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DISTRIBUTIONALLY ROBUST GAUSSIAN PROCESS REGRESSION AND BAYESIAN INVERSE PROBLEMS

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We study a distributionally robust optimization formulation (i.e., a min-max game) for two representative problems in Bayesian nonparametric estimation: Gaussian process regression and, more generally, linear inverse problems. Our formulation seeks the best mean-squared error predictor in an infinite-dimensional space against an adversary who chooses the worst-case model in a Wasserstein ball around a nominal infinite-dimensional Bayesian model. The transport cost is chosen to control features such as the degree of roughness of the sample paths that the adversary is allowed to inject. We show that strong duality holds in the sense that max-min equals min-max, and that there exists a unique Nash equilibrium that can be computed by a sequence of finite-dimensional approximations. Crucially, the worst-case distribution is itself Gaussian. We explore the properties of the Nash equilibrium and the effects of hyperparameters through numerical experiments, demonstrating the versatility of our modeling framework.

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MEAN FIELD GAMES WITH COMMON NOISE AND DEGENERATE IDIOSYNCRATIC NOISE

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We study the forward-backward system of stochastic partial differential equations describing a mean field game for a large population of small players subject to both idiosyncratic and common noise. The unique feature of the problem is that the idiosyncratic noise coefficient may be degenerate, so that the system does not admit smooth solutions in general. We develop a new notion of weak solutions for backward stochastic Hamilton–Jacobi–Bellman equations, and use this to build probabilistically weak solutions of the mean field game system. Under an additional monotonicity assumption, we prove the uniqueness of a strong solution.

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ERGODIC RISK SENSITIVE CONTROL OF MARKOVIAN MULTICLASS MANY-SERVER QUEUES WITH ABANDONMENT

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We study the optimal scheduling problem for a Markovian multiclass queueing network with abandonment in the Halfin–Whitt regime, under the long run average (ergodic) risk sensitive cost criterion. The objective is to prove asymptotic optimality for the optimal control arising from the corresponding ergodic risk sensitive control (ERSC) problem for the limiting diffusion. In particular, we show that the optimal ERSC value associated with the diffusion-scaled queueing process converges to that of the limiting diffusion in the asymptotic regime. The challenge that ERSC poses is that one cannot express the ERSC cost as an expectation over the mean empirical measure associated with the queueing process, unlike in the usual case of a long run average (ergodic) cost. We develop a novel approach by exploiting the variational representations of the limiting diffusion and the Poisson-driven queueing dynamics, which both involve certain auxiliary controls. The ERSC costs for both the diffusion-scaled queueing process and the limiting diffusion can be represented as the integrals of an extended running cost over a mean empirical measure associated with the corresponding extended processes using these auxiliary controls. For the lower bound proof, we exploit the connections of the ERSC problem for the limiting diffusion with a two-person zero-sum stochastic differential game. We also make use of the mean empirical measures associated with the extended limiting diffusion and diffusion-scaled processes with the auxiliary controls. One major technical challenge in both lower and upper bound proofs, is to establish the tightness of the aforementioned mean empirical measures for the extended processes. We identify nearly optimal controls appropriately in both cases so that the existing ergodicity properties of the limiting diffusion and diffusion-scaled queueing processes can be used.

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STRONG SOLUTIONS TO SUBMODULAR MEAN FIELD GAMES WITH COMMON NOISE AND RELATED MCKEAN–VLASOV FBSDES

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This paper studies multidimensional mean field games with common noise and the related system of McKean–Vlasov forward-backward stochastic differential equations deriving from the stochastic maximum principle. We first propose some structural conditions which are related to the submodularity of the underlying mean field game and are a sort of opposite version of the well known Lasry–Lions monotonicity. By reformulating the representative player minimization problem via the stochastic maximum principle, the submodularity conditions allow to prove comparison principles for the forward-backward system, which correspond to the monotonicity of the best reply map. Building on this property, existence of strong solutions is shown via Tarski's fixed point theorem, both for the mean field game and for the related McKean–Vlasov forward-backward system. In both cases, the set of solutions enjoys a lattice structure, with minimal and maximal solutions which can be constructed by iterating the best reply map or via the *fictitious play* algorithm.

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HESSIAN SPECTRUM AT THE GLOBAL MINIMUM AND TOPOLOGY TRIVIALIZATION OF LOCALLY ISOTROPIC GAUSSIAN RANDOM FIELDS

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We study the energy landscape near the ground state of a model of a single particle in a random potential with trivial topology. More precisely, we find the large-dimensional limit of the Hessian spectrum at the global minimum of the Hamiltonian $X_N(x) + \frac{\mu}{2}\|x\|^2, x \in \mathbb{R}^N$, when μ is above the phase transition threshold so that the system has only one critical point. Here X_N is a locally isotropic Gaussian random field. The same idea is also applied to study the more general model of elastic manifold. In the replica symmetric regime, our results confirm the predictions of Fyodorov and Le Doussal made in 2018 and 2020 using the replica method.

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QUANTITATIVE CLTS ON THE POISSON SPACE VIA SKOROHOD ESTIMATES AND p-POINCARÉ INEQUALITIES

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We establish new explicit bounds on the Gaussian approximation of Poisson functionals based on novel estimates of moments of Skorohod integrals. Combining these with the Malliavin–Stein method, we derive bounds in the Wasserstein and Kolmogorov distances whose application requires minimal moment assumptions on add-one cost operators—thereby extending the results from (Last, Peccati and Schulte, 2016). Our paper also removes a redundant term from the bound in Kolmogorov distance and replaces a complicated term by a simpler one, without requiring additional assumptions. Our applications include a central limit theorem (CLT) for the online nearest neighbour graph, whose validity was conjectured in (Wade, 2009; Penrose and Wade, 2009). We also apply our techniques to derive quantitative CLTs for edge functionals of the Gilbert graph, of the k-nearest neighbour graph and of the radial spanning tree. In most cases, even the qualitative CLTs are new.

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CONTINUOUS TIME RANDOM MATCHING

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Continuous-time random matching with a large (continuum) population is widely exploited in the literature, but has not had a rigorous formulation, nor a demonstration of its key assumed properties. This paper provides the first probabilistic foundation for this approach by presenting a mathematical model of continuous-time random matching and showing its existence and properties. The agents' types, which can change due to random matching and random mutation, form a continuum of independent continuous-time Markov chains. Using the exact law of large numbers, we show how the cross-sectional distribution of agent types evolves deterministically according to an explicit ordinary differential equation. Nonstandard analysis is used in proving the main theorem.

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CONDITIONS FOR EXISTENCE AND UNIQUENESS OF THE INVERSE FIRST-PASSAGE TIME PROBLEM APPLICABLE FOR LÉVY PROCESSES AND DIFFUSIONS

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For a real-valued stochastic process $(X_t)_{t\geq 0}$ we establish conditions under which the inverse first-passage time problem has a solution for any random variable $\xi > 0$. For Markov processes we give additional conditions under which the solutions are unique and solutions corresponding to ordered initial states fulfill a comparison principle. As examples we show that these conditions include Lévy processes with infinite activity or unbounded variation and diffusions on an interval with appropriate behavior at the boundaries. Our methods are based on the techniques used in the case of Brownian motion and rely on discrete approximations of solutions via Γ-convergence from (*Theory Probab. Appl.* **25** (1980) 362–366) and (*Ann. Appl. Probab.* **21** (2011) 1663–1693) combined with stochastic ordering arguments adapted from (*Theory Probab. Appl.* **67** (2023) 570–592).

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ON A GENERAL KAC–RICE FORMULA FOR THE MEASURE OF A LEVEL SET

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Let $X(\cdot)$ be a random field from $\mathbb{R}^D$ to $\mathbb{R}^d$, where $D \geq d$. We study the level set $X^{-1}(u)$, where $u \in \mathbb{R}^d$. Specifically, we provide a weak condition for this level set to be rectifiable. Next, we establish the Kac–Rice formula to compute the $(D-d)$-dimensional Hausdorff measure. Our results extend previous work, particularly in the non-Gaussian case where we obtain a very general result. We conclude with several extensions and examples of applications, including functions of Gaussian random fields, zeros of the likelihood functions, gravitational microlensing, and shot-noise.

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THE MORAN MODEL WITH RANDOM RESAMPLING RATES

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In this paper we consider the two-type Moran model with N individuals. Each individual is assigned a resampling rate, drawn independently from a probability distribution $\mathbb{P}$ on $\mathbb{R}_+$, and a type, either 1 or 0. Each individual resamples its type at its assigned rate, by adopting the type of an individual drawn uniformly at random. Let $Y^N(t)$ denote the empirical distribution of the resampling rates of the individuals with type 1 at time Nt. We show that if $\mathbb{P}$ has countable support and satisfies certain tail and moment conditions, then in the limit as $N \to \infty$ the process $(Y^N(t))_{t\geq 0}$ converges in law to the process $(S(t)\mathbb{P})_{t\geq 0}$, in the so-called Meyer–Zheng topology, where $(S(t))_{t\geq 0}$ is the Fisher–Wright diffusion with diffusion constant D given by $1/D = \int_{\mathbb{R}_+}(1/r)\mathbb{P}(\mathrm{d}r)$.

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EULER SCHEME FOR SDES DRIVEN BY FRACTIONAL BROWNIAN MOTIONS: INTEGRABILITY AND CONVERGENCE IN LAW

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We prove that the Euler scheme for stochastic differential equations driven by fractional Brownian motions (fBm) with Hurst parameter $H > 1/3$ and its Malliavin derivatives are integrable uniformly in step size n. Then we use the integrability results to derive the weak convergence rate $n^{1-4H+\varepsilon}$ for the Euler scheme. The proof for integrability is based on an application of the argument of (*Ann. Probab.* **41** (2013) 3026–3050) to a quadratic functional of the fBm. The proof of weak convergence applies Malliavin calculus and some upper-bound estimates for weighted random sums.

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L^∞-OPTIMAL TRANSPORT OF ANISOTROPIC LOG-CONCAVE MEASURES AND EXPONENTIAL CONVERGENCE IN FISHER'S INFINITESIMAL MODEL

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We prove upper bounds on the L^∞-Wasserstein distance from optimal transport between strongly log-concave probability densities and log-Lipschitz perturbations. In the simplest setting, such a bound amounts to a transport-information inequality involving the L^∞-Wasserstein metric and the relative L^∞-Fisher information. We show that this inequality can be sharpened significantly in situations where the involved densities are anisotropic. Our proof is based on probabilistic techniques using Langevin dynamics. As an application of these results, we obtain sharp exponential rates of convergence in Fisher's infinitesimal model from quantitative genetics, generalising recent results by Calvez, Poyato, and Santambrogio in dimension 1 to arbitrary dimensions.

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ASYMPTOTIC EXPANSIONS FOR HIGH-FREQUENCY OPTION DATA

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We derive a nonparametric higher-order asymptotic expansion for small-time changes of conditional characteristic functions of Itô semimartingale increments. The asymptotics setup is of joint type: both the length of the time interval of the increment of the underlying process and the time gap between evaluating the conditional characteristic function are shrinking. The spot semimartingale characteristics of the underlying process as well as their spot semimartingale characteristics appear as leading terms in the derived asymptotic expansions. The analysis applies to a general class of Itô semimartingales that includes in particular Lévy-driven SDEs and time-changed Lévy processes. The asymptotic expansion results are subsequently used to construct a test for whether volatility jumps are of finite or infinite variation. In an application to high-frequency data of options written on the S&P 500 index, we find evidence for infinite variation volatility jumps.

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SHARP NONUNIQUENESS OF SOLUTIONS TO 2D NAVIER–STOKES EQUATIONS WITH SPACE-TIME WHITE NOISE

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In this paper we are concerned with the 2D incompressible Navier–Stokes equations driven by space-time white noise. We establish existence of infinitely many global-in-time probabilistically strong and analytically weak solutions u for every divergence free initial condition $u_0 \in L^p \cup C^{-1+\delta}$, $p \in (1,2)$, $\delta > 0$. More precisely, there exist infinitely many solutions such that $u - z \in C([0,\infty); L^p) \cap L^2_{\mathrm{loc}}([0,\infty); H^\zeta) \cap L^1_{\mathrm{loc}}([0,\infty); W^{\frac{1}{3},1})$ for some $\zeta \in (0,1)$, where z is the solution to the linear equation. This result in particular implies nonuniqueness in law. Our result is sharp in the sense that the solution satisfying $u - z \in C([0,\infty); L^2) \cap L^2_{\mathrm{loc}}([0,\infty); H^\zeta)$ for some $\zeta \in (0,1)$ is unique.

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THE ANISOTROPY OF 2D OR 3D GAUSSIAN RANDOM FIELDS THROUGH THEIR LIPSCHITZ–KILLING CURVATURE DENSITIES

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We are interested here in modeling and estimating the anisotropy of Gaussian random fields through the geometry of their excursion sets. In order to do this, we use Lipschitz–Killing curvatures of the level sets as functions of the levels and see them as generalized processes for which we are able to obtain a joint functional central limit theorem. For 2D and 3D stationary Gaussian fields we provide explicit formulas for the Lipschitz–Killing curvature densities. Then, we can deduce geometrical equivalent of second spectral moments and anisotropy ratios that allow the estimation of the anisotropy of the underlying Gaussian field.

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CUTOFF FOR RANDOM WALK ON RANDOM GRAPHS WITH A COMMUNITY STRUCTURE

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We consider a variant of the configuration model with an embedded community structure and study the mixing properties of a simple random walk on it. Every vertex has an internal $\deg^{\text{int}} \geq 3$ and an outgoing $\deg^{\text{out}}$ number of half-edges. Given a stochastic matrix Q, we pick a random perfect matching of the half-edges subject to the constraint that each vertex v has $\deg^{\text{int}}(v)$ neighbours inside its community and the proportion of outgoing half-edges from community i matched to a half-edge from community j is $Q(i, j)$. Assuming the number of communities is constant and they all have comparable sizes, we prove the following dichotomy: simple random walk on the resulting graph exhibits cutoff if and only if the product of the Cheeger constant of Q times $\log n$ (where n is the number of vertices) diverges.

In (*Ann. Appl. Probab.* **30** (2020) 1824–1846), Ben-Hamou established a dichotomy for cutoff for a nonbacktracking random walk on a similar random graph model with two communities. We prove that the same characterisation of cutoff holds for simple random walk.

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MEAN-FIELD BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS AND NONLOCAL PDES WITH QUADRATIC GROWTH

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In this paper, we study general mean-field backward stochastic differential equations (BSDEs, for short) with quadratic growth. First, using some new ideas, we prove the existence and uniqueness of local and global solutions for a one-dimensional mean-field BSDE when the generator $g(t, Y, Z, \mathbb{P}_Y, \mathbb{P}_Z)$ has quadratic growth in Z and the terminal value is bounded. Second, we derive a comparison theorem for general mean-field BSDEs by applying the Girsanov transform. Third, within this framework, we use the mean-field BSDE to provide a probabilistic representation of the viscosity solution for a nonlocal partial differential equation (PDE, for short) as an extended nonlinear Feynman–Kac formula, which yields the existence and uniqueness of the solution to the PDE. Finally, we prove the convergence of the particle systems to general mean-field BSDEs with quadratic growth and give the corresponding convergence rate.

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STOCHASTIC EPIDEMIC MODELS WITH VARYING INFECTIVITY AND WANING IMMUNITY

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We study an individual-based stochastic epidemic model in which infected individuals become susceptible again, after each infection. In contrast to classical compartment models, after each infection, the infectivity is a random function of the time elapsed since infection. Similarly, recovered individuals become gradually susceptible after some time according to a random susceptibility function.

We study the large population asymptotic behaviour of the model: we prove a functional law of large numbers (FLLN) and investigate the endemic equilibria of the limiting deterministic model. The limit depends on the law of the susceptibility random functions but only on the mean infectivity functions. The FLLN is proved by constructing a sequence of i.i.d. auxiliary processes and adapting the approach from the theory of propagation of chaos. The limit is a generalisation of a PDE model introduced by Kermack and McKendrick, and we show how this PDE model can be obtained as a special case of our FLLN limit.

If R_0 is less than (or equal to) some threshold, the epidemic does not last forever and eventually disappears from the population, while if R_0 is larger than this threshold, the epidemic will not go extinct and there exists an endemic equilibrium. The value of this threshold turns out to be the harmonic mean of the susceptibility a long time after an infection.

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DRIFT CONTROL WITH DISCRETIONARY STOPPING FOR A DIFFUSION

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We consider stochastic control with discretionary stopping for the drift of a diffusion process over an infinite time horizon. The objective is to choose a control process and a stopping time to minimize the expectation of a convex terminal cost in the presence of a fixed operating cost and a control-dependent running cost per unit of elapsed time. Under appropriate conditions on the coefficients of the controlled diffusion, an optimal pair of control and stopping rules is shown to exist. Moreover, under the same assumptions, it is shown that the optimal control is a constant which can be computed fairly explicitly; and that it is optimal to stop the first time an appropriate interval is visited. We consider also a constrained version of the above problem, in which an upper bound on the expectation of available stopping times is imposed; we show that this constrained problem can be reduced to an unconstrained problem with some appropriate change of parameters and, as a result, solved by similar arguments.

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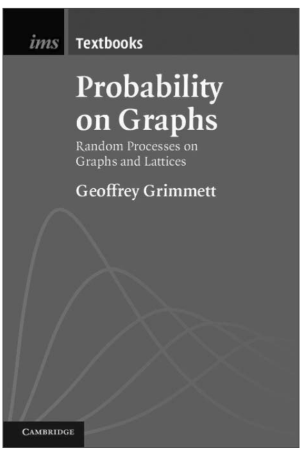

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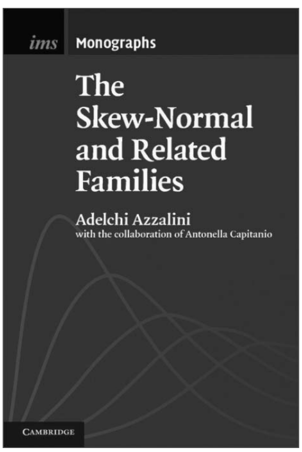

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