and SALOFF-COSTE, L. (2023). The 4-player Gambler's ruin problem. In *From Classical Analysis to Analysis on Fractals. Vol.* 1. *A Tribute to Robert Strichartz. Appl. Numer. Harmon. Anal.* 83–106. Birkhäuser, Cham. MR4676384 https://doi.org/10.1007/978-3-031-37800-3_5

[14] SPITZER, F. (1964). *Principles of Random Walk*. *The University Series in Higher Mathematics*. Van Nostrand, Princeton. MR0171290

MSC2020 subject classifications. Primary 60G50; secondary 60F17, 60G40, 60J45.
Key words and phrases. Random walk, Brownian motion, first-passage time, overshoot, moving boundary.

The Annals of Applied Probability
2024, Vol. 34, No. 5, 4408–4454
https://doi.org/10.1214/24-AAP2070
This research was funded, in whole or in part, by EPSRC, EP/K032208/1. A CC BY 4.0 license is applied to this article arising from this submission, in accordance with the grant's open access conditions.

SIR EPIDEMICS IN POPULATIONS WITH LARGE SUB-COMMUNITIES

BY FRANK BALL[1,a], DAVID SIRL[1,b] AND PIETER TRAPMAN[2,c]

[1]*School of Mathematical Sciences, University of Nottingham,* [a]*frank.ball@nottingham.ac.uk,* [b]*david.sirl.uk@gmail.com*
[2]*Department of Mathematics, Stockholm University,* [c]*j.p.trapman@rug.nl*

We investigate final outcome properties of an SIR (susceptible → infective → recovered) epidemic model defined on a population of large sub-communities in which there is stronger disease transmission within the communities than between them. Our analysis involves approximation of the epidemic process by a chain of within-community large outbreaks spreading between the communities. We derive law of large numbers and central limit type results for the number of individuals and the number of communities affected and the so-called severity of the outbreak. These results are valid as the size of communities tends to infinity, with the number of communities either fixed or also tending to infinity. The weaker between-community connections lead to randomness even in the law of large numbers type limit. As part of our proofs we also obtain a new result concerning the rate of convergence of the expected fraction infected in a standard SIR epidemic to its large-population limit.

REFERENCES

ABBE, E. (2018). Community detection and stochastic block models. *Found. Trends Commun. Inf. Theory* **14** 1–162.

AIZENMAN, M. and NEWMAN, C. M. (1986). Discontinuity of the percolation density in one-dimensional $1/|x-y|^2$ percolation models. *Comm. Math. Phys.* **107** 611–647. MR0868738

ANDERSSON, H. and BRITTON, T. (2000). *Stochastic Epidemic Models and Their Statistical Analysis. Lecture Notes in Statistics* **151**. Springer, New York. MR1784822 https://doi.org/10.1007/978-1-4612-1158-7

BAILEY, N. T. J. (1975). *The Mathematical Theory of Infectious Diseases and Its Applications*, 2nd ed. Hafner, New York. MR0452809

VON BAHR, B. and MARTIN-LÖF, A. (1980). Threshold limit theorems for some epidemic processes. *Adv. in Appl. Probab.* **12** 319–349. MR0569431 https://doi.org/10.2307/1426600

BALL, F. (1986). A unified approach to the distribution of total size and total area under the trajectory of infectives in epidemic models. *Adv. in Appl. Probab.* **18** 289–310. MR0840096 https://doi.org/10.2307/1427301

BALL, F. (2021). Central limit theorems for SIR epidemics and percolation on configuration model random graphs. *Ann. Appl. Probab.* **31** 2091–2142. MR4332692 https://doi.org/10.1214/20-aap1642

BALL, F. and BRITTON, T. (2005). An epidemic model with exposure-dependent severities. *J. Appl. Probab.* **42** 932–949. MR2203813 https://doi.org/10.1239/jap/1134587807

BALL, F. and CLANCY, D. (1993). The final size and severity of a generalised stochastic multitype epidemic model. *Adv. in Appl. Probab.* **25** 721–736. MR1241925 https://doi.org/10.2307/1427788

BALL, F. and DONNELLY, P. (1995). Strong approximations for epidemic models. *Stochastic Process. Appl.* **55** 1–21. MR1312145 https://doi.org/10.1016/0304-4149(94)00034-Q

BALL, F. and LYNE, O. D. (2001). Stochastic multitype SIR epidemics among a population partitioned into households. *Adv. in Appl. Probab.* **33** 99–123. MR1825318 https://doi.org/10.1239/aap/999187899

BALL, F., MOLLISON, D. and SCALIA-TOMBA, G. (1997). Epidemics with two levels of mixing. *Ann. Appl. Probab.* **7** 46–89. MR1428749 https://doi.org/10.1214/aoap/1034625252

BALL, F. and NEAL, P. (2002). A general model for stochastic SIR epidemics with two levels of mixing *Math. Biosci.* **180** 73–102. MR1950749 https://doi.org/10.1016/S0025-5564(02)00125-6

BALL, F., SIRL, D. and TRAPMAN, P. (2009). Threshold behaviour and final outcome of an epidemic on a random network with household structure. *Adv. in Appl. Probab.* **41** 765–796. MR2571316 https://doi.org/10.1239/aap/1253281063

MSC2020 subject classifications. Primary 92D30; secondary 60F99, 60K99.
Key words and phrases. Stochastic epidemic, final size, basic reproduction number, branching process, coupling, embedded process, weak convergence, central limit theorem.

BARBOUR, A. and MOLLISON, D. (1990). Epidemics and random graphs. In *Stochastic Processes in Epidemic Theory* 86–89. Springer, Berlin.

BILLINGSLEY, P. (1968). *Convergence of Probability Measures*. Wiley, New York. MR0233396

BLACK, A. J., HOUSE, T., KEELING, M. J. and ROSS, J. V. (2014). The effect of clumped population structure on the variability of spreading dynamics. *J. Theoret. Biol.* **359** 45–53. MR3248418 https://doi.org/10.1016/j.jtbi.2014.05.042

BOLLOBÁS, B., JANSON, S. and RIORDAN, O. (2007). The phase transition in inhomogeneous random graphs. *Random Structures Algorithms* **31** 3–122. MR2337396 https://doi.org/10.1002/rsa.20168

BRITTON, T., JANSON, S. and MARTIN-LÖF, A. (2007). Graphs with specified degree distributions, simple epidemics, and local vaccination strategies. *Adv. in Appl. Probab.* **39** 922–948. MR2381582

CORLESS, R. M., GONNET, G. H., HARE, D. E. G., JEFFREY, D. J. and KNUTH, D. E. (1996). On the Lambert *W* function. *Adv. Comput. Math.* **5** 329–359. MR1414285 https://doi.org/10.1007/BF02124750

DALY, F. (1979). Collapsing supercritical branching processes. *J. Appl. Probab.* **16** 732–739. MR0549553 https://doi.org/10.1017/s002190020003343x

DAWSON, D. A. and GOROSTIZA, L. G. (2013). Percolation in an ultrametric space. *Electron. J. Probab.* **18** 1–26. MR3035740 https://doi.org/10.1214/EJP.v18-1789

DIEKMANN, O., HEESTERBEEK, H. and BRITTON, T. (2013). *Mathematical Tools for Understanding Infectious Disease Dynamics*. *Princeton Series in Theoretical and Computational Biology*. Princeton Univ. Press, Princeton, NJ. MR3015083

DURRETT, R. (2010). *Probability: Theory and Examples*, 4th ed. *Cambridge Series in Statistical and Probabilistic Mathematics* **31**. Cambridge Univ. Press, Cambridge. MR2722836 https://doi.org/10.1017/CBO9780511779398

ETHIER, S. N. and KURTZ, T. G. (1986). *Markov Processes: Characterization and Convergence*. *Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics*. Wiley, New York. MR0838085 https://doi.org/10.1002/9780470316658

GRIMMETT, G. R. and STIRZAKER, D. R. (2001). *Probability and Random Processes*, 3rd ed. Oxford Univ. Press, New York. MR2059709

VAN DER HOFSTAD, R. (2021). *Random Graphs and Complex Networks II*, Available at: https://www.win.tue.nl/~rhofstad/NotesRGCNII.pdf.

JAGERS, P. (1975). *Branching Processes with Biological Applications*. *Wiley Series in Probability and Mathematical Statistics—Applied Probability and Statistics*. Wiley-Interscience, New York. MR0488341

LUDWIG, D. (1975). Final size distributions for epidemics. *Math. Biosci.* **23** 33–46. MR0403727 https://doi.org/10.1016/0025-5564(75)90119-4

MOSSEL, E., NEEMAN, J. and SLY, A. (2015). Reconstruction and estimation in the planted partition model. *Probab. Theory Related Fields* **162** 431–461. MR3383334 https://doi.org/10.1007/s00440-014-0576-6

NAGAEV, A. V. and STARTSEV, A. V. (1968). Threshold theorem for an epidemic model. *Math. Notes* **3** 115–119.

NAGAEV, A. V. and STARTSEV, A. N. (1970). The asymptotic analysis of a stochastic model of an epidemic. *Theory Probab. Appl.* **15** 98–107.

NEAL, P. (2006). Multitype randomized Reed–Frost epidemics and epidemics upon random graphs. *Ann. Appl. Probab.* **16** 1166–1189. MR2260061 https://doi.org/10.1214/105051606000000123

NEWMAN, M. E. J. (2002). Spread of epidemic disease on networks. *Phys. Rev. E* (3) **66** 016128, 11. MR1919737 https://doi.org/10.1103/PhysRevE.66.016128

NORROS, I. and REITTU, H. (2006). On a conditionally Poissonian graph process. *Adv. in Appl. Probab.* **38** 59–75. MR2213964 https://doi.org/10.1239/aap/1143936140

PELLIS, L., FERGUSON, N. M. and FRASER, C. (2008). The relationship between real-time and discrete-generation models of epidemic spread. *Math. Biosci.* **216** 63–70. MR2474745 https://doi.org/10.1016/j.mbs.2008.08.009

PÓLYA, G. and SZEGŐ, G. (1978). *Problems and Theorems in Analysis. I*, 4th German ed. *Grundlehren der Mathematischen Wissenschaften* **193**. Springer, Berlin. Series, integral calculus, theory of functions, Translated from the German by D. Aeppli. MR0580154

SCALIA-TOMBA, G. (1985). Asymptotic final-size distribution for some chain-binomial processes. *Adv. in Appl. Probab.* **17** 477–495. MR0798872 https://doi.org/10.2307/1427116

SCALIA-TOMBA, G. (1986). Asymptotic final size distribution of the multitype Reed–Frost process. *J. Appl. Probab.* **23** 563–584. MR0855366 https://doi.org/10.1017/s0021900200111751

SCALIA-TOMBA, G. (1990). On the asymptotic final size distribution of epidemics in heterogeneous populations. In *Stochastic Processes in Epidemic Theory* 189–196. Springer, Berlin.

SELLKE, T. (1983). On the asymptotic distribution of the size of a stochastic epidemic. *J. Appl. Probab.* **20** 390–394. MR0698541 https://doi.org/10.1017/s0021900200023536

SERFLING, R. J. (1980). *Approximation Theorems of Mathematical Statistics*. *Wiley Series in Probability and Mathematical Statistics*. Wiley, New York. MR0595165

VAN DER VAART, A. W. and WELLNER, J. A. (1996). *Weak Convergence and Empirical Processes*: *With Applications to Statistics*. *Springer Series in Statistics*. Springer, New York. MR1385671 https://doi.org/10.1007/978-1-4757-2545-2

WATSON, R. K. (1972). On an epidemic in a stratified population. *J. Appl. Probab.* **9** 659–666. MR0350916 https://doi.org/10.1017/s0021900200035956

The Annals of Applied Probability
2024, Vol. 34, No. 5, 4455–4492
https://doi.org/10.1214/24-AAP2071

THE MODIFIED MSA, A GRADIENT FLOW AND CONVERGENCE

BY DEVEN SETHI[a] AND DAVID ŠIŠKA[b]

School of Mathematics, University of Edinburgh, [a]*D.Sethi-1@sms.ed.ac.uk,* [b]*D.Siska@ed.ac.uk*

The modified method of successive approximations (MSA) is an iterative scheme for approximating solutions to stochastic control problems in continuous time based on Pontryagin optimality principle which, starting with an initial open loop control, solves the forward equation, the backward adjoint equation and then performs a static minimization step. We observe that this is an implicit Euler scheme for a gradient flow system. We prove that appropriate interpolations of the iterates of the modified MSA converge to a gradient flow with rate τ. We then study the convergence of this gradient flow as time goes to infinity. In the general (nonconvex) case we prove that the gradient term itself converges to zero. This is a consequence of an energy identity which shows that the optimization objective decreases along the gradient flow. Moreover, in the convex case, when Pontryagin optimality principle provides a sufficient condition for optimality, we prove that the optimization objective converges at rate $\frac{1}{S}$ to its optimal value and at exponential rate under strong convexity. The main technical difficulties lie in obtaining appropriate properties of the Hamiltonian (growth, continuity). These are obtained by utilising the theory of bounded mean oscillation (BMO) martingales required for estimates on the adjoint backward stochastic differential equation (BSDE).

REFERENCES

[1] BARLES, G. and JAKOBSEN, E. R. (2002). On the convergence rate of approximation schemes for Hamilton-Jacobi-Bellman equations. *Math. Model. Numer. Anal.* **36** 33–54. MR1916291 https://doi.org/10.1051/m2an:2002002

[2] BELLMAN, R. (1957). *Dynamic Programming*. Princeton Univ. Press, Princeton, NJ. MR0090477

[3] CARMONA, R. (2016). *Lectures on BSDEs, Stochastic Control, and Stochastic Differential Games with Financial Applications*. *Financial Mathematics* **1**. SIAM, Philadelphia, PA. MR3629171 https://doi.org/10.1137/1.9781611974249

[4] DONG, H. and KRYLOV, N. V. (2007). The rate of convergence of finite-difference approximations for parabolic Bellman equations with Lipschitz coefficients in cylindrical domains. *Appl. Math. Optim.* **56** 37–66. MR2334605 https://doi.org/10.1007/s00245-007-0879-4

[5] DONG, Y. (2022). Randomized optimal stopping problem in continuous time and reinforcement learning algorithm. Available at arXiv:2208.02409.

[6] E, W., HAN, J. and JENTZEN, A. (2017). Deep learning-based numerical methods for high-dimensional parabolic partial differential equations and backward stochastic differential equations. *Commun. Math. Stat.* **5** 349–380. MR3736669 https://doi.org/10.1007/s40304-017-0117-6

[7] FLEMING, W. H. and SONER, H. M. (2006). *Controlled Markov Processes and Viscosity Solutions*, 2nd ed. *Stochastic Modelling and Applied Probability* **25**. Springer, New York. MR2179357

[8] GONON, L., GROHS, P., JENTZEN, A., KOFLER, D. and ŠIŠKA, D. (2022). Uniform error estimates for artificial neural network approximations for heat equations. *IMA J. Numer. Anal.* **42** 1991–2054. MR4454915 https://doi.org/10.1093/imanum/drab027

[9] GROHS, P., HORNUNG, F., JENTZEN, A. and VON WURSTEMBERGER, P. (2023). A proof that artificial neural networks overcome the curse of dimensionality in the numerical approximation of Black-Scholes partial differential equations. *Mem. Amer. Math. Soc.* MR4574851

[10] GYÖNGY, I. and ŠIŠKA, D. (2009). On finite-difference approximations for normalized Bellman equations. *Appl. Math. Optim.* **60** 297–339. MR2551485 https://doi.org/10.1007/s00245-009-9082-0

MSC2020 subject classifications. Primary 93E20, 60H30; secondary 37N40, 65K99.
Key words and phrases. Stochastic control, gradient flow, iterative methods, convergence.

[11] HARTER, J. and RICHOU, A. (2019). A stability approach for solving multidimensional quadratic BSDEs. *Electron. J. Probab.* **24** Paper No. 4. MR3916324 https://doi.org/10.1214/18-EJP260

[12] HU, K., KAZEYKINA, A. and REN, Z. (2019). Mean-field langevin system, optimal control and deep neural networks. Available at arXiv:1909.07278.

[13] HUANG, Y.-J., WANG, Z. and ZHOU, Z. (2022). Convergence of policy improvement for entropy-regularized stochastic control problems. Available at arXiv:2209.07059.

[14] JABIR, J.-F., ŠIŠKA, D. and SZPRUCH, Ł. (2019). Mean-field neural ODEs via relaxed optimal control. Available at arXiv:1912.05475.

[15] JACKA, S. D. and MIJATOVIĆ, A. (2017). On the policy improvement algorithm in continuous time. *Stochastics* **89** 348–359. MR3574706 https://doi.org/10.1080/17442508.2016.1187609

[16] JACKA, S. D., MIJATOVIC, A. and SIRAJ, D. (2017). Coupling and a generalised policy iteration algorithm in continuous time. Available at arXiv:1707.07834.

[17] JENTZEN, A., SALIMOVA, D. and WELTI, T. (2021). A proof that deep artificial neural networks overcome the curse of dimensionality in the numerical approximation of Kolmogorov partial differential equations with constant diffusion and nonlinear drift coefficients. *Commun. Math. Sci.* **19** 1167–1205. MR4283528 https://doi.org/10.4310/CMS.2021.v19.n5.a1

[18] JI, S. and XU, R. (2022). A modified method of successive approximations for stochastic recursive optimal control problems. *SIAM J. Control Optim.* **60** 2759–2786. MR4480633 https://doi.org/10.1137/21M1418204

[19] KAZAMAKI, N. (1994). *Continuous Exponential Martingales and BMO. Lecture Notes in Math.* **1579**. Springer, Berlin. MR1299529 https://doi.org/10.1007/BFb0073585

[20] KERIMKULOV, B., ŠIŠKA, D. and SZPRUCH, L. (2020). Exponential convergence and stability of Howard's policy improvement algorithm for controlled diffusions. *SIAM J. Control Optim.* **58** 1314–1340. MR4096123 https://doi.org/10.1137/19M1236758

[21] KERIMKULOV, B., ŠIŠKA, D. and SZPRUCH, L. (2021). A modified MSA for stochastic control problems. *Appl. Math. Optim.* **84** 3417–3436. MR4308232 https://doi.org/10.1007/s00245-021-09750-2

[22] KRYLOV, I. A. and ČERNOUS'KO, F. L. (1962). On the method of successive approximations for solution of optimal control problems (in Russian). *USSR Comput. Math. Math. Phys.* **2** 1371–1382.

[23] KRYLOV, N. V. (1980). *Controlled Diffusion Processes. Applications of Mathematics* **14**. Springer, New York-Berlin. Translated from the Russian by A. B. Aries. MR0601776

[24] MAEDA, J. and JACKA, S. D. (2017). Evaluation of the rate of convergence in the PIA. Available at arXiv:1709.06466.

[25] PARDOUX, É. and PENG, S. G. (1990). Adapted solution of a backward stochastic differential equation. *Systems Control Lett.* **14** 55–61. MR1037747 https://doi.org/10.1016/0167-6911(90)90082-6

[26] PENG, S. G. (1990). A general stochastic maximum principle for optimal control problems. *SIAM J. Control Optim.* **28** 966–979. MR1051633 https://doi.org/10.1137/0328054

[27] PHAM, H. (2009). *Continuous-time Stochastic Control and Optimization with Financial Applications. Stochastic Modelling and Applied Probability* **61**. Springer, Berlin. MR2533355 https://doi.org/10.1007/978-3-540-89500-8

[28] PUTERMAN, M. L. (1981). On the convergence of policy iteration for controlled diffusions. *J. Optim. Theory Appl.* **33** 137–144. MR0608899 https://doi.org/10.1007/BF00935182

[29] REISINGER, C., STOCKINGER, W. and ZHANG, Y. (2021). A fast iterative PDE-based algorithm for feedback controls of nonsmooth mean-field control problems. Available at arXiv:2108.06740.

[30] REISINGER, C., STOCKINGER, W. and ZHANG, Y. (2023). Linear convergence of a policy gradient method for some finite horizon continuous time control problems. *SIAM J. Control Optim.* **61** 3526–3558. MR4679972 https://doi.org/10.1137/22M1492180

[31] SABATE VIDALES, M., ŠIŠKA, D. and SZPRUCH, L. (2021). Unbiased deep solvers for linear parametric PDEs. *Appl. Math. Finance* **28** 299–329. MR4440962 https://doi.org/10.1080/1350486X.2022.2030773

[32] SANTAMBROGIO, F. (2017). {Euclidean, metric, and Wasserstein} gradient flows: An overview. *Bull. Math. Sci.* **7** 87–154. MR3625852 https://doi.org/10.1007/s13373-017-0101-1

[33] SIRIGNANO, J. and SPILIOPOULOS, K. (2018). DGM: A deep learning algorithm for solving partial differential equations. *J. Comput. Phys.* **375** 1339–1364. MR3874585 https://doi.org/10.1016/j.jcp.2018.08.029

[34] ŠIŠKA, D. and SZPRUCH, Ł. (2020). Gradient flows for regularized stochastic control problems. Available at arXiv:2006.05956.

[35] ZHANG, J. (2017). Backward stochastic differential equations. In *Backward Stochastic Differential Equations* 79–99. Springer, Berlin.

The Annals of Applied Probability
2024, Vol. 34, No. 5, 4493–4512
https://doi.org/10.1214/24-AAP2072

A SINGULAR TWO-PHASE STEFAN PROBLEM AND PARTICLES INTERACTING THROUGH THEIR HITTING TIMES

BY GRAEME BAKER[1,a] AND MYKHAYLO SHKOLNIKOV[2,b]

[1]*Department of Statistics, Columbia University,* [a]*g.baker@columbia.edu*
[2]*Department of Mathematical Sciences, Carnegie Mellon University,* [b]*mshkolni@gmail.com*

We consider a probabilistic formulation of a singular *two-phase* Stefan problem in one space dimension, which amounts to a coupled system of two McKean–Vlasov stochastic differential equations. In the financial context of systemic risk, this system models two competing regions with a large number of interconnected banks or firms at risk of default. Our main result shows the existence of a solution whose discontinuities obey the natural physicality condition for the problem at hand. Thus, this work extends the recent series of existence results for singular *one-phase* Stefan problems in one space dimension. As for the one-phase problems, our existence result is obtained via a large system limit of a finite particle system approximation in the Skorokhod M1 topology. But, unlike for the previously studied one-phase case, the free boundary herein is not necessarily monotone, so that the large system limit is obtained by a novel argument.

REFERENCES

[1] AÏT-SAHALIA, Y., CACHO-DIAZ, J. and LAEVEN, R. J. A. (2015). Modeling financial contagion using mutually exciting jump processes. *J. Financ. Econ.* **117** 585–606. https://doi.org/10.1016/j.jfineco.2015.03.002

[2] AÏT-SAHALIA, Y. and HURD, T. R. (2016). Portfolio choice in markets with contagion. *J. Financ. Econom.* **14** 28.

[3] ANTUNOVIĆ, T., BURDZY, K., PERES, Y. and RUSCHER, J. (2011). Isolated zeros for Brownian motion with variable drift. *Electron. J. Probab.* **16** 1793–1814. MR2842087 https://doi.org/10.1214/EJP.v16-927

[4] BAYRAKTAR, E., GUO, G., TANG, W. and ZHANG, Y. P. (2023). Systemic robustness: a mean-field particle system approach. arXiv:2212.08518 [math, q-fin].

[5] BAYRAKTAR, E., GUO, G., TANG, W. and ZHANG, Y. P. (2024). McKean–Vlasov equations involving hitting times: Blow-ups and global solvability. *Ann. Appl. Probab.* **34** 1600–1622. MR4700266 https://doi.org/10.1214/23-aap1999

[6] CONT, R., MOUSSA, A. and SANTOS, E. A. (2011). Network structure and systemic risk in banking systems. In *Handbook on Systemic Risk*, 1st ed. (J.-P. Fouque and J. A. Langsam, eds.) 327–368. Cambridge Univ. Press, Cambridge. MR2941723 https://doi.org/10.1017/CBO9781139151184.018

[7] CUCHIERO, C., REISINGER, C. and RIGGER, S. (2023). Optimal bailout strategies resulting from the drift controlled supercooled Stefan problem. *Ann. Oper. Res.* **336** 1315–1349. MR4745659 https://doi.org/10.1007/s10479-023-05293-7

[8] CUCHIERO, C., RIGGER, S. and SVALUTO-FERRO, S. (2023). Propagation of minimality in the supercooled Stefan problem. *Ann. Appl. Probab.* **33** 1588–1618. MR4564435 https://doi.org/10.1214/22-aap1850

[9] DELARUE, F., INGLIS, J., RUBENTHALER, S. and TANRÉ, E. (2015). Global solvability of a networked integrate-and-fire model of McKean–Vlasov type. *Ann. Appl. Probab.* **25** 2096–2133. MR3349003 https://doi.org/10.1214/14-AAP1044

[10] DELARUE, F., INGLIS, J., RUBENTHALER, S. and TANRÉ, E. (2015). Particle systems with a singular mean-field self-excitation. Application to neuronal networks. *Stochastic Process. Appl.* **125** 2451–2492. MR3322871 https://doi.org/10.1016/j.spa.2015.01.007

MSC2020 subject classifications. 82C22, 60H30, 35K10, 35K60, 91G80.
Key words and phrases. Interacting particle systems, mean-field models, singular interactions, systemic risk, Stefan problems, free boundary problems.

[11] DELARUE, F., NADTOCHIY, S. and SHKOLNIKOV, M. (2022). Global solutions to the supercooled Stefan problem with blow-ups: Regularity and uniqueness. *Probab. Math. Phys.* **3** 171–213. MR4420299 https://doi.org/10.2140/pmp.2022.3.171

[12] DEMBO, A. and TSAI, L.-C. (2019). Criticality of a randomly-driven front. *Arch. Ration. Mech. Anal.* **233** 643–699. MR3951690 https://doi.org/10.1007/s00205-019-01365-w

[13] GIACOMIN, G. and LEBOWITZ, J. L. (1998). Phase segregation dynamics in particle systems with long range interactions. II. Interface motion. *SIAM J. Appl. Math.* **58** 1707–1729. MR1638739 https://doi.org/10.1137/S0036139996313046

[14] GÖTZ, I. G. and ZALTZMAN, B. (1995). Two-phase Stefan problem with supercooling. *SIAM J. Math. Anal.* **26** 694–714. MR1325910 https://doi.org/10.1137/S003614109223605X

[15] HAMBLY, B., LEDGER, S. and SØJMARK, A. (2019). A McKean–Vlasov equation with positive feedback and blow-ups. *Ann. Appl. Probab.* **29** 2338–2373. MR3983340 https://doi.org/10.1214/18-AAP1455

[16] HAMBLY, B. and SØJMARK, A. (2019). An SPDE model for systemic risk with endogenous contagion. *Finance Stoch.* **23** 535–594. MR3968278 https://doi.org/10.1007/s00780-019-00396-1

[17] MEĬRMANOV, A. M. (1994). The Stefan problem with surface tension in the three-dimensional case with spherical symmetry: Nonexistence of the classical solution. *European J. Appl. Math.* **5** 1–19. MR1270785 https://doi.org/10.1017/S0956792500001261

[18] NADTOCHIY, S. and SHKOLNIKOV, M. (2019). Particle systems with singular interaction through hitting times: Application in systemic risk modeling. *Ann. Appl. Probab.* **29** 89–129. MR3910001 https://doi.org/10.1214/18-AAP1403

[19] NADTOCHIY, S. and SHKOLNIKOV, M. (2020). Mean field systems on networks, with singular interaction through hitting times. *Ann. Probab.* **48** 1520–1556. MR4112723 https://doi.org/10.1214/19-AOP1403

[20] SZNITMAN, A.-S. (1991). Topics in propagation of chaos. In *École D'Été de Probabilités de Saint-Flour XIX*—1989. *Lecture Notes in Math.* **1464** 165–251. Springer, Berlin. MR1108185 https://doi.org/10.1007/BFb0085169

[21] VISINTIN, A. (1998). Introduction to the models of phase transitions. *Boll. Unione Mat. Ital.* **1** 1–47. MR1619027

[22] WHITT, W. (2002). *Stochastic-Process Limits: An Introduction to Stochastic-Process Limits and Their Application to Queues. Springer Series in Operations Research*. Springer, New York. MR1876437

The Annals of Applied Probability
2024, Vol. 34, No. 5, 4513–4553
https://doi.org/10.1214/24-AAP2073

STOCHASTIC PROCESSES WITH COMPETING REINFORCEMENTS

BY DIRK ERHARD[1,a] AND GUILHERME REIS[2,b]

[1]*Instituto de Matemática, UFBA,* [a]*erharddirk@gmail.com*
[2]*National Institute of Applied Mathematics, IMPA,* [b]*guilherme.reis.mat@gmail.com*

We introduce a simple but powerful strategy to study processes driven by two or more reinforcement mechanisms in competition. We apply our method to two types of models: to nonconservative zero range processes on finite graphs, and to multi-particle random walks with positive and negative reinforcement on the edges. The results hold for a broad class of reinforcement functions, including those with superlinear growth. Our strategy consists in a comparison of the original processes with suitable reference models. To implement the comparison we estimate an object reminiscent to the Radon–Nikodym derivative on a carefully chosen set of trajectories. Our results describe the almost sure long time behaviour of the processes. We also prove a phase transition depending on the strength of the reinforcement functions.

REFERENCES

[1] ANGEL, A. G., EVANS, M. R., LEVINE, E. and MUKAMEL, D. (2007). Criticality and condensation in a non-conserving zero-range process. *J. Stat. Mech. Theory Exp.* **2007** P08017–P08017. https://doi.org/10.1088/1742-5468/2007/08/p08017

[2] ARNOLD, L. (1981). Mathematical models of chemical reactions. In *Stochastic Systems: The Mathematics of Filtering and Identification and Applications* (M. Hazewinkel and J. C. Willems, eds.) 111–134. Springer, Netherlands. MR0674319

[3] BLOUNT, D. (1991). Comparison of stochastic and deterministic models of a linear chemical reaction with diffusion. *Ann. Probab.* **19** 1440–1462. MR1127711

[4] BLOUNT, D. (1992). Law of large numbers in the supremum norm for a chemical reaction with diffusion. *Ann. Appl. Probab.* **2** 131–141. MR1143396

[5] COTAR, C. and THACKER, D. (2017). Edge- and vertex-reinforced random walks with super-linear reinforcement on infinite graphs. *Ann. Probab.* **45** 2655–2706. MR3693972 https://doi.org/10.1214/16-AOP1122

[6] DAVIS, B. (1990). Reinforced random walk. *Probab. Theory Related Fields* **84** 203–229. MR1030727 https://doi.org/10.1007/BF01197845

[7] ERHARD, D., FRANCO, T. and REIS, G. (2023). The directed edge reinforced random walk: The ant mill phenomenon. *J. Stat. Phys.* **190** Paper No. 18, 18. MR4510455 https://doi.org/10.1007/s10955-022-03031-0

[8] FRANCO, T. and GROISMAN, P. (2012). A particle system with explosions: Law of large numbers for the density of particles and the blow-up time. *J. Stat. Phys.* **149** 629–642. MR2998594 https://doi.org/10.1007/s10955-012-0621-8

[9] KIOUS, D., MAILLER, C. and SCHAPIRA, B. (2022). Finding geodesics on graphs using reinforcement learning. *Ann. Appl. Probab.* **32** 3889–3929. MR4497861 https://doi.org/10.1214/21-aap1777

[10] LIMIC, V. (2003). Attracting edge property for a class of reinforced random walks. *Ann. Probab.* **31** 1615–1654. MR1989445 https://doi.org/10.1214/aop/1055425792

[11] LIMIC, V. and TARRÈS, P. (2007). Attracting edge and strongly edge reinforced walks. *Ann. Probab.* **35** 1783–1806. MR2349575 https://doi.org/10.1214/009117906000001097

[12] MA, T., XIA, Z. and YANG, F. (2017). An ant colony random walk algorithm for overlapping community detection. In *Lecture Notes in Computer Science Intelligent Data Engineering and Automated Learning–IDEAL* 2017 20–26. https://doi.org/10.1007/978-3-319-68935-7_3

[13] MERKL, F. and ROLLES, S. W. W. (2006). Linearly edge-reinforced random walks. In *Dynamics & Stochastics. Institute of Mathematical Statistics Lecture Notes—Monograph Series* **48** 66–77. IMS, Beachwood, OH. MR2306189 https://doi.org/10.1214/074921706000000103

MSC2020 subject classifications. Primary 60K35, 60K37; secondary 60G50.
Key words and phrases. Reinforced random walk, ant random walk.

[14] OLIVEIRA, R. (2008). Balls-in-bins processes with feedback and Brownian motion. *Combin. Probab. Comput.* **17** 87–110. MR2376425 https://doi.org/10.1017/S0963548307008577

[15] PEMANTLE, R. (1988). Phase transition in reinforced random walk and RWRE on trees. *Ann. Probab.* **16** 1229–1241. MR0942765

[16] PEMANTLE, R. (2007). A survey of random processes with reinforcement. *Probab. Surv.* **4** 1–79. MR2282181 https://doi.org/10.1214/07-PS094

[17] QUITTNER, P. and SOUPLET, P. (2007). Superlinear parabolic problems. In *Blow-up, Global Existence and Steady States*. *Birkhäuser Advanced Texts: Basler Lehrbücher.* [*Birkhäuser Advanced Texts: Basel Textbooks*]. Birkhäuser, Basel. MRMR2346798

[18] TARRÈS, P. (2004). Vertex-reinforced random walk on $\mathbb{Z}$ eventually gets stuck on five points. *Ann. Probab.* **32** 2650–2701. MR2078554 https://doi.org/10.1214/009117907000000694

[19] VOLKOV, S. (2001). Vertex-reinforced random walk on arbitrary graphs. *Ann. Probab.* **29** 66–91. MR1825142 https://doi.org/10.1214/aop/1008956322

The Annals of Applied Probability
2024, Vol. 34, No. 5, 4554–4594
https://doi.org/10.1214/24-AAP2074

ASYMPTOTIC BEHAVIOUR OF THE NOISY VOTER MODEL DENSITY PROCESS

BY RICHARD PYMAR[1,a] AND NICOLÁS RIVERA[2,b]

[1]*School of Computing and Mathematical Sciences, Birkbeck, University of London,* [a]*r.pymar@bbk.ac.uk*
[2]*Instituto de Estadística, Facultad de Ciencias, Universidad de Valparaíso,* [b]*n.a.rivera.aburto@gmail.com*

Given a transition matrix P indexed by a finite set V of vertices, the voter model is a discrete-time Markov chain in $\{0,1\}^V$ where at each time-step a randomly chosen vertex x imitates the opinion of vertex y with probability $P(x,y)$. The noisy voter model is a variation of the voter model in which vertices may change their opinions by the action of an external noise. The strength of this noise is measured by an extra parameter $p \in [0,1]$.

In this work we analyse the density process, defined as the stationary mass of vertices with opinion 1, that is, $S_t = \sum_{x \in V} \pi(x)\xi_t(x)$, where π is the stationary distribution of P, and $\xi_t(x)$ is the opinion of vertex x at time t. We investigate the asymptotic behaviour of S_t when t tends to infinity for different values of the noise parameter p. In particular, by allowing P and p to be functions of the size $|V|$, we show that, under appropriate conditions and small enough p a normalised version of S_t converges to a Gaussian random variable, while for large enough p, S_t converges to a Bernoulli random variable. We provide further analysis of the noisy voter model on a variety of specific graphs including the complete graph, cycle, torus, and hypercube, where we identify the critical rate p (depending on the size $|V|$) that separates these two asymptotic behaviours.

REFERENCES

[1] ADDARIO-BERRY, L., OLIVEIRA, R. I., PERES, Y. and SOUSI, P. (2016). Random walks colliding before getting trapped. *Electron. J. Probab.* **21** Paper No. 42, 19. MR3530319 https://doi.org/10.1214/16-EJP4414

[2] ALDOUS, D. and FILL, J. A. (2002). Reversible Markov Chains and random walks on graphs. Unfinished monograph, recompiled 2014, available at http://www.stat.berkeley.edu/~aldous/RWG/book.html.

[3] BASU, R. and SLY, A. (2017). Evolving voter model on dense random graphs. *Ann. Appl. Probab.* **27** 1235–1288. MR3655865 https://doi.org/10.1214/16-AAP1230

[4] BERENBRINK, P., GIAKKOUPIS, G., KERMARREC, A.-M. and MALLMANN-TRENN, F. (2016). Bounds on the voter model in dynamic networks. In 43*rd International Colloquium on Automata, Languages, and Programming. LIPIcs. Leibniz Int. Proc. Inform.* **55** Art. No. 146, 15. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern. MR3577207

[5] BRESLER, G. and NAGARAJ, D. (2019). Stein's method for stationary distributions of Markov chains and application to Ising models. *Ann. Appl. Probab.* **29** 3230–3265. MR4019887 https://doi.org/10.1214/19-AAP1479

[6] CARRO, A., TORAL, R. and SAN MIGUEL, M. (2016). The noisy voter model on complex networks. *Sci. Rep.* **6** 1–14.

[7] CASTELLANO, C., FORTUNATO, S. and LORETO, V. (2007). Statistical physics of social dynamics. *Rev. Modern Phys.* **81**. https://doi.org/10.1103/RevModPhys.81.591

[8] CHATTERJEE, S. and DEY, P. S. (2010). Applications of Stein's method for concentration inequalities. *Ann. Probab.* **38** 2443–2485. MR2683635 https://doi.org/10.1214/10-AOP542

[9] CHATTERJEE, S. and SHAO, Q.-M. (2011). Nonnormal approximation by Stein's method of exchangeable pairs with application to the Curie–Weiss model. *Ann. Appl. Probab.* **21** 464–483. MR2807964 https://doi.org/10.1214/10-AAP712

MSC2020 subject classifications. 60K35, 60J10.
Key words and phrases. Voter model, interacting particle systems.

[10] CHEN, Y.-T., CHOI, J. and COX, J. T. (2016). On the convergence of densities of finite voter models to the Wright–Fisher diffusion. *Ann. Inst. Henri Poincaré Probab. Stat.* **52** 286–322. MR3449304 https://doi.org/10.1214/14-AIHP639

[11] CLIFFORD, P. and SUDBURY, A. (1973). A model for spatial conflict. *Biometrika* **60** 581–588. MR0343950 https://doi.org/10.1093/biomet/60.3.581

[12] COOPER, C., ELSÄSSER, R., ONO, H. and RADZIK, T. (2013). Coalescing random walks and voting on connected graphs. *SIAM J. Discrete Math.* **27** 1748–1758. MR3116640 https://doi.org/10.1137/120900368

[13] COOPER, C., FRIEZE, A. and RADZIK, T. (2010). Multiple random walks in random regular graphs. *SIAM J. Discrete Math.* **23** 1738–1761. MR2570201 https://doi.org/10.1137/080729542

[14] COX, J. T. (1989). Coalescing random walks and voter model consensus times on the torus in $\mathbf{Z}^d$. *Ann. Probab.* **17** 1333–1366. MR1048930

[15] COX, J. T., PERES, Y. and STEIF, J. E. (2016). Cutoff for the noisy voter model. *Ann. Appl. Probab.* **26** 917–932. MR3476629 https://doi.org/10.1214/15-AAP1108

[16] EICHELSBACHER, P. and LÖWE, M. (2010). Stein's method for dependent random variables occurring in statistical mechanics. *Electron. J. Probab.* **15** 962–988. MR2659754 https://doi.org/10.1214/EJP.v15-777

[17] EICHELSBACHER, P. and REINERT, G. (2008). Stein's method for discrete Gibbs measures. *Ann. Appl. Probab.* **18** 1588–1618. MR2434182 https://doi.org/10.1214/07-AAP0498

[18] FERNÁNDEZ-GRACIA, J., SUCHECKI, K., RAMASCO, J. J., SAN MIGUEL, M. and EGUÍLUZ, V. M. (2014). Is the voter model a model for voters? *Phys. Rev. Lett.* **112** 158701. https://doi.org/10.1103/PhysRevLett.112.158701

[19] GALAM, S. (2005). Local dynamics vs. social mechanisms: A unifying frame. 70. https://doi.org/10.1209/epl/i2004-10526-5

[20] GOLDSTEIN, L. and WIROONSRI, N. (2018). Stein's method for positively associated random variables with applications to the Ising and voter models, bond percolation, and contact process. *Ann. Inst. Henri Poincaré Probab. Stat.* **54** 385–421. MR3765894 https://doi.org/10.1214/16-AIHP808

[21] GRANOVSKY, B. L. and MADRAS, N. (1995). The noisy voter model. *Stochastic Process. Appl.* **55** 23–43. MR1312146 https://doi.org/10.1016/0304-4149(94)00035-R

[22] HASSIN, Y. and PELEG, D. (2001). Distributed probabilistic polling and applications to proportionate agreement. *Inform. and Comput.* **171** 248–268. MR1872781 https://doi.org/10.1006/inco.2001.3088

[23] HOLLEY, R. A. and LIGGETT, T. M. (1975). Ergodic theorems for weakly interacting infinite systems and the voter model. *Ann. Probab.* **3** 643–663. MR0402985 https://doi.org/10.1214/aop/1176996306

[24] HUO, R. and DURRETT, R. (2019). The zealot voter model. *Ann. Appl. Probab.* **29** 3128–3154. MR4019884 https://doi.org/10.1214/19-AAP1476

[25] KANADE, V., MALLMANN-TRENN, F. and SAUERWALD, T. (2019). On coalescence time in graphs: When is coalescing as fast as meeting? In *Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms* 956–965. SIAM, Philadelphia, PA. MR3909528 https://doi.org/10.1137/1.9781611975482.59

[26] KIRMAN, A. (1993). Ants, rationality, and recruitment. *Q. J. Econ.* **108** 137–156.

[27] KRAPIVSKY, P. L. (1992). Kinetics of monomer-monomer surface catalytic reactions. *Phys. Rev. A* **45** 1067–1072. https://doi.org/10.1103/PhysRevA.45.1067

[28] LANCHIER, N. (2012). The Axelrod model for the dissemination of culture revisited. *Ann. Appl. Probab.* **22** 860–880. MR2953571 https://doi.org/10.1214/11-AAP790

[29] LAWLER, G. F. (2013). *Intersections of Random Walks. Modern Birkhäuser Classics*. Birkhäuser, New York. Reprint of the 1996 edition. MR2985195 https://doi.org/10.1007/978-1-4614-5972-9

[30] LEVIN, D. A., PERES, Y. and WILMER, E. L. (1999). *Markov Chains and Mixing Times*. Amer. Math. Soc..

[31] LIGGETT, T. M. (2005). *Interacting Particle Systems. Classics in Mathematics*. Springer, Berlin. Reprint of the 1985 original. MR2108619 https://doi.org/10.1007/b138374

[32] MARINOV, R. (2013). Counting vertices in a Voter-type model. arXiv preprint. Available at arXiv:1311.4807.

[33] OLIVEIRA, R. I. (2012). On the coalescence time of reversible random walks. *Trans. Amer. Math. Soc.* **364** 2109–2128. MR2869200 https://doi.org/10.1090/S0002-9947-2011-05523-6

[34] OLIVEIRA, R. I. and PERES, Y. (2019). Random walks on graphs: New bounds on hitting, meeting, coalescing and returning. In 2019 *Proceedings of the Sixteenth Workshop on Analytic Algorithmics and Combinatorics* (*ANALCO*) 119–126. SIAM, Philadelphia, PA. MR3909448 https://doi.org/10.1137/1.9781611975505.13

[35] PERALTA, A. F., CARRO, A., SAN MIGUEL, M. and TORAL, R. (2018). Analytical and numerical study of the non-linear noisy voter model on complex networks. *Chaos* **28** 075516, 14. MR3831054 https://doi.org/10.1063/1.5030112

[36] PERALTA, A. F., CARRO, A., SAN MIGUEL, M. and TORAL, R. (2018). Stochastic pair approximation treatment of the noisy voter model. *New J. Phys.* **20** 103045.

[37] RAMADAS, H. (2014). Mixing of the noisy voter model. *Electron. Commun. Probab.* **19** no. 17, 9. MR3183570 https://doi.org/10.1214/ECP.v19-2968

[38] RAVASZ, M., SZABÓ, G. and SZOLNOKI, A. (2004). Spreading of families in cyclic predator-prey models. *Phys. Rev. E* **70** 012901. https://doi.org/10.1103/PhysRevE.70.012901

[39] RINOTT, Y. and ROTAR, V. (1997). On coupling constructions and rates in the CLT for dependent summands with applications to the antivoter model and weighted U-statistics. *Ann. Appl. Probab.* **7** 1080–1105. MR1484798 https://doi.org/10.1214/aoap/1043862425

[40] RÖLLIN, A. (2008). A note on the exchangeability condition in Stein's method. *Statist. Probab. Lett.* **78** 1800–1806. MR2453918 https://doi.org/10.1016/j.spl.2008.01.043

[41] ROSS, N. (2011). Fundamentals of Stein's method. *Probab. Surv.* **8** 210–293. MR2861132 https://doi.org/10.1214/11-PS182

[42] SCHNEIDER-MIZELL, C. M. and SANDER, L. M. (2009). A generalized voter model on complex networks. *J. Stat. Phys.* **136** 59–71. MR2525227 https://doi.org/10.1007/s10955-009-9757-6

[43] STARNINI, M., BARONCHELLI, A. and PASTOR-SATORRAS, R. (2012). Ordering dynamics of the multi-state voter model. *J. Stat. Mech. Theory Exp.* P10027, 12. MR3036072 https://doi.org/10.1088/1742-5468/2012/10/p10027

[44] STEIN, C. (1972). A bound for the error in the normal approximation to the distribution of a sum of dependent random variables. In *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability* (*Univ. California, Berkeley, Calif.*, 1970/1971), *Vol. II: Probability Theory* 583–602. Univ. California Press, Berkeley, CA. MR0402873

[45] SUCHECKI, K., EGUÍLUZ, V. M. and MIGUEL, M. S. (2004). Conservation laws for the voter model in complex networks. *Europhys. Lett.* **69** 228. https://doi.org/10.1209/epl/i2004-10329-8

[46] VAN DER HOFSTAD, R. and KEANE, M. (2008). An elementary proof of the hitting time theorem. *Amer. Math. Monthly* **115** 753–756. MR2456097 https://doi.org/10.1080/00029890.2008.11920588

[47] YILDIZ, E., OZDAGLAR, A., ACEMOGLU, D., SABERI, A. and SCAGLIONE, A. (2013). Binary opinion dynamics with stubborn agents. *ACM Trans. Econ. Comput.* **1** 19:1–19:30. https://doi.org/10.1145/2538508

The Annals of Applied Probability
2024, Vol. 34, No. 5, 4595–4634
https://doi.org/10.1214/24-AAP2075

APPROXIMATION OF STOCHASTIC INTEGRALS WITH JUMPS VIA WEIGHTED BMO APPROACH

BY NGUYEN TRAN THUAN[a]

Department of Mathematics, Saarland University, [a]*nguyen@math.uni-saarland.de*

This article investigates discrete-time approximations of stochastic integrals driven by semimartingales with jumps via weighted bounded mean oscillation (BMO) approach. This approach enables L_p-estimates, $p \in (2, \infty)$, for the approximation error depending on the weight, and it allows a change of the underlying measure which leaves the error estimates unchanged. To take advantage of this approach, we propose a new approximation scheme obtained from an adjustment for the Riemann approximation based on tracking jumps of the underlying semimartingale. We discuss a way to optimize the approximation and also illustrate the sharpness of the obtained convergence rates. When the small jump activity of the semimartingale behaves like an α-stable process with $\alpha \in (1, 2)$, our scheme achieves under a regular regime the same convergence rate for the error as in Rosenbaum and Tankov [*Ann. Appl. Probab.* **24** (2014) 1002–1048]. Moreover, our approach extends to the case $\alpha \in (0, 1]$ and to the L_p-setting which are not treated there. As an application, we apply the methods in the special case where the semimartingale is an exponential Lévy process to mean-variance hedging of European type options.

REFERENCES

[1] APPLEBAUM, D. (2009). *Lévy Processes and Stochastic Calculus*, 2nd ed. *Cambridge Studies in Advanced Mathematics* **116**. Cambridge Univ. Press, Cambridge. MR2512800 https://doi.org/10.1017/CBO9780511809781

[2] BENTH, F. E., DI NUNNO, G., LØKKA, A., ØKSENDAL, B. and PROSKE, F. (2003). Explicit representation of the minimal variance portfolio in markets driven by Lévy processes. *Math. Finance* **13** 55–72. Conference on Applications of Malliavin Calculus in Finance (Rocquencourt, 2001). MR1968096 https://doi.org/10.1111/1467-9965.t01-1-00005

[3] BLUMENTHAL, R. M. and GETOOR, R. K. (1961). Sample functions of stochastic processes with stationary independent increments. *J. Math. Mech.* **10** 493–516. MR0123362

[4] BRODÉN, M. and TANKOV, P. (2011). Tracking errors from discrete hedging in exponential Lévy models. *Int. J. Theor. Appl. Finance* **14** 803–837. MR2845218 https://doi.org/10.1142/S0219024911006760

[5] CHOULLI, T., KRAWCZYK, L. and STRICKER, C. (1998). $\mathscr{E}$-martingales and their applications in mathematical finance. *Ann. Probab.* **26** 853–876. MR1626523 https://doi.org/10.1214/aop/1022855653

[6] CONT, R. and TANKOV, P. (2004). *Financial Modelling with Jump Processes*. *Chapman & Hall/CRC Financial Mathematics Series*. CRC Press, Boca Raton, FL. MR2042661

[7] CONT, R., TANKOV, P. and VOLTCHKOVA, E. (2007). Hedging with options in models with jumps. In *Stochastic Analysis and Applications*. *Abel Symp*. **2** 197–217. Springer, Berlin. MR2397788 https://doi.org/10.1007/978-3-540-70847-6_8

[8] DELBAEN, F., MONAT, P., SCHACHERMAYER, W., SCHWEIZER, M. and STRICKER, C. (1997). Weighted norm inequalities and hedging in incomplete markets. *Finance Stoch.* **1** 181–227.

[9] DELLACHERIE, C. and MEYER, P.-A. (1982). *Probabilities and Potential*. *B*: *Theory of Martingales*. *North-Holland Mathematics Studies* **72**. North-Holland, Amsterdam. Translated from the French by J. P. Wilson. MR0745449

[10] DEREICH, S. and HEIDENREICH, F. (2011). A multilevel Monte Carlo algorithm for Lévy-driven stochastic differential equations. *Stochastic Process. Appl.* **121** 1565–1587. MR2802466 https://doi.org/10.1016/j.spa.2011.03.015

MSC2020 subject classifications. Primary 60H05, 91G20; secondary 60H07, 60G51.

Key words and phrases. Approximation of stochastic integral, convergence rate, Lévy process, semimartingale, weighted bounded mean oscillation.

[11] FUKASAWA, M. (2011). Asymptotically efficient discrete hedging. In *Stochastic Analysis with Financial Applications*. *Progress in Probability* **65** 331–346. Birkhäuser, Basel. MR3050797 https://doi.org/10.1007/978-3-0348-0097-6_19

[12] FUKASAWA, M. (2014). Efficient discretization of stochastic integrals. *Finance Stoch*. **18** 175–208. MR3146491 https://doi.org/10.1007/s00780-013-0215-6

[13] GEISS, C. and GEISS, S. (2004). On approximation of a class of stochastic integrals and interpolation. *Stoch*. *Stoch*. *Rep*. **76** 339–362. MR2075477 https://doi.org/10.1080/10451120410001728445

[14] GEISS, C., GEISS, S. and LAUKKARINEN, E. (2013). A note on Malliavin fractional smoothness for Lévy processes and approximation. *Potential Anal*. **39** 203–230. MR3102985 https://doi.org/10.1007/s11118-012-9326-5

[15] GEISS, S. (2002). Quantitative approximation of certain stochastic integrals. *Stoch*. *Stoch*. *Rep*. **73** 241–270. MR1932161 https://doi.org/10.1080/1045112021000025934

[16] GEISS, S. (2005). Weighted BMO and discrete time hedging within the Black–Scholes model. *Probab. Theory Related Fields* **132** 13–38. MR2136865 https://doi.org/10.1007/s00440-004-0389-0

[17] GEISS, S. and THUAN, N. T. (2021). On Riemann–Liouville operators, BMO, gradient estimates in the Lévy–Itô space, and approximation. Available at arXiv:2009.00899v2.

[18] GEISS, S. and TOIVOLA, A. (2009). Weak convergence of error processes in discretizations of stochastic integrals and Besov spaces. *Bernoulli* **15** 925–954. MR2597578 https://doi.org/10.3150/09-BEJ197

[19] GEISS, S. and TOIVOLA, A. (2015). On fractional smoothness and L_p-approximation on the Gaussian space. *Ann*. *Probab*. **43** 605–638. MR3306001 https://doi.org/10.1214/13-AOP884

[20] GEISS, S. and YLINEN, J. (2020). Weighted bounded mean oscillation applied to backward stochastic differential equations. *Stochastic Process*. *Appl*. **130** 3711–3752. MR4092418 https://doi.org/10.1016/j.spa.2019.10.007

[21] GOBET, E. and LANDON, N. (2014). Almost sure optimal hedging strategy. *Ann*. *Appl*. *Probab*. **24** 1652–1690. MR3211007 https://doi.org/10.1214/13-AAP959

[22] GOBET, E. and TEMAM, E. (2001). Discrete time hedging errors for options with irregular payoffs. *Finance Stoch*. **5** 357–367. MR1849426 https://doi.org/10.1007/PL00013539

[23] HUBALEK, F., KALLSEN, J. and KRAWCZYK, L. (2006). Variance-optimal hedging for processes with stationary independent increments. *Ann*. *Appl*. *Probab*. **16** 853–885. MR2244435 https://doi.org/10.1214/105051606000000178

[24] JACOD, J., MÉLÉARD, S. and PROTTER, P. (2000). Explicit form and robustness of martingale representations. *Ann*. *Probab*. **28** 1747–1780. MR1813842 https://doi.org/10.1214/aop/1019160506

[25] JACOD, J. and SHIRYAEV, A. N. (2003). *Limit Theorems for Stochastic Processes*, 2nd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **288**. Springer, Berlin. MR1943877 https://doi.org/10.1007/978-3-662-05265-5

[26] KOHATSU-HIGA, A. and TANKOV, P. (2010). Jump-adapted discretization schemes for Lévy-driven SDEs. *Stochastic Process*. *Appl*. **120** 2258–2285. MR2684745 https://doi.org/10.1016/j.spa.2010.07.001

[27] MIKULEVICIUS, R. (2012). On the rate of convergence of simple and jump-adapted weak Euler schemes for Lévy driven SDEs. *Stochastic Process*. *Appl*. **122** 2730–2757. MR2926173 https://doi.org/10.1016/j.spa.2012.04.013

[28] PLATEN, E. and BRUTI-LIBERATI, N. (2010). *Numerical Solution of Stochastic Differential Equations with Jumps in Finance*. *Stochastic Modelling and Applied Probability* **64**. Springer, Berlin. MR2723480 https://doi.org/10.1007/978-3-642-13694-8

[29] PROTTER, P. E. (2004). *Stochastic Integration and Differential Equations*: *Stochastic Modelling and Applied Probability*, 2nd ed. *Applications of Mathematics* (*New York*) **21**. Springer, Berlin. MR2020294

[30] ROSENBAUM, M. and TANKOV, P. (2014). Asymptotically optimal discretization of hedging strategies with jumps. *Ann*. *Appl*. *Probab*. **24** 1002–1048. MR3199979 https://doi.org/10.1214/13-AAP940

[31] SATO, K. (2013). *Lévy Processes and Infinitely Divisible Distributions*. *Cambridge Studies in Advanced Mathematics* **68**. Cambridge Univ. Press, Cambridge. Translated from the 1990 Japanese original, Revised edition of the 1999 English translation. MR3185174

[32] SCHOUTENS, W. (2003). *Lévy Processes in Finance*: *Pricing Financial Derivatives*. Wiley, New York.

[33] SCHWEIZER, M. (2001). A guided tour through quadratic hedging approaches. In *Option Pricing*, *Interest Rates and Risk Management*. *Handb*. *Math*. *Finance* 538–574. Cambridge Univ. Press, Cambridge. MR1848562 https://doi.org/10.1017/CBO9780511569708.016

[34] TANKOV, P. (2011). Pricing and hedging in exponential Lévy models: Review of recent results. In *Paris–Princeton Lectures on Mathematical Finance* 2010. *Lecture Notes in Math*. **2003** 319–359. Springer, Berlin. MR2762364 https://doi.org/10.1007/978-3-642-14660-2_5

[35] TANKOV, P. and VOLTCHKOVA, E. (2009). Asymptotic analysis of hedging errors in models with jumps. *Stochastic Process*. *Appl*. **119** 2004–2027. MR2519354 https://doi.org/10.1016/j.spa.2008.10.002

[36] THUAN, N. T. (2022). Explicit Föllmer–Schweizer decomposition and discretization with jump correction in exponential Lévy models. Available at arXiv:2009.04328v2.

[37] THUAN, N. T. (2024). Supplement to "Approximation of stochastic integrals with jumps via weighted BMO approach." https://doi.org/10.1214/24-AAP2075SUPP

The Annals of Applied Probability
2024, Vol. 34, No. 5, 4635–4693
https://doi.org/10.1214/24-AAP2076
This research was funded, in whole or in part, by FWF, P 34142. A CC BY 4.0 license is applied to this article arising from this submission, in accordance with the grant's open access conditions.

PHASE TRANSITIONS OF COMPOSITION SCHEMES: MITTAG-LEFFLER AND MIXED POISSON DISTRIBUTIONS

BY CYRIL BANDERIER[1], MARKUS KUBA[2,a] AND MICHAEL WALLNER[3,b]

This article is kindly devoted to Alois Panholzer, on the occasion of his 50th birthday.

[1]*Laboratoire d'Informatique de Paris-Nord, Université Sorbonne Paris Nord*
[2]*Dept. of Applied Mathematics and Physics, University of Applied Sciences Technikum Wien,* [a]*kuba@technikum-wien.at*
[3]*Institute of Discrete Mathematics and Geometry, Technische Universität Wien,* [b]*michael.wallner@tuwien.ac.at*

Multitudinous probabilistic and combinatorial objects are associated with generating functions satisfying a composition scheme $F(z) = G(H(z))$. The analysis becomes challenging when this scheme is critical (i.e., G and H are simultaneously singular). Motivated by many examples (random mappings, planar maps, directed lattice paths), we consider a natural extension of this scheme, namely $F(z, u) = G(uH(z))M(z)$. We also consider a variant of this scheme, which allows us to analyse the number of H-components of a given size in F.

We prove that these two models lead to a rich world of limit laws, where we identify the key role played by a new universal law introduced in this article: the three-parameter Mittag-Leffler distribution, which is essentially the product of a beta and a Mittag-Leffler distribution. We also prove (double) phase transitions, additionally involving Boltzmann and mixed Poisson distributions, bringing a unified explanation of the associated thresholds. In all cases we obtain moment convergence and local limit theorems. We end with extensions of the critical composition scheme to a cycle scheme and to the multivariate case, leading to product distributions. Applications are presented for random walks, trees (supertrees of trees, increasingly labelled trees, preferential attachment trees), triangular Pólya urns, and the Chinese restaurant process.

REFERENCES

[1] ADDARIO-BERRY, L. (2019). A probabilistic approach to block sizes in random maps. *ALEA Lat. Amer. J. Probab. Math. Stat.* **16** 1–13. MR3903022 https://doi.org/10.30757/alea.v16-01
[2] ADELL, J. A. and DE LA CAL, J. (1993). On the uniform convergence of normalized Poisson mixtures to their mixing distribution. *Statist. Probab. Lett.* **18** 227–232. MR1241619 https://doi.org/10.1016/0167-7152(93)90221-4
[3] ALDOUS, D. J. (1985). Exchangeability and related topics. In *École d'été de Probabilités de Saint-Flour, XIII*—1983. *Lecture Notes in Math.* **1117** 1–198. Springer, Berlin. MR0883646 https://doi.org/10.1007/BFb0099421
[4] ARRATIA, R., BARBOUR, A. D. and TAVARÉ, S. (2003). *Logarithmic Combinatorial Structures: A Probabilistic Approach. EMS Monographs in Mathematics*. Eur. Math. Soc., Zürich. MR2032426 https://doi.org/10.4171/000
[5] BANDERIER, C. and DRMOTA, M. (2015). Formulae and asymptotics for coefficients of algebraic functions. *Combin. Probab. Comput.* **24** 1–53. MR3318039 https://doi.org/10.1017/S0963548314000728
[6] BANDERIER, C. and FLAJOLET, P. (2002). Basic analytic combinatorics of directed lattice paths. *Theoret. Comput. Sci.* **281** 37–80. Selected papers in honour of Maurice Nivat. MR1909568 https://doi.org/10.1016/S0304-3975(02)00007-5
[7] BANDERIER, C., FLAJOLET, P., SCHAEFFER, G. and SORIA, M. (2001). Random maps, coalescing saddles, singularity analysis, and Airy phenomena. *Random Structures Algorithms* **19** 194–246. MR1871555 https://doi.org/10.1002/rsa.10021

MSC2020 subject classifications. Primary 60C05; secondary 60E07, 60G50, 05A15.
Key words and phrases. Mixed Poisson distributions, Mittag-Leffler distributions, stable laws, Boltzmann distributions, analytic combinatorics, generating functions, singularity analysis, critical composition schemes.

[8] Banderier, C. and Hitczenko, P. (2012). Enumeration and asymptotics of restricted compositions having the same number of parts. *Discrete Appl. Math.* **160** 2542–2554. MR2971338 https://doi.org/10.1016/j.dam.2011.12.011

[9] Banderier, C., Kuba, M. and Panholzer, A. (2009). Analysis of three graph parameters for random trees. *Random Structures Algorithms* **35** 42–69. MR2532873 https://doi.org/10.1002/rsa.20259

[10] Banderier, C., Kuba, M., Wagner, S. and Wallner, M. (2024). Composition Schemes: Q-Enumerations and Phase Transitions in Gibbs Models. In *35th International Conference on Probabilistic, Combinatorial and Asymptotic Methods for the Analysis of Algorithms* (*AofA* 2024). *LIPIcs. Leibniz Int. Proc. Inform.* **302** Art. No. 7, 18. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern. MR4774655 https://doi.org/10.4230/lipics.aofa.2024.7

[11] Banderier, C., Kuba, M. and Wallner, M. (2024). Phase transitions of composition schemes II: Stable laws and Boltzmann distributions. In *Preparation*.

[12] Banderier, C., Marchal, P. and Wallner, M. (2020). Periodic Pólya urns, the density method and asymptotics of Young tableaux. *Ann. Probab.* **48** 1921–1965. MR4124529 https://doi.org/10.1214/19-AOP1411

[13] Banderier, C. and Wallner, M. (2014). Some reflections on directed lattice paths. In *Proceedings of the 25th International Conference on Probabilistic, Combinatorial and Asymptotic Methods for the Analysis of Algorithms*. *Discrete Math. Theor. Comput. Sci. Proc., BA* 25–36. Assoc. Discrete Math. Theor. Comput. Sci., Nancy. MR3525961

[14] Banderier, C. and Wallner, M. (2017). Lattice paths with catastrophes. *Discrete Math. Theor. Comput. Sci.* **19** Paper No. 23, 32. MR3711680

[15] Banderier, C. and Wallner, M. (2018). Local time for lattice paths and the associated limit laws. In *GASCom 2018—Random and Exhaustive Generation of Combinatorial Structures* 69–78. CEUR Workshop Proceedings, Aachen. MR4636548

[16] Banderier, C. and Wallner, M. (2019). The kernel method for lattice paths below a line of rational slope. In *Lattice Path Combinatorics and Applications*. *Dev. Math.* **58** 119–154. Springer, Cham. MR3930454

[17] Bender, E. A. (1973). Central and local limit theorems applied to asymptotic enumeration. *J. Combin. Theory Ser. A* **15** 91–111. MR0375433 https://doi.org/10.1016/0097-3165(73)90038-1

[18] Bergeron, F., Flajolet, P. and Salvy, B. (1992). Varieties of increasing trees. In *CAAP '92* (*Rennes*, 1992). *Lecture Notes in Computer Science* **581** 24–48. Springer, Berlin. MR1251994 https://doi.org/10.1007/3-540-55251-0_2

[19] Bernstein, M. and Sloane, N. J. A. (1995). Some canonical sequences of integers. *Linear Algebra Appl.* **226/228** 57–72. MR1344554 https://doi.org/10.1016/0024-3795(94)00245-9

[20] Bodini, O., Dien, M., Fontaine, X., Genitrini, A. and Hwang, H.-K. (2016). Increasing diamonds. In *LATIN 2016: Theoretical Informatics*. *Lecture Notes in Computer Science* **9644** 207–219. Springer, Berlin. MR3492529 https://doi.org/10.1007/978-3-662-49529-2_16

[21] Bodini, O., Roussel, O. and Soria, M. (2012). Boltzmann samplers for first-order differential specifications. *Discrete Appl. Math.* **160** 2563–2572. MR2971340 https://doi.org/10.1016/j.dam.2012.05.022

[22] Bousquet-Mélou, M. (2006). Limit laws for embedded trees: Applications to the integrated superBrownian excursion. *Random Structures Algorithms* **29** 475–523. MR2268233 https://doi.org/10.1002/rsa.20136

[23] Capelas de Oliveira, E., Mainardi, F. and Vaz, J. Jr. (2011). Models based on Mittag-Leffler functions for anomalous relaxation in dielectrics. *Eur. Phys. J. Spec. Top.* **193** 161–171.

[24] Carleman, T. (1923). Sur les équations intégrales singulières à noyau réel et symétrique. Uppsala Universitets Årsskrift.

[25] Darling, D. A. and Kac, M. (1957). On occupation times for Markoff processes. *Trans. Amer. Math. Soc.* **84** 444–458. MR0084222 https://doi.org/10.2307/1992825

[26] DeVries, T. (2011). Algorithms for Bivariate Singularity Analysis. PhD thesis, Univ. of Pennsylvania. MR2898631

[27] Dolera, E. and Favaro, S. (2020). A Berry–Esseen theorem for Pitman's α-diversity. *Ann. Appl. Probab.* **30** 847–869. MR4108124 https://doi.org/10.1214/19-AAP1518

[28] Drmota, M. (2009). *Random Trees: An Interplay Between Combinatorics and Probability*. Springer, Vienna. MR2484382 https://doi.org/10.1007/978-3-211-75357-6

[29] Drmota, M. and Soria, M. (1995). Marking in combinatorial constructions: Generating functions and limiting distributions. *Theoret. Comput. Sci.* **144** 67–99. Special volume on mathematical analysis of algorithms. MR1337754 https://doi.org/10.1016/0304-3975(94)00294-S

[30] Drmota, M. and Soria, M. (1997). Images and preimages in random mappings. *SIAM J. Discrete Math.* **10** 246–269. MR1445035 https://doi.org/10.1137/S0895480194268421

[31] DUBOURDIEU, J. (1939). *Les Principes Fondamentaux du Calcul des Probabilités et la Théorie de L'assurance-maladie*. Gauthier-Villars, Paris.

[32] DUCHON, P., FLAJOLET, P., LOUCHARD, G. and SCHAEFFER, G. (2004). Boltzmann samplers for the random generation of combinatorial structures. *Combin. Probab. Comput.* **13** 577–625. MR2095975 https://doi.org/10.1017/S0963548304006315

[33] FELLER, W. (1949). Fluctuation theory of recurrent events. *Trans. Amer. Math. Soc.* **67** 98–119. MR0032114 https://doi.org/10.2307/1990420

[34] FELLER, W. (1971). *An Introduction to Probability Theory and Its Applications*, *Vol. II*, 2nd ed. Wiley, New York. MR0386954

[35] FERRARI, A., LETAC, G. and TOURNERET, J.-Y. (2004). Multivariate mixed Poisson distributions. In 12*th European Signal Processing Conference*, *IEEE* 1067–1070.

[36] FILL, J. A., FLAJOLET, P. and KAPUR, N. (2005). Singularity analysis, Hadamard products, and tree recurrences. *J. Comput. Appl. Math.* **174** 271–313. MR2106441 https://doi.org/10.1016/j.cam.2004.04.014

[37] FLAJOLET, P. (2006). Some exactly solvable models of urn process theory. Slides for the conference Math-info in Nancy.

[38] FLAJOLET, P., DUMAS, P. and PUYHAUBERT, V. (2006). Some exactly solvable models of urn process theory. In *Fourth Colloquium on Mathematics and Computer Science Algorithms*, *Trees*, *Combinatorics and Probabilities*. *Discrete Math. Theor. Comput. Sci. Proc.*, *AG* 59–118. Assoc. Discrete Math. Theor. Comput. Sci., Nancy. MR2509623

[39] FLAJOLET, P., GABARRÓ, J. and PEKARI, H. (2005). Analytic urns. *Ann. Probab.* **33** 1200–1233. MR2135318 https://doi.org/10.1214/009117905000000026

[40] FLAJOLET, P. and ODLYZKO, A. (1990). Singularity analysis of generating functions. *SIAM J. Discrete Math.* **3** 216–240. MR1039294 https://doi.org/10.1137/0403019

[41] FLAJOLET, P. and SORIA, M. (1990). Gaussian limiting distributions for the number of components in combinatorial structures. *J. Combin. Theory Ser. A* **53** 165–182. MR1041444 https://doi.org/10.1016/0097-3165(90)90056-3

[42] FLAJOLET, P. and SORIA, M. (1993). General combinatorial schemas: Gaussian limit distributions and exponential tails. *Discrete Math.* **114** 159–180. MR1217750 https://doi.org/10.1016/0012-365X(93)90364-Y

[43] FRÉCHET, M. and SHOHAT, J. (1931). A proof of the generalized second-limit theorem in the theory of probability. *Trans. Amer. Math. Soc.* **33** 533–543. MR1501604 https://doi.org/10.2307/1989421

[44] FUCHS, M. (2008). Subtree sizes in recursive trees and binary search trees: Berry–Esseen bounds and Poisson approximations. *Combin. Probab. Comput.* **17** 661–680. MR2454562 https://doi.org/10.1017/S0963548308009243

[45] FUCHS, M. (2012). Limit theorems for subtree size profiles of increasing trees. *Combin. Probab. Comput.* **21** 412–441. MR2912789 https://doi.org/10.1017/S096354831100071X

[46] FUCHS, M., HWANG, H.-K. and NEININGER, R. (2006). Profiles of random trees: Limit theorems for random recursive trees and binary search trees. *Algorithmica* **46** 367–407. MR2291961 https://doi.org/10.1007/s00453-006-0109-5

[47] GARRA, R. and GARRAPPA, R. (2018). The Prabhakar or three parameter Mittag-Leffler function: Theory and application. *Commun. Nonlinear Sci. Numer. Simul.* **56** 314–329. MR3709831 https://doi.org/10.1016/j.cnsns.2017.08.018

[48] GIMÉNEZ, O. and NOY, M. (2009). Asymptotic enumeration and limit laws of planar graphs. *J. Amer. Math. Soc.* **22** 309–329. MR2476775 https://doi.org/10.1090/S0894-0347-08-00624-3

[49] GIMÉNEZ, O., NOY, M. and RUÉ, J. (2013). Graph classes with given 3-connected components: Asymptotic enumeration and random graphs. *Random Structures Algorithms* **42** 438–479. MR3068033 https://doi.org/10.1002/rsa.20421

[50] GOLDSCHMIDT, C. and HAAS, B. (2015). A line-breaking construction of the stable trees. *Electron. J. Probab.* **20** no. 16, 24. MR3317158 https://doi.org/10.1214/EJP.v20-3690

[51] GOLDSCHMIDT, C., HAAS, B. and SÉNIZERGUES, D. (2022). Stable graphs: Distributions and line-breaking construction. *Ann. Henri Lebesgue* **5** 841–904. MR4526241 https://doi.org/10.5802/ahl.138

[52] GORENFLO, R., KILBAS, A. A., MAINARDI, F. and ROGOSIN, S. V. (2014). *Mittag-Leffler Functions*, *Related Topics and Applications*. *Springer Monographs in Mathematics*. Springer, Heidelberg. MR3244285 https://doi.org/10.1007/978-3-662-43930-2

[53] GÓRSKA, K., HORZELA, A., BRATEK, Ł., DATTOLI, G. and PENSON, K. A. (2018). The Havriliak–Negami relaxation and its relatives: The response, relaxation and probability density functions. *J. Phys. A* **51** 135202, 15. MR3780344 https://doi.org/10.1088/1751-8121/aaafc0

[54] GÓRSKA, K., HORZELA, A., LATTANZI, A. and POGÁNY, T. K. (2021). On complete monotonicity of three parameter Mittag-Leffler function. *Appl. Anal. Discrete Math.* **15** 118–128. MR4250089 https://doi.org/10.2298/aadm190226025g

[55] GOUET, R. (1993). Martingale functional central limit theorems for a generalized Pólya urn. *Ann. Probab.* **21** 1624–1639. MR1235432

[56] GOURDON, X. (1998). Largest component in random combinatorial structures. *Discrete Math.* **180** 185–209. MR1603725 https://doi.org/10.1016/S0012-365X(97)00115-5

[57] GRANDELL, J. (1997). *Mixed Poisson Processes. Monographs on Statistics and Applied Probability* **77**. CRC Press, London. MR1463943 https://doi.org/10.1007/978-1-4899-3117-7

[58] HARRIS, T. E. (1963). *The Theory of Branching Processes. Die Grundlehren der Mathematischen Wissenschaften, Band* 119. Springer, Berlin. MR0163361

[59] HUMBERT, P. (1945). Nouvelles correspondances symboliques. *Bull. Sci. Math.* **69** 121–129. MR0015564

[60] HWANG, H.-K. (1999). Asymptotics of Poisson approximation to random discrete distributions: An analytic approach. *Adv. in Appl. Probab.* **31** 448–491. MR1724562 https://doi.org/10.1239/aap/1029955143

[61] HWANG, H.-K. (2003). Second phase changes in random m-ary search trees and generalized quicksort: Convergence rates. *Ann. Probab.* **31** 609–629. MR1964943 https://doi.org/10.1214/aop/1048516530

[62] HWANG, H.-K. (2004). Phase changes in random recursive structures and algorithms. In *Probability, Finance and Insurance* 82–97. World Scientific, River Edge, NJ. MR2189200

[63] HWANG, H.-K. and ZACHAROVAS, V. (2010). Uniform asymptotics of Poisson approximation to the Poisson-binomial distribution. *Teor. Veroyatn. Primen.* **55** 305–334. MR2768906 https://doi.org/10.1137/S0040585X97984784

[64] IBRAGIMOV, I. A. and ČERNIN, K. E. (1959). On the unimodality of stable laws. *Teor. Veroyatn. Primen.* **4** 453–456. MR0116382

[65] JAMES, L. F. (2010). Lamperti-type laws. *Ann. Appl. Probab.* **20** 1303–1340. MR2676940 https://doi.org/10.1214/09-AAP660

[66] JAMES, L. F. (2013). Stick-breaking $PG(\alpha, \zeta)$-generalized gamma processes. Available at arXiv:1308.6570.

[67] JAMES, L. F. (2015). Generalized Mittag Leffler distributions arising as limits in preferential attachment models. Available at arXiv:1509.07150.

[68] JANSON, S. (2004). Functional limit theorems for multitype branching processes and generalized Pólya urns. *Stochastic Process. Appl.* **110** 177–245. MR2040966 https://doi.org/10.1016/j.spa.2003.12.002

[69] JANSON, S. (2005). Asymptotic degree distribution in random recursive trees. *Random Structures Algorithms* **26** 69–83. MR2116576 https://doi.org/10.1002/rsa.20046

[70] JANSON, S. (2006). Limit theorems for triangular urn schemes. *Probab. Theory Related Fields* **134** 417–452. MR2226887 https://doi.org/10.1007/s00440-005-0442-7

[71] JANSON, S. (2006). Random cutting and records in deterministic and random trees. *Random Structures Algorithms* **29** 139–179. MR2245498 https://doi.org/10.1002/rsa.20086

[72] JANSON, S. (2010). Addendum to Moments of Gamma type and the Brownian supremum process area [MR2645216]. *Probab. Surv.* **7** 207–208. MR2771251 https://doi.org/10.1214/10-PS169

[73] JANSON, S., KNUTH, D. E., ŁUCZAK, T. and PITTEL, B. (1993). The birth of the giant component. *Random Structures Algorithms* **4** 231–358. MR1220220 https://doi.org/10.1002/rsa.3240040303

[74] JANSON, S., KUBA, M. and PANHOLZER, A. (2011). Generalized Stirling permutations, families of increasing trees and urn models. *J. Combin. Theory Ser. A* **118** 94–114. MR2737187 https://doi.org/10.1016/j.jcta.2009.11.006

[75] JANSON, S. and POUYANNE, N. (2018). Moment convergence of balanced Pólya processes. *Electron. J. Probab.* **23** Paper No. 34, 13. MR3798244 https://doi.org/10.1214/17-EJP80

[76] KARLIS, D. (2005). EM algorithm for mixed Poisson and other discrete distributions. *Astin Bull.* **35** 3–24. MR2142681 https://doi.org/10.2143/AST.35.1.583163

[77] KNUTH, D. E. (1998). *The Art of Computer Programming, Vol.* 3: *Sorting and Searching*. Addison-Wesley, Reading, MA. Second edition [of MR0445948]. MR3077154

[78] KOLCHIN, V. F. (1999). *Random Graphs. Encyclopedia of Mathematics and Its Applications* **53**. Cambridge Univ. Press, Cambridge. MR1728076

[79] KUBA, M. and PANHOLZER, A. (2006). Analysis of label-based parameters in increasing trees. In *Fourth Colloquium on Mathematics and Computer Science Algorithms, Trees, Combinatorics and Probabilities. Discrete Math. Theor. Comput. Sci. Proc., AG* 321–330. Assoc. Discrete Math. Theor. Comput. Sci., Nancy. MR2509642

[80] KUBA, M. and PANHOLZER, A. (2006). Descendants in increasing trees. *Electron. J. Combin.* **13** Research Paper 8, 14. MR2200536 https://doi.org/10.37236/1034

[81] KUBA, M. and PANHOLZER, A. (2007). On the degree distribution of the nodes in increasing trees. *J. Combin. Theory Ser. A* **114** 597–618. MR2319165 https://doi.org/10.1016/j.jcta.2006.08.003
[82] KUBA, M. and PANHOLZER, A. (2010). A combinatorial approach to the analysis of bucket recursive trees. *Theoret. Comput. Sci.* **411** 3255–3273. MR2676868 https://doi.org/10.1016/j.tcs.2010.05.030
[83] KUBA, M. and PANHOLZER, A. (2011). Analysis of statistics for generalized Stirling permutations. *Combin. Probab. Comput.* **20** 875–910. MR2847273 https://doi.org/10.1017/S0963548311000381
[84] KUBA, M. and PANHOLZER, A. (2012). Bilabelled increasing trees and hook-length formulae. *European J. Combin.* **33** 248–258. MR2854646 https://doi.org/10.1016/j.ejc.2011.09.043
[85] KUBA, M. and PANHOLZER, A. (2012). Limiting distributions for a class of diminishing urn models. *Adv. in Appl. Probab.* **44** 87–116. MR2951548 https://doi.org/10.1239/aap/1331216646
[86] KUBA, M. and PANHOLZER, A. (2012). On death processes and urn models. In 23*rd Intern. Meeting on Probabilistic, Combinatorial, and Asymptotic Methods for the Analysis of Algorithms* (*AofA'*12). *Discrete Math. Theor. Comput. Sci. Proc.*, *AQ* 29–41. Assoc. Discrete Math. Theor. Comput. Sci., Nancy. MR2957318
[87] KUBA, M. and PANHOLZER, A. (2016). Combinatorial families of multilabelled increasing trees and hook-length formulas. *Discrete Math.* **339** 227–254. MR3404485 https://doi.org/10.1016/j.disc.2015.08.010
[88] KUBA, M. and PANHOLZER, A. (2016). On moment sequences and mixed Poisson distributions. *Probab. Surv.* **13** 89–155. MR3550936 https://doi.org/10.1214/14-PS244
[89] KUBA, M. and PANHOLZER, A. (2022). On bucket increasing trees, clustered increasing trees and increasing diamonds. *Combin. Probab. Comput.* **31** 629–661. MR4439775 https://doi.org/10.1017/s0963548321000493
[90] KUBA, M. and SULZBACH, H. (2017). On martingale tail sums in affine two-color urn models with multiple drawings. *J. Appl. Probab.* **54** 96–117. MR3632608 https://doi.org/10.1017/jpr.2016.89
[91] LOÈVE, M. (1977). *Probability Theory. I*, 4th ed. *Graduate Texts in Mathematics* **45**. Springer, New York. MR0651017
[92] MAHMOUD, H. M. (2009). *Pólya Urn Models. Texts in Statistical Science Series*. CRC Press, Boca Raton, FL. MR2435823
[93] MAHMOUD, H. M., SMYTHE, R. T. and SZYMAŃSKI, J. (1993). On the structure of random plane-oriented recursive trees and their branches. *Random Structures Algorithms* **4** 151–176. MR1206674 https://doi.org/10.1002/rsa.3240040204
[94] MARCHAL, P. (2003). Constructing a sequence of random walks strongly converging to Brownian motion. In *Discrete Random Walks* (*Paris*, 2003). *Discrete Math. Theor. Comput. Sci. Proc.*, *AC* 181–190. Assoc. Discrete Math. Theor. Comput. Sci., Nancy. MR2042386
[95] MARCHAL, P. (2008). A note on the fragmentation of a stable tree. In *Fifth Colloquium on Mathematics and Computer Science*. *Discrete Math. Theor. Comput. Sci. Proc.*, *AI* 489–499. Assoc. Discrete Math. Theor. Comput. Sci., Nancy. MR2508809
[96] MARCKERT, J.-F. (2023). Growing random uniform d-ary trees. *Ann. Comb.* **27** 51–66. MR4557916 https://doi.org/10.1007/s00026-022-00621-3
[97] MASSÉ, J.-C. and THEODORESCU, R. (2005). Neyman type A distribution revisited. *Stat. Neerl.* **59** 206–213. MR2153615 https://doi.org/10.1111/j.1467-9574.2005.00287.x
[98] MEIR, A. and MOON, J. W. (1970). Cutting down random trees. *J. Aust. Math. Soc.* **11** 313–324. MR0284370
[99] MIERMONT, G. (2009). Random maps and their scaling limits. In *Fractal Geometry and Stochastics IV. 4th Conference* 197–224. Birkhäuser, Basel.
[100] MITTAG-LEFFLER, G. (1903). Sur la nouvelle fonction $E\_\alpha(x)$. *C. R. Math. Acad. Sci. Paris* **137** 554–558.
[101] MITTAG-LEFFLER, G. (1903). Une généralisation de l'intégrale de Laplace–Abel. *C. R. Math. Acad. Sci. Paris* **136** 537–539.
[102] MÖHLE, M. (2021). A restaurant process with cocktail bar and relations to the three-parameter Mittag-Leffler distribution. *J. Appl. Probab.* **58** 978–1006. MR4342591 https://doi.org/10.1017/jpr.2021.10
[103] MONASSON, R. (2005). A generating function method for the average-case analysis of DPLL. In *Approximation, Randomization and Combinatorial Optimization. Lecture Notes in Computer Science* **3624** 402–413. Springer, Berlin. MR2193704 https://doi.org/10.1007/11538462_34
[104] MÓRI, T. F. (2005). The maximum degree of the Barabási-Albert random tree. *Combin. Probab. Comput.* **14** 339–348. MR2138118 https://doi.org/10.1017/S0963548304006133
[105] NEYMAN, J. (1939). On a new class of "contagious" distributions applicable in entomology and bacteriology. *Ann. Math. Stat.* **10** 35–57.
[106] NIST Digital Library of Mathematical Functions. Available at http://dlmf.nist.gov/, release 1.1.3 of 2021-09-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds.

[107] PANAGIOTOU, K., RAMZEWS, L. and STUFLER, B. (2023). Exact-size sampling of enriched trees in linear time. *SIAM J. Comput.* **52** 1097–1131. MR4649394 https://doi.org/10.1137/21M1459733

[108] PANHOLZER, A. (2003). Non-crossing trees revisited: Cutting down and spanning subtrees. In *Discrete Random Walks* (*Paris*, 2003). *Discrete Math. Theor. Comput. Sci. Proc., AC* 265–276. Assoc. Discrete Math. Theor. Comput. Sci., Nancy. MR2042393

[109] PANHOLZER, A. (2006). Cutting down very simple trees. *Quaest. Math.* **29** 211–227. MR2233368 https://doi.org/10.2989/16073600609486160

[110] PANHOLZER, A. and PRODINGER, H. (2007). Level of nodes in increasing trees revisited. *Random Structures Algorithms* **31** 203–226. MR2343719 https://doi.org/10.1002/rsa.20161

[111] PANHOLZER, A. and SEITZ, G. (2012). Limiting distributions for the number of inversions in labelled tree families. *Ann. Comb.* **16** 847–870. MR3000449 https://doi.org/10.1007/s00026-012-0164-3

[112] PANHOLZER, A. and SEITZ, G. (2014). Ancestors and descendants in evolving k-tree models. *Random Structures Algorithms* **44** 465–489. MR3214201 https://doi.org/10.1002/rsa.20474

[113] PEKÖZ, E. A., RÖLLIN, A. and ROSS, N. (2013). Degree asymptotics with rates for preferential attachment random graphs. *Ann. Appl. Probab.* **23** 1188–1218. MR3076682 https://doi.org/10.1214/12-AAP868

[114] PEKÖZ, E. A., RÖLLIN, A. and ROSS, N. (2016). Generalized gamma approximation with rates for urns, walks and trees. *Ann. Probab.* **44** 1776–1816. MR3502594 https://doi.org/10.1214/15-AOP1010

[115] PEMANTLE, R. and WILSON, M. C. (2013). *Analytic Combinatorics in Several Variables. Cambridge Studies in Advanced Mathematics* **140**. Cambridge Univ. Press, Cambridge. MR3088495 https://doi.org/10.1017/CBO9781139381864

[116] PHILIPPE, F. and ROBERT, S. (2011). *Analytic Combinatorics*. Cambridge Univ. Press, Cambridge.

[117] PILLAI, R. N. (1990). On Mittag-Leffler functions and related distributions. *Ann. Inst. Statist. Math.* **42** 157–161. MR1054728 https://doi.org/10.1007/BF00050786

[118] PITMAN, J. (1995). Exchangeable and partially exchangeable random partitions. *Probab. Theory Related Fields* **102** 145–158. MR1337249 https://doi.org/10.1007/BF01213386

[119] PITMAN, J. (2006). *Combinatorial Stochastic Processes. Lecture Notes in Math.* **1875**. Springer, Berlin. Lectures from the 32nd Summer School on Probability Theory held in Saint-Flour, July 7–24, 2002. MR2245368

[120] PITMAN, J. and YOR, M. (1997). The two-parameter Poisson–Dirichlet distribution derived from a stable subordinator. *Ann. Probab.* **25** 855–900. MR1434129 https://doi.org/10.1214/aop/1024404422

[121] POLLARD, H. (1946). The representation of $e^{-x^{\lambda}}$ as a Laplace integral. *Bull. Amer. Math. Soc.* **52** 908–910. MR0018286 https://doi.org/10.1090/S0002-9904-1946-08672-3

[122] PRABHAKAR, T. R. (1971). A singular integral equation with a generalized Mittag Leffler function in the kernel. *Yokohama Math. J.* **19** 7–15. MR0293349

[123] PUYHAUBERT, V. (2005). Modèles d'urnes et phénomènes de seuils en combinatoire analytique. PhD thesis, École polytechnique, Palaiseau.

[124] RÖSLER, U. and RÜSCHENDORF, L. (2001). The contraction method for recursive algorithms. *Algorithmica* **29** 3–33. MR1887296 https://doi.org/10.1007/BF02679611

[125] SIBISI, N. K. (2024). Convolutions and mixtures of gamma, stable and Mittag-Leffler distributions. Available at arXiv:2402.15228.

[126] SKELLAM, J. G. and SHENTON, L. R. (1957). Distributions associated with random walk and recurrent events. *J. Roy. Statist. Soc. Ser. B* **19** 64–111 (discussion 111–118). MR0091570

[127] SPORTIELLO, A. (2021). Boltzmann sampling of irreducible context-free structures in linear time. Available at arXiv:2105.12881.

[128] STUFLER, B. (2024). Gibbs partitions: A comprehensive phase diagram. *Ann. Inst. Henri Poincaré Probab. Stat.* To appear.

[129] SU, C., FENG, Q. and HU, Z. (2006). Uniform recursive trees: Branching structure and simple random downward walk. *J. Math. Anal. Appl.* **315** 225–243. MR2196543 https://doi.org/10.1016/j.jmaa.2005.05.004

[130] UCHAIKIN, V. V. and ZOLOTAREV, V. M. (1999). *Chance and Stability: Stable Distributions and Their Applications. Modern Probability and Statistics*. VSP, Utrecht. MR1745764 https://doi.org/10.1515/9783110935974

[131] WALLNER, M. (2020). A half-normal distribution scheme for generating functions. *European J. Combin.* **87** 103138, 21. MR4090824 https://doi.org/10.1016/j.ejc.2020.103138

[132] WHITTAKER, E. T. and WATSON, G. N. (1996). *A Course of Modern Analysis. Cambridge Mathematical Library*. Cambridge Univ. Press, Cambridge. Reprint of the fourth (1927) edition. MR1424469 https://doi.org/10.1017/CBO9780511608759

[133] WILLMOT, G. E. (1986). Mixed compound Poisson distributions. *Astin Bull.* **16** S59–S79.

[134] WILLMOT, G. E. and LIN, X. S. (2001). *Lundberg Approximations for Compound Distributions with Insurance Applications*. *Lecture Notes in Statistics* **156**. Springer, New York. MR1794566 https://doi.org/10.1007/978-1-4613-0111-0

[135] WIMAN, A. (1905). Über den Fundamentalsatz in der Teorie der Funktionen $E_a(x)$. *Acta Math*. **29** 191–201. MR1555014 https://doi.org/10.1007/BF02403202

The Annals of Applied Probability
2024, Vol. 34, No. 5, 4694–4708
https://doi.org/10.1214/24-AAP2077

IDENTIFYING THE DEVIATOR

BY NOGA ALON[1,a], BENJAMIN GUNBY[2,c], XIAOYU HE[1,b], ERAN SHMAYA[3,d] AND EILON SOLAN[4,e]

[1]*Department of Mathematics, Princeton University,* [a]*nalon@math.princeton.edu,* [b]*xiaoyuh@princeton.edu*
[2]*Department of Mathematics, Rutgers University,* [c]*bg570@connect.rutgers.edu*
[3]*Department of Economics, Stony Brook University,* [d]*eran.shmaya@stonybrook.edu*
[4]*The School of Mathematical Sciences, Tel Aviv University,* [e]*eilons@tauex.tau.ac.il*

A group of players are supposed to follow a prescribed profile of strategies. If they follow this profile, they will reach a given target. We show that if the target is not reached because some player deviates, then an outside observer can identify the deviator. We also construct identification methods in two nontrivial cases.

REFERENCES

[1] ABRAMOVICH, F. and RITOV, Y. (2013). *Statistical Theory*: *A Concise Introduction*. *Texts in Statistical Science Series*. CRC Press, Boca Raton, FL. MR3114871

[2] ASHKENAZI-GOLAN, G., FLESCH, J., PREDTETCHINSKI, A. and SOLAN, E. (2022). Existence of equilibria in repeated games with long-run payoffs. *Proc*. *Natl*. *Acad*. *Sci*. *USA* **119** Paper No. e2105867119, 8. MR4417583

[3] ASHKENAZI-GOLAN, G., FLESCH, J., PREDTETCHINSKI, A. and SOLAN, E. (2023). Regularity of the minmax value and equilibria in multiplayer Blackwell games. *Israel J*. *Math*. forthcoming.

[4] AUMANN, R. J. and SHAPLEY, L. S. (1994). Long-term competition—a game-theoretic analysis. In *Essays in Game Theory* (*Stony Brook*, *NY*, 1992) 1–15. Springer, New York. MR1300862

[5] BLACKWELL, D. (1983). A hypothesis-testing game without a value. In *A Festschrift for Erich L*. *Lehmann*. *Wadsworth Statist*./*Probab*. *Ser*. 79–82. Wadsworth, Belmont, CA. MR0689739

[6] DURRETT, R. (2019). *Probability—Theory and Examples*. 2th ed. *Cambridge Series in Statistical and Probabilistic Mathematics* **49**. Cambridge Univ. Press, Cambridge. MR3930614 https://doi.org/10.1017/9781108591034

[7] DVORETZKY, A. and ERDÖS, P. (1951). Some problems on random walk in space. In *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, 1950 353–367. Univ. California Press, Berkeley. MR0047272

[8] ERDŐS, P. and TAYLOR, S. J. (1963). Some problems concerning the structure of random walk paths. *Acta Math*. *Acad*. *Sci*. *Hung*. **11** 137–162. MR0121870 https://doi.org/10.1007/BF02020631

[9] FLESCH, J. and SOLAN, E. (2023). Repeated games with tail-measurable payoffs. *David Gale: Mathematical economist: Essays in appreciation on His 100th Birthday*. Monographs in Mathematical Economics. (Springer), forthcoming.

[10] FLESCH, J. and SOLAN, E. (2023). Stochastic games with general payoff functions. *Math*. *Oper*. *Res*. forthcoming.

[11] HA, C. W. (1980). Minimax and fixed point theorems. *Math*. *Ann*. **248** 73–77. MR0569411 https://doi.org/10.1007/BF01349255

[12] MAITRA, A. and SUDDERTH, W. (1998). Finitely additive stochastic games with Borel measurable payoffs. *Internat*. *J*. *Game Theory* **27** 257–267. MR1645524 https://doi.org/10.1007/s001820050071

[13] MARTIN, D. A. (1998). The determinacy of Blackwell games. *J*. *Symbolic Logic* **63** 1565–1581. MR1665779 https://doi.org/10.2307/2586667

[14] NASH, J. F. JR. (1950). Equilibrium points in n-person games. *Proc*. *Natl*. *Acad*. *Sci*. *USA* **36** 48–49. MR0031701 https://doi.org/10.1073/pnas.36.1.48

[15] RÉVÉSZ, P. (2013). *Random Walk in Random and Non-random Environments*, 3rd ed. World Scientific, Hackensack, NJ. MR3060348 https://doi.org/10.1142/8678

[16] SOLAN, E. (1999). Three-player absorbing games. *Math*. *Oper*. *Res*. **24** 669–698. MR1854248 https://doi.org/10.1287/moor.24.3.669

MSC2020 subject classifications. Primary 91A35; secondary 62C05, 62F07.
Key words and phrases. multiplayer games, detecting deviations, random walk, zero-sum games.

[17] VIEILLE, N. (2000). Solvable states in n-player stochastic games. *SIAM J. Control Optim.* **38** 1794–1804. MR1776656 https://doi.org/10.1137/S0363012998345937

The Annals of Applied Probability
2024, Vol. 34, No. 5, 4709–4757
https://doi.org/10.1214/24-AAP2078

VISCOSITY SOLUTIONS TO SECOND ORDER ELLIPTIC HAMILTON-JACOBI–BELLMAN EQUATIONS WITH INFINITE DELAY

BY JIANJUN ZHOU[a]

College of Science, Northwest A&F University, [a]*zhoujianjun@nwsuaf.edu.cn*

This paper introduces a notion of viscosity solutions for second order elliptic Hamilton–Jacobi–Bellman (HJB) equations with infinite delay associated with infinite-horizon optimal control problems for stochastic differential equations with infinite delay. We identify the value functional of optimal control problems as unique viscosity solution to associated second order elliptic HJB equations with infinite delay. We also show that our notion of viscosity solutions is consistent with the corresponding notion of classical solutions, and satisfies a stability property.

REFERENCES

[1] BORWEIN, J. M. and ZHU, Q. J. (2005). *Techniques of Variational Analysis*. *CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC* **20**. Springer, New York. MR2144010

[2] CARLIER, G. and TAHRAOUI, R. (2010). Hamilton–Jacobi–Bellman equations for the optimal control of a state equation with memory. *ESAIM Control Optim*. *Calc*. *Var.* **16** 744–763. MR2674635 https://doi.org/10.1051/cocv/2009024

[3] CONT, R. and FOURNIÉ, D.-A. (2010). Change of variable formulas for non-anticipative functionals on path space. *J. Funct*. *Anal*. **259** 1043–1072. MR2652181 https://doi.org/10.1016/j.jfa.2010.04.017

[4] CONT, R. and FOURNIÉ, D.-A. (2013). Functional Itô calculus and stochastic integral representation of martingales. *Ann*. *Probab.* **41** 109–133. MR3059194 https://doi.org/10.1214/11-AOP721

[5] CRANDALL, M. G., ISHII, H. and LIONS, P.-L. (1992). User's guide to viscosity solutions of second order partial differential equations. *Bull*. *Amer. Math*. *Soc*. (*N.S.*) **27** 1–67. MR1118699 https://doi.org/10.1090/S0273-0979-1992-00266-5

[6] CRANDALL, M. G. and LIONS, P.-L. (1983). Viscosity solutions of Hamilton–Jacobi equations. *Trans*. *Amer. Math*. *Soc*. **277** 1–42. MR0690039 https://doi.org/10.2307/1999343

[7] DUPIRE, B. (2019). Functional Itô calculus. *Quant*. *Finance* **19** 721–729. MR3939653 https://doi.org/10.1080/14697688.2019.1575974

[8] EKREN, I. (2017). Viscosity solutions of obstacle problems for fully nonlinear path-dependent PDEs. *Stochastic Process*. *Appl*. **127** 3966–3996. MR3718103 https://doi.org/10.1016/j.spa.2017.03.016

[9] EKREN, I., KELLER, C., TOUZI, N. and ZHANG, J. (2014). On viscosity solutions of path dependent PDEs. *Ann*. *Probab.* **42** 204–236. MR3161485 https://doi.org/10.1214/12-AOP788

[10] EKREN, I., TOUZI, N. and ZHANG, J. (2016). Viscosity solutions of fully nonlinear parabolic path dependent PDEs: Part I. *Ann*. *Probab.* **44** 1212–1253. MR3474470 https://doi.org/10.1214/14-AOP999

[11] EKREN, I., TOUZI, N. and ZHANG, J. (2016). Viscosity solutions of fully nonlinear parabolic path dependent PDEs: Part II. *Ann*. *Probab.* **44** 2507–2553. MR3531674 https://doi.org/10.1214/15-AOP1027

[12] ELSANOSI, I., ØKSENDAL, B. and SULEM, A. (2000). Some solvable stochastic control problems with delay. *Stoch*. *Stoch*. *Rep*. **71** 69–89. MR1813507 https://doi.org/10.1080/17442500008834259

[13] FABBRI, G., GOZZI, F. and ŚWIĘCH, A. (2017). Stochastic optimal control in infinite dimension. Dynamic programming and HJB equations. In *Probability Theory and Stochastic Modelling* **82**. Springer, Berlin.

[14] FEDERICO, S. (2011). A stochastic control problem with delay arising in a pension fund model. *Finance Stoch*. **15** 421–459. MR2833095 https://doi.org/10.1007/s00780-010-0146-4

[15] FEDERICO, S., GOLDYS, B. and GOZZI, F. (2010). HJB equations for the optimal control of differential equations with delays and state constraints, I: Regularity of viscosity solutions. *SIAM J. Control Optim*. **48** 4910–4937. MR2735510 https://doi.org/10.1137/09076742X

MSC2020 subject classifications. 35D40, 49L20, 49L25, 60H30, 93C23, 93E20.

Key words and phrases. Second order elliptic Hamilton–Jacobi–Bellman equations, infinite delay, viscosity solutions, optimal control, stochastic differential equations.

[16] FENGYING, W. and KE, W. (2007). The existence and uniqueness of the solution for stochastic functional differential equations with infinite delay. *J. Math. Anal. Appl.* **331** 516–531. MR2306021 https://doi.org/10.1016/j.jmaa.2006.09.020

[17] FLEMING, W. H. and SONER, H. M. (2006). *Controlled Markov Processes and Viscosity Solutions*, 2nd ed. *Stochastic Modelling and Applied Probability* **25**. Springer, New York. MR2179357

[18] GABAY, D. and GRASSELLI, M. (2012). Fair demographic risk sharing in defined contribution pension systems. *J. Econom. Dynam. Control* **36** 657–669. MR2882088 https://doi.org/10.1016/j.jedc.2011.12.002

[19] GOZZI, F. and MARINELLI, C. (2006). Stochastic optimal control of delay equations arising in advertising models. In *Stochastic Partial Differential Equations and Applications—VII*. *Lect. Notes Pure Appl. Math.* **245** 133–148. CRC Press/CRC, Boca Raton, FL. MR2227226 https://doi.org/10.1201/9781420028720.ch13

[20] GOZZI, F., MARINELLI, C. and SAVIN, S. (2009). On controlled linear diffusions with delay in a model of optimal advertising under uncertainty with memory effects. *J. Optim. Theory Appl.* **142** 291–321. MR2525792 https://doi.org/10.1007/s10957-009-9524-5

[21] GOZZI, F., ROUY, E. and ŚWIĘCH, A. (2000). Second order Hamilton–Jacobi equations in Hilbert spaces and stochastic boundary control. *SIAM J. Control Optim.* **38** 400–430. MR1741146 https://doi.org/10.1137/S0363012997324909

[22] HINO, Y., MURAKAMI, S. and NAITO, T. (1991). *Functional-Differential Equations with Infinite Delay*. *Lecture Notes in Math.* **1473**. Springer, Berlin. MR1122588 https://doi.org/10.1007/BFb0084432

[23] KOLMANOVSKIĬ, V. B. and SHAĬKHET, L. E. (1996). *Control of Systems with Aftereffect*. *Translations of Mathematical Monographs* **157**. Amer. Math. Soc., Providence, RI. MR1415834 https://doi.org/10.1090/mmono/157

[24] LIONS, P.-L. (1983). On the Hamilton–Jacobi–Bellman equations. *Acta Appl. Math.* **1** 17–41. MR0702045 https://doi.org/10.1007/BF02433840

[25] LIONS, P.-L. (1983). Optimal control of diffusion processes and Hamilton–Jacobi–Bellman equations. I. The dynamic programming principle and applications. *Comm. Partial Differential Equations* **8** 1101–1174. MR0709164 https://doi.org/10.1080/03605308308820297

[26] LIONS, P.-L. (1983). Optimal control of diffusion processes and Hamilton–Jacobi–Bellman equations. II. Viscosity solutions and uniqueness. *Comm. Partial Differential Equations* **8** 1229–1276. MR0709162 https://doi.org/10.1080/03605308308820301

[27] LIONS, P.-L. (1988). Viscosity solutions of fully nonlinear second-order equations and optimal stochastic control in infinite dimensions. I. The case of bounded stochastic evolutions. *Acta Math.* **161** 243–278. MR0971797 https://doi.org/10.1007/BF02392299

[28] LIONS, P.-L. (1989). Viscosity solutions of fully nonlinear second order equations and optimal stochastic control in infinite dimensions. II. Optimal control of Zakai's equation. In *Stochastic Partial Differential Equations and Applications*, *II* (*Trento*, 1988). *Lecture Notes in Math.* **1390** 147–170. Springer, Berlin. MR1019600 https://doi.org/10.1007/BFb0083943

[29] LIONS, P.-L. (1989). Viscosity solutions of fully nonlinear second-order equations and optimal stochastic control in infinite dimensions. III. Uniqueness of viscosity solutions for general second-order equations. *J. Funct. Anal.* **86** 1–18. MR1013931 https://doi.org/10.1016/0022-1236(89)90062-1

[30] MOHAMMED, S. E. A. (1984). *Stochastic Functional Differential Equations*. *Research Notes in Mathematics* **99**. Pitman, Boston, MA. MR0754561

[31] ØKSENDAL, B. and SULEM, A. (2001). A maximum principle for optimal control of stochastic systems with delay, with applications to finance. In *Optimal Control and Partial Differential Equations* 64–79. IOS, Amsterdam. MR3586761

[32] ØKSENDAL, B., SULEM, A. and ZHANG, T. (2011). Optimal control of stochastic delay equations and time-advanced backward stochastic differential equations. *Adv. in Appl. Probab.* **43** 572–596. MR2848391 https://doi.org/10.1239/aap/1308662493

[33] REN, Z. (2016). Viscosity solutions of fully nonlinear elliptic path dependent partial differential equations. *Ann. Appl. Probab.* **26** 3381–3414. MR3582806 https://doi.org/10.1214/16-AAP1178

[34] REN, Z. and ROSESTOLATO, M. (2020). Viscosity solutions of path-dependent PDEs with randomized time. *SIAM J. Math. Anal.* **52** 1943–1979. MR4088380 https://doi.org/10.1137/18M122666X

[35] REN, Z., TOUZI, N. and ZHANG, J. (2017). Comparison of viscosity solutions of fully nonlinear degenerate parabolic path-dependent PDEs. *SIAM J. Math. Anal.* **49** 4093–4116. MR3715377 https://doi.org/10.1137/16M1090338

[36] ŚWIĘCH, A. (1994). "Unbounded" second order partial differential equations in infinite-dimensional Hilbert spaces. *Comm. Partial Differential Equations* **19** 1999–2036. MR1301180 https://doi.org/10.1080/03605309408821080

[37] ŚWIĘCH, A. (2020). Viscosity solutions to HJB equations for boundary-noise and boundary-control problems. *SIAM J. Control Optim.* **58** 303–326. MR4053860 https://doi.org/10.1137/19M1257469

[38] YONG, J. and ZHOU, X. Y. (1999). *Stochastic Controls: Hamiltonian Systems and HJB Equations*. *Applications of Mathematics* (*New York*) **43**. Springer, New York. MR1696772 https://doi.org/10.1007/978-1-4612-1466-3

[39] ZHOU, J. (2018). A class of infinite-horizon stochastic delay optimal control problems and a viscosity solution to the associated HJB equation. *ESAIM Control Optim. Calc. Var.* **24** 639–676. MR3816408 https://doi.org/10.1051/cocv/2017042

[40] ZHOU, J. (2023). Viscosity solutions to second order path-dependent Hamilton–Jacobi–Bellman equations and applications. *Ann. Appl. Probab.* **33** 5564–5612. MR4677740 https://doi.org/10.1214/23-aap1954

The Annals of Applied Probability
2024, Vol. 34, No. 5, 4758–4798
https://doi.org/10.1214/24-AAP2079

DIFFERENTIABILITY OF QUADRATIC FORWARD-BACKWARD SDES WITH ROUGH DRIFT

BY PETER IMKELLER[1,a], RHOSS LIKIBI PELLAT[2,b] AND OLIVIER MENOUKEU-PAMEN[3,c]

[1]*Institut für Mathematik, Humboldt-Universität zu Berlin,* [a]*imkeller@mathematik.hu-berlin.de*
[2]*Research Center, African Institute for Mathematical Sciences (AIMS-Ghana),* [b]*rhoss@aims.edu.gh*
[3]*IFAM, Department of Mathematical Sciences, University of Liverpool,* [c]*menoukeu@liverpool.ac.uk*

In this paper, we consider quadratic forward-backward SDEs (QFBSDEs), for which the drift in the forward equation does not satisfy the standard globally Lipschitz condition and the driver of the backward system possesses nonlinearity of type $f(|y|)|z|^2$, where f is any locally integrable function. We prove both the Malliavin and classical differentiability of solutions to this type of QFBSDEs and provide representations of these derivatives processes. As a by-product, we derive a representation formula of the control variable Z_t as a conditional expectation of the terminal value, the driver and the Malliavin weights, when the drift term is only bounded and Hölder continuous. We study a numerical approximation of this system in the sense of Imkeller and Dos Reis (*Stochastic Process. Appl.* **120** (2010) 2286–2288) in which the authors assume that the drift is Lipschitz and the driver of the BSDE is globally Lipschitz in y and quadratic in the traditional sense in z (i.e., f is a positive constant). We show that the rate of convergence is the same as in (*Stochastic Process. Appl.* **120** (2010) 2286–2288).

REFERENCES

[1] ANKIRCHNER, S., IMKELLER, P. and DOS REIS, G. (2007). Classical and variational differentiability of BSDEs with quadratic growth. *Electron. J. Probab.* **12** 1418–1453. MR2354164 https://doi.org/10.1214/EJP.v12-462

[2] BAHLALI, K. (2020). A domination method for solving unbounded quadratic BSDEs. *Grad. J. Math.* **5** 20–36. MR4099031

[3] BAHLALI, K. (2021). BSDEs driven by $|z|^2/y$ and applicatioins to PDEs and decision theory, 10. Available at arXiv:1810.05664v3.

[4] BAHLALI, K., EDDAHBI, M. and OUKNINE, Y. (2017). Quadratic BSDE with $\mathbb{L}^2$-terminal data: Krylov's estimate, Itô-Krylov's formula and existence results. *Ann. Probab.* **45** 2377–2397. MR3693965 https://doi.org/10.1214/16-AOP1115

[5] BAÑOS, D., MEYER-BRANDIS, T., PROSKE, F. and DUEDAHL, S. (2017). Computing deltas without derivatives. *Finance Stoch.* **21** 509–549. MR3626624 https://doi.org/10.1007/s00780-016-0321-3

[6] BARRIEU, P. and EL KAROUI, N. (2013). Monotone stability of quadratic semimartingales with applications to unbounded general quadratic BSDEs. *Ann. Probab.* **41** 1831–1863. MR3098060 https://doi.org/10.1214/12-AOP743

[7] BOUCHARD, B. and TOUZI, N. (2004). Discrete-time approximation and Monte-Carlo simulation of backward stochastic differential equations. *Stochastic Process. Appl.* **111** 175–206. MR2056536 https://doi.org/10.1016/j.spa.2004.01.001

[8] BRIAND, P. and CONFORTOLA, F. (2008). BSDEs with stochastic Lipschitz condition and quadratic PDEs in Hilbert spaces. *Stochastic Process. Appl.* **118** 818–838. MR2411522 https://doi.org/10.1016/j.spa.2007.06.006

[9] BRIAND, P. and HU, Y. (2006). BSDE with quadratic growth and unbounded terminal value. *Probab. Theory Related Fields* **136** 604–618. MR2257138 https://doi.org/10.1007/s00440-006-0497-0

MSC2020 subject classifications. Primary 60H10; secondary 60H07, 65C30.
Key words and phrases. Quadratic forward-backward SDEs, BMO martingales, Malliavin calculus, stochastic flows, rough drift.

[10] BRIAND, P. and HU, Y. (2008). Quadratic BSDEs with convex generators and unbounded terminal conditions. *Probab. Theory Related Fields* **141** 543–567. MR2391164 https://doi.org/10.1007/s00440-007-0093-y

[11] CHASSAGNEUX, J.-F. and RICHOU, A. (2016). Numerical simulation of quadratic BSDEs. *Ann. Appl. Probab.* **26** 262–304. MR3449318 https://doi.org/10.1214/14-AAP1090

[12] CHEVANCED, D. (1997). Numerical methods for backward stochastic differential equations. In *Numerical Methods in Finance* (L. C. G. Rogers and D. Talay, eds.) 232-244. Cambridge Univ. Press, Cambridge. MR1470517

[13] DELBAEN, F., HU, Y. and RICHOU, A. (2011). On the uniqueness of solutions to quadratic BSDEs with convex generators and unbounded terminal conditions. *Ann. Inst. Henri Poincaré Probab. Stat.* **47** 559–574. MR2814423 https://doi.org/10.1214/10-AIHP372

[14] DI NUNNO, G., ØKSENDAL, B. and PROSKE, F. (2009). *Malliavin Calculus for Lévy Processes with Applications to Finance*. *Universitext*. Springer, Berlin. MR2460554 https://doi.org/10.1007/978-3-540-78572-9

[15] DOS REIS, G. (2010). On some properties of solutions to quadratic growth BSDE and applications to finance and insurance. PhD thesis, Humboldt Univ.

[16] EL KAROUI, N., PENG, S. and QUENEZ, M. C. (1997). Backward stochastic differential equations in finance. *Math. Finance* **7** 1–71. MR1434407 https://doi.org/10.1111/1467-9965.00022

[17] FLANDOLI, F., GUBINELLI, M. and PRIOLA, E. (2010). Well-posedness of the transport equation by stochastic perturbation. *Invent. Math.* **180** 1–53. MR2593276 https://doi.org/10.1007/s00222-009-0224-4

[18] FLANDOLI, F., GUBINELLI, M. and PRIOLA, E. (2010). Flow of diffeomorphisms for SDEs with unbounded Hölder continuous drift. *Bull. Sci. Math.* **134** 405–422. MR2651899 https://doi.org/10.1016/j.bulsci.2010.02.003

[19] FREI, C. and DOS REIS, G. (2013). Quadratic FBSDE with generalized Burgers' type nonlinearities, perturbations and large deviations. *Stoch. Dyn.* **13** 1250015, 37. MR3039415 https://doi.org/10.1142/S0219493712500153

[20] GEISS, C., GEISS, S. and GOBET, E. (2012). Generalized fractional smoothness and L_p-variation of BSDEs with non-Lipschitz terminal condition. *Stochastic Process. Appl.* **122** 2078–2116. MR2921973 https://doi.org/10.1016/j.spa.2012.02.006

[21] GEISS, S. and YLINEN, J. (2021). Decoupling on the Wiener space, related Besov spaces, and applications to BSDEs. *Mem. Amer. Math. Soc.* **272** v + 112. MR4320771 https://doi.org/10.1090/memo/1335

[22] GOBET, E. and MAKHLOUF, A. (2010). $\boldsymbol{L}_2$-time regularity of BSDEs with irregular terminal functions. *Stochastic Process. Appl.* **120** 1105–1132. MR2639740 https://doi.org/10.1016/j.spa.2010.03.003

[23] GOBET, E. and TURKEDJIEV, P. (2016). Approximation of backward stochastic differential equations using Malliavin weights and least-squares regression. *Bernoulli* **22** 530–562. MR3449792 https://doi.org/10.3150/14-BEJ667

[24] HU, Y., IMKELLER, P. and MÜLLER, M. (2005). Utility maximization in incomplete markets. *Ann. Appl. Probab.* **15** 1691–1712. MR2152241 https://doi.org/10.1214/105051605000000188

[25] HU, Y., NUALART, D. and SONG, X. (2011). Malliavin calculus for backward stochastic differential equations and application to numerical solutions. *Ann. Appl. Probab.* **21** 2379–2423. MR2895419 https://doi.org/10.1214/11-AAP762

[26] IMKELLER, P. and DOS REIS, G. (2010). Corrigendum to "Path regularity and explicit convergence rate for BSDE with truncated quadratic growth" [Stochastic Process. Appl. 120 (2010) 348–379] [MR2584898]. *Stochastic Process. Appl.* **120** 2286–2288. MR2684746 https://doi.org/10.1016/j.spa.2010.06.008

[27] IMKELLER, P. and DOS REIS, G. (2010). Path regularity and explicit convergence rate for BSDE with truncated quadratic growth. *Stochastic Process. Appl.* **120** 348–379. MR2584898 https://doi.org/10.1016/j.spa.2009.11.004

[28] IMKELLER, P., RÉVEILLAC, A. and RICHTER, A. (2012). Differentiability of quadratic BSDEs generated by continuous martingales. *Ann. Appl. Probab.* **22** 285–336. MR2932548 https://doi.org/10.1214/11-AAP769

[29] KAZAMAKI, N. (1994). *Continuous Exponential Martingales and BMO*. *Lecture Notes in Math.* **1579**. Springer, Berlin. MR1299529 https://doi.org/10.1007/BFb0073585

[30] KOBYLANSKI, M. (2000). Backward stochastic differential equations and partial differential equations with quadratic growth. *Ann. Probab.* **28** 558–602. MR1782267 https://doi.org/10.1214/aop/1019160253

[31] KUNITA, H. (1990). *Stochastic Flows and Stochastic Differential Equations*. *Cambridge Studies in Advanced Mathematics* **24**. Cambridge Univ. Press, Cambridge. MR1070361

[32] LÊ, K. (2020). A stochastic sewing lemma and applications. *Electron. J. Probab.* **25** Paper No. 38, 55. MR4089788 https://doi.org/10.1214/20-ejp442

[33] LIKIBI PELLAT, R., MENOUKEU PAMEN, O. and OUKNINE, Y. (2022). A class of quadratic forward-backward stochastic differential equations. *J. Math. Anal. Appl.* **514** Paper No. 126100, 39. MR4429584 https://doi.org/10.1016/j.jmaa.2022.126100

[34] MA, J. and ZHANG, J. (2002). Path regularity for solutions of backward stochastic differential equations. *Probab. Theory Related Fields* **122** 163–190. MR1894066 https://doi.org/10.1007/s004400100144

[35] MA, J. and ZHANG, J. (2002). Representation theorems for backward stochastic differential equations. *Ann. Appl. Probab.* **12** 1390–1418. MR1936598 https://doi.org/10.1214/aoap/1037125868

[36] MASTROLIA, T., POSSAMAÏ, D. and RÉVEILLAC, A. (2017). On the Malliavin differentiability of BSDEs. *Ann. Inst. Henri Poincaré Probab. Stat.* **53** 2817–2857. MR3531681 https://doi.org/10.1214/15-AOP1035

[37] MENOUKEU-PAMEN, O., MEYER-BRANDIS, T., NILSSEN, T., PROSKE, F. and ZHANG, T. (2013). A variational approach to the construction and Malliavin differentiability of strong solutions of SDE's. *Math. Ann.* **357** 761–799. MR3096525 https://doi.org/10.1007/s00208-013-0916-3

[38] MENOUKEU-PAMEN, O. and TANGPI, L. (2019). Strong solutions of some one-dimensional SDEs with random and unbounded drifts. *SIAM J. Math. Anal.* **51** 4105–4141. MR4021273 https://doi.org/10.1137/18M1218662

[39] MILLET, A., NUALART, D. and SANZ, M. (1989). Integration by parts and time reversal for diffusion processes. *Ann. Probab.* **17** 208–238. MR0972782

[40] MOHAMMED, S.-E. A., NILSSEN, T. K. and PROSKE, F. N. (2015). Sobolev differentiable stochastic flows for SDEs with singular coefficients: Applications to the transport equation. *Ann. Probab.* **43** 1535–1576. MR3342670 https://doi.org/10.1214/14-AOP909

[41] MORLAIS, M.-A. (2009). Quadratic BSDEs driven by a continuous martingale and applications to the utility maximization problem. *Finance Stoch.* **13** 121–150. MR2465489 https://doi.org/10.1007/s00780-008-0079-3

[42] NEGYESI, B., ANDERSSON, K. and OOSTERLEE, C. (2021). The one step Malliavin scheme: New discretization of BSDEs implemented with deep learning regressions, 04. Available at arXiv:2110.05421v1.

[43] NUALART, D. (2006). *The Malliavin Calculus and Related Topics*, 2nd ed. *Probability and Its Applications* (*New York*). Springer, Berlin. MR2200233

[44] PARDOUX, É. and PENG, S. (1992). Backward stochastic differential equations and quasilinear parabolic partial differential equations. In *Stochastic Partial Differential Equations and Their Applications* (*Charlotte*, *NC*, 1991). *Lect. Notes Control Inf. Sci.* **176** 200–217. Springer, Berlin. MR1176785 https://doi.org/10.1007/BFb0007334

[45] REVUZ, D. and YOR, M. (1999). *Continuous Martingales and Brownian Motion*, 3rd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **293**. Springer, Berlin. MR1725357 https://doi.org/10.1007/978-3-662-06400-9

[46] RICHOU, A. (2011). Numerical simulation of BSDEs with drivers of quadratic growth. *Ann. Appl. Probab.* **21** 1933–1964. MR2884055 https://doi.org/10.1214/10-AAP744

[47] TIAN, D. (2023). Pricing principle via Tsallis relative entropy in incomplete markets. *SIAM J. Financial Math.* **14** 250–278. MR4544543 https://doi.org/10.1137/22M1471614

[48] VERETENNIKOV, A. Y. (1979). Strong solutions of stochastic differential equations. *Teor. Veroyatn. Primen.* **24** 348–360. MR0532447

[49] VERETENNIKOV, A. Y. (1980). Strong solutions and explicit formulas for solutions of stochastic integral equations. *Mat. Sb.* (*N.S.*) **111** 434–452, 480. MR0568986

[50] ZHANG, J. (2004). A numerical scheme for BSDEs. *Ann. Appl. Probab.* **14** 459–488. MR2023027 https://doi.org/10.1214/aoap/1075828058

[51] ZHENG, S., ZHANG, L. and FENG, L. (2021). On the backward stochastic differential equation with generator $f(y)|z|^2$. *J. Math. Anal. Appl.* **500** Paper No. 125102, 20. MR4222383 https://doi.org/10.1016/j.jmaa.2021.125102

The Annals of Applied Probability
2024, Vol. 34, No. 5, 4799–4835
https://doi.org/10.1214/24-AAP2082

CONVERGENCE OF ADAPTED EMPIRICAL MEASURES ON $\mathbb{R}^d$

BY BEATRICE ACCIAIO[a] AND SONGYAN HOU[b]

Department of Mathematics, ETH Zürich, [a]*beatrice.acciaio@math.ethz.ch,* [b]*songyan.hou@math.ethz.ch*

We consider empirical measures of $\mathbb{R}^d$-valued stochastic process in finite discrete-time. We show that the adapted empirical measure introduced in the recent work (*Ann. Appl. Probab.* **32** (2022) 529–550) by Backhoff et al. in compact spaces can be defined analogously on $\mathbb{R}^d$, and that it converges almost surely to the underlying measure under the adapted Wasserstein distance. Moreover, we quantitatively analyze the convergence of the adapted Wasserstein distance between those two measures. We establish convergence rates of the expected error as well as the deviation error under different moment conditions. Under suitable integrability and kernel assumptions, we recover the optimal convergence rates of both expected error and deviation error. Furthermore, we propose a modification of the adapted empirical measure with projection on a nonuniform grid, which obtains the same convergence rate but under weaker assumptions.

REFERENCES

[1] ACCIAIO, B., BACKHOFF-VERAGUAS, J. and ZALASHKO, A. (2020). Causal optimal transport and its links to enlargement of filtrations and continuous-time stochastic optimization. *Stochastic Process. Appl.* **130** 2918–2953. MR4080734 https://doi.org/10.1016/j.spa.2019.08.009

[2] ADLER, J. and LUNZ, S. (2018). Banach Wasserstein GAN. *Adv. Neural Inf. Process. Syst.* **31**.

[3] BACKHOFF, J., BARTL, D., BEIGLBÖCK, M. and WIESEL, J. (2022). Estimating processes in adapted Wasserstein distance. *Ann. Appl. Probab.* **32** 529–550. MR4386535 https://doi.org/10.1214/21-aap1687

[4] BACKHOFF, J., BEIGLBÖCK, M., LIN, Y. and ZALASHKO, A. (2017). Causal transport in discrete time and applications. *SIAM J. Optim.* **27** 2528–2562. MR3738324 https://doi.org/10.1137/16M1080197

[5] BACKHOFF-VERAGUAS, J., BARTL, D., BEIGLBÖCK, M. and EDER, M. (2020). Adapted Wasserstein distances and stability in mathematical finance. *Finance Stoch.* **24** 601–632. MR4118990 https://doi.org/10.1007/s00780-020-00426-3

[6] BION-NADAL, J. and TALAY, D. (2019). On a Wasserstein-type distance between solutions to stochastic differential equations. *Ann. Appl. Probab.* **29** 1609–1639. MR3914552 https://doi.org/10.1214/18-AAP1423

[7] BOISSARD, E. (2011). Simple bounds for convergence of empirical and occupation measures in 1-Wasserstein distance. *Electron. J. Probab.* **16** 2296–2333. MR2861675 https://doi.org/10.1214/EJP.v16-958

[8] BOISSARD, E. and LE GOUIC, T. (2014). On the mean speed of convergence of empirical and occupation measures in Wasserstein distance. *Ann. Inst. Henri Poincaré Probab. Stat.* **50** 539–563. MR3189084 https://doi.org/10.1214/12-AIHP517

[9] BOLLEY, F., GUILLIN, A. and VILLANI, C. (2007). Quantitative concentration inequalities for empirical measures on non-compact spaces. *Probab. Theory Related Fields* **137** 541–593. MR2280433 https://doi.org/10.1007/s00440-006-0004-7

[10] BRÉMAUD, P. and YOR, M. (1978). Changes of filtrations and of probability measures. *Z. Wahrsch. Verw. Gebiete* **45** 269–295. MR0511775 https://doi.org/10.1007/BF00537538

[11] DEDECKER, J. and FAN, X. (2015). Deviation inequalities for separately Lipschitz functionals of iterated random functions. *Stochastic Process. Appl.* **125** 60–90. MR3274692 https://doi.org/10.1016/j.spa.2014.08.001

[12] DEREICH, S., SCHEUTZOW, M. and SCHOTTSTEDT, R. (2013). Constructive quantization: Approximation by empirical measures. *Ann. Inst. Henri Poincaré Probab. Stat.* **49** 1183–1203. MR3127919 https://doi.org/10.1214/12-AIHP489

MSC2020 subject classifications. Primary 60B10, 62G30; secondary 49Q22.
Key words and phrases. Adapted Wasserstein distance, empirical measure, convergence rate.

[13] FOURNIER, N. and GUILLIN, A. (2015). On the rate of convergence in Wasserstein distance of the empirical measure. *Probab. Theory Related Fields* **162** 707–738. MR3383341 https://doi.org/10.1007/s00440-014-0583-7

[14] GIGLI, N. (2008). On the geometry of the space of probability measures in $\mathbb{R}^n$ endowed with the quadratic optimal transport distance. Ph.D. thesis, Scuola Normale Superiore.

[15] GLANZER, M., PFLUG, G. C. and PICHLER, A. (2019). Incorporating statistical model error into the calculation of acceptability prices of contingent claims. *Math. Program.* **174** 499–524. MR3935089 https://doi.org/10.1007/s10107-018-1352-7

[16] GOZLAN, N. and LÉONARD, C. (2007). A large deviation approach to some transportation cost inequalities. *Probab. Theory Related Fields* **139** 235–283. MR2322697 https://doi.org/10.1007/s00440-006-0045-y

[17] HAN, B. (2022). Distributionally robust risk evaluation with a causality constraint and structural information. ArXiv preprint. Available at arXiv:2203.10571.

[18] IBRAGIMOV, R. and SHARAKHMETOV, S. (1998). On an exact constant for the Rosenthal inequality. *Theory Probab. Appl.* **42** 294–302.

[19] IBRAGIMOV, R. and SHARAKHMETOV, S. (2001). The best constant in the Rosenthal inequality for nonnegative random variables. *Statist. Probab. Lett.* **55** 367–376. MR1877641 https://doi.org/10.1016/S0167-7152(01)00134-1

[20] KLEMMER, K., XU, T., ACCIAIO, B. and NEILL, D. B. (2022). SPATE-GAN: Improved generative modeling of dynamic spatio-temporal patterns with an autoregressive embedding loss. In *AAAI*.

[21] LASSALLE, R. (2018). Causal transport plans and their Monge-Kantorovich problems. *Stoch. Anal. Appl.* **36** 452–484. MR3784142 https://doi.org/10.1080/07362994.2017.1422747

[22] LEI, J. (2020). Convergence and concentration of empirical measures under Wasserstein distance in unbounded functional spaces. *Bernoulli* **26** 767–798. MR4036051 https://doi.org/10.3150/19-BEJ1151

[23] MCDIARMID, C. (1989). On the method of bounded differences. In *Surveys in Combinatorics*, 1989 (*Norwich*, 1989). *London Mathematical Society Lecture Note Series* **141** 148–188. Cambridge Univ. Press, Cambridge. MR1036755

[24] OBŁÓJ, J. and WIESEL, J. (2021). Robust estimation of superhedging prices. *Ann. Statist.* **49** 508–530. MR4206688 https://doi.org/10.1214/20-AOS1966

[25] PFLUG, G. C. and PICHLER, A. (2012). A distance for multistage stochastic optimization models. *SIAM J. Optim.* **22** 1–23. MR2902682 https://doi.org/10.1137/110825054

[26] PFLUG, G. C. and PICHLER, A. (2014). *Multistage Stochastic Optimization*. *Springer Series in Operations Research and Financial Engineering*. Springer, Cham. MR3288310 https://doi.org/10.1007/978-3-319-08843-3

[27] PFLUG, G. C. and PICHLER, A. (2015). Dynamic generation of scenario trees. *Comput. Optim. Appl.* **62** 641–668. MR3426120 https://doi.org/10.1007/s10589-015-9758-0

[28] PFLUG, G. C. and PICHLER, A. (2016). From empirical observations to tree models for stochastic optimization: Convergence properties. *SIAM J. Optim.* **26** 1715–1740. MR3543169 https://doi.org/10.1137/15M1043376

[29] PICHLER, A. (2013). Evaluations of risk measures for different probability measures. *SIAM J. Optim.* **23** 530–551. MR3033118 https://doi.org/10.1137/110857088

[30] RÜSCHENDORF, L. (1985). The Wasserstein distance and approximation theorems. *Z. Wahrsch. Verw. Gebiete* **70** 117–129. MR0795791 https://doi.org/10.1007/BF00532240

[31] VERSHYNIN, R. (2018). *High-Dimensional Probability*: *An Introduction with Applications in Data Science*. *Cambridge Series in Statistical and Probabilistic Mathematics* **47**. Cambridge Univ. Press, Cambridge. MR3837109 https://doi.org/10.1017/9781108231596

[32] WIESEL, J. C. W. (2022). Measuring association with Wasserstein distances. *Bernoulli* **28** 2816–2832. MR4474563 https://doi.org/10.3150/21-bej1438

[33] XU, T., WENLIANG, L. K., MUNN, M. and ACCIAIO, B. (2020). Cot-gan: Generating sequential data via causal optimal transport. *Adv. Neural Inf. Process. Syst.* **33** 8798–8809.

The Annals of Applied Probability
2024, Vol. 34, No. 5, 4836–4862
https://doi.org/10.1214/24-AAP2083

MULTILEVEL PATH BRANCHING FOR DIGITAL OPTIONS

BY MICHAEL B. GILES[1,a] AND ABDUL-LATEEF HAJI-ALI[2,b]

[1] *Mathematical Institute, University of Oxford,* [a] *mike.giles@maths.ox.ac.uk*
[2] *Maxwell Institute, School of Mathematical and Computer Sciences, Heriot-Watt University,* [b] *a.hajiali@hw.ac.uk*

We propose a new Monte Carlo-based estimator for digital options with assets modelled by a stochastic differential equation (SDE). The new estimator is based on repeated path splitting and relies on the correlation of approximate paths of the underlying SDE that share parts of a Brownian path. Combining this new estimator with multilevel Monte Carlo (MLMC) leads to an estimator with a computational complexity that is similar to the complexity of a MLMC estimator when applied to options with Lipschitz payoffs.

REFERENCES

[1] ACHTSIS, N., COOLS, R. and NUYENS, D. (2013). Conditional sampling for barrier option pricing under the LT method. *SIAM J. Financial Math.* **4** 327–352. MR3035469 https://doi.org/10.1137/110855909

[2] BAYER, C., BEN HAMMOUDA, C. and TEMPONE, R. (2024). Multilevel Monte Carlo with numerical smoothing for robust and efficient computation of probabilities and densities. *SIAM J. Sci. Comput.* **46** A1514–A1548. MR4741082 https://doi.org/10.1137/22M1495718

[3] BURGOS, S. and GILES, M. B. (2012). Computing Greeks using multilevel path simulation. In *Monte Carlo and Quasi-Monte Carlo Methods* 2010. *Springer Proc. Math. Stat.* **23** 281–296. Springer, Heidelberg. MR3173839 https://doi.org/10.1007/978-3-642-27440-4_13

[4] DUQUESNE, T. and LE GALL, J.-F. (2002). Random trees, Lévy processes and spatial branching processes. *Astérisque* **281** vi+147. MR1954248

[5] ETHERIDGE, A. M. (2000). *An Introduction to Superprocesses*. *University Lecture Series* **20**. Amer. Math. Soc., Providence, RI. MR1779100 https://doi.org/10.1090/ulect/020

[6] FRIEDMAN, A. (1964). *Partial Differential Equations of Parabolic Type*. Prentice-Hall, Inc., Englewood Cliffs, NJ. MR0181836

[7] GILES, M. B. (2008). Multilevel Monte Carlo path simulation. *Oper. Res.* **56** 607–617. MR2436856 https://doi.org/10.1287/opre.1070.0496

[8] GILES, M. B. (2015). Multilevel Monte Carlo methods. *Acta Numer.* **24** 259–328. MR3349310 https://doi.org/10.1017/S096249291500001X

[9] GILES, M. B. (2024). MLMC Techniques for Discontinuous Functions. In *Monte Carlo and Quasi-Monte Carlo Methods*. *Springer Proc. Math. Stat.* **460** 33–47. Springer, Cham. MR4774914 https://doi.org/10.1007/978-3-031-59762-6_2

[10] GILES, M. B., DEBRABANT, K. and RÖSSLER, A. (2019). Analysis of multilevel Monte Carlo path simulation using the Milstein discretisation. *Discrete Contin. Dyn. Syst. Ser. B* **24** 3881–3903. MR3986260 https://doi.org/10.3934/dcdsb.2018335

[11] GILES, M. B., HIGHAM, D. J. and MAO, X. (2009). Analysing multi-level Monte Carlo for options with non-globally Lipschitz payoff. *Finance Stoch.* **13** 403–413. MR2519838 https://doi.org/10.1007/s00780-009-0092-1

[12] GILES, M. B., NAGAPETYAN, T. and RITTER, K. (2015). Multilevel Monte Carlo approximation of distribution functions and densities. *SIAM/ASA J. Uncertain. Quantificat.* **3** 267–295. MR3338004 https://doi.org/10.1137/140960086

[13] GILES, M. B. and SZPRUCH, L. (2014). Antithetic multilevel Monte Carlo estimation for multi-dimensional SDEs without Lévy area simulation. *Ann. Appl. Probab.* **24** 1585–1620. MR3211005 https://doi.org/10.1214/13-AAP957

[14] GOBET, E. and LABART, C. (2008). Sharp estimates for the convergence of the density of the Euler scheme in small time. *Electron. Commun. Probab.* **13** 352–363. MR2415143 https://doi.org/10.1214/ECP.v13-1393

MSC2020 subject classifications. Primary 65C05, 65C30; secondary 65B99, 60J85.
Key words and phrases. Monte Carlo, multilevel, path splitting, computational complexity, branching processes.

[15] GRIEBEL, M., KUO, F. Y. and SLOAN, I. H. (2013). The smoothing effect of integration in $\mathbb{R}^d$ and the ANOVA decomposition. *Math*. *Comp*. **82** 383–400. MR2983028 https://doi.org/10.1090/S0025-5718-2012-02578-6

[16] GRIEWANK, A., KUO, F. Y., LEÖVEY, H. and SLOAN, I. H. (2018). High dimensional integration of kinks and jumps—smoothing by preintegration. *J*. *Comput*. *Appl*. *Math*. **344** 259–274. MR3825515 https://doi.org/10.1016/j.cam.2018.04.009

[17] HAJI-ALI, A.-L., NOBILE, F., VON SCHWERIN, E. and TEMPONE, R. (2016). Optimization of mesh hierarchies in multilevel Monte Carlo samplers. *Stoch*. *Partial Differ. Equ*. *Anal*. *Comput*. **4** 76–112. MR3475836 https://doi.org/10.1007/s40072-015-0049-7

[18] HAJI-ALI, A.-L., SPENCE, J. and TECKENTRUP, A. L. (2022). Adaptive multilevel Monte Carlo for probabilities. *SIAM J*. *Numer. Anal*. **60** 2125–2149. MR4467537 https://doi.org/10.1137/21M1447064

[19] KLOEDEN, P. E. and PLATEN, E. (1992). *Numerical Solution of Stochastic Differential Equations*. *Applications of Mathematics* (*New York*) **23**. Springer, Berlin. MR1214374 https://doi.org/10.1007/978-3-662-12616-5

[20] MCKEAN, H. P. (1975). Application of Brownian motion to the equation of Kolmogorov–Petrovskii–Piskunov. *Comm*. *Pure Appl*. *Math*. **28** 323–331. MR0400428 https://doi.org/10.1002/cpa.3160280302

The Annals of Applied Probability
2024, Vol. 34, No. 5, 4863–4898
https://doi.org/10.1214/24-AAP2084

LIMIT THEOREMS FOR EXPONENTIAL RANDOM GRAPHS

BY ALESSANDRA BIANCHI[1,a], FRANCESCA COLLET[2,b] AND ELENA MAGNANINI[3,c]

[1]*Dipartimento di Matematica "Tullio Levi-Civita", Università degli Studi di Padova,* [a]*alessandra.bianchi@unipd.it*
[2]*Dipartimento di Informatica, Università degli Studi di Verona,* [b]*francesca.collet@univr.it*
[3]*WIAS,* [c]*elena.magnanini@wias-berlin.de*

We consider the edge-triangle model, a two-parameter family of exponential random graphs in which dependence between edges is introduced through triangles. In the so-called replica symmetric regime, the limiting free energy exists together with a complete characterization of the phase diagram of the model. We borrow tools from statistical mechanics to obtain limit theorems for the edge density. First, we investigate the asymptotic distribution of this quantity, as the graph size tends to infinity, in the various phases. Then, we study the fluctuations of the edge density around its average value off the critical curve and formulate conjectures about the behavior at criticality based on the analysis of a mean-field approximation of the model. Some of our results can be extended with no substantial changes to more general classes of exponential random graphs.

REFERENCES

[1] ARISTOFF, D. and ZHU, L. (2018). On the phase transition curve in a directed exponential random graph model. *Adv. in Appl. Probab.* **50** 272–301. MR3781986 https://doi.org/10.1017/apr.2018.13

[2] BIANCHI, A., COLLET, F. and MAGNANINI, E. (2022). The GHS and other correlation inequalities for the two-star model. *ALEA Lat. Am. J. Probab. Math. Stat.* **19** 1679–1695. MR4517735 https://doi.org/10.30757/alea.v19-64

[3] BILLINGSLEY, P. (1986). *Probability and Measure*, 2nd ed. *Wiley Series in Probability and Mathematical Statistics*: *Probability and Mathematical Statistics*. Wiley, New York. MR0830424

[4] BORGS, C., CHAYES, J., LOVÁSZ, L., SÓS, V. T. and VESZTERGOMBI, K. (2006). Counting graph homomorphisms. In *Topics in Discrete Mathematics*. *Algorithms Combin.* **26** 315–371. Springer, Berlin. MR2249277 https://doi.org/10.1007/3-540-33700-8_18

[5] BORGS, C., CHAYES, J. T., LOVÁSZ, L., SÓS, V. T. and VESZTERGOMBI, K. (2008). Convergent sequences of dense graphs. I. Subgraph frequencies, metric properties and testing. *Adv. Math.* **219** 1801–1851. MR2455626 https://doi.org/10.1016/j.aim.2008.07.008

[6] BORGS, C., CHAYES, J. T., LOVÁSZ, L., SÓS, V. T. and VESZTERGOMBI, K. (2012). Convergent sequences of dense graphs II. Multiway cuts and statistical physics. *Ann. of Math.* (2) **176** 151–219. MR2925382 https://doi.org/10.4007/annals.2012.176.1.2

[7] CHATTERJEE, S. (2017). *Large Deviations for Random Graphs*. *Lecture Notes in Math.* **2197**. Springer, Cham. MR3700183 https://doi.org/10.1007/978-3-319-65816-2

[8] CHATTERJEE, S. and DEY, P. S. (2010). Applications of Stein's method for concentration inequalities. *Ann. Probab.* **38** 2443–2485. MR2683635 https://doi.org/10.1214/10-AOP542

[9] CHATTERJEE, S. and DIACONIS, P. (2013). Estimating and understanding exponential random graph models. *Ann. Statist.* **41** 2428–2461. MR3127871 https://doi.org/10.1214/13-AOS1155

[10] CHATTERJEE, S. and VARADHAN, S. R. S. (2011). The large deviation principle for the Erdős-Rényi random graph. *European J. Combin.* **32** 1000–1017. MR2825532 https://doi.org/10.1016/j.ejc.2011.03.014

[11] ELLIS, R. S. (1985). *Entropy*, *Large Deviations*, *and Statistical Mechanics*. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **271**. Springer, New York. MR0793553 https://doi.org/10.1007/978-1-4613-8533-2

[12] ERDŐS, P. and RÉNYI, A. (1961). On the evolution of random graphs. *Bull. Inst. Int. Stat.* **38** 343–347. MR0148055

MSC2020 subject classifications. Primary 60F05, 60F15, 05C80; secondary 60F10, 60B10.
Key words and phrases. Exponential random graphs, mean-field approximation, large deviations, phase transition, standard and nonstandard limit theorems, Yang–Lee theorem.

[13] FIENBERG, S. E. (2010). Introduction to papers on the modeling and analysis of network data. *Ann. Appl. Stat.* **4** 1–4. MR2758081 https://doi.org/10.1214/10-AOAS346

[14] FIENBERG, S. E. (2010). Introduction to papers on the modeling and analysis of network data—II. *Ann. Appl. Stat.* **4** 533–534. MR2744531 https://doi.org/10.1214/10-AOAS365

[15] GIARDINÀ, C., GIBERTI, C. and MAGNANINI, E. (2021). Approximating the cumulant generating function of triangles in the Erdös-Rényi random graph. *J. Stat. Phys.* **182** 23. MR4205284 https://doi.org/10.1007/s10955-021-02707-3

[16] GRIFFITHS, R. B., HURST, C. A. and SHERMAN, S. (1970). Concavity of magnetization of an Ising ferromagnet in a positive external field. *J. Math. Phys.* **11** 790–795. MR0266507 https://doi.org/10.1063/1.1665211

[17] LOVÁSZ, L. (2012). *Large Networks and Graph Limits*. Amer. Math. Soc., Providence, RI.

[18] LOVÁSZ, L. and SZEGEDY, B. (2006). Limits of dense graph sequences. *J. Combin. Theory Ser. B* **96** 933–957. MR2274085 https://doi.org/10.1016/j.jctb.2006.05.002

[19] LOVÁSZ, L. and SZEGEDY, B. (2007). Szemerédi's lemma for the analyst. *Geom. Funct. Anal.* **17** 252–270. MR2306658 https://doi.org/10.1007/s00039-007-0599-6

[20] MAGNANINI, E. (2019). A Monte Carlo method for large deviations applied to Erdős–Rényi random graphs Ph.D. thesis, Department of Pure and Applied Mathematics, University of Modena and Reggio Emilia.

[21] MUKHERJEE, S. and XU, Y. (2023). Statistics of the two star ERGM. *Bernoulli* **29** 24–51. MR4497238 https://doi.org/10.3150/21-bej1448

[22] PARK, J. and NEWMAN, M. E. J. (2004). Statistical mechanics of networks. *Phys. Rev. E* (3) **70** 066117–066122. MR2133807 https://doi.org/10.1103/PhysRevE.70.066117

[23] PARK, J. and NEWMAN, M. EJ. (2005). Solution for the properties of a clustered network. *Phys. Rev. E* **72** 026136.

[24] RADIN, C. and YIN, M. (2013). Phase transitions in exponential random graphs. *Ann. Appl. Probab.* **23** 2458–2471. MR3127941 https://doi.org/10.1214/12-AAP907

[25] RINALDO, A., FIENBERG, S. E. and ZHOU, Y. (2009). On the geometry of discrete exponential families with application to exponential random graph models. *Electron. J. Stat.* **3** 446–484. MR2507456 https://doi.org/10.1214/08-EJS350

[26] STRAUSS, D. (1986). On a general class of models for interaction. *SIAM Rev.* **28** 513–527. MR0867682 https://doi.org/10.1137/1028156

[27] YANG, C. N. and LEE, T. D. (1952). Statistical theory of equations of state and phase transitions. I. Theory of condensation. *Phys. Rev.* (2) **87** 404–409. MR0053028

The Annals of Applied Probability
2024, Vol. 34, No. 5, 4899–4920
https://doi.org/10.1214/24-AAP2085

SPECTRAL GAP OF THE SYMMETRIC INCLUSION PROCESS

BY SEONWOO KIM[1,a] AND FEDERICO SAU[2,b]

[1]*June E Huh Center for Mathematical Challenges, Korea Institute for Advanced Study,* [a]*seonwookim@kias.re.kr*
[2]*Dipartimento di Matematica, Informatica e Geoscienze, Università degli Studi di Trieste,* [b]*federico.sau@units.it*

We consider the symmetric inclusion process on a general finite graph. Our main result establishes universal upper and lower bounds for the spectral gap of this interacting particle system in terms of the spectral gap of the random walk on the same graph. In the regime in which the gamma-like reversible measures of the particle systems are log-concave, our bounds match, yielding a version for the symmetric inclusion process of the celebrated Aldous' spectral gap conjecture originally formulated for the interchange process. Finally, by means of duality techniques, we draw analogous conclusions for an interacting diffusion-like unbounded conservative spin system known as Brownian energy process.

REFERENCES

[1] ALDOUS, D. and LANOUE, D. (2012). A lecture on the averaging process. *Probab. Surv.* **9** 90–102. MR2908618 https://doi.org/10.1214/11-PS184

[2] BARTHE, F. and WOLFF, P. (2009). Remarks on non-interacting conservative spin systems: The case of gamma distributions. *Stochastic Process. Appl.* **119** 2711–2723. MR2532220 https://doi.org/10.1016/j.spa.2009.02.004

[3] BIANCHI, A., DOMMERS, S. and GIARDINÀ, C. (2017). Metastability in the reversible inclusion process. *Electron. J. Probab.* **22** Paper No. 70, 34. MR3698739 https://doi.org/10.1214/17-EJP98

[4] BRISTIEL, A. and CAPUTO, P. (2024). Entropy inequalities for random walks and permutations. *Ann. Inst. Henri Poincaré Probab. Stat.* **60** 54–81. MR4718374 https://doi.org/10.1214/22-aihp1267

[5] CAPUTO, P. (2003). Uniform Poincaré inequalities for unbounded conservative spin systems: The non-interacting case. *Stochastic Process. Appl.* **106** 223–244. MR1989628 https://doi.org/10.1016/S0304-4149(03)00044-9

[6] CAPUTO, P. (2004). Spectral gap inequalities in product spaces with conservation laws. In *Stochastic Analysis on Large Scale Interacting Systems. Adv. Stud. Pure Math.* **39** 53–88. Math. Soc. Japan, Tokyo. MR2073330 https://doi.org/10.2969/aspm/03910053

[7] CAPUTO, P. (2008). On the spectral gap of the Kac walk and other binary collision processes. *ALEA Lat. Am. J. Probab. Math. Stat.* **4** 205–222. MR2429910

[8] CAPUTO, P., LABBÉ, C. and LACOIN, H. (2020). Mixing time of the adjacent walk on the simplex. *Ann. Probab.* **48** 2449–2493. MR4152648 https://doi.org/10.1214/20-AOP1428

[9] CAPUTO, P., LABBÉ, C. and LACOIN, H. (2022). Spectral gap and cutoff phenomenon for the Gibbs sampler of $\nabla\varphi$ interfaces with convex potential. *Ann. Inst. Henri Poincaré Probab. Stat.* **58** 794–826. MR4421608 https://doi.org/10.1214/21-aihp1174

[10] CAPUTO, P., LIGGETT, T. M. and RICHTHAMMER, T. (2010). Proof of Aldous' spectral gap conjecture. *J. Amer. Math. Soc.* **23** 831–851. MR2629990 https://doi.org/10.1090/S0894-0347-10-00659-4

[11] CARINCI, G., GIARDINÀ, C. and REDIG, F. (2021). Consistent particle systems and duality. *Electron. J. Probab.* **26** Paper No. 125, 31. MR4320949 https://doi.org/10.1214/21-ejp684

[12] CORUJO, J. (2023). On the spectrum and ergodicity of a neutral multi-allelic Moran model. *ALEA Lat. Am. J. Probab. Math. Stat.* **20** 505–546. MR4567719 https://doi.org/10.30757/alea.v20-18

[13] DELLO SCHIAVO, L., PORTINALE, L. and SAU, F. (2024). Scaling limits of random walks, harmonic profiles, and stationary nonequilibrium states in Lipschitz domains. *Ann. Appl. Probab.* **34** 1789–1845. MR4728158 https://doi.org/10.1214/23-aap2007

[14] DIACONIS, P. and SALOFF-COSTE, L. (1993). Comparison theorems for reversible Markov chains. *Ann. Appl. Probab.* **3** 696–730. MR1233621

MSC2020 subject classifications. Primary 60K35; secondary 60J27, 05C50.
Key words and phrases. Interacting particle systems, unbounded conservative spin systems, spectral gap, symmetric inclusion process, Brownian energy process.

[15] DONNELLY, P. and KURTZ, T. G. (1996). A countable representation of the Fleming-Viot measure-valued diffusion. *Ann. Probab.* **24** 698–742. MR1404525 https://doi.org/10.1214/aop/1039639359

[16] ETHERIDGE, A. M. and KURTZ, T. G. (2019). Genealogical constructions of population models. *Ann. Probab.* **47** 1827–1910. MR3980910 https://doi.org/10.1214/18-AOP1266

[17] ETHERIDGE, A. M., KURTZ, T. G., LETTER, I., RALPH, P. L. and LUNG, T. T. H. (2024). Looking forwards and backwards: Dynamics and genealogies of locally regulated populations. *Electron. J. Probab.* **29** Paper No. 28, 85. MR4705167 https://doi.org/10.1214/24-ejp1075

[18] FLOREANI, S., REDIG, F. and SAU, F. (2022). Orthogonal polynomial duality of boundary driven particle systems and non-equilibrium correlations. *Ann. Inst. Henri Poincaré Probab. Stat.* **58** 220–247. MR4374677 https://doi.org/10.1214/21-aihp1163

[19] FRANCESCHINI, C., GONÇALVES, P. and SAU, F. (2022). Symmetric inclusion process with slow boundary: Hydrodynamics and hydrostatics. *Bernoulli* **28** 1340–1381. MR4388941 https://doi.org/10.3150/21-bej1390

[20] FRASSEK, R. and GIARDINÀ, C. (2022). Exact solution of an integrable non-equilibrium particle system. *J. Math. Phys.* **63** Paper No. 103301, 35. MR4493574 https://doi.org/10.1063/5.0086715

[21] GIARDINÀ, C., KURCHAN, J. and REDIG, F. (2007). Duality and exact correlations for a model of heat conduction. *J. Math. Phys.* **48** 033301, 15. MR2314497 https://doi.org/10.1063/1.2711373

[22] GIARDINÀ, C., KURCHAN, J., REDIG, F. and VAFAYI, K. (2009). Duality and hidden symmetries in interacting particle systems. *J. Stat. Phys.* **135** 25–55. MR2505724 https://doi.org/10.1007/s10955-009-9716-2

[23] GIARDINÀ, C., REDIG, F. and VAFAYI, K. (2010). Correlation inequalities for interacting particle systems with duality. *J. Stat. Phys.* **141** 242–263. MR2726642 https://doi.org/10.1007/s10955-010-0055-0

[24] GONÇALVES, P. and JARA, M. (2014). Nonlinear fluctuations of weakly asymmetric interacting particle systems. *Arch. Ration. Mech. Anal.* **212** 597–644. MR3176353 https://doi.org/10.1007/s00205-013-0693-x

[25] GRIFFITHS, R. C. (2014). The Λ-Fleming-Viot process and a connection with Wright-Fisher diffusion. *Adv. in Appl. Probab.* **46** 1009–1035. MR3290427 https://doi.org/10.1239/aap/1418396241

[26] GROSSKINSKY, S., REDIG, F. and VAFAYI, K. (2011). Condensation in the inclusion process and related models. *J. Stat. Phys.* **142** 952–974. MR2781716 https://doi.org/10.1007/s10955-011-0151-9

[27] GROSSKINSKY, S., REDIG, F. and VAFAYI, K. (2013). Dynamics of condensation in the symmetric inclusion process. *Electron. J. Probab.* **18** no. 66, 23. MR3078025 https://doi.org/10.1214/EJP.v18-2720

[28] HERMON, J. and SALEZ, J. (2019). A version of Aldous' spectral-gap conjecture for the zero range process. *Ann. Appl. Probab.* **29** 2217–2229. MR3984254 https://doi.org/10.1214/18-AAP1449

[29] HERMON, J. and SALEZ, J. (2023). Modified log-Sobolev inequalities for strong-Rayleigh measures. *Ann. Appl. Probab.* **33** 1301–1314. MR4564432 https://doi.org/10.1214/22-aap1847

[30] JANVRESSE, E., LANDIM, C., QUASTEL, J. and YAU, H. T. (1999). Relaxation to equilibrium of conservative dynamics. I. Zero-range processes. *Ann. Probab.* **27** 325–360. MR1681098 https://doi.org/10.1214/aop/1022677265

[31] JATUVIRIYAPORNCHAI, W., CHLEBOUN, P. and GROSSKINSKY, S. (2020). Structure of the condensed phase in the inclusion process. *J. Stat. Phys.* **178** 682–710. MR4059956 https://doi.org/10.1007/s10955-019-02451-9

[32] KIM, S. (2021). Second time scale of the metastability of reversible inclusion processes. *Probab. Theory Related Fields* **180** 1135–1187. MR4288339 https://doi.org/10.1007/s00440-021-01036-6

[33] KIM, S. and SEO, I. (2021). Condensation and metastable behavior of non-reversible inclusion processes. *Comm. Math. Phys.* **382** 1343–1401. MR4227174 https://doi.org/10.1007/s00220-021-04016-y

[34] KIPNIS, C. and LANDIM, C. (1999). *Scaling Limits of Interacting Particle Systems. Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **320**. Springer, Berlin. MR1707314 https://doi.org/10.1007/978-3-662-03752-2

[35] LABBÉ, C. and PETIT, E. (2022). Hydrodynamic limit and cutoff for the biased adjacent walk on the simplex. Available at arXiv:2203.04008.

[36] LANDIM, C., MARCONDES, D. and SEO, I. (2023). Metastable behavior of weakly mixing Markov chains: The case of reversible, critical zero-range processes. *Ann. Probab.* **51** 157–227. MR4515693 https://doi.org/10.1214/22-aop1593

[37] LANDIM, C., PANIZO, G. and YAU, H. T. (2002). Spectral gap and logarithmic Sobolev inequality for unbounded conservative spin systems. *Ann. Inst. Henri Poincaré Probab. Stat.* **38** 739–777. MR1931585 https://doi.org/10.1016/S0246-0203(02)01108-1

[38] LANDIM, C., SETHURAMAN, S. and VARADHAN, S. (1996). Spectral gap for zero-range dynamics. *Ann. Probab.* **24** 1871–1902. MR1415232 https://doi.org/10.1214/aop/1041903209

[39] LIGGETT, T. M. (2005). *Interacting Particle Systems. Classics in Mathematics*. Springer, Berlin. Reprint of the 1985 original. MR2108619 https://doi.org/10.1007/b138374

[40] LU, S. L. and YAU, H.-T. (1993). Spectral gap and logarithmic Sobolev inequality for Kawasaki and Glauber dynamics. *Comm. Math. Phys.* **156** 399–433. MR1233852

[41] MONTENEGRO, R. and TETALI, P. (2006). Mathematical aspects of mixing times in Markov chains. *Found. Trends Theor. Comput. Sci.* **1** x+121. MR2341319 https://doi.org/10.1561/0400000003

[42] PYMAR, R. and RIVERA, N. (2023). Mixing of the symmetric beta-binomial splitting process on arbitrary graphs. Available at arXiv:2307.02406.

[43] QUASTEL, J. (1992). Diffusion of color in the simple exclusion process. *Comm. Pure Appl. Math.* **45** 623–679. MR1162368 https://doi.org/10.1002/cpa.3160450602

[44] QUATTROPANI, M. and SAU, F. (2023). Mixing of the averaging process and its discrete dual on finite-dimensional geometries. *Ann. Appl. Probab.* **33** 936–971. MR4564423 https://doi.org/10.1214/22-aap1838

[45] REDIG, F. and SAU, F. (2018). Factorized duality, stationary product measures and generating functions. *J. Stat. Phys.* **172** 980–1008. MR3830295 https://doi.org/10.1007/s10955-018-2090-1

[46] SALEZ, J. (2023). Universality of cutoff for exclusion with reservoirs. *Ann. Probab.* **51** 478–494. MR4546624 https://doi.org/10.1214/22-aop1600

[47] SALEZ, J. (2024). Spectral gap and curvature of monotone Markov chains. *Ann. Probab.* **52** 1153–1161. MR4736702 https://doi.org/10.1214/24-aop1688

[48] SALEZ, J., TIKHOMIROV, K. and YOUSSEF, P. (2023). Upgrading MLSI to LSI for reversible Markov chains. *J. Funct. Anal.* **285** Paper No. 110076, 15. MR4620718 https://doi.org/10.1016/j.jfa.2023.110076

[49] SALOFF-COSTE, L. (1997). Lectures on finite Markov chains. In *Lectures on Probability Theory and Statistics* (*Saint-Flour*, 1996). *Lecture Notes in Math.* **1665** 301–413. Springer, Berlin. MR1490046 https://doi.org/10.1007/BFb0092621

[50] SASADA, M. (2013). On the spectral gap of the Kac walk and other binary collision processes on d-dimensional lattice. In *Symmetries, Integrable Systems and Representations*. *Springer Proc. Math. Stat.* **40** 543–560. Springer, Heidelberg. MR3077700 https://doi.org/10.1007/978-1-4471-4863-0_23

[51] SHIMAKURA, N. (1977). Équations différentielles provenant de la génétique des populations. *Tohoku Math. J.* (2) **29** 287–318. MR0504058 https://doi.org/10.2748/tmj/1178240658

[52] VAN GINKEL, B., REDIG, F. and SAU, F. (2016). Duality and stationary distributions of the "immediate exchange model" and its generalizations. *J. Stat. Phys.* **163** 92–112. MR3472095 https://doi.org/10.1007/s10955-016-1478-z

The Annals of Applied Probability
2024, Vol. 34, No. 5, 4921–4939
https://doi.org/10.1214/24-AAP2086

THE IMPOSSIBILITY REGION FOR DETECTING SPARSE MIXTURES USING THE HIGHER CRITICISM

BY DAVID L. DONOHO[1,a] AND ALON KIPNIS[2,b]

[1] *Department of Statistics, Stanford University,* [a] *donoho@stanford.edu*
[2] *School of Computer Science, Reichman University,* [b] *alon.kipnis@runi.ac.il*

Consider a multiple hypothesis testing setting involving rare/weak effects: relatively few tests, out of possibly many, deviate from their null hypothesis behavior. Summarizing the significance of each test by a p-value, we construct a global test against the joint null using the higher criticism (HC) statistics of these p-values. We calibrate the rare/weak model using parameters controlling the asymptotic distribution of nonnull p-values near zero. We derive a region in the parameter space where the HC test is asymptotically powerless. Our derivation involves very different tools than previously used to show powerlessness of HC, relying on properties of the empirical processes underlying HC. In particular, our result applies to situations where HC is not asymptotically optimal, or when the asymptotically detectable region of the parameter space is unknown.

REFERENCES

ARIAS-CASTRO, E., CANDÈS, E. J. and PLAN, Y. (2011). Global testing under sparse alternatives: ANOVA, multiple comparisons and the higher criticism. *Ann. Statist.* **39** 2533–2556. MR2906877 https://doi.org/10.1214/11-AOS910

ARIAS-CASTRO, E. and WANG, M. (2015). The sparse Poisson means model. *Electron. J. Stat.* **9** 2170–2201. MR3406276 https://doi.org/10.1214/15-EJS1066

ARIAS-CASTRO, E. and WANG, M. (2017). Distribution-free tests for sparse heterogeneous mixtures. *TEST* **26** 71–94. MR3613606 https://doi.org/10.1007/s11749-016-0499-x

BAIRAMOV, I., BERRED, A. and STEPANOV, A. (2010). Limit results for ordered uniform spacings. *Statist. Papers* **51** 227–240. MR2556597 https://doi.org/10.1007/s00362-008-0134-3

CAI, T. T., JENG, X. J. and JIN, J. (2011). Optimal detection of heterogeneous and heteroscedastic mixtures. *J. R. Stat. Soc. Ser. B. Stat. Methodol.* **73** 629–662. MR2867452 https://doi.org/10.1111/j.1467-9868.2011.00778.x

CAI, T. T. and WU, Y. (2014). Optimal detection of sparse mixtures against a given null distribution. *IEEE Trans. Inf. Theory* **60** 2217–2232. MR3181520 https://doi.org/10.1109/TIT.2014.2304295

DELAIGLE, A. and HALL, P. (2009). Higher criticism in the context of unknown distribution, non-independence and classification. In *Perspectives in Mathematical Sciences I. Stat. Sci. Interdiscip. Res.* **7** 109–138. World Sci. Publ., Hackensack, NJ. MR2581742 https://doi.org/10.1142/9789814273633_0006

DONOHO, D. and JIN, J. (2004). Higher criticism for detecting sparse heterogeneous mixtures. *Ann. Statist.* **32** 962–994. MR2065195 https://doi.org/10.1214/009053604000000265

DONOHO, D. and JIN, J. (2009). Feature selection by higher criticism thresholding achieves the optimal phase diagram. *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **367** 4449–4470. With electronic supplementary materials available online. MR2546396 https://doi.org/10.1098/rsta.2009.0129

DONOHO, D. L. and JIN, J. (2009). Higher criticism thresholding: Optimal feature selection when useful features are rare and weak. *Proc. Natl. Acad. Sci.* **105** 14790–14795. MR2520682 https://doi.org/10.1073/pnas.0903931106

DONOHO, D. L. and KIPNIS, A. (2022). Higher criticism to compare two large frequency tables, with sensitivity to possible rare and weak differences. *Ann. Statist.* **50** 1447–1472. MR4441127 https://doi.org/10.1214/21-aos2158

GONTSCHARUK, V., LANDWEHR, S. and FINNER, H. (2016). Goodness of fit tests in terms of local levels with special emphasis on higher criticism tests. *Bernoulli* **22** 1331–1363. MR3474818 https://doi.org/10.3150/14-BEJ694

MSC2020 subject classifications. 62G10, 62H17, 62G30, 62H15.
Key words and phrases. Multiple testing, empirical process, higher criticism, phase transition.

HALL, P. and JIN, J. (2010). Innovated higher criticism for detecting sparse signals in correlated noise. *Ann. Statist.* **38** 1686–1732. MR2662357 https://doi.org/10.1214/09-AOS764

JIN, J. and KE, Z. T. (2016). Rare and weak effects in large-scale inference: Methods and phase diagrams. *Statist. Sinica* **26** 1–34. MR3468343

KIPNIS, A. (2023). Unification of rare/weak detection models using moderate deviations analysis and log-chisquared p-values. *Statist. Sinica*. To appear. https://doi.org/10.5705/ss.202023.0128

LI, J. and SIEGMUND, D. (2015). Higher criticism: p-values and criticism. *Ann. Statist.* **43** 1323–1350. MR3346705 https://doi.org/10.1214/15-AOS1312

MITZENMACHER, M. and UPFAL, E. (2017). *Probability and Computing: Randomization and Probabilistic Techniques in Algorithms and Data Analysis*, 2nd ed. Cambridge Univ. Press, Cambridge. MR3674428

MOSCOVICH, A., NADLER, B. and SPIEGELMAN, C. (2016). On the exact Berk–Jones statistics and their p-value calculation. *Electron. J. Stat.* **10** 2329–2354. MR3544289 https://doi.org/10.1214/16-EJS1172

MUKHERJEE, R., PILLAI, N. S. and LIN, X. (2015). Hypothesis testing for high-dimensional sparse binary regression. *Ann. Statist.* **43** 352–381. MR3311863 https://doi.org/10.1214/14-AOS1279

The Annals of Applied Probability
2024, Vol. 34, No. 5, 4940–4997
https://doi.org/10.1214/24-AAP2087

MEETING, COALESCENCE AND CONSENSUS TIME ON RANDOM DIRECTED GRAPHS

BY LUCA AVENA[1,a], FEDERICO CAPANNOLI[2,b], RAJAT SUBHRA HAZRA[2,c] AND MATTEO QUATTROPANI[3,d]

[1]*Dipartimento di Matematica e Informatica "Ulisse Dini"—Università degli Studi di Firenze,* [a]*luca.avena@unifi.it*
[2]*Mathematisch Instituut, Universiteit Leiden,* [b]*f.capannoli@math.leidenuniv.nl,* [c]*r.s.hazra@math.leidenuniv.nl*
[3]*Dipartimento di Matematica "Guido Castelnuovo", Sapienza Università di Roma,* [d]*matteo.quattropani@uniroma1.it*

We consider the so-called directed configuration model (DCM), that is, a random directed graph with prescribed in- and out-degrees. In this random geometry, we study the meeting time of two random walks starting at stationarity, the coalescence time for a system of coalescent random walks, and the consensus time of the classical voter model. Indeed, it is known that the latter three quantities are related to each other under certain *mean field conditions* requiring fast enough mixing time and not too concentrated stationary distribution. Our main result shows that, for a typical large graph from the DCM ensemble, the meeting time is well-approximated by an exponential random variable for which we provide the first-order asymptotics of its expectation, showing that the latter is linear in the size of the graph, and its pre-constant depends on some explicit statistics of the degree sequence. As a byproduct, we explore the effect of the degree sequence in changing the meeting, coalescence, and consensus time by discussing several classes of examples of interest also from an applied perspective. Our approach follows the classical idea of converting meeting into hitting times of a proper collapsed chain, which we control by the so-called first visit time lemma. The main technical challenge is related to the fact that in such a directed setting the stationary distribution is random, and it depends on the whole realization of the graph. As a consequence, a good share of the proof focuses on showing that certain functions of the stationary distribution concentrate around their expectations, and on their characterization, via proper annealing arguments.

REFERENCES

[1] ALDOUS, D. (1989). *Probability Approximations via the Poisson Clumping Heuristic*. *Applied Mathematical Sciences* **77**. Springer, New York. MR0969362 https://doi.org/10.1007/978-1-4757-6283-9

[2] ALDOUS, D. (2013). Interacting particle systems as stochastic social dynamics. *Bernoulli* **19** 1122–1149. MR3102546 https://doi.org/10.3150/12-BEJSP04

[3] ALDOUS, D. and FILL, J. A. (2002). Reversible Markov Chains and Random Walks on Graphs. Unfinished monograph, recompiled 2014. Available at http://www.stat.berkeley.edu/~aldous/RWG/book.html.

[4] ALDOUS, D. J. and BROWN, M. (1992). Inequalities for rare events in time-reversible Markov chains. I. In *Stochastic Inequalities* (*Seattle*, *WA*, 1991). *Institute of Mathematical Statistics Lecture Notes—Monograph Series* **22** 1–16. IMS, Hayward, CA. MR1228050 https://doi.org/10.1214/lnms/1215461937

[5] AVENA, L., BALDASSO, R., HAZRA, R. S., DEN HOLLANDER, F. and QUATTROPANI, M. (2024). Discordant edges for the voter model on regular random graphs. *ALEA Lat. Am. J. Probab. Math. Stat.* **21** 431–464. MR4715547 https://doi.org/10.30757/alea.v21-18

[6] BOLLOBÁS, B. (1980). A probabilistic proof of an asymptotic formula for the number of labelled regular graphs. *European J. Combin.* **1** 311–316. MR0595929 https://doi.org/10.1016/S0195-6698(80)80030-8

[7] BORDENAVE, C., CAPUTO, P. and SALEZ, J. (2018). Random walk on sparse random digraphs. *Probab. Theory Related Fields* **170** 933–960. MR3773804 https://doi.org/10.1007/s00440-017-0796-7

MSC2020 subject classifications. Primary 05C80; secondary 05C82, 82C22.
Key words and phrases. Random directed graphs, voter model, meeting times, coalescent random walks.

[8] BORDENAVE, C., CAPUTO, P. and SALEZ, J. (2019). Cutoff at the "entropic time" for sparse Markov chains. *Probab. Theory Related Fields* **173** 261–292. MR3916108 https://doi.org/10.1007/s00440-018-0834-0

[9] BROWN, M. (1999). Interlacing eigenvalues in time reversible Markov chains. *Math. Oper. Res.* **24** 847–864. MR1854356 https://doi.org/10.1287/moor.24.4.847

[10] CAI, X. S., CAPUTO, P., PERARNAU, G. and QUATTROPANI, M. (2023). Rankings in directed configuration models with heavy tailed in-degrees. *Ann. Appl. Probab.* **33** 5613–5667. MR4677741 https://doi.org/10.1214/23-aap1955

[11] CAI, X. S. and PERARNAU, G. (2020). Minimum stationary values of sparse random directed graphs. ArXiv preprint. Available at arXiv:2010.07246.

[12] CAI, X. S. and PERARNAU, G. (2021). The giant component of the directed configuration model revisited. *ALEA Lat. Am. J. Probab. Math. Stat.* **18** 1517–1528. MR4282197 https://doi.org/10.30757/ALEA.v18-55

[13] CAPUTO, P. and QUATTROPANI, M. (2020). Stationary distribution and cover time of sparse directed configuration models. *Probab. Theory Related Fields* **178** 1011–1066. MR4168393 https://doi.org/10.1007/s00440-020-00995-6

[14] CAPUTO, P. and QUATTROPANI, M. (2021). Mixing time of PageRank surfers on sparse random digraphs. *Random Structures Algorithms* **59** 376–406. MR4295568 https://doi.org/10.1002/rsa.21009

[15] CAPUTO, P. and QUATTROPANI, M. (2021). Mixing time trichotomy in regenerating dynamic digraphs. *Stochastic Process. Appl.* **137** 222–251. MR4244192 https://doi.org/10.1016/j.spa.2021.03.003

[16] CHEN, Y.-T. (2013). Sharp benefit-to-cost rules for the evolution of cooperation on regular graphs. *Ann. Appl. Probab.* **23** 637–664. MR3059271 https://doi.org/10.1214/12-AAP849

[17] CHEN, Y.-T. (2018). Wright–Fisher diffusions in stochastic spatial evolutionary games with death-birth updating. *Ann. Appl. Probab.* **28** 3418–3490. MR3861817 https://doi.org/10.1214/18-AAP1390

[18] CHEN, Y.-T. (2021). Precise asymptotics of some meeting times arising from the voter model on large random regular graphs. *Electron. Commun. Probab.* **26** Paper No. 5, 13. MR4236675

[19] CHEN, Y.-T. (2023). The replicator equation in stochastic spatial evolutionary games. *Stochastic Process. Appl.* **157** 94–139. MR4521765 https://doi.org/10.1016/j.spa.2022.11.013

[20] CHEN, Y.-T., CHOI, J. and COX, J. T. (2016). On the convergence of densities of finite voter models to the Wright–Fisher diffusion. *Ann. Inst. Henri Poincaré Probab. Stat.* **52** 286–322. MR3449304 https://doi.org/10.1214/14-AIHP639

[21] CHEN, Y.-T. and COX, J. T. (2018). Weak atomic convergence of finite voter models toward Fleming–Viot processes. *Stochastic Process. Appl.* **128** 2463–2488. MR3804801 https://doi.org/10.1016/j.spa.2017.09.015

[22] CLIFFORD, P. and SUDBURY, A. (1973). A model for spatial conflict. *Biometrika* **60** 581–588. MR0343950 https://doi.org/10.1093/biomet/60.3.581

[23] COOPER, C. and FRIEZE, A. (2004). The size of the largest strongly connected component of a random digraph with a given degree sequence. *Combin. Probab. Comput.* **13** 319–337. MR2056402 https://doi.org/10.1017/S096354830400611X

[24] COOPER, C. and FRIEZE, A. (2005). The cover time of random regular graphs. *SIAM J. Discrete Math.* **18** 728–740. MR2157821 https://doi.org/10.1137/S0895480103428478

[25] COOPER, C. and FRIEZE, A. (2007). The cover time of sparse random graphs. *Random Structures Algorithms* **30** 1–16. MR2283218 https://doi.org/10.1002/rsa.20151

[26] COOPER, C. and FRIEZE, A. (2008). The cover time of the giant component of a random graph. *Random Structures Algorithms* **32** 401–439. MR2422388 https://doi.org/10.1002/rsa.20201

[27] COOPER, C., FRIEZE, A. and RADZIK, T. (2009/10). Multiple random walks in random regular graphs. *SIAM J. Discrete Math.* **23** 1738–1761. MR2570201 https://doi.org/10.1137/080729542

[28] COX, J. T. (1989). Coalescing random walks and voter model consensus times on the torus in $\mathbf{Z}^d$. *Ann. Probab.* **17** 1333–1366. MR1048930

[29] DONNELLY, P. and WELSH, D. (1983). Finite particle systems and infection models. *Math. Proc. Cambridge Philos. Soc.* **94** 167–182. MR0704809 https://doi.org/10.1017/S0305004100060989

[30] DURRETT, R. (2007). *Random Graph Dynamics*. *Cambridge Series in Statistical and Probabilistic Mathematics* **20**. Cambridge Univ. Press, Cambridge. MR2271734

[31] FERNLEY, J. and ORTGIESE, M. (2023). Voter models on subcritical scale-free random graphs. *Random Structures Algorithms* **62** 376–429. MR4545864

[32] FILL, J. A. and LYZINSKI, V. (2014). Hitting times and interlacing eigenvalues: A stochastic approach using intertwinings. *J. Theoret. Probab.* **27** 954–981. MR3245993 https://doi.org/10.1007/s10959-012-0457-9

[33] GRAHAM, R. L., KNUTH, D. E. and PATASHNIK, O. (1994). *Concrete Mathematics*: *A Foundation for Computer Science*, 2nd ed. Addison-Wesley Company, Reading, MA. MR1397498

[34] HERMON, J., LI, S., YAO, D. and ZHANG, L. (2022). Mean field behavior during the Big Bang regime for coalescing random walks. *Ann. Probab.* **50** 1813–1884. MR4474503 https://doi.org/10.1214/22-aop1571

[35] HOFSTAD, R. V. D. (2016). *Random Graphs and Complex Networks*. Cambridge University Press, Cambridge.

[36] HOLLEY, R. A. and LIGGETT, T. M. (1975). Ergodic theorems for weakly interacting infinite systems and the voter model. *Ann. Probab.* **3** 643–663. MR0402985 https://doi.org/10.1214/aop/1176996306

[37] KINGMAN, J. F. C. (1982). The coalescent. *Stochastic Process. Appl.* **13** 235–248. MR0671034 https://doi.org/10.1016/0304-4149(82)90011-4

[38] FRENCH, D. R. and LARCOMBE, P. J. (2003). The Catalan number k-fold self-convolution identity: The original formulation. *J. Combin. Math. Combin. Comput.* **46**.

[39] LIGGETT, T. M. (1985). *Interacting Particle Systems. Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **276**. Springer, New York. MR0776231 https://doi.org/10.1007/978-1-4613-8542-4

[40] LIGGETT, T. M. (1999). *Stochastic Interacting Systems: Contact, Voter and Exclusion Processes. Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **324**. Springer, Berlin. MR1717346 https://doi.org/10.1007/978-3-662-03990-8

[41] MANZO, F., QUATTROPANI, M. and SCOPPOLA, E. (2021). A probabilistic proof of Cooper & Frieze's "first visit time lemma". *ALEA Lat. Am. J. Probab. Math. Stat.* **18** 1739–1758. MR4332217 https://doi.org/10.30757/alea.v18-64

[42] OLIVEIRA, R. I. (2013). Mean field conditions for coalescing random walks. *Ann. Probab.* **41** 3420–3461. MR3127887 https://doi.org/10.1214/12-AOP813

[43] QUATTROPANI, M. and SAU, F. (2023). On the meeting of random walks on random DFA. *Stochastic Process. Appl.* **166** Paper No. 104225. MR4650886 https://doi.org/10.1016/j.spa.2023.104225

[44] SOOD, V., ANTAL, T. and REDNER, S. (2008). Voter models on heterogeneous networks. *Phys. Rev. E* (3) **77** 041121. MR2495459 https://doi.org/10.1103/PhysRevE.77.041121