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# CONVERGENCE OF POPULATION PROCESSES WITH SMALL AND FREQUENT MUTATIONS TO THE CANONICAL EQUATION OF ADAPTIVE DYNAMICS

BY NICOLAS CHAMPAGNAT[1,a] AND VINCENT HASS[2,b]

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In this article, a stochastic individual-based model describing Darwinian evolution of asexual, phenotypic trait-structured population, is studied. We consider a large population with constant population size characterised by a resampling rate modeling competition pressure driving selection and a mutation rate where mutations occur during life. In this model, the population state at fixed time is given as a measure on the space of phenotypes and the evolution of the population is described by a continuous time, measure-valued Markov process. We investigate the asymptotic behaviour of the system, where mutations are frequent, in the double simultaneous limit of large population ($K \to +\infty$) and small mutational effects ($\sigma_K \to 0$) proving convergence to an ODE known as the canonical equation of adaptive dynamics. This result holds only for a certain range of $\sigma_K$ parameters (as a function of $K$) which must be small enough but not too small either. The canonical equation describes the evolution in time of the dominant trait in the population driven by a fitness gradient. This result is based on an slow-fast asymptotic analysis. We use an averaging method, inspired by Kurtz (In *Applied Stochastic Analysis* (1992) 186–209, Springer), which exploits a martingale approach and compactness-uniqueness arguments. The contribution of the fast component, which converges to the centered Fleming–Viot process, is obtained by averaging according to its invariant measure, recently characterised in Champagnat and Hass (*Stochastic Process. Appl.* **166** (2023) 104219).

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# WEAK ERROR ESTIMATES FOR ROUGH VOLATILITY MODELS

BY PETER K. FRIZ[1,3,b], WILLIAM SALKELD[2,a] AND THOMAS WAGENHOFER[3,c]

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We consider a class of stochastic processes with rough stochastic volatility, examples of which include the rough Bergomi and rough Stein–Stein model, that have gained considerable importance in quantitative finance.

A basic question for such (non-Markovian) models concerns efficient numerical schemes. While strong rates are well understood (order $H$), we tackle here the intricate question of weak rates. Our main result asserts that the weak rate, for a reasonably large class of test function, is essentially of order $\min\{3H+\frac{1}{2}, 1\}$ where $H \in (0, 1/2]$ is the Hurst parameter of the fractional Brownian motion that underlies the rough volatility process.

Interestingly, the phase transition at $H = 1/6$ is related to the correlation between the two driving factors, and thus gives additional meaning to a quantity already of central importance in stochastic volatility modelling. Our results are complemented by a lower bound which show that the obtained weak rate is indeed optimal.

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# ASYMPTOTICS FOR THE SITE FREQUENCY SPECTRUM ASSOCIATED WITH THE GENEALOGY OF A BIRTH AND DEATH PROCESS

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Consider a birth and death process started from one individual in which each individual gives birth at rate $\lambda$ and dies at rate $\mu$, so that the population size grows at rate $r = \lambda - \mu$. Lambert (*Theor. Popul. Biol.* **122** (2018) 30–35) and Harris, Johnston, and Roberts (*Ann. Appl. Probab.* **30** (2020) 1368–1414) came up with methods for constructing the exact genealogy of a sample of size $n$ taken from this population at time $T$. We use the construction of Lambert, which is based on the coalescent point process, to obtain asymptotic results for the site frequency spectrum associated with this sample. In the supercritical case $r > 0$, our results extend results of Durrett (*Ann. Appl. Probab.* **23** (2013) 230–250) for exponentially growing populations. In the critical case $r = 0$, our results parallel those that Dahmer and Kersting (*Ann. Appl. Probab.* **25** (2015) 1325–1348) obtained for Kingman's coalescent.

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# SOLIDARITY OF GIBBS SAMPLERS: THE SPECTRAL GAP

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Gibbs samplers are preeminent Markov chain Monte Carlo algorithms used in computational physics and statistical computing. Yet, their most fundamental properties, such as relations between convergence characteristics of their various versions, are not well understood.

In this paper we prove the solidarity principle of the spectral gap for the Gibbs sampler: if any of the random scan or $d!$ deterministic scans has a spectral gap then all of them have. Our methods rely on geometric interpretation of the Gibbs samplers as alternating projection algorithms and analysis of the rate of convergence in the von Neumann–Halperin method of cyclic alternating projections. As a byproduct of our analysis, we also establish that deterministic scan Gibbs, despite being nonreversible, share many robustness properties with reversible chains, including exponential inequalities and central limit theorems under the same conditions.

In addition, we provide a quantitative result: if the spectral gap of the random scan Gibbs sampler decays with dimension at a polynomial rate $\beta$, then the rate is no worse than $2\beta + 2$ for any deterministic scan.

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# THE CRITICAL BETA-SPLITTING RANDOM TREE I: HEIGHTS AND RELATED RESULTS

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In the critical beta-splitting model of a random $n$-leaf binary tree, leaf-sets are recursively split into subsets, and a set of $m$ leaves is split into subsets containing $i$ and $m-i$ leaves with probabilities proportional to $1/i(m-i)$. We study the continuous-time model in which the holding time before that split is exponential with rate $h_{m-1}$, the harmonic number. We (sharply) evaluate the first two moments of the time-height $D_n$ and of the edge-height $L_n$ of a uniform random leaf (i.e., the length of the path from the root to the leaf), and prove the corresponding CLTs. We study the correlation between the heights of two random leaves of the same tree realization, and analyze the expected number of splits necessary for a set of $t$ leaves to partially or completely break away from each other. We give tail bounds for the time-height and the edge-height of the *tree*, that is, the maximal leaf heights. We show that there is a limit distribution for the size of a uniform random subtree, and derive the asymptotics of the mean size. Our proofs are based on asymptotic analysis of the attendant (sum-type) recurrences. The essential idea is to replace such a recursive equality by a pair of recursive inequalities for which matching asymptotic solutions can be found, allowing one to bound, both ways, the elusive explicit solution of the recursive equality. This reliance on recursive inequalities necessitates usage of Laplace transforms rather than Fourier characteristic functions.

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# CONTRACTIVE COUPLING RATES AND CURVATURE LOWER BOUNDS FOR MARKOV CHAINS

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Contractive coupling rates have been recently introduced by Conforti as a tool to establish convex Sobolev inequalities (including modified log-Sobolev and Poincaré inequality) for some classes of Markov chains. In this work, for most of the examples discussed by Conforti, we use contractive coupling rates to prove stronger inequalities, in the form of curvature lower bounds (in entropic and discrete Bakry–Émery sense) and geodesic convexity of some entropic functionals. In addition, we recall and give straightforward generalizations of some notions of coarse Ricci curvature, and we discuss some of their properties and relations with the concepts of couplings and coupling rates: as an application, we show exponential contraction of the $p$-Wasserstein distance for the heat flow in the aforementioned examples.

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# APPROXIMATELY OPTIMAL DISTRIBUTED STOCHASTIC CONTROLS BEYOND THE MEAN FIELD SETTING

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We study high-dimensional stochastic optimal control problems in which many agents cooperate to minimize a convex cost functional. We consider both the full-information problem, in which each agent observes the states of all other agents, and the distributed problem, in which each agent observes only its own state. Our main results are sharp nonasymptotic bounds on the gap between these two problems, measured both in terms of their value functions and optimal states. Along the way, we develop theory for distributed optimal stochastic control in parallel with the classical setting, by characterizing optimizers in terms of an associated stochastic maximum principle and a Hamilton–Jacobi-type equation. By specializing these results to the setting of mean field control, in which costs are (symmetric) functions of the empirical distribution of states, we derive the optimal rate for the convergence problem in the displacement convex regime.

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# SAMPLE-PATH LARGE DEVIATION PRINCIPLE FOR A 2-D STOCHASTIC INTERACTING VORTEX DYNAMICS WITH SINGULAR KERNEL

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We consider a stochastic interacting vortex system of $N$ particles, approximating the vorticity formulation of the 2-D Navier–Stokes equation on a torus. The singular interaction kernel is given by the Biot–Savart law. Assuming only that the initial state has finite energy, we derive a sample-path large deviation principle for the empirical measure as the number of vortices tends to infinity. This contrasts with previous studies that require the initial state to have finite entropy. The rate function is characterized by an explicit formula that takes finite values only on sample paths with finite energy and finite integrals of $L^2$ norms over time. The proof employs a symmetrization technique for the representation of the singular kernel, originating from Delort (*J. Amer. Math. Soc.* **4** (1991) 553–586), and a carefully modified energy dissipation structure to establish a sharp prior estimate for an auxiliary functional as a modification of the rate function. The key step is to prove that the singular term after symmetrization can be bounded by the integral of $L^2$ norms along sample paths.

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# UNADJUSTED HAMILTONIAN MCMC WITH STRATIFIED MONTE CARLO TIME INTEGRATION

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A randomized time integrator is suggested for unadjusted Hamiltonian Monte Carlo (uHMC) which involves a very minor modification to the usual Verlet time integrator, and hence, is easy to implement. For target distributions of the form $\mu(dx) \propto e^{-U(x)}\,dx$ where $U : \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ is $K$-strongly convex but only $L$-gradient Lipschitz, and initial distributions $\nu$ with finite second moment, coupling proofs reveal that an $\varepsilon$-accurate approximation of the target distribution in $L^2$-Wasserstein distance $\boldsymbol{\mathcal{W}}^2$ can be achieved by the uHMC algorithm with randomized time integration using $O((d/K)^{1/3}(L/K)^{5/3}\varepsilon^{-2/3}\log(\boldsymbol{\mathcal{W}}^2(\mu,\nu)/\varepsilon)^+)$ gradient evaluations; whereas for such rough target densities the corresponding complexity of the uHMC algorithm with Verlet time integration is in general $O((d/K)^{1/2}(L/K)^2\varepsilon^{-1}\log(\boldsymbol{\mathcal{W}}^2(\mu,\nu)/\varepsilon)^+)$. Metropolis-adjustable randomized time integrators are also provided.

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# Λ-WRIGHT–FISHER PROCESSES WITH GENERAL SELECTION AND OPPOSING ENVIRONMENTAL EFFECTS: FIXATION AND COEXISTENCE

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Our results characterize the long-term behavior for a broad class of $\Lambda$-Wright–Fisher processes (on $[0, 1]$) with frequency-dependent and environmental selection. In particular, we reveal a rich variety of parameter-dependent behaviors and provide explicit criteria to discriminate between them. That includes the situation in which both boundary points, 0 and 1, are repelling—a new phenomenon in this context. This has significant biological implications, because it means that selection alone can maintain genetic variation, that is, coexistence. If a boundary point is attractive, we derive polynomial/exponential decay rates for the probability of not being polynomially/exponentially close to that boundary, depending on some weak/strong integrability conditions. Moreover, we provide a handy representation of the fixation probability. In our proofs we make use of Siegmund duality. The dual process can be sandwiched near the boundaries in-between transformed Lévy processes. In this way we relate the boundary behavior of the dual process to fluctuation properties of these Lévy processes and shed new light on previously established conditions for attractive/repelling boundary points. Our method allows us to treat models that so far could not be analyzed by means of moment or Bernstein duality. This closes an existing gap in the literature.

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# INDEPENDENCE PROPERTY OF THE BUSEMANN FUNCTION IN EXACTLY SOLVABLE KPZ MODELS

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The study of Kadar–Parsi–Zhang (KPZ) universality class has been a subject of great interest among mathematicians and physicists over the past three decades. A notably successful approach for analyzing KPZ models is the coupling method, which hinges on understanding random growth from stationary initial conditions defined by Busemann functions. To advance in this direction, we investigate the independence property of the Busemann function across multiple directions in various exactly solvable KPZ models. These models encompass the corner growth model, the inverse-gamma polymer, Brownian last-passage percolation, the O'Connell–Yor polymer, the KPZ equation, and the directed landscape. In the context of the corner growth model, our result states that disjoint Busemann increments in different directions along a down-right path are independent, as long as their associated semi-infinite geodesics have nonempty intersections almost surely. The proof for the independence utilizes the queueing representation of the Busemann process developed by Seppäläinen et al. As an application, our independence result yields a near-optimal probability upper bound (missing by a logarithmic factor) for the rare event where the endpoint of a point-to-line inverse-gamma polymer is close to the diagonal.

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# STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS ARISING IN SELF-ORGANIZED CRITICALITY

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We study scaling limits of the weakly driven Zhang and the Bak–Tang–Wiesenfeld (BTW) model for self-organized criticality. We show that the weakly driven Zhang model converges to a stochastic partial differential equation (PDE) with singular-degenerate diffusion. In addition, the deterministic BTW model is shown to converge to a singular-degenerate PDE. Alternatively, the proof of the scaling limit can be understood as a convergence proof of a finite-difference discretization for singular-degenerate stochastic PDEs. This extends recent work on finite difference approximation of (deterministic) quasilinear diffusion equations to discontinuous diffusion coefficients and stochastic PDEs. In addition, we perform numerical simulations illustrating key features of the considered models and the convergence to stochastic PDEs in spatial dimension $d = 1, 2$.

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# SHARP THRESHOLDS IN INFERENCE OF PLANTED SUBGRAPHS

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We connect the study of phase transitions in high-dimensional statistical inference to the study of threshold phenomena in random graphs.

A major question in the study of the Erdős–Rényi random graph $G(n, p)$ is to understand the probability, as a function of $p$, that $G(n, p)$ contains a given subgraph $H = H_n$. This was studied for many specific examples of $H$, starting with classical work of Erdős and Rényi (1960). More recent work studies this question for general $H$, both in building a general theory of sharp versus coarse transitions (Friedgut and Bourgain (1999); Hatami (2012)) and in results on the location of the transition (Kahn and Kalai (2007); Talagrand (2010); Frankston, Kahn, Narayanan, Park (2019); Park and Pham (2022)).

In inference problems, one often studies the optimal accuracy of inference as a function of the amount of noise. In a variety of sparse recovery problems, an "all-or-nothing (AoN) phenomenon" has been observed: Informally, as the amount of noise is gradually increased, at some critical threshold the inference problem undergoes a sharp jump from near-perfect recovery to near-zero accuracy (Gamarnik and Zadik (2017); Reeves, Xu, Zadik (2021)). We can regard AoN as the natural inference analogue of the sharp threshold phenomenon in random graphs. In contrast with the general theory developed for sharp thresholds of random graph properties, the AoN phenomenon has only been studied so far in specific inference settings, and a general theory behind its appearance remains elusive.

In this paper we study the general problem of inferring a graph $H = H_n$ planted in an Erdős–Rényi random graph, thus naturally connecting the two lines of research mentioned above. We show that questions of AoN are closely connected to first moment thresholds, and to a generalization of the so-called Kahn–Kalai expectation threshold that scans over subgraphs of $H$ of edge density at least $q$. In a variety of settings we characterize AoN, by showing that AoN occurs *if and only if* this "generalized expectation threshold" is roughly constant in $q$. Our proofs combine techniques from random graph theory and Bayesian inference.

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# RATES IN THE CENTRAL LIMIT THEOREM FOR RANDOM PROJECTIONS OF MARTINGALES

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In this paper, we consider partial sums of martingale differences weighted by random variables drawn uniformly on the sphere, and globally independent of the martingale differences. Combining Lindeberg's method and a series of arguments due to Bobkov, Chistyakov and Götze, we show that the Kolmogorov distance between the distribution of these weighted sums and the limiting Gaussian is "super-fast" of order $(\log n)^2/n$, under conditions allowing us to control the higher-order conditional moments of the martingale differences. We also show that the same rate is achieved if we consider a quantity very close to these weighted sums, and give an application of this result to the least squares estimator of the slope in the linear model with Gaussian design.

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# WILSON LINES IN THE LATTICE HIGGS MODEL AT STRONG COUPLING

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We consider the 4D fixed length lattice Higgs model with Wilson action for the gauge field and structure group $\mathbb{Z}_n$. We study Wilson line observables in the strong coupling regime and compute their asymptotic behavior with error estimates. Our analysis is based on a high-temperature representation of the lattice Higgs measure combined with Poisson approximation. We also give a short proof of the folklore result that Wilson line (and loop) expectations exhibit perimeter law decay whenever the Higgs field coupling constant is positive.

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# THE STOCHASTIC PRIMITIVE EQUATIONS WITH NONISOTHERMAL TURBULENT PRESSURE

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In this paper, we introduce and study the primitive equations with *non*-isothermal turbulent pressure and transport noise. They are derived from the Navier–Stokes equations by employing stochastic versions of the Boussinesq and the hydrostatic approximations. The temperature dependence of the turbulent pressure can be seen as a consequence of an additive noise acting on the small vertical dynamics. For such a model we prove global well-posedness in $H^1$ where the noise is considered in both the Itô and Stratonovich formulations. Compared to previous variants of the primitive equations, the one considered here presents a more intricate coupling between the velocity field and the temperature. The corresponding analysis is seriously more involved than in the deterministic setting. Finally, the continuous dependence on the initial data and the energy estimates proven here are new, even in the case of isothermal turbulent pressure.

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# THE EFFECTIVE STRENGTH OF SELECTION IN RANDOM ENVIRONMENT

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We analyse a family of two-types Wright–Fisher models with selection in a random environment and skewed offspring distribution. We provide a calculable criterion to quantify the impact of different shapes of selection on the fate of the weakest allele, and thus compare them. The main mathematical tool is duality, which we prove to hold, also in presence of random environment (quenched and in some cases annealed), between the population's allele frequencies and genealogy, both in the case of finite population size and in the scaling limit for large size. Duality also yields new insight on properties of branching-coalescing processes in random environment, such as their long-term behaviour.

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