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HIGH-DIMENSIONAL SCALING LIMITS AND FLUCTUATIONS OF ONLINE LEAST-SQUARES SGD WITH SMOOTH COVARIANCE

BY KRISHNAKUMAR BALASUBRAMANIAN[1,a], PROMIT GHOSAL[2,b] AND YE HE[3,c]

[1]*Department of Statistics, University of California, Davis,* [a]*kbala@ucdavis.edu*
[2]*Department of Statistics, University of Chicago,* [b]*promit@uchicago.edu*
[3]*School of Mathematics, Georgia Institute of Technology,* [c]*yhe367@gatech.edu*

We derive high-dimensional scaling limits and fluctuations for the online least-squares stochastic gradient descent (SGD) algorithm by taking the properties of the data generating model explicitly into consideration. Our approach treats the SGD iterates as an interacting particle system, where the expected interaction is characterized by the covariance structure of the input. Assuming smoothness conditions on moments of order up to eight orders, and without explicitly assuming Gaussianity, we establish the high-dimensional scaling limits and fluctuations in the form of infinite-dimensional ordinary differential equations (ODEs) or stochastic differential equations (SDEs). Our results reveal a precise three-step phase transition of the iterates; it goes from being ballistic, to diffusive, and finally to purely random behavior, as the noise variance goes from low, to moderate, and finally to very-high noise setting. In the low-noise setting, we further characterize the precise fluctuations of the (scaled) iterates as infinite-dimensional SDEs. We also show the existence and uniqueness of solutions to the derived limiting ODEs and SDEs. Our results have several applications, including characterization of the limiting mean-square estimation or prediction errors and their fluctuations, which can be obtained by analytically or numerically solving the limiting equations.

REFERENCES

[1] ABBE, E., ADSERA, E. B. and MISIAKIEWICZ, T. (2022). The merged-staircase property: A necessary and nearly sufficient condition for SGD learning of sparse functions on two-layer neural networks. In *Conference on Learning Theory* 4782–4887. PMLR.

[2] ADAMS, R. A. and FOURNIER, J. J. F. (2003). *Sobolev Spaces*, 2nd ed. *Pure and Applied Mathematics* (*Amsterdam*) **140**. Elsevier/Academic Press, Amsterdam. MR2424078

[3] ADLER, R. J. and TAYLOR, J. E. (2007). *Random Fields and Geometry*. *Springer Monographs in Mathematics*. Springer, New York. MR2319516

[4] AGRAWALLA, B., BALASUBRAMANIAN, K. and GHOSAL, P. (2023). High-dimensional Central Limit Theorems for Linear Functionals of Online Least-Squares SGD. arXiv preprint. Available at arXiv:2302.09727.

[5] ANASTASIOU, A., BALASUBRAMANIAN, K. and ERDOGDU, M. A. (2019). Normal approximation for stochastic gradient descent via non-asymptotic rates of martingale CLT. In *Conference on Learning Theory* PMLR.

[6] ARNOL'D, V. I. (1992). *Ordinary Differential Equations*. *Springer Textbook*. Springer, Berlin. MR1162307

[7] ASI, H. and DUCHI, J. C. (2019). Stochastic (approximate) proximal point methods: Convergence, optimality, and adaptivity. *SIAM J. Optim.* **29** 2257–2290. MR4000228 https://doi.org/10.1137/18M1230323

[8] BA, J., ERDOGDU, M. A., SUZUKI, T., WANG, Z., WU, D. and YANG, G. (2022). High-dimensional asymptotics of feature learning: How one gradient step improves the representation. In *Advances in Neural Information Processing Systems*.

[9] BALÁZS, M. and SEPPÄLÄINEN, T. (2010). Order of current variance and diffusivity in the asymmetric simple exclusion process. *Ann. of Math.* (2) **171** 1237–1265. MR2630064 https://doi.org/10.4007/annals.2010.171.1237

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[10] BARAKAT, A., BIANCHI, P., HACHEM, W. and SCHECHTMAN, S. (2021). Stochastic optimization with momentum: Convergence, fluctuations, and traps avoidance. *Electron. J. Stat.* **15** 3892–3947. MR4298987 https://doi.org/10.1214/21-ejs1880

[11] BARTLETT, P. L., LONG, P. M., LUGOSI, G. and TSIGLER, A. (2020). Benign overfitting in linear regression. *Proc. Natl. Acad. Sci. USA* **117** 30063–30070. MR4263288 https://doi.org/10.1073/pnas.1907378117

[12] BEN AROUS, G., GHEISSARI, R. and JAGANNATH, A. (2021). Online stochastic gradient descent on nonconvex losses from high-dimensional inference. *J. Mach. Learn. Res.* **22** Paper No. 106. MR4279757

[13] BEN AROUS, G., GHEISSARI, R. and JAGANNATH, A. (2024). High-dimensional limit theorems for SGD: Effective dynamics and critical scaling. *Comm. Pure Appl. Math.* **77** 2030–2080. MR4692886 https://doi.org/10.1002/cpa.22169

[14] BENVENISTE, A., MÉTIVIER, M. and PRIOURET, P. (1990). *Adaptive Algorithms and Stochastic Approximations*. *Applications of Mathematics* (*New York*) **22**. Springer, Berlin. MR1082341 https://doi.org/10.1007/978-3-642-75894-2

[15] BERTINI, L. and GIACOMIN, G. (1997). Stochastic Burgers and KPZ equations from particle systems. *Comm. Math. Phys.* **183** 571–607. MR1462228 https://doi.org/10.1007/s002200050044

[16] BILLINGSLEY, P. (2009). *Convergence of Probability Measures*. Wiley, New York. MR0233396

[17] BOTTOU, L., CURTIS, F. E. and NOCEDAL, J. (2018). Optimization methods for large-scale machine learning. *SIAM Rev.* **60** 223–311. MR3797719 https://doi.org/10.1137/16M1080173

[18] CAPONNETTO, A. and DE VITO, E. (2007). Optimal rates for the regularized least-squares algorithm. *Found. Comput. Math.* **7** 331–368. MR2335249 https://doi.org/10.1007/s10208-006-0196-8

[19] CELENTANO, M., CHENG, C. and MONTANARI, A. (2021). The high-dimensional asymptotics of first order methods with random data. arXiv preprint. Available at arXiv:2112.07572.

[20] CHANDRASEKHER, K. A., LOU, M. and PANANJADY, A. (2024). Alternating minimization for generalized rank-1 matrix sensing: Sharp predictions from a random initialization. *Inf. Inference* **13** Paper No. iaae025. MR4798949 https://doi.org/10.1093/imaiai/iaae025

[21] CHANDRASEKHER, K. A., PANANJADY, A. and THRAMPOULIDIS, C. (2023). Sharp global convergence guarantees for iterative nonconvex optimization with random data. *Ann. Statist.* **51** 179–210. MR4564853 https://doi.org/10.1214/22-aos2246

[22] CHEN, H., LU, W. and SONG, R. (2021). Statistical inference for online decision making via stochastic gradient descent. *J. Amer. Statist. Assoc.* **116** 708–719. MR4270018 https://doi.org/10.1080/01621459.2020.1826325

[23] CHIZAT, L. and BACH, F. (2018). On the global convergence of gradient descent for over-parameterized models using optimal transport. *Adv. Neural Inf. Process. Syst.* **31**.

[24] CORWIN, I. (2012). The Kardar–Parisi–Zhang equation and universality class. *Random Matrices Theory Appl.* **1** 1130001. MR2930377 https://doi.org/10.1142/S2010326311300014

[25] CORWIN, I., GHOSAL, P., SHEN, H. and TSAI, L.-C. (2020). Stochastic PDE limit of the six vertex model. *Comm. Math. Phys.* **375** 1945–2038. MR4091508 https://doi.org/10.1007/s00220-019-03678-z

[26] CORWIN, I., SHEN, H. and TSAI, L.-C. (2018). ASEP(q, j) converges to the KPZ equation. *Ann. Inst. Henri Poincaré Probab. Stat.* **54** 995–1012. MR3795074 https://doi.org/10.1214/17-AIHP829

[27] DA PRATO, G. and ZABCZYK, J. (2014). *Stochastic Equations in Infinite Dimensions*, 2nd ed. *Encyclopedia of Mathematics and Its Applications* **152**. Cambridge Univ. Press, Cambridge. MR3236753 https://doi.org/10.1017/CBO9781107295513

[28] DARLING, R. W. R. and NORRIS, J. R. (2008). Differential equation approximations for Markov chains. *Probab. Surv.* **5** 37–79. MR2395153 https://doi.org/10.1214/07-PS121

[29] DAUVERGNE, D., ORTMANN, J. and VIRÁG, B. (2022). The directed landscape. *Acta Math.* **229** 201–285. MR4554223 https://doi.org/10.4310/acta.2022.v229.n2.a1

[30] DAVIS, D., DRUSVYATSKIY, D. and JIANG, L. (2024). Asymptotic normality and optimality in nonsmooth stochastic approximation. *Ann. Statist.* **52** 1485–1508. MR4804817 https://doi.org/10.1214/24-aos2401

[31] DEIFT, P. and TROGDON, T. (2019). Universality in numerical computation with random data: Case studies and analytical results. *J. Math. Phys.* **60** 103306. MR4024931 https://doi.org/10.1063/1.5117151

[32] DIEULEVEUT, A., DURMUS, A. and BACH, F. (2020). Bridging the gap between constant step size stochastic gradient descent and Markov chains. *Ann. Statist.* **48** 1348–1382. MR4124326 https://doi.org/10.1214/19-AOS1850

[33] DIEULEVEUT, A., FLAMMARION, N. and BACH, F. (2017). Harder, better, faster, stronger convergence rates for least-squares regression. *J. Mach. Learn. Res.* **18** Paper No. 101. MR3725440

[34] DING, X. and TROGDON, T. (2022). The conjugate gradient algorithm on a general class of spiked covariance matrices. *Quart. Appl. Math.* **80** 99–155. MR4360552 https://doi.org/10.1090/qam/1605

[35] DING, Z., CHEN, S., LI, Q. and WRIGHT, S. J. (2022). Overparameterization of deep ResNet: Zero loss and mean-field analysis. *J. Mach. Learn. Res.* **23** Paper No. [48]. MR4420773

[36] DUCHI, J. C. and RUAN, F. (2021). Asymptotic optimality in stochastic optimization. *Ann. Statist.* **49** 21–48. MR4206668 https://doi.org/10.1214/19-AOS1831

[37] DURMUS, A., MOULINES, E., NAUMOV, A. and SAMSONOV, S. (2025). Finite-time high-probability bounds for Polyak-Ruppert averaged iterates of linear stochastic approximation. *Math. Oper. Res.* **50** 935–964. MR4902778 https://doi.org/10.1287/moor.2022.0179

[38] DURMUS, A., MOULINES, E., NAUMOV, A., SAMSONOV, S., SCAMAN, K. and WAI, H.-T. (2021). Tight high probability bounds for linear stochastic approximation with fixed stepsize. *Adv. Neural Inf. Process. Syst.* **34** 30063–30074.

[39] FANG, Y., XU, J. and YANG, L. (2018). Online bootstrap confidence intervals for the stochastic gradient descent estimator. *J. Mach. Learn. Res.* **19** Paper No. 78. MR3899780

[40] FONTAINE, X., DE BORTOLI, V. and DURMUS, A. (2021). Convergence rates and approximation results for SGD and its continuous-time counterpart. In *Conference on Learning Theory* 1965–2058. PMLR.

[41] GERBELOT, C., TROIANI, E., MIGNACCO, F., KRZAKALA, F. and ZDEBOROVÁ, L. (2024). Rigorous dynamical mean-field theory for stochastic gradient descent methods. *SIAM J. Math. Data Sci.* **6** 400–427. MR4741502 https://doi.org/10.1137/23M1594388

[42] GESS, B., GVALANI, R. S. and KONAROVSKYI, V. (2022). Conservative SPDEs as fluctuating mean field limits of stochastic gradient descent. arXiv preprint. Available at arXiv:2207.05705.

[43] GHOSAL, P. (2017). Hall–Littlewood-PushTASEP and its KPZ limit. arXiv e-prints. Available at arXiv:1701.07308.

[44] GRAY, R. M. et al. (2006). Toeplitz and circulant matrices: A review. *Found. Trends Commun. Inf. Theory* **2** 155–239.

[45] GUPTA, A., CHEN, H., PI, J. and TENDOLKAR, G. (2020). Some limit properties of Markov chains induced by recursive stochastic algorithms. *SIAM J. Math. Data Sci.* **2** 967–1003. MR4162310 https://doi.org/10.1137/19M1258104

[46] KALLIANPUR, G. and XIONG, J. (1995). *Stochastic Differential Equations in Infinite-Dimensional Spaces*. *Institute of Mathematical Statistics Lecture Notes—Monograph Series* **26**. IMS, Hayward, CA. MR1465436

[47] KARATZAS, I. and SHREVE, S. E. (1991). *Brownian Motion and Stochastic Calculus*, 2nd ed. *Graduate Texts in Mathematics* **113**. Springer, New York. MR1121940 https://doi.org/10.1007/978-1-4612-0949-2

[48] KHAMARU, K., DESHPANDE, Y., LATTIMORE, T., MACKEY, L. and WAINWRIGHT, M. J. (2021). Near-optimal inference in adaptive linear regression. arXiv preprint. Available at arXiv:2107.02266v3.

[49] KIPNIS, C. and LANDIM, C. (1999). *Scaling Limits of Interacting Particle Systems*. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **320**. Springer, Berlin. MR1707314 https://doi.org/10.1007/978-3-662-03752-2

[50] KRICHENE, W. and BARTLETT, P. L. (2017). Acceleration and averaging in stochastic descent dynamics. *Adv. Neural Inf. Process. Syst.* **30**.

[51] KUSHNER, H. J. and CLARK, D. S. (1978). *Stochastic Approximation Methods for Constrained and Unconstrained Systems*. *Applied Mathematical Sciences* **26**. Springer, New York. MR0499560

[52] KUSHNER, H. J. and YIN, G. G. (2003). *Stochastic Approximation and Recursive Algorithms and Applications*: *Stochastic Modelling and Applied Probability*, 2nd ed. *Applications of Mathematics* (*New York*) **35**. Springer, New York. MR1993642

[53] LAN, G. (2020). *First-Order and Stochastic Optimization Methods for Machine Learning*. *Springer Series in the Data Sciences*. Springer, Cham. MR4219819 https://doi.org/10.1007/978-3-030-39568-1

[54] LI, C. J., WANG, M., LIU, H. and ZHANG, T. (2018). Near-optimal stochastic approximation for online principal component estimation. *Math. Program.* **167** 75–97. MR3748346 https://doi.org/10.1007/s10107-017-1182-z

[55] LI, Q., TAI, C. and E, W. (2019). Stochastic modified equations and dynamics of stochastic gradient algorithms I: Mathematical foundations. *J. Mach. Learn. Res.* **20** Paper No. 40. MR3948080

[56] LIGGETT, T. M. (1999). *Stochastic Interacting Systems*: *Contact*, *Voter and Exclusion Processes*. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **324**. Springer, Berlin. MR1717346 https://doi.org/10.1007/978-3-662-03990-8

[57] LIN, J. and ROSASCO, L. (2017). Optimal rates for multi-pass stochastic gradient methods. *J. Mach. Learn. Res.* **18** Paper No. 97. MR3714260

[58] LIU, J. and YUAN, Y. (2022). On almost sure convergence rates of stochastic gradient methods. In *Conference on Learning Theory* 2963–2983. PMLR.

[59] LJUNG, L., PFLUG, G. and WALK, H. (1992). *Stochastic Approximation and Optimization of Random Systems*. *DMV Seminar* **17**. Birkhäuser, Basel. MR1162311 https://doi.org/10.1007/978-3-0348-8609-3

[60] LORD, G. J., POWELL, C. E. and SHARDLOW, T. (2014). *An Introduction to Computational Stochastic PDEs*. *Cambridge Texts in Applied Mathematics*. Cambridge Univ. Press, New York. MR3308418 https://doi.org/10.1017/CBO9781139017329

[61] MATETSKI, K., QUASTEL, J. and REMENIK, D. (2021). The KPZ fixed point. *Acta Math*. **227** 115–203. MR4346267 https://doi.org/10.4310/acta.2021.v227.n1.a3

[62] MATTINGLY, J. C., PILLAI, N. S. and STUART, A. M. (2012). Diffusion limits of the random walk Metropolis algorithm in high dimensions. *Ann*. *Appl*. *Probab*. **22** 881–930. MR2977981 https://doi.org/10.1214/10-AAP754

[63] MEI, S., MONTANARI, A. and NGUYEN, P.-M. (2018). A mean field view of the landscape of two-layer neural networks. *Proc*. *Natl*. *Acad*. *Sci*. *USA* **115** E7665–E7671. MR3845070 https://doi.org/10.1073/pnas.1806579115

[64] MERTIKOPOULOS, P., HALLAK, N., KAVIS, A. and CEVHER, V. (2020). On the almost sure convergence of stochastic gradient descent in non-convex problems. *Adv*. *Neural Inf*. *Process*. *Syst*. **33** 1117–1128.

[65] MEYN, S. (2022). *Control Systems and Reinforcement Learning*. Cambridge Univ. Press, Cambridge.

[66] MÜCKE, N., NEU, G. and ROSASCO, L. (2019). Beating SGD saturation with tail-averaging and minibatching. *Adv*. *Neural Inf*. *Process*. *Syst*. **32**.

[67] MUKHERJEE, S. and SEN, S. (2022). Variational inference in high-dimensional linear regression. *J*. *Mach*. *Learn*. *Res*. **23** Paper No. [304]. MR4577743

[68] PAQUETTE, C., LEE, K., PEDREGOSA, F. and PAQUETTE, E. (2021). SGD in the large: Average-case analysis, asymptotics, and stepsize criticality. In *Conference on Learning Theory* 3548–3626. PMLR.

[69] PAQUETTE, C. and PAQUETTE, E. (2021). Dynamics of stochastic momentum methods on large-scale, quadratic models. *Adv*. *Neural Inf*. *Process*. *Syst*. **34** 9229–9240.

[70] PAQUETTE, C., PAQUETTE, E., ADLAM, B. and PENNINGTON, J. (2022). Homogenization of SGD in high-dimensions: Exact dynamics and generalization properties. arXiv preprint. Available at arXiv:2205.07069.

[71] PAQUETTE, C., PAQUETTE, E., ADLAM, B. and PENNINGTON, J. (2022). Implicit regularization or implicit conditioning? Exact risk trajectories of SGD in high dimensions. In *Advances in Neural Information Processing Systems*.

[72] PAQUETTE, C., VAN MERRIËNBOER, B., PAQUETTE, E. and PEDREGOSA, F. (2023). Halting time is predictable for large models: A universality property and average-case analysis. *Found*. *Comput*. *Math*. **23** 597–673. MR4568200 https://doi.org/10.1007/s10208-022-09554-y

[73] PHAM, H. T. and NGUYEN, P.-M. (2021). Limiting fluctuation and trajectorial stability of multilayer neural networks with mean field training. *Adv*. *Neural Inf*. *Process*. *Syst*. **34** 4843–4855.

[74] PILLAI, N. S., STUART, A. M. and THIÉRY, A. H. (2012). Optimal scaling and diffusion limits for the Langevin algorithm in high dimensions. *Ann*. *Appl*. *Probab*. **22** 2320–2356. MR3024970 https://doi.org/10.1214/11-AAP828

[75] PILLAUD-VIVIEN, L., RUDI, A. and BACH, F. (2018). Statistical optimality of stochastic gradient descent on hard learning problems through multiple passes. *Adv*. *Neural Inf*. *Process*. *Syst*. **31**.

[76] POLYAK, B. T. and JUDITSKY, A. B. (1992). Acceleration of stochastic approximation by averaging. *SIAM J*. *Control Optim*. **30** 838–855. MR1167814 https://doi.org/10.1137/0330046

[77] QIU, J. and SEN, S. (2023). The TAP free energy for high-dimensional linear regression. *Ann*. *Appl*. *Probab*. **33** 2643–2680. MR4612652 https://doi.org/10.1214/22-aap1874

[78] QUASTEL, J. (2012). Introduction to KPZ. In *Current Developments in Mathematics*, 2011 125–194. Int. Press, Somerville, MA. MR3098078

[79] QUASTEL, J. and SARKAR, S. (2023). Convergence of exclusion processes and the KPZ equation to the KPZ fixed point. *J*. *Amer*. *Math*. *Soc*. **36** 251–289. MR4495842 https://doi.org/10.1090/jams/999

[80] REVUZ, D. and YOR, M. (1999). *Continuous Martingales and Brownian Motion*, 3rd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **293**. Springer, Berlin. MR1725357 https://doi.org/10.1007/978-3-662-06400-9

[81] ROBBINS, H. and MONRO, S. (1951). A stochastic approximation method. *Ann*. *Math*. *Stat*. **22** 400–407. MR0042668 https://doi.org/10.1214/aoms/1177729586

[82] ROTSKOFF, G. M. and VANDEN-EIJNDEN, E. (2022). Trainability and accuracy of artificial neural networks: An interacting particle system approach. *Comm*. *Pure Appl*. *Math*. **75** 1889–1935. MR4465905 https://doi.org/10.1002/cpa.22074

[83] RUPPERT, D. (1988). Efficient estimations from a slowly convergent Robbins-Monro process. Technical Report, Cornell Univ. Operations Research and Industrial Engineering.

[84] SEBBOUH, O., GOWER, R. M. and DEFAZIO, A. (2021). Almost sure convergence rates for stochastic gradient descent and stochastic heavy ball. In *Conference on Learning Theory* 3935–3971. PMLR.

[85] SHAO, Q.-M. and ZHANG, Z.-S. (2022). Berry-Esseen bounds for multivariate nonlinear statistics with applications to M-estimators and stochastic gradient descent algorithms. *Bernoulli* **28** 1548–1576. MR4411502 https://doi.org/10.3150/21-bej1336

[86] SIRIGNANO, J. and SPILIOPOULOS, K. (2020). Mean field analysis of neural networks: A law of large numbers. *SIAM J. Appl. Math.* **80** 725–752. MR4074020 https://doi.org/10.1137/18M1192184

[87] SIRIGNANO, J. and SPILIOPOULOS, K. (2020). Mean field analysis of neural networks: A central limit theorem. *Stochastic Process. Appl.* **130** 1820–1852. MR4058290 https://doi.org/10.1016/j.spa.2019.06.003

[88] SIRIGNANO, J. and SPILIOPOULOS, K. (2022). Mean field analysis of deep neural networks. *Math. Oper. Res.* **47** 120–152. MR4403748 https://doi.org/10.1287/moor.2020.1118

[89] SPOHN, H. (2012). *Large Scale Dynamics of Interacting Particles*. Springer, Berlin.

[90] SZNITMAN, A.-S. (2004). Topics in random walks in random environment. In *School and Conference on Probability Theory*. *ICTP Lect. Notes*, *XVII* 203–266. Abdus Salam Int. Cent. Theoret. Phys., Trieste. MR2198849

[91] TAN, Y. S. and VERSHYNIN, R. (2023). Online stochastic gradient descent with arbitrary initialization solves non-smooth, non-convex phase retrieval. *J. Mach. Learn. Res.* **24** Paper No. [58]. MR4582480

[92] TOULIS, P. and AIROLDI, E. M. (2017). Asymptotic and finite-sample properties of estimators based on stochastic gradients. *Ann. Statist.* **45** 1694–1727. MR3670193 https://doi.org/10.1214/16-AOS1506

[93] UDELL, M. and TOWNSEND, A. (2019). Why are big data matrices approximately low rank? *SIAM J. Math. Data Sci.* **1** 144–160. MR3949704 https://doi.org/10.1137/18M1183480

[94] VEIGA, R., STEPHAN, L., LOUREIRO, B., KRZAKALA, F. and ZDEBOROVÁ, L. (2023). Phase diagram of stochastic gradient descent in high-dimensional two-layer neural networks. *J. Stat. Mech. Theory Exp.* **11** Paper No. 114008. MR4725421 https://doi.org/10.1088/1742-5468/ad01b1

[95] VIGNAT, C. (2012). A generalized Isserlis theorem for location mixtures of Gaussian random vectors. *Statist. Probab. Lett.* **82** 67–71. MR2863025 https://doi.org/10.1016/j.spl.2011.09.008

[96] WANG, C. and LU, Y. M. (2019). The scaling limit of high-dimensional online independent component analysis. *J. Stat. Mech. Theory Exp.* **12** 124011. MR4063609 https://doi.org/10.1088/1742-5468/ab39d6

[97] WANG, C., MATTINGLY, J. and LU, Y. M. (2017). Scaling limit: Exact and tractable analysis of online learning algorithms with applications to regularized regression and PCA. arXiv preprint. Available at arXiv:1712.04332.

[98] YU, L., BALASUBRAMANIAN, K., VOLGUSHEV, S. and ERDOGDU, M. A. (2021). An analysis of constant step size SGD in the non-convex regime: Asymptotic normality and bias. *Adv. Neural Inf. Process.*

[99] ZEITOUNI, O. (2004). Random walks in random environment. In *Lectures on Probability Theory and Statistics*. *Lecture Notes in Math.* **1837** 189–312. Springer, Berlin. MR2071631 https://doi.org/10.1007/978-3-540-39874-5_2

[100] ZUO, B., YIN, C. and BALAKRISHNAN, N. (2021). Expressions for joint moments of elliptical distributions. *J. Comput. Appl. Math.* **391** Paper No. 113418. MR4209729 https://doi.org/10.1016/j.cam.2021.113418

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INVARIANCE PRINCIPLE AND MCKEAN–VLASOV LIMIT FOR RANDOMIZED LOAD BALANCING IN HEAVY TRAFFIC

BY RAMI ATAR[1,a] AND GERSHON WOLANSKY[2,b]

[1]*Viterbi Faculty of Electrical and Computer Engineering, Technion—Israel Institute of Technology,* [a]*rami@technion.ac.il*
[2]*Department of Mathematics, Technion—Israel Institute of Technology,* [b]*gershonw@technion.ac.il*

We consider a load balancing model where a Poisson stream of jobs arrive at a system of many servers whose service time distribution possesses a finite second moment. A small fraction of arrivals pass through the power-of-choice algorithm, which assigns a job to the shortest among ℓ, $\ell \geq 2$, randomly chosen queues, and the remaining jobs are assigned to queues chosen uniformly at random. The system is analyzed at critical load in an asymptotic regime where both the number of servers and the usual heavy traffic parameter associated with individual queue lengths grow to infinity. The first main result is a hydrodynamic limit stating that the empirical measure of the diffusively normalized queue lengths converges to a path in measure space whose density is given by the unique solution of a parabolic PDE with nonlocal coefficients. The PDE has a stationary solution expressed explicitly, which in particular provides a quantification of the balance achieved by the algorithm. For fixed ℓ, as the intensity of the load balancing stream varies between its limits, this solution varies from exponential distribution to a Dirac distribution, demonstrating that the result covers the whole range from independence to state space collapse.

Further, two forms of an invariance principle are proved, one under a rather general initial distribution and the other under exchangeability of the initial conditions. In the latter, the limit of individual normalized queue length is given by a McKean–Vlasov SDE, and propagation of chaos holds. The McKean–Vlasov limit is closely related to limit results for Brownian particles on $\mathbb{R}_+$ interacting through their rank (with a specific interaction). However, an entirely different set of tools is required, as the collection of n prelimit particles does not obey a Markovian evolution on $\mathbb{R}_+^n$.

REFERENCES

[1] AGARWAL, P. and RAMANAN, K. (2025). Invariant states of hydrodynamic limits of randomized load balancing networks. *Math. Oper. Res.* arXiv preprint. Available at arXiv:2008.08510.

[2] AGHAJANI, R., LI, X. and RAMANAN, K. (2017). The PDE method for the analysis of randomized load balancing networks. *Proc. ACM Meas. Anal. Comput. Syst.* **1** 1–28.

[3] AGHAJANI, R. and RAMANAN, K. (2019). The hydrodynamic limit of a randomized load balancing network. *Ann. Appl. Probab.* **29** 2114–2174. MR3984253 https://doi.org/10.1214/18-AAP1444

[4] ATAR, R., KESLASSY, I., MENDELSON, G., ORDA, A. and VARGAFTIK, S. (2020). Persistent-idle load-distribution. *Stoch. Syst.* **10** 152–169. MR4119248 https://doi.org/10.1287/stsy.2019.0054

[5] ATAR, R. and LIPSHUTZ, D. (2021). Heavy traffic limits for join-the-shortest-estimated-queue policy using delayed information. *Math. Oper. Res.* **46** 268–300. MR4224428 https://doi.org/10.1287/moor.2020.1056

[6] BANERJEE, S., BUDHIRAJA, A. and ESTEVEZ, B. (2025). Load balancing in parallel queues and rank-based diffusions. *Math. Oper. Res.* arXiv preprint. Available at arXiv:2302.10317.

[7] BANERJEE, S. and MUKHERJEE, D. (2019). Join-the-shortest queue diffusion limit in Halfin–Whitt regime: Tail asymptotics and scaling of extrema. *Ann. Appl. Probab.* **29** 1262–1309. MR3910028 https://doi.org/10.1214/18-AAP1436

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[8] BANERJEE, S. and MUKHERJEE, D. (2020). Join-the-shortest queue diffusion limit in Halfin–Whitt regime: Sensitivity on the heavy-traffic parameter. *Ann. Appl. Probab.* **30** 80–144. MR4068307 https://doi.org/10.1214/19-AAP1496
[9] BAYRAKTAR, E., BUDHIRAJA, A. and COHEN, A. (2019). Rate control under heavy traffic with strategic servers. *Ann. Appl. Probab.* **29** 1–35. MR3909999 https://doi.org/10.1214/17-AAP1349
[10] BILLINGSLEY, P. (2013). *Convergence of Probability Measures*. Wiley, New York.
[11] BRAMSON, M., LU, Y. and PRABHAKAR, B. (2010). Randomized load balancing with general service time distributions. *ACM SIGMETRICS Perform. Eval. Rev.* **38** 275–286.
[12] BRAMSON, M., LU, Y. and PRABHAKAR, B. (2012). Asymptotic independence of queues under randomized load balancing. *Queueing Syst.* **71** 247–292. MR2943660 https://doi.org/10.1007/s11134-012-9311-0
[13] BRAMSON, M., LU, Y. and PRABHAKAR, B. (2013). Decay of tails at equilibrium for FIFO join the shortest queue networks. *Ann. Appl. Probab.* **23** 1841–1878. MR3114919 https://doi.org/10.1214/12-AAP888
[14] BRAVERMAN, A. (2020). Steady-state analysis of the join-the-shortest-queue model in the Halfin–Whitt regime. *Math. Oper. Res.* **45** 1069–1103. MR4135843 https://doi.org/10.1287/moor.2019.1023
[15] CHEN, H. and YE, H.-Q. (2012). Asymptotic optimality of balanced routing. *Oper. Res.* **60** 163–179. MR2911665 https://doi.org/10.1287/opre.1110.0998
[16] CHUNG, K. L. and WILLIAMS, R. J. (1990). *Introduction to Stochastic Integration*, 2nd ed. *Probability and Its Applications*. Birkhäuser, Boston, MA. MR1102676 https://doi.org/10.1007/978-1-4612-4480-6
[17] DALEY, D. J. and MIYAZAWA, M. (2019). A martingale view of Blackwell's renewal theorem and its extensions to a general counting process. *J. Appl. Probab.* **56** 602–623. MR3986955 https://doi.org/10.1017/jpr.2019.35
[18] ESCHENFELDT, P. and GAMARNIK, D. (2016). Supermarket queueing system in the heavy traffic regime. Short queue dynamics. arXiv preprint. Available at arXiv:1610.03522.
[19] ESCHENFELDT, P. and GAMARNIK, D. (2018). Join the shortest queue with many servers. The heavy-traffic asymptotics. *Math. Oper. Res.* **43** 867–886. MR3846076 https://doi.org/10.1287/moor.2017.0887
[20] FOSS, S. and STOLYAR, A. L. (2017). Large-scale join-idle-queue system with general service times. *J. Appl. Probab.* **54** 995–1007. MR3731281 https://doi.org/10.1017/jpr.2017.49
[21] GÄRTNER, J. (1988). On the McKean–Vlasov limit for interacting diffusions. *Math. Nachr.* **137** 197–248. MR0968996 https://doi.org/10.1002/mana.19881370116
[22] GILBARG, D. and TRUDINGER, N. S. (1983). *Elliptic Partial Differential Equations of Second Order*, 2nd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **224**. Springer, Berlin. MR0737190 https://doi.org/10.1007/978-3-642-61798-0
[23] GODLEWSKI, E. and RAVIART, P.-A. (1991). *Hyperbolic Systems of Conservation Laws*. *Mathématiques & Applications* (*Paris*) [*Mathematics and Applications*] **3/4**. Ellipses, Paris. MR1304494
[24] GRAHAM, C. (2000). Chaoticity on path space for a queueing network with selection of the shortest queue among several. *J. Appl. Probab.* **37** 198–211. MR1761670 https://doi.org/10.1017/s0021900200015345
[25] GRAHAM, C. (2005). Functional central limit theorems for a large network in which customers join the shortest of several queues. *Probab. Theory Related Fields* **131** 97–120. MR2105045 https://doi.org/10.1007/s00440-004-0372-9
[26] GUPTA, V. and WALTON, N. (2019). Load balancing in the nondegenerate slowdown regime. *Oper. Res.* **67** 281–294. MR3919870 https://doi.org/10.1287/opre.2018.1768
[27] JACOD, J. and SHIRYAEV, A. N. (1987). *Limit Theorems for Stochastic Processes*. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **288**. Springer, Berlin. MR0959133 https://doi.org/10.1007/978-3-662-02514-7
[28] KRICHAGINA, E. V. and TAKSAR, M. I. (1992). Diffusion approximation for $GI/G/1$ controlled queues. *Queueing Syst. Theory Appl.* **12** 333–367. MR1200872 https://doi.org/10.1007/BF01158808
[29] LACKER, D. (2018). On a strong form of propagation of chaos for McKean–Vlasov equations. *Electron. Commun. Probab.* **23** Paper No. 45. MR3841406 https://doi.org/10.1214/18-ECP150
[30] LIU, X., GONG, K. and YING, L. (2022). Large-system insensitivity of zero-waiting load balancing algorithms. *ACM SIGMETRICS Perform. Eval. Rev.* **50** 101–102.
[31] LUCZAK, M. J. and MCDIARMID, C. (2006). On the maximum queue length in the supermarket model. *Ann. Probab.* **34** 493–527. MR2223949 https://doi.org/10.1214/00911790500000710
[32] LUCZAK, M. J. and NORRIS, J. (2005). Strong approximation for the supermarket model. *Ann. Appl. Probab.* **15** 2038–2061. MR2152252 https://doi.org/10.1214/105051605000000368
[33] MENDELSON, G. and XU, K. (2025). Care: Resource allocation using sparse communication. *Oper. Res.* arXiv preprint. Available at arXiv:2206.02410.
[34] MITZENMACHER, M. (2001). The power of two choices in randomized load balancing. *IEEE Trans. Parallel Distrib. Syst.* **12** 1094–1104.

[35] MUKHERJEE, D., BORST, S. C., VAN LEEUWAARDEN, J. S. H. and WHITING, P. A. (2018). Universality of power-of-d load balancing in many-server systems. *Stoch. Syst.* **8** 265–292. MR3899726 https://doi.org/10.1287/stsy.2018.0016

[36] MUKHOPADHYAY, A. and MAZUMDAR, R. R. (2016). Analysis of randomized join-the-shortest-queue (JSQ) schemes in large heterogeneous processor-sharing systems. *IEEE Trans. Control Netw. Syst.* **3** 116–126. MR3514587 https://doi.org/10.1109/TCNS.2015.2428331

[37] PINCHOVER, Y. and RUBINSTEIN, J. (2005). *An Introduction to Partial Differential Equations*. Cambridge Univ. Press, Cambridge. MR2164768 https://doi.org/10.1017/CBO9780511801228

[38] PROTTER, P. (1990). *Stochastic Integration and Differential Equations: A New Approach. Applications of Mathematics (New York)* **21**. Springer, Berlin. MR1037262 https://doi.org/10.1007/978-3-662-02619-9

[39] SHKOLNIKOV, M. (2012). Large systems of diffusions interacting through their ranks. *Stochastic Process. Appl.* **122** 1730–1747. MR2914770 https://doi.org/10.1016/j.spa.2012.01.011

[40] STOLYAR, A. L. (2015). Pull-based load distribution in large-scale heterogeneous service systems. *Queueing Syst.* **80** 341–361. MR3367704 https://doi.org/10.1007/s11134-015-9448-8

[41] STOLYAR, A. L. (2017). Pull-based load distribution among heterogeneous parallel servers: The case of multiple routers. *Queueing Syst.* **85** 31–65. MR3604117 https://doi.org/10.1007/s11134-016-9508-8

[42] SZNITMAN, A.-S. (1991). Topics in propagation of chaos. In *École D'Été de Probabilités de Saint-Flour XIX—1989. Lecture Notes in Math.* **1464** 165–251. Springer, Berlin. MR1108185 https://doi.org/10.1007/BFb0085169

[43] VAN DER BOOR, M., BORST, S. C., VAN LEEUWAARDEN, J. S. H. and MUKHERJEE, D. (2022). Scalable load balancing in networked systems: A survey of recent advances. *SIAM Rev.* **64** 554–622. MR4461562 https://doi.org/10.1137/20M1323746

[44] VAN DER BOOR, M., ZUBELDIA, M. and BORST, S. (2020). Zero-wait load balancing with sparse messaging. *Oper. Res. Lett.* **48** 368–375. MR4093088 https://doi.org/10.1016/j.orl.2020.04.006

[45] VVEDENSKAYA, N. D., DOBRUSHIN, R. L. and KARPELEVICH, F. I. (1996). A queueing system with a choice of the shorter of two queues—an asymptotic approach. *Problemy Peredachi Informatsii* **32** 20–34. MR1384927

[46] WOLANSKY, G. (1995). Comparison between two models of self-gravitating clusters: Conditions for gravitational collapse. *Nonlinear Anal.* **24** 1119–1129. MR1321749 https://doi.org/10.1016/0362-546X(94)E0028-F

[47] YING, L., SRIKANT, R. and KANG, X. (2017). The power of slightly more than one sample in randomized load balancing. *Math. Oper. Res.* **42** 692–722. MR3685262 https://doi.org/10.1287/moor.2016.0823

[48] ZHAO, Z., BANERJEE, S. and MUKHERJEE, D. (2025). Many-server asymptotics for join-the-shortest queue in the super-Halfin–Whitt scaling window. *Math. Oper. Res.* arXiv preprint. Available at arXiv:2106.00121.

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DYNAMIC SPATIAL MATCHING

BY YASH KANORIA[a]

Graduate Business School, Columbia University, [a]*ykanoria@columbia.edu*

Motivated by a variety of online matching platforms, we consider demand and supply units which are located i.i.d. in $[0, 1]^d$, and each demand unit needs to be matched with a supply unit. The goal is to minimize the expected average distance between matched pairs (the "cost"). We model dynamic arrivals of one or both of demand and supply with uncertain locations of future arrivals, and characterize the scaling behavior of the achievable cost in terms of system size (number of supply units), as a function of the dimension d. Our achievability results are backed by concrete matching algorithms. Across cases, we find that the platform can achieve cost (nearly) as low as that achievable if the locations of future arrivals had been known beforehand. Furthermore, in all cases except one, cost nearly as low in terms of scaling as the expected distance to the nearest neighboring supply unit is achievable, that is, the matching constraint does not cause an increase in cost either. The aberrant case is where only demand arrivals are dynamic, and $d = 1$; excess supply significantly reduces cost in this case.

REFERENCES

[1] AJTAI, M., KOMLÓS, J. and TUSNÁDY, G. (1984). On optimal matchings. *Combinatorica* **4** 259–264. MR0779885 https://doi.org/10.1007/BF02579135

[2] AKBARPOUR, M., ALIMOHAMMADI, Y., LI, S. and SABERI, A. (2021). The value of excess supply in spatial matching markets. arXiv preprint. Available at arXiv:2104.03219.

[3] BALKANSKI, E., FAENZA, Y. and PÉRIVIER, N. (2023). The power of greedy for online minimum cost matching on the line. In *Proceedings of the* 24*th ACM Conference on Economics and Computation* 185–205.

[4] BERTSIMAS, D. J. and VAN RYZIN, G. (1993). Stochastic and dynamic vehicle routing in the Euclidean plane with multiple capacitated vehicles. *Oper. Res.* **41** 60–76. MR1205094 https://doi.org/10.1287/opre.41.1.60

[5] BESBES, O., CASTRO, F. and LOBEL, I. (2018). Spatial capacity planning. Available at SSRN 3292651.

[6] CHATTERJEE, S., PELED, R., PERES, Y. and ROMIK, D. (2010). Gravitational allocation to Poisson points. *Ann. of Math.* (2) **172** 617–671. MR2680428 https://doi.org/10.4007/annals.2010.172.617

[7] HALFIN, S. and WHITT, W. (1981). Heavy-traffic limits for queues with many exponential servers. *Oper. Res.* **29** 567–588. MR0629195 https://doi.org/10.1287/opre.29.3.567

[8] HOFFMAN, C., HOLROYD, A. E. and PERES, Y. (2006). A stable marriage of Poisson and Lebesgue. *Ann. Probab.* **34** 1241–1272. MR2257646 https://doi.org/10.1214/009117906000000098

[9] HOLDEN, N., PERES, Y. and ZHAI, A. (2021). Gravitational allocation for uniform points on the sphere. *Ann. Probab.* **49** 287–321. MR4203339 https://doi.org/10.1214/20-AOP1452

[10] HOLROYD, A. E. and PERES, Y. (2005). Extra heads and invariant allocations. *Ann. Probab.* **33** 31–52. MR2118858 https://doi.org/10.1214/009117904000000603

[11] LEIGHTON, T. and SHOR, P. (1989). Tight bounds for minimax grid matching with applications to the average case analysis of algorithms. *Combinatorica* **9** 161–187. MR1030371 https://doi.org/10.1007/BF02124678

[12] LEVIN, D. A. and PERES, Y. (2017). *Markov Chains and Mixing Times*, 2nd ed. **107**. Am. Math. Soc., Providence, RI. MR3726904 https://doi.org/10.1090/mbk/107

[13] MARKÓ, R. and TIMÁR, Á. (2016). A Poisson allocation of optimal tail. *Ann. Probab.* **44** 1285–1307. MR3474472 https://doi.org/10.1214/15-AOP1001

[14] SHOR, P. W. (1986). The average-case analysis of some on-line algorithms for bin packing. *Combinatorica* **6** 179–200. MR0875840 https://doi.org/10.1007/BF02579171

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[15] SHOR, P. W. (1991). How to pack better than best fit: Tight bounds for average-case online bin packing. In [1991] *Proceedings* 32*nd Annual Symposium of Foundations of Computer Science* 752–759. IEEE Comput. Soc.

[16] TALAGRAND, M. (1992). Matching random samples in many dimensions. *Ann. Appl. Probab.* **2** 846–856. MR1189420

[17] TALAGRAND, M. (2021). *Upper and Lower Bounds for Stochastic Processes—Decomposition Theorems*. *Ergebnisse der Mathematik und Ihrer Grenzgebiete*. 3. *Folge*. *A Series of Modern Surveys in Mathematics* [*Results in Mathematics and Related Areas*. 3*rd Series*. *A Series of Modern Surveys in Mathematics*] **60**. Springer, Cham. Second edition [of 3184689]. MR4381414 https://doi.org/10.1007/978-3-030-82595-9

[18] TALLURI, K. T. and VAN RYZIN, G. J. (2004). *The Theory and Practice of Revenue Management*. *International Series in Operations Research & Management Science* **68**. Kluwer Academic, Boston, MA. MR2089816 https://doi.org/10.1007/b139000

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ANOMALOUS AND TOTAL DISSIPATION DUE TO ADVECTION BY SOLUTIONS OF RANDOMLY FORCED NAVIER–STOKES EQUATIONS

BY MARTINA HOFMANOVÁ[1,a], UMBERTO PAPPALETTERA[1,b], RONGCAHN ZHU[2,c] AND XIANGCHAN ZHU[3,d]

[1]*Fakultät für Mathematik, Universität Bielefeld,* [a]*hofmanova@math.uni-bielefeld.de,* [b]*upappale@math.uni-bielefeld.de*

[2]*Department of Mathematics, Beijing Institute of Technology,* [c]*zhurongchan@126.com*

[3]*State Key Laboratory of Mathematical Sciences, Academy of Mathematics and Systems Science, Chinese Academy of Sciences,* [d]*zhuxiangchan@126.com*

We propose a novel approach to induce anomalous dissipation through advection driven by turbulent fluid flows. Specifically, we establish the existence of a velocity field v satisfying randomly forced Navier–Stokes equations, leading to total dissipation of kinetic energy in finite time when advecting a passive scalar. This dissipation phenomenon is uniform across viscosity parameters and initial conditions, representing a case of anomalous dissipation. We further explore dissipation induced by individual realizations of v. Our results extend to scenarios where the passive scalar is replaced by solutions to two or three-dimensional deterministic Navier–Stokes equations advected by v.

REFERENCES

[1] AMBROSIO, L., FUSCO, N. and PALLARA, D. (2000). *Functions of Bounded Variation and Free Discontinuity Problems*. *Oxford Mathematical Monographs*. The Clarendon Press, New York. MR1857292

[2] ARMSTRONG, S. and VICOL, V. (2025). Anomalous diffusion by fractal homogenization. *Ann. PDE* **11** Paper No. 2, 145. MR4846255 https://doi.org/10.1007/s40818-024-00189-6

[3] ASSING, S., FLANDOLI, F. and PAPPALETTERA, U. (2021). Stochastic model reduction: Convergence and applications to climate equations. *J. Evol. Equ.* **21** 3813–3848. MR4350564 https://doi.org/10.1007/s00028-021-00708-z

[4] BATCHELOR, G. K. (1959). Small-scale variation of convected quantities like temperature in turbulent fluid. I. General discussion and the case of small conductivity. *J. Fluid Mech.* **5** 113–133. MR0100457 https://doi.org/10.1017/S002211205900009X

[5] BEDROSSIAN, J. and COTI ZELATI, M. (2017). Enhanced dissipation, hypoellipticity, and anomalous small noise inviscid limits in shear flows. *Arch. Ration. Mech. Anal.* **224** 1161–1204. MR3621820 https://doi.org/10.1007/s00205-017-1099-y

[6] BERNARD, D., GAWĘDZKI, K. and KUPIAINEN, A. (1998). Slow modes in passive advection. *J. Stat. Phys.* **90** 519–569. MR1616898 https://doi.org/10.1023/A:1023212600779

[7] BONICATTO, P., CIAMPA, G. and CRIPPA, G. (2022). On the advection-diffusion equation with rough coefficients: Weak solutions and vanishing viscosity. *J. Math. Pures Appl.* (9) **167** 204–224. MR4496901 https://doi.org/10.1016/j.matpur.2022.09.005

[8] BONICATTO, P., CIAMPA, G. and CRIPPA, G. (2024). Weak and parabolic solutions of advection-diffusion equations with rough velocity field. *J. Evol. Equ.* **24** Paper No. 1, 16. MR4678868 https://doi.org/10.1007/s00028-023-00919-6

[9] BRUÉ, E., COLOMBO, M., CRIPPA, G., DE LELLIS, C. and SORELLA, M. (2024). Onsager critical solutions of the forced Navier-Stokes equations. *Commun. Pure Appl. Anal.* **23** 1350–1366. MR4799447 https://doi.org/10.3934/cpaa.2023071

[10] BRUÈ, E. and DE LELLIS, C. (2023). Anomalous dissipation for the forced 3D Navier-Stokes equations. *Comm. Math. Phys.* **400** 1507–1533. MR4595604 https://doi.org/10.1007/s00220-022-04626-0

[11] BUCKMASTER, T. and VICOL, V. (2019). Nonuniqueness of weak solutions to the Navier-Stokes equation. *Ann. of Math.* (2) **189** 101–144. MR3898708 https://doi.org/10.4007/annals.2019.189.1.3

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[12] BURCZAK, J., MODENA, S. and SZÉKELYHIDI, L. (2021). Non uniqueness of power-law flows. *Comm. Math. Phys.* **388** 199–243. MR4328053 https://doi.org/10.1007/s00220-021-04231-7

[13] BURCZAK, J., SZÉKELYHIDI, L. and WU, B. (2023). Anomalous dissipation and Euler flows. Available at arXiv:2310.02934.

[14] COLOMBO, M., CRIPPA, G. and SORELLA, M. (2023). Anomalous dissipation and lack of selection in the Obukhov-Corrsin theory of scalar turbulence. *Ann. PDE* **9** Paper No. 21, 48. MR4662772 https://doi.org/10.1007/s40818-023-00162-9

[15] CONSTANTIN, P., KISELEV, A., RYZHIK, L. and ZLATOŠ, A. (2008). Diffusion and mixing in fluid flow. *Ann. of Math.* (2) **168** 643–674. MR2434887 https://doi.org/10.4007/annals.2008.168.643

[16] COTI ZELATI, M. (2020). Stable mixing estimates in the infinite Péclet number limit. *J. Funct. Anal.* **279** 108562, 25. MR4095800 https://doi.org/10.1016/j.jfa.2020.108562

[17] DA PRATO, G. and ZABCZYK, J. (1992). A note on stochastic convolution. *Stoch. Anal. Appl.* **10** 143–153. MR1154532 https://doi.org/10.1080/07362999208809260

[18] DEBUSSCHE, A. and PAPPALETTERA, U. (2024). Second order perturbation theory of two-scale systems in fluid dynamics. *J. Eur. Math. Soc.* (*JEMS*). https://doi.org/10.4171/JEMS/1501

[19] DIPERNA, R. J. and LIONS, P.-L. (1989). Ordinary differential equations, transport theory and Sobolev spaces. *Invent. Math.* **98** 511–547. MR1022305 https://doi.org/10.1007/BF01393835

[20] DRIVAS, T. D., ELGINDI, T. M., IYER, G. and JEONG, I.-J. (2022). Anomalous dissipation in passive scalar transport. *Arch. Ration. Mech. Anal.* **243** 1151–1180. MR4381138 https://doi.org/10.1007/s00205-021-01736-2

[21] DRIVAS, T. D. and EYINK, G. L. (2017). A Lagrangian fluctuation-dissipation relation for scalar turbulence. Part I. Flows with no bounding walls. *J. Fluid Mech.* **829** 153–189. MR3698051 https://doi.org/10.1017/jfm.2017.567

[22] FLANDOLI, F. (1995). *Regularity Theory and Stochastic Flows for Parabolic SPDEs*. *Stochastics Monographs* **9**. Gordon & Breach, Yverdon. MR1347450

[23] FLANDOLI, F., GALEATI, L. and LUO, D. (2022). Eddy heat exchange at the boundary under white noise turbulence. *Philos. Trans. R. Soc. A* **380** Paper No. 20210096, 13. MR4402221

[24] FLANDOLI, F., GALEATI, L. and LUO, D. (2024). Quantitative convergence rates for scaling limit of SPDEs with transport noise. *J. Differ. Equ.* **394** 237–277. MR4714774 https://doi.org/10.1016/j.jde.2024.02.053

[25] FLANDOLI, F. and GATAREK, D. (1995). Martingale and stationary solutions for stochastic Navier-Stokes equations. *Probab. Theory Related Fields* **102** 367–391. MR1339739 https://doi.org/10.1007/BF01192467

[26] FLANDOLI, F. and LUO, D. (2021). High mode transport noise improves vorticity blow-up control in 3D Navier-Stokes equations. *Probab. Theory Related Fields* **180** 309–363. MR4265023 https://doi.org/10.1007/s00440-021-01037-5

[27] FLANDOLI, F. and ROMITO, M. (2008). Markov selections for the 3D stochastic Navier-Stokes equations. *Probab. Theory Related Fields* **140** 407–458. MR2365480 https://doi.org/10.1007/s00440-007-0069-y

[28] GALEATI, L. (2020). On the convergence of stochastic transport equations to a deterministic parabolic one. *Stoch. Partial Differ. Equ. Anal. Comput.* **8** 833–868. MR4174071 https://doi.org/10.1007/s40072-019-00162-6

[29] GESS, B. and YAROSLAVTSEV, I. (2025). Stabilization by transport noise and enhanced dissipation in the Kraichnan model. *J. Evol. Equ.* **25** Paper No. 42, 63. MR4894224 https://doi.org/10.1007/s00028-025-01066-w

[30] JEONG, I.-J. and YONEDA, T. (2021). Vortex stretching and enhanced dissipation for the incompressible 3D Navier-Stokes equations. *Math. Ann.* **380** 2041–2072. MR4297205 https://doi.org/10.1007/s00208-020-02019-z

[31] JEONG, I.-J. and YONEDA, T. (2022). Quasi-streamwise vortices and enhanced dissipation for incompressible 3D Navier-Stokes equations. *Proc. Amer. Math. Soc.* **150** 1279–1286. MR4375721 https://doi.org/10.1090/proc/15754

[32] KRAICHNAN, R. H. (1968). Small-scale structure of a scalar field convected by turbulence. *Phys. Fluids* **11** 945–953.

[33] MAJDA, A. J. and KRAMER, P. R. (1999). Simplified models for turbulent diffusion: Theory, numerical modelling, and physical phenomena. *Phys. Rep.* **314** 237–574. MR1699757 https://doi.org/10.1016/S0370-1573(98)00083-0

[34] OBUHOV, A. M. (1949). The structure of the temperature field in a turbulent flow. *Izv. Akad. Nauk SSSR, Ser. Geogr. Geofiz.* **13** 58–69. MR0034164

[35] PAPPALETTERA, U. (2022). Quantitative mixing and dissipation enhancement property of Ornstein-Uhlenbeck flow. *Comm. Partial Differential Equations* **47** 2309–2340. MR4526894 https://doi.org/10.1080/03605302.2022.2122835

[36] PAPPALETTERA, U. (2024). Global existence and non-uniqueness for the Cauchy problem associated to 3D Navier-Stokes equations perturbed by transport noise. *Stoch. Partial Differ. Equ. Anal. Comput.* **12** 1769–1804. MR4781796 https://doi.org/10.1007/s40072-023-00318-5

[37] ROWAN, K. (2024). Accelerated relaxation enhancing flows cause total dissipation. *Nonlinearity* **37** Paper No. 095010, 7. MR4780774 https://doi.org/10.1088/1361-6544/ad6054

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TWO-DIMENSIONAL DELTA-BOSE GAS: SKEW-PRODUCT RELATIVE MOTIONS

BY YU-TING CHEN[a]

Department of Mathematics and Statistics, University of Victoria, [a]*chenyuting@uvic.ca*

We prove a Feynman–Kac-type formula for the relative motion of the two-body delta-Bose gas in two dimensions. The multiplicative functional is not exponential, and the process is a skew-product diffusion uniquely extended in law, in the sense of Erickson (*Probab. Theory Related Fields* **85** (1990) 73–89), from $\mathrm{BES}(0,\beta\downarrow)$ of Donati-Martin and Yor (*Publ. Res. Inst. Math. Sci.* **42** (2006) 879–895) as the radial part. We give two different proofs of the formula. The first uses the original excursion characterization of $\mathrm{BES}(0,\beta\downarrow)$, and the second is via the lower-dimensional Bessel processes at the expectation level. The latter proof contrasts the long-standing approach for delta-function interactions by adding mollifiers to the Laplacians since the present approximations are from "lower, fractional dimensions." Moreover, the second proof conducts a new study of $\mathrm{BES}(0,\beta\downarrow)$ as an SDE since we handle the drift via certain ratios of the Macdonald functions. The properties proven include the strong well-posedness and comparison of the SDE of $\mathrm{BES}(0,\beta\downarrow)$ for all initial conditions. In particular, this well-posedness contrasts the fact that the skew-product diffusion for the Feynman–Kac-type formula has a singular drift of L^p_{loc}-integrability only for $p \leq 2$.

REFERENCES

[1] ALBERTS, T., KHANIN, K. and QUASTEL, J. (2014). The continuum directed random polymer. *J. Stat. Phys.* **154** 305–326. MR3162542 https://doi.org/10.1007/s10955-013-0872-z

[2] ALBEVERIO, S., BRZEŹNIAK, Z. and DĄBROWSKI, L. (1995). Fundamental solution of the heat and Schrödinger equations with point interaction. *J. Funct. Anal.* **130** 220–254. MR1331982 https://doi.org/10.1006/jfan.1995.1068

[3] ALBEVERIO, S., FENSTAD, J. E. and HØEGH-KROHN, R. (1979). Singular perturbations and nonstandard analysis. *Trans. Amer. Math. Soc.* **252** 275–295. MR0534122 https://doi.org/10.2307/1998089

[4] ALBEVERIO, S., GESZTESY, F., HØEGH-KROHN, R. and HOLDEN, H. (1987). Point interactions in two dimensions: Basic properties, approximations and applications to solid state physics. *J. Reine Angew. Math.* **380** 87–107. MR0916201 https://doi.org/10.1515/crll.1987.380.87

[5] ALBEVERIO, S., GESZTESY, F., HØEGH-KROHN, R. and HOLDEN, H. (1988). *Solvable Models in Quantum Mechanics. Texts and Monographs in Physics*. Springer, New York. MR0926273 https://doi.org/10.1007/978-3-642-88201-2

[6] ALBEVERIO, S., HØEGH-KROHN, R. and STREIT, L. (1977). Energy forms, Hamiltonians, and distorted Brownian paths. *J. Math. Phys.* **18** 907–917. MR0446236 https://doi.org/10.1063/1.523359

[7] ARFKEN, G. B., WEBER, H. J. and HARRIS, F. E. (2012). *Mathematical Methods for Physicists. A Comprehensive Guide*, 7th ed. Elsevier, Amsterdam. https://doi.org/10.1016/C2009-0-30629-7

[8] BAHOURI, H., CHEMIN, J.-Y. and DANCHIN, R. (2011). *Fourier Analysis and Nonlinear Partial Differential Equations. Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **343**. Springer, Heidelberg. MR2768550 https://doi.org/10.1007/978-3-642-16830-7

[9] BEREZIN, F. A. and FADDEEV, L. D. (1961). Remark on the Schrödinger equation with singular potential. *Dokl. Akad. Nauk SSSR* **137** 1011–1014. MR0129309

[10] BERTINI, L. and CANCRINI, N. (1995). The stochastic heat equation: Feynman–Kac formula and intermittence. *J. Stat. Phys.* **78** 1377–1401. MR1316109 https://doi.org/10.1007/BF02180136

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[11] BERTINI, L. and CANCRINI, N. (1998). The two-dimensional stochastic heat equation: Renormalizing a multiplicative noise. *J. Phys. A* **31** 615–622. MR1629198 https://doi.org/10.1088/0305-4470/31/2/019

[12] BERTOIN, J. (1996). *Lévy Processes*. *Cambridge Tracts in Mathematics* **121**. Cambridge Univ. Press, Cambridge. MR1406564

[13] BEZNEA, L. and BOBOC, N. (2009). Feynman–Kac formula for left continuous additive functionals and extended Kato class measures. *Potential Anal*. **30** 139–164. MR2471145 https://doi.org/10.1007/s11118-008-9109-1

[14] BLUMENTHAL, R. M. (1992). *Excursions of Markov Processes*. *Probability and Its Applications*. Birkhäuser, Boston, MA. MR1138461 https://doi.org/10.1007/978-1-4684-9412-9

[15] CARAVENNA, F., SUN, R. and ZYGOURAS, N. (2019). On the moments of the $(2+1)$-dimensional directed polymer and stochastic heat equation in the critical window. *Comm. Math. Phys*. **372** 385–440. MR4032870 https://doi.org/10.1007/s00220-019-03527-z

[16] CARAVENNA, F., SUN, R. and ZYGOURAS, N. (2023). The critical 2d stochastic heat flow. *Invent. Math*. **233** 325–460. MR4602000 https://doi.org/10.1007/s00222-023-01184-7

[17] CARAVENNA, F., SUN, R. and ZYGOURAS, N. (2023). The critical $2d$ stochastic heat flow is not a Gaussian multiplicative chaos. *Ann. Probab.* **51** 2265–2300. MR4666296 https://doi.org/10.1214/23-aop1648

[18] CHAN, T. (1994). Occupation times of compact sets by planar Brownian motion. *Ann. Inst. Henri Poincaré Probab. Stat*. **30** 317–329. MR1277003

[19] CHEN, Y.-T. (2021). The critical 2D delta-Bose gas as mixed-order asymptotics of planar Brownian motion. arXiv e-prints 1–56. Available at arXiv:2105.05154.

[20] CHEN, Y.-T. (2024). Delta-Bose gas from the viewpoint of the two-dimensional stochastic heat equation. *Ann. Probab.* **52** 127–187. MR4698027 https://doi.org/10.1214/23-aop1649

[21] CHEN, Y.-T. (2024). Stochastic motions of the two-dimensional many-body delta-Bose gas. arXiv e-prints 1–122. Available at arXiv:2401.17243.

[22] COMETS, F. (2017). *Directed Polymers in Random Environments*: *École d'Étéde Probabilités de Saint-Flour XLVI*–2016. *Lecture Notes in Math*. **2175**. Springer, Cham. MR3444835 https://doi.org/10.1007/978-3-319-50487-2

[23] CSÁKI, E., FÖLDES, A. and HU, Y. (2004). Strong approximations of additive functionals of a planar Brownian motion. *Stochastic Process. Appl*. **109** 263–293. MR2031770 https://doi.org/10.1016/j.spa.2003.09.007

[24] DALANG, R. C. (1999). Extending the martingale measure stochastic integral with applications to spatially homogeneous s.p.d.e.'s. *Electron. J. Probab.* **4** no. 6, 29. MR1684157 https://doi.org/10.1214/EJP.v4-43

[25] DELL'ANTONIO, G. F., FIGARI, R. and TETA, A. (1994). Hamiltonians for systems of N particles interacting through point interactions. *Ann. Inst. H. Poincaré Phys. Théor.* **60** 253–290. MR1281647

[26] DIMOCK, J. and RAJEEV, S. G. (2004). Multi-particle Schrödinger operators with point interactions in the plane. *J. Phys. A* **37** 9157–9173. MR2090290 https://doi.org/10.1088/0305-4470/37/39/008

[27] DONATI-MARTIN, C., ROYNETTE, B., VALLOIS, P. and YOR, M. (2008). On constants related to the choice of the local time at 0, and the corresponding Itô measure for Bessel processes with dimension $d = 2(1-\alpha)$, $0 < \alpha < 1$. *Studia Sci. Math. Hungar.* **45** 207–221. MR2417969 https://doi.org/10.1556/SScMath.2007.1033

[28] DONATI-MARTIN, C. and YOR, M. (2006). Some explicit Krein representations of certain subordinators, including the gamma process. *Publ. Res. Inst. Math. Sci*. **42** 879–895. MR2289152

[29] EBERLE, A. (1999). *Uniqueness and Non-uniqueness of Semigroups Generated by Singular Diffusion Operators*. *Lecture Notes in Math*. **1718**. Springer, Berlin. MR1734956 https://doi.org/10.1007/BFb0103045

[30] ERICKSON, K. B. (1990). Continuous extensions of skew product diffusions. *Probab. Theory Related Fields* **85** 73–89. MR1044301 https://doi.org/10.1007/BF01377630

[31] EZAWA, H., KLAUDER, J. R. and SHEPP, L. A. (1974). A path space picture for Feynman–Kac averages. *Ann. Physics* **88** 588–620. MR0384018 https://doi.org/10.1016/0003-4916(74)90182-1

[32] FENG, Z. S. (2016). *Rescaled Directed Random Polymer in Random Environment in Dimension* $1+2$. ProQuest LLC, Ann Arbor, MI. Thesis (Ph.D.)–University of Toronto (Canada). MR3563962

[33] GILBARG, D. and TRUDINGER, N. S. (2001). *Elliptic Partial Differential Equations of Second Order*. *Classics in Mathematics*. Springer, Berlin. Reprint of the 1998 edition. MR1814364

[34] GRIESEMER, M. and HOFACKER, M. (2022). From short-range to contact interactions in two-dimensional many-body quantum systems. *Ann. Henri Poincaré* **23** 2769–2818. MR4454364 https://doi.org/10.1007/s00023-021-01149-7

[35] GU, Y., QUASTEL, J. and TSAI, L.-C. (2021). Moments of the 2D SHE at criticality. *Probab. Math. Phys*. **2** 179–219. MR4404819 https://doi.org/10.2140/pmp.2021.2.179

[36] HAMANA, Y. and MATSUMOTO, H. (2013). The probability distributions of the first hitting times of Bessel processes. *Trans. Amer. Math. Soc.* **365** 5237–5257. MR3074372 https://doi.org/10.1090/S0002-9947-2013-05799-6

[37] HENRY, D. (1999). *Geometric Theory of Semilinear Parabolic Equations. Lecture Notes in Math.* **840**. Springer, Berlin. MR0610244 https://doi.org/10.1007/BFb0089647

[38] IKEDA, N. and WATANABE, S. (1989). *Stochastic Differential Equations and Diffusion Processes*, 2nd ed. *North-Holland Mathematical Library* **24**. North-Holland, Amsterdam. MR1011252

[39] ISMAIL, M. E. H. and MULDOON, M. E. (1978). Monotonicity of the zeros of a cross-product of Bessel functions. *SIAM J. Math. Anal.* **9** 759–767. MR0486686 https://doi.org/10.1137/0509055

[40] ITÔ, K. and MCKEAN, H. P. JR. (1974). *Diffusion Processes and Their Sample Paths. Die Grundlehren der Mathematischen Wissenschaften, Band* 125. Springer, Berlin. Second printing, corrected. MR0345224

[41] KALLIANPUR, G. and ROBBINS, H. (1953). Ergodic property of the Brownian motion process. *Proc. Nat. Acad. Sci. U.S.A.* **39** 525–533. MR0056233 https://doi.org/10.1073/pnas.39.6.525

[42] KARATZAS, I. and SHREVE, S. E. (1998). *Brownian Motion and Stochastic Calculus*, 2nd ed. *Graduate Texts in Mathematics* **113**. Springer, New York. MR1121940 https://doi.org/10.1007/978-1-4612-0949-2

[43] KARDAR, M., PARISI, G. and ZHANG, Y.-C. (1986). Dynamic scaling of growing interfaces. *Phys. Rev. Lett.* **56** 889–892. https://doi.org/10.1103/PhysRevLett.56.889

[44] KASAHARA, Y. and KOTANI, S. (1979). On limit processes for a class of additive functionals of recurrent diffusion processes. *Z. Wahrsch. Verw. Gebiete* **49** 133–153. MR0543989 https://doi.org/10.1007/BF00534253

[45] KHAS'MINSKII, R. Z. (1959). On positive solutions of the equation $\mathfrak{A}u + Vu = 0$. *Theory Probab. Appl.* **IV** 309–318. https://doi.org/10.1137/110403

[46] KNIGHT, F. B. (1981). Characterization of the Levy measures of inverse local times of gap diffusion. In *Seminar on Stochastic Processes*, 1981 (*Evanston, Ill.*, 1981) (E. Çinlar, K. L. Chung and R. K. Getoor, eds.). *Progr. Probab. Statist.* **1** 53–78. Birkhäuser, Boston, MA. MR0647781 https://doi.org/10.1007/978-1-4612-3938-3_3

[47] KOTANI, S. and WATANABE, S. (2006). Kreĭn's spectral theory of strings and generalized diffusion processes. In *Functional Analysis in Markov Processes* (*Katata/Kyoto*, 1981) (M. Fukushima, ed.). *Lecture Notes in Math.* **923** 235–259. Springer, Berlin. MR0661628

[48] KRONIG, R. and PENNEY, W. G. (1931). Quantum mechanics of electrons in crystal lattices. *Proc. R. Soc. A, Math. Phys. Eng. Sci.* **130** 499–513. https://doi.org/10.1098/rspa.1931.0019

[49] KRYLOV, N. V. and RÖCKNER, M. (2005). Strong solutions of stochastic equations with singular time dependent drift. *Probab. Theory Related Fields* **131** 154–196. MR2117951 https://doi.org/10.1007/s00440-004-0361-z

[50] LEBEDEV, N. N. (1972). *Special Functions and Their Applications*. Dover, New York. Revised edition, translated from the Russian and edited by Richard A. Silverman, Unabridged and corrected republication. MR0350075

[51] MARCUS, M. B. and ROSEN, J. (2010). *Markov Processes, Gaussian Processes, and Local Times. Cambridge Studies in Advanced Mathematics* **100**. Cambridge Univ. Press, Cambridge. MR2250510 https://doi.org/10.1017/CBO9780511617997

[52] MIJATOVIĆ, A. and URUSOV, M. (2012). Convergence of integral functionals of one-dimensional diffusions. *Electron. Commun. Probab.* **17** no. 61, 13. MR3005734 https://doi.org/10.1214/ECP.v17-1825

[53] NELSON, E. (1977). Internal set theory: A new approach to nonstandard analysis. *Bull. Amer. Math. Soc.* **83** 1165–1198. MR0469763 https://doi.org/10.1090/S0002-9904-1977-14398-X

[54] PITMAN, J. and YOR, M. (1981). Bessel processes and infinitely divisible laws. In *Stochastic Integrals* (*Proc. Sympos., Univ. Durham, Durham*, 1980) (D. Williams, ed.). *Lecture Notes in Math.* **851** 285–370. Springer, Berlin. MR0620995 https://doi.org/10.1007/BFb0088732

[55] PORTENKO, N. I. (1975). Diffusion processes with an unbounded drift coefficient. *Teor. Verojatnost. i Primenen.* **20** 27–37. https://doi.org/10.1137/1120003

[56] QUASTEL, J. (2011). Introduction to KPZ. *Current Developments in Mathematics* **1** 125–194. https://doi.org/10.4310/CDM.2011.v2011.n1.a3

[57] REVUZ, D. and YOR, M. (1999). *Continuous Martingales and Brownian Motion*, 3rd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **293**. Springer, Berlin. MR1725357 https://doi.org/10.1007/978-3-662-06400-9

[58] SCHILLING, R. L., SONG, R. and VONDRAČEK, Z. (2012). *Bernstein Functions: Theory and Applications*, 2nd ed. *De Gruyter Studies in Mathematics* **37**. de Gruyter, Berlin. MR2978140 https://doi.org/10.1515/9783110269338

[59] SPITZER, F. (1958). Some theorems concerning 2-dimensional Brownian motion. *Trans. Amer. Math. Soc.* **87** 187–197. MR0104296 https://doi.org/10.2307/1993096

[60] STREIT, L. (1981). Energy forms: Schroedinger theory, processes. *Phys*. *Rep*. **77** 363–375. MR0639038 https://doi.org/10.1016/0370-1573(81)90084-3

[61] TONINELLI, F. (2018). $(2+1)$-dimensional interface dynamics: Mixing time, hydrodynamic limit and anisotropic KPZ growth. In *Proceedings of the International Congress of Mathematicians—Rio de Janeiro* 2018. *Invited Lectures* **III** 2733–2758. World Scientific, Hackensack, NJ. MR3966870

[62] VERETENNIKOV, A. J. (1980). On strong solutions and explicit formulas for solutions of stochastic integral equations. *Mathematics of the USSR-Sbornik* **39** 387–403. https://doi.org/10.1070/SM1981v039n03ABEH001522

[63] WALSH, J. B. (1986). An introduction to stochastic partial differential equations. In *École D'été de Probabilités de Saint-Flour*, *XIV*—1984 (P. Hennequin, ed.). *Lecture Notes in Math*. **1180** 265–439. Springer, Berlin. MR0876085 https://doi.org/10.1007/BFb0074920

[64] WATANABE, S. (1975). On time inversion of one-dimensional diffusion processes. *Z*. *Wahrsch*. *Verw*. *Gebiete* **31** 115–124. MR0365731 https://doi.org/10.1007/BF00539436

[65] YAMADA, T. and WATANABE, S. (1971). On the uniqueness of solutions of stochastic differential equations. *J*. *Math*. *Kyoto Univ*. **11** 155–167. MR0278420 https://doi.org/10.1215/kjm/1250523691

[66] YING, J. (1996). Bivariate Revuz measures and the Feynman–Kac formula. *Ann*. *Inst*. *Henri Poincaré Probab*. *Stat*. **32** 251–287. MR1386221

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LARGE SAMPLE THEORY FOR BURES–WASSERSTEIN BARYCENTRES

BY LEONARDO V. SANTORO[a] AND VICTOR M. PANARETOS[b]

[1]*Institut de Mathématiques, École Polytechnique Fédérale de Lausanne,* [a]*leonardo.santoro@epfl.ch,* [b]*victor.panaretos@epfl.ch*

We establish a strong law of large numbers and a central limit theorem in the Bures–Wasserstein space of covariance operators—or equivalently centred Gaussian measures—over a general separable Hilbert space. Specifically, we show that empirical barycentre sequences indexed by sample size are almost certainly relatively compact, with accumulation points comprising population barycentres. We give a sufficient regularity condition for the limit to be unique. When the limit is unique, we also establish a central limit theorem under a refined pair of moment and regularity conditions. Finally, we prove strong operator convergence of the empirical optimal transport maps to their population counterparts. Though our results naturally extend finite-dimensional counterparts, including associated regularity conditions, our techniques are distinctly different owing to the functional nature of the problem in the general setting. A key element is the characterisation of compact sets in the Bures–Wasserstein topology that reflects an *ordered* Heine–Borel property of the Bures–Wasserstein space.

REFERENCES

[1] AFSARI, B. (2011). Riemannian L^p center of mass: Existence, uniqueness, and convexity. *Proc. Amer. Math. Soc.* **139** 655–673.
[2] AGUEH, M. and CARLIER, G. (2013). Barycenters in the Wasserstein space. *SIAM J. Math. Anal.* **43** 904–924.
[3] ÁLVAREZ-ESTEBAN, P. C., DEL BARRIO, E., CUESTA-ALBERTOS, J. A. and MATRÁN, C. (2018). Wide consensus aggregation in the Wasserstein space: Application to location-scatter families. *Bernoulli* **24** 3147–3179.
[4] ÁLVAREZ-ESTEBAN, P. C., DEL BARRIO, E., CUESTA-ALBERTOS, J. and MATRÁN, C. (2016). A fixed-point approach to barycenters in Wasserstein space. *J. Math. Anal. Appl.* **441** 744–762.
[5] AMBROSIO, L., GIGLI, N. and SAVARÉ, G. (2005). *Gradient Flows: In Metric Spaces and in the Space of Probability Measures*. Springer, Berlin.
[6] BAKER, C. R. (1970). On covariance operators technical report, North Carolina state university. Dept. of statistics.
[7] BENKO, M., HÄRDLE, W. and KNEIP, A. (2009). Common functional principal components.
[8] BHATIA, R., JAIN, T. and LIM, Y. (2019). On the Bures–Wasserstein distance between positive definite matrices. *Expo. Math.* **37** 165–191.
[9] BHATTACHARYA, R. and PATRANGENARU, V. (2003). Large sample theory of intrinsic and extrinsic sample means on manifolds. *Ann. Statist.* **31** 1–29.
[10] BOGACHEV, V. I. (1998). *Gaussian Measures* **62** 1447–1476. Math. Assoc. of America, Washington.
[11] BRÉZIS, H. (2011). *Functional Analysis, Sobolev Spaces and Partial Differential Equations* **2**. Springer, Berlin.
[12] CHEN, Y., LIN, Z. and MÜLLER, H.-G. (2021). Wasserstein regression. *J. Amer. Statist. Assoc.* 1–14.
[13] CUESTA-ALBERTOS, J., MATRÁN-BEA, C. and TUERO-DIAZ, A. (1996). On lower bounds for the L2-Wasserstein metric in a Hilbert space. *J. Theor. Probab.* **9** 263–283.
[14] DAI, M., ZHANG, Z. and SRIVASTAVA, A. (2019). Analyzing dynamical brain functional connectivity as trajectories on space of covariance matrices. IEEE Trans. Med. Imag. **39** 611–620.
[15] DEL MORAL, P. and NICLAS, A. (2018). A Taylor expansion of the square root matrix function. *J. Math. Anal. Appl.* **465** 259–266.

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[16] DRYDEN, I. L., KOLOYDENKO, A. and ZHOU, D. (2009). Non-Euclidean statistics for covariance matrices, with applications to diffusion tensor imaging. *Ann. Appl. Stat.* **3** 1102–1123.

[17] EVANS, S. N. and JAFFE, A. Q. (2023). Strong laws of large numbers for Fréchet means. Bernoulli (to appear. Available at arXiv:2012.12859).

[18] FERRATY, F. and VIEU, P. (2006). *Nonparametric Functional Data Analysis*. Springer, Berlin.

[19] FRÉCHET, M. (1948). Les éléments aléatoires de nature quelconque dans un espace distancié. *Ann. Inst. Henri Poincaré* **10** 215–310.

[20] GILL, R. D. (2004). Teleportation into quantum statistics. arXiv preprint. Available at arXiv:math/0405572.

[21] HSING, T. and EUBANK, R. (2015). *Theoretical Foundations of Functional Data Analysis, with an Introduction to Linear Operators* **997**. Wiley, New York.

[22] KANTOROVICH, L. V. (1942). On the translocation of masses. *Dokl. Akad. Nauk. USSR (NS)* **37** 199–201.

[23] KENDALL, W. S. and LE, H. (2011). Limit theorems for empirical Fréchet means of independent and non-identically distributed manifold-valued random variables.

[24] KONER, S. and STAICU, A.-M. (2023). Second-generation functional data. *Annu. Rev. Stat. Appl.* **10** 547–572.

[25] KROSHNIN, A., SPOKOINY, V. and SUVORIKOVA, A. (2021). Statistical inference for Bures–Wasserstein barycenters. *Ann. Appl. Probab.* **31** 1264–1298.

[26] LE GOUIC, T. and LOUBES, J.-M. (2017). Existence and consistency of Wasserstein barycenters. *Probab. Theory Related Fields* **168** 901–917.

[27] LE GOUIC, T., PARIS, Q., RIGOLLET, P. and STROMME, A. J. (2022). Fast convergence of empirical barycenters in Alexandrov spaces and the Wasserstein space. *J. Eur. Math. Soc.*

[28] LEDOUX, M. and TALAGRAND, M. (2013). *Probability in Banach Spaces: Isoperimetry and Processes*. Springer, Berlin.

[29] LINDQUIST, M. A. (2008). The statistical analysis of fMRI data. *Statist. Sci.* **23** 439–464.

[30] MASAROTTO, V., PANARETOS, V. M. and ZEMEL, Y. (2019). Procrustes metrics on covariance operators and optimal transportation of Gaussian processes. *Sankhya A* **81** 172–213.

[31] MASAROTTO, V., PANARETOS, V. M. and ZEMEL, Y. (2022). Transportation-based functional ANOVA and PCA for covariance operators. arXiv preprint. Available at arXiv:2212.04797.

[32] MINH, H. Q. (2022). Entropic regularization of Wasserstein distance between infinite-dimensional Gaussian measures and Gaussian processes. *J. Theoret. Probab.* 1–96.

[33] OLKIN, I. and PUKELSHEIM, F. (1982). The distance between two random vectors with given dispersion matrices. *Linear Algebra Appl.* **48** 257–263.

[34] PANARETOS, V. M., KRAUS, D. and MADDOCKS, J. H. (2010). Second-order comparison of Gaussian random functions and the geometry of DNA minicircles. *J. Amer. Statist. Assoc.* **105** 670–682.

[35] PANARETOS, V. M. and TAVAKOLI, S. (2013). Fourier analysis of stationary time series in function space. *Ann. Statist.* **41** 568–603.

[36] PEDERSEN, G. K. (1972). Some operator monotone functions. *Proc. Amer. Math. Soc.* **36** 309–310.

[37] PETERSEN, A. and MÜLLER, H.-G. (2019). Fréchet regression for random objects with Euclidean predictors. *Ann. Statist.* **47** 691–719.

[38] PIGOLI, D., ASTON, J. A., DRYDEN, I. L. and SECCHI, P. (2014). Distances and inference for covariance operators. *Biometrika* **101** 409–422.

[39] POWERS, R. T. and STØRMER, E. (1970). Free states of the canonical anticommutation relations. *Comm. Math. Phys.* **16** 1–33. https://doi.org/cmp/1103842028.

[40] SANTORO, L. V. and PANARETOS, V. M. (2024). Statistical inference for Bures-Wasserstein flows. Preprint.

[41] SCHACHERMAYER, W. and TEICHMANN, J. (2009). Characterization of optimal transport plans for the Monge-Kantorovich problem. *Proc. Amer. Math. Soc.* **137** 519–529.

[42] TAKATSU, A. (2010). On Wasserstein geometry of Gaussian measures. In *Probabilistic Approach to Geometry* 463–472. Mathematical Society of Japan.

[43] THANWERDAS, Y. and PENNEC, X. (2023). Bures–Wasserstein minimizing geodesics between covariance matrices of different ranks. *SIAM J. Matrix Anal. Appl.* **44** 1447–1476. https://doi.org/10.1137/22M149168X

[44] VILLANI, C. (2009). *Optimal Transport: Old and New* 338. Springer, Berlin.

[45] ZEMEL, Y. (2023). Non-injectivity of Bures–Wasserstein barycentres in infinite dimensions. arXiv preprint. Available at arXiv:2311.09385.

[46] ZEMEL, Y. and PANARETOS, V. M. (2019). Fréchet means and procrustes analysis in Wasserstein space. *Bernoulli* **25** 932–976.

[47] ZIEZOLD, H. (1977). On expected figures and a strong law of large numbers for random elements in quasi-metric spaces. In *Transactions of the Seventh Prague Conference on Information Theory, Statistical Decision Functions, Random Processes and of the* 1974 *European Meeting of Statisticians* 591–602. Springer, Berlin.

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LIMITS OF OPEN ASEP STATIONARY MEASURES NEAR A BOUNDARY

BY ZONGRUI YANG[a]

Department of Mathematics, Columbia University, [a]*zy2417@columbia.edu*

Consider the stationary measure of open asymmetric simple exclusion process (ASEP) on the lattice $\{1, \dots, n\}$. Taking n to infinity while fixing the jump rates, this measure converges to a measure on the semi-infinite lattice. In the high and low density phases, we characterize the limiting measure and provide bounds on the convergence rates in total variation distance. Our approach involves bounding the total variation distance using generating functions, which are further estimated through a subtle analysis of the atom masses of Askey–Wilson signed measures.

REFERENCES

[1] ASKEY, R. and WILSON, J. (1985). Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials. *Mem. Amer. Math. Soc.* **54** iv+55. MR0783216 https://doi.org/10.1090/memo/0319

[2] BARRAQUAND, G., CORWIN, I. and YANG, Z. (2024). Stationary measures for integrable polymers on a strip. *Invent. Math.* **237** 1567–1641. MR4777093 https://doi.org/10.1007/s00222-024-01277-x

[3] BARRAQUAND, G. and LE DOUSSAL, P. (2022). Steady state of the KPZ equation on an interval and Liouville quantum mechanics. *Europhys. Lett.* **137** 61003.

[4] BLYTHE, R. A. and EVANS, M. R. (2007). Nonequilibrium steady states of matrix-product form: A solver's guide. *J. Phys. A: Math. Theor.* **40** R333–R441. MR2437671 https://doi.org/10.1088/1751-8113/40/46/R01

[5] BRYC, W. and KUZNETSOV, A. (2022). Markov limits of steady states of the KPZ equation on an interval. *ALEA Lat. Amer. J. Probab. Math. Stat.* **19** 1329–1351. MR4512149 https://doi.org/10.30757/alea.v19-53

[6] BRYC, W., KUZNETSOV, A., WANG, Y. and WESOŁOWSKI, J. (2023). Markov processes related to the stationary measure for the open KPZ equation. *Probab. Theory Related Fields* **185** 353–389. MR4528972 https://doi.org/10.1007/s00440-022-01110-7

[7] BRYC, W. and WANG, Y. (2019). Limit fluctuations for density of asymmetric simple exclusion processes with open boundaries. *Ann. Inst. Henri Poincaré Probab. Stat.* **55** 2169–2194. MR4029151 https://doi.org/10.1214/18-AIHP945

[8] BRYC, W. and WESOŁOWSKI, J. (2010). Askey–Wilson polynomials, quadratic harnesses and martingales. *Ann. Probab.* **38** 1221–1262. MR2674998 https://doi.org/10.1214/09-AOP503

[9] BRYC, W. and WESOŁOWSKI, J. (2017). Asymmetric simple exclusion process with open boundaries and quadratic harnesses. *J. Stat. Phys.* **167** 383–415. MR3626634 https://doi.org/10.1007/s10955-017-1747-5

[10] CORTEEL, S. and WILLIAMS, L. K. (2011). Tableaux combinatorics for the asymmetric exclusion process and Askey–Wilson polynomials. *Duke Math. J.* **159** 385–415. MR2831874 https://doi.org/10.1215/00127094-1433385

[11] CORWIN, I. (2022). Some recent progress on the stationary measure for the open KPZ equation. In *Toeplitz Operators and Random Matrices—in Memory of Harold Widom. Oper. Theory Adv. Appl.* **289** 321–360. Birkhäuser, Cham. MR4573955 https://doi.org/10.1007/978-3-031-13851-5_15

[12] CORWIN, I. and KNIZEL, A. (2024). Stationary measure for the open KPZ equation. *Comm. Pure Appl. Math.* **77** 2183–2267. MR4705293 https://doi.org/10.1002/cpa.22174

[13] DERRIDA, B., ENAUD, C. and LEBOWITZ, J. L. (2004). The asymmetric exclusion process and Brownian excursions. *J. Stat. Phys.* **115** 365–382. MR2070099 https://doi.org/10.1023/B:JOSS.0000019833.35328.b4

[14] DERRIDA, B., EVANS, M. R., HAKIM, V. and PASQUIER, V. (1993). Exact solution of a 1D asymmetric exclusion model using a matrix formulation. *J. Phys. A: Math. Theor.* **26** 1493–1517. MR1219679

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[15] DERRIDA, B., LEBOWITZ, J. L. and SPEER, E. R. (2002). Exact free energy functional for a driven diffusive open stationary nonequilibrium system. *Phys. Rev. Lett.* **89** 030601.

[16] DERRIDA, B., LEBOWITZ, J. L. and SPEER, E. R. (2003). Exact large deviation functional of a stationary open driven diffusive system: The asymmetric exclusion process. *J. Stat. Phys.* **110** 775–810. Special issue in honor of Michael E. Fisher's 70th birthday (Piscataway, NJ, 2001). MR1964689 https://doi.org/10.1023/A:1022111919402

[17] DUHART, H. G. (2015). Large deviations for boundary driven exclusion processes. Ph.D. thesis, Univ. Bath.

[18] DUHART, H. G., MÖRTERS, P. and ZIMMER, J. (2018). The semi-infinite asymmetric exclusion process: Large deviations via matrix products. *Potential Anal.* **48** 301–323. MR3779090 https://doi.org/10.1007/s11118-017-9635-9

[19] ENAUD, C. and DERRIDA, B. (2004). Large deviation functional of the weakly asymmetric exclusion process. *J. Stat. Phys.* **114** 537–562. MR2035624 https://doi.org/10.1023/B:JOSS.0000012501.43746.cf

[20] ESSLER, F. H. L. and RITTENBERG, V. (1996). Representations of the quadratic algebra and partially asymmetric diffusion with open boundaries. *J. Phys. A: Math. Theor.* **29** 3375–3407. MR1400161 https://doi.org/10.1088/0305-4470/29/13/013

[21] GANTERT, N., NESTORIDI, E. and SCHMID, D. (2023). Mixing times for the simple exclusion process with open boundaries. *Ann. Appl. Probab.* **33** 972–1012. MR4564424 https://doi.org/10.1214/22-aap1839

[22] GROSSKINSKY, S. (2004). Phase transitions in nonequilibrium stochastic particle systems with local conservation laws. Ph.D. thesis, Technische Universität München.

[23] JAFARPOUR, F. and MASHARIAN, S. (2007). Matrix product steady states as superposition of product shock measures in 1D driven systems. *J. Stat. Mech.* **2007**. P10013.

[24] LIGGETT, T. M. (1975). Ergodic theorems for the asymmetric simple exclusion process. *Trans. Amer. Math. Soc.* **213** 237–261. MR0410986 https://doi.org/10.2307/1998046

[25] MALLICK, K. and SANDOW, S. (1997). Finite-dimensional representations of the quadratic algebra: Applications to the exclusion process. *J. Phys. A: Math. Theor.* **30** 4513–4526. MR1479787 https://doi.org/10.1088/0305-4470/30/13/008

[26] NESTORIDI, E. and SCHMID, D. (2024). Approximating the stationary distribution of the ASEP with open boundaries. *Comm. Math. Phys.* **405** Paper No. 176, 64. MR4777077 https://doi.org/10.1007/s00220-024-05033-3

[27] RUDIN, W. (1987). *Real and Complex Analysis*, 3rd ed. McGraw-Hill, New York. MR0924157

[28] SASAMOTO, T. (2000). Density profile of the one-dimensional partially asymmetric simple exclusion process with open boundaries. *J. Phys. Soc. Jpn.* **69** 1055–1067.

[29] SASAMOTO, T. and WILLIAMS, L. (2014). Combinatorics of the asymmetric exclusion process on a semi-infinite lattice. *J. Comb.* **5** 419–434. MR3324896 https://doi.org/10.4310/JOC.2014.v5.n4.a1

[30] SCHÜTZ, G. and DOMANY, E. (1993). Phase transitions in an exactly soluble one-dimensional exclusion process. *J. Stat. Phys.* **72** 277–296.

[31] UCHIYAMA, M., SASAMOTO, T. and WADATI, M. (2004). Asymmetric simple exclusion process with open boundaries and Askey–Wilson polynomials. *J. Phys. A: Math. Theor.* **37** 4985–5002. MR2065218 https://doi.org/10.1088/0305-4470/37/18/006

[32] WANG, Y., WESOŁOWSKI, J. and YANG, Z. (2024). Askey–Wilson signed measures and open ASEP in the shock region. *Int. Math. Res. Not. IMRN* **2024** 11104–11134. MR4782793 https://doi.org/10.1093/imrn/rnae116

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DICKMAN APPROXIMATION OF WEIGHTED SUMS OF INDEPENDENT RANDOM VARIABLES IN THE KOLMOGOROV DISTANCE

BY CHINMOY BHATTACHARJEE[1,a] AND MATTHIAS SCHULTE[2,b]

[1]*Department of Mathematics, University of Hamburg,* [a]*chinmoy.bhattacharjee@uni-hamburg.de*
[2]*Institute of Mathematics, Hamburg University of Technology,* [b]*matthias.schulte@tuhh.de*

We consider distributional approximation by generalized Dickman distributions, which appear in number theory, perpetuities, logarithmic combinatorial structures and many other areas. We prove bounds in the Kolmogorov distance for the approximation of certain weighted sums of Bernoulli and Poisson random variables by members of this family. While such results have previously been shown in Bhattacharjee and Goldstein (2019) for distances based on smoother test functions and for a special case of the random variables considered in this paper, results in the Kolmogorov distance are new. We also establish optimality of our rates of convergence by deriving lower bounds. As a result, some interesting phase transitions emerge depending on the choice of the underlying parameters. The proofs of our results mainly rely on the use of Stein's method. In particular, we study the solutions of the Stein equation corresponding to the test functions associated to the Kolmogorov distance, and establish their smoothness properties. As applications, we study the runtime of the Quickselect algorithm, an edge-length statistic of a long-range percolation model, and the weighted depth in randomly grown simple increasing trees.

REFERENCES

[1] ARRAS, B., MIJOULE, G., POLY, G. and SWAN, Y. (2017). A new approach to the Stein-Tikhomirov method: with applications to the second Wiener chaos and Dickman convergence. Available at https://arxiv.org/abs/1605.06819.

[2] ARRATIA, R., BARBOUR, A. D. and TAVARÉ, S. (2000). Limits of logarithmic combinatorial structures. *Ann. Probab.* **28** 1620–1644. MR1813836 https://doi.org/10.1214/aop/1019160500

[3] ARRATIA, R., BARBOUR, A. D. and TAVARÉ, S. (2003). *Logarithmic Combinatorial Structures: A Probabilistic Approach. EMS Monographs in Mathematics*. European Mathematical Society (EMS), Zürich. MR2032426 https://doi.org/10.4171/000

[4] BARBOUR, A. D. and NIETLISPACH, B. (2011). Approximation by the Dickman distribution and quasi-logarithmic combinatorial structures. *Electron. J. Probab.* **16** 880–902. MR2793242 https://doi.org/10.1214/EJP.v16-881

[5] BHATTACHARJEE, C. and GOLDSTEIN, L. (2019). Dickman approximation in simulation, summations and perpetuities. *Bernoulli* **25** 2758–2792. MR4003564 https://doi.org/10.3150/18-BEJ1070

[6] CHATTERJEE, S. (2014). A short survey of Stein's method. In *Proceedings of the International Congress of Mathematicians—Seoul* 2014. *Vol. IV* 1–24. Kyung Moon Sa, Seoul. MR3727600

[7] CHEN, L. H. Y., GOLDSTEIN, L. and SHAO, Q.-M. (2011). *Normal Approximation by Stein's Method. Probability and Its Applications (New York)*. Springer, Heidelberg. MR2732624 https://doi.org/10.1007/978-3-642-15007-4

[8] DE LA BRETÈCHE, R. and TENENBAUM, G. (2021). On strong and almost sure local limit theorems for a probabilistic model of the Dickman distribution. *Lith. Math. J.* **61** 301–311. MR4313593 https://doi.org/10.1007/s10986-021-09529-6

[9] DICKMAN, K. (1930). On the frequency of numbers containing prime factors of a certain relative magnitude. *Ark. Mat. Astron. Fys.* **22** 1–14.

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[10] EBNER, B. and SWAN, Y. (2024). *Independent Additive Weighted Bias Distributions and Associated Goodness-of-Fit Tests* (M. Barigozzi, S. Hörmann and D. Paindaveine, eds.). *Recent Advances in Econometrics and Statistics*. Springer, Cham.

[11] GOLDSTEIN, L. (2017). Non asymptotic distributional bounds for the Dickman approximation of the running time of the Quickselect algorithm. Available at https://arxiv.org/abs/1703.00505v1.

[12] GOLDSTEIN, L. (2018). Non-asymptotic distributional bounds for the Dickman approximation of the running time of the Quickselect algorithm. *Electron. J. Probab.* **23** 100. MR3862615 https://doi.org/10.1214/18-EJP227

[13] GONČAROV, V. (1962). On the field of combinatory analysis. *Amer. Math. Soc. Transl. Ser.* 2 **19** 1–46. MR0131369 https://doi.org/10.1090/trans2/019/01

[14] GRIMMETT, G. (1999). *Percolation*, 2nd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **321**. Springer, Berlin. MR1707339 https://doi.org/10.1007/978-3-662-03981-6

[15] HWANG, H.-K. and TSAI, T.-H. (2002). Quickselect and the Dickman function. *Combin. Probab. Comput.* **11** 353–371. MR1918722 https://doi.org/10.1017/S0963548302005138

[16] KUBA, M. and PANHOLZER, A. (2007). On weighted path lengths and distances in increasing trees. *Probab. Engrg. Inform. Sci.* **21** 419–433. MR2348066 https://doi.org/10.1017/S0269964807000058

[17] MANSTAVICHYUS, E. (1992). Remarks on elements of semigroups that are free of large prime factors. *Liet. Mat. Rink.* **32** 512–525. MR1246042 https://doi.org/10.1007/BF00970673

[18] MOLCHANOV, S. A. and PANOV, V. A. (2020). The Dickman-Goncharov distribution. *Uspekhi Mat. Nauk* **75** 107–152. MR4181059 https://doi.org/10.4213/rm9976

[19] PANHOLZER, A. and PRODINGER, H. (2007). Level of nodes in increasing trees revisited. *Random Structures Algorithms* **31** 203–226. MR2343719 https://doi.org/10.1002/rsa.20161

[20] PEKÖZ, E. A., RÖLLIN, A. and ROSS, N. (2013). Degree asymptotics with rates for preferential attachment random graphs. *Ann. Appl. Probab.* **23** 1188–1218. MR3076682 https://doi.org/10.1214/12-AAP868

[21] PENROSE, M. D. and WADE, A. R. (2004). Random minimal directed spanning trees and Dickman-type distributions. *Adv. Appl. Probab.* **36** 691–714. MR2079909 https://doi.org/10.1239/aap/1093962229

[22] PINSKY, R. G. (2018). On the strange domain of attraction to generalized Dickman distribution for sums of independent random variables. *Electron. J. Probab.* **23** 3. MR3751078 https://doi.org/10.1214/17-EJP126

[23] PINSKY, R. G. (2019). A natural probabilistic model on the integers and its relation to Dickman-type distributions and Buchstab's function. In *Probability and Analysis in Interacting Physical Systems*. *Springer Proc. Math. Stat.* **283** 267–294. Springer, Cham. MR3968515 https://doi.org/10.1007/978-3-030-15338-0_10

[24] RÉNYI, A. (1962). Théorie des éléments saillants d'une suite d'observations. *Ann. Fac. Sci. Univ. Clermont-Ferrand* **8** 7–13. MR0286162

[25] ROSS, N. (2011). Fundamentals of Stein's method. *Probab. Surv.* **8** 210–293. MR2861132 https://doi.org/10.1214/11-PS182

[26] STEIN, C. (1972). A bound for the error in the normal approximation to the distribution of a sum of dependent random variables. In *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability* (*Univ. California*, *Berkeley*, *Calif.*, 1970/1971), *Vol. II*: *Probability Theory* 583–602. Univ. California Press, Berkeley, CA. MR0402873

[27] STEIN, C. (1986). *Approximate Computation of Expectations*. *Institute of Mathematical Statistics Lecture Notes—Monograph Series* **7**. IMS, Hayward, CA. MR0882007

[28] VERVAAT, W. (1979). On a stochastic difference equation and a representation of nonnegative infinitely divisible random variables. *Adv. Appl. Probab.* **11** 750–783. MR0544194 https://doi.org/10.2307/1426858

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OPTIMAL STOPPING WITH NONLINEAR EXPECTATION: GEOMETRIC AND ALGORITHMIC SOLUTIONS

BY TOMASZ KOSMALA[1,a] AND JOHN MORIARTY[2,b]

[1]*Department of Mathematics, Imperial College London and School of Mathematical Sciences, Queen Mary University of London,* [a]*t.kosmala@imperial.ac.uk*

[2]*School of Mathematical Sciences, Queen Mary University of London,* [b]*j.moriarty@qmul.ac.uk*

We use the geometry of suitably generalised potentials to solve risk-sensitive Markovian optimal stopping problems. As in the linear case due to Dynkin and Yushkievich (1967), the value function is the pointwise infimum of those functions which dominate the gain function. An emphasis is placed on geometric and pathwise arguments, rather than exploiting convexity, positive homogeneity or related analytical properties. An algorithm is provided to construct the value function at the computational cost of a two-dimensional search.

REFERENCES

[1] ALVAREZ, L. (2007). Knightian Uncertainty, k-Ignorance, and Optimal Timing Discussion Papers No. 25, Aboa Centre for Economics.

[2] BARRIEU, P. and EL KAROUI, N. (2009). Pricing, hedging, and designing derivatives with risk measures. In *Indifference Pricing*: *Theory and Applications* 77–146.

[3] BARRON, E. N., CARDALIAGUET, P. and JENSEN, R. R. (2000). Radon–Nikodym theorem in L^∞. *Appl*. *Math*. *Optim*. **42** 103–126. MR1784171 https://doi.org/10.1007/s002450010006

[4] BAYRAKTAR, E. and YAO, S. (2011). Optimal stopping for non-linear expectations—Part I. *Stochastic Process*. *Appl*. **121** 185–211. MR2746173 https://doi.org/10.1016/j.spa.2010.10.001

[5] BAYRAKTAR, E. and YAO, S. (2011). Optimal stopping for non-linear expectations—Part II. *Stochastic Process*. *Appl*. **121** 212–264. MR2746174 https://doi.org/10.1016/j.spa.2010.10.002

[6] BEIBEL, M. and LERCHE, H. R. (1997). A new look at optimal stopping problems related to mathematical finance *Statist*. *Sinica* **7** 93–108.

[7] BION-NADAL, J. (2009). Time consistent dynamic risk processes. *Stochastic Process*. *Appl*. **119** 633–654. MR2494007 https://doi.org/10.1016/j.spa.2008.02.011

[8] CHEN, Z. and EPSTEIN, L. (2002). Ambiguity, risk, and asset returns in continuous time. *Econometrica* **70** 1403–1443. MR1929974 https://doi.org/10.1111/1468-0262.00337

[9] CHENG, X. and RIEDEL, F. (2013). Optimal stopping under ambiguity in continuous time. *Math*. *Financ*. *Econ*. **7** 29–68. MR3023890 https://doi.org/10.1007/s11579-012-0081-6

[10] CHERIDITO, P., DELBAEN, F. and KUPPER, M. (2006). Dynamic monetary risk measures for bounded discrete-time processes. *Electron*. *J*. *Probab*. **11** no. 3, 57–106. MR2199055 https://doi.org/10.1214/EJP.v11-302

[11] CHO, M. J. and STOCKBRIDGE, R. H. (2002). Linear programming formulation for optimal stopping problems. *SIAM J*. *Control Optim*. **40** 1965–1982. MR1897205 https://doi.org/10.1137/S0363012900377663

[12] CHONG, W. F., HU, Y., LIANG, G. and ZARIPHOPOULOU, T. (2019). An ergodic BSDE approach to forward entropic risk measures: Representation and large-maturity behavior. *Finance Stoch*. **23** 239–273. MR3898406 https://doi.org/10.1007/s00780-018-0377-3

[13] CHRISTENSEN, S. (2013). Optimal decision under ambiguity for diffusion processes. *Math*. *Methods Oper*. *Res*. **77** 207–226. MR3070026 https://doi.org/10.1007/s00186-012-0425-2

[14] CHRISTENSEN, S., CROCCE, F., MORDECKI, E. and SALMINEN, P. (2019). On optimal stopping of multidimensional diffusions. *Stochastic Process*. *Appl*. **129** 2561–2581. MR3958442 https://doi.org/10.1016/j.spa.2018.07.014

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[15] CROCCE, F. and MORDECKI, E. (2022). An algorithm to solve optimal stopping problems for one-dimensional diffusions. *ALEA Lat. Amer. J. Probab. Math. Stat.* **19** 1353–1375. MR4512150 https://doi.org/10.30757/alea.v19-54

[16] DAYANIK, S. and KARATZAS, I. (2003). On the optimal stopping problem for one-dimensional diffusions. *Stochastic Process. Appl.* **107** 173–212. MR1999788 https://doi.org/10.1016/S0304-4149(03)00076-0

[17] DELBAEN, F., HU, Y. and RICHOU, A. (2011). On the uniqueness of solutions to quadratic BSDEs with convex generators and unbounded terminal conditions. *Ann. Inst. Henri Poincaré Probab. Stat.* **47** 559–574. MR2814423 https://doi.org/10.1214/10-AIHP372

[18] DELBAEN, F., PENG, S. and ROSAZZA GIANIN, E. (2010). Representation of the penalty term of dynamic concave utilities. *Finance Stoch.* **14** 449–472. MR2670421 https://doi.org/10.1007/s00780-009-0119-7

[19] DENIS, L., HU, M. and PENG, S. (2011). Function spaces and capacity related to a sublinear expectation: Application to G-Brownian motion paths. *Potential Anal.* **34** 139–161. MR2754968 https://doi.org/10.1007/s11118-010-9185-x

[20] DYNKIN, E. B. (1963). The optimum choice of the instant for stopping a Markov process. *Adv. Sov. Math.* **4** 627–629.

[21] DYNKIN, E. B. and YUSHKEVICH, A. A. (1967). *Markov Processes: Theorems and Problems*. Nauka, Moscow.

[22] FÖLLMER, H. and SCHIED, A. (2016). *Stochastic Finance: An Introduction in Discrete Time. De Gruyter Graduate*. de Gruyter, Berlin. Fourth revised and extended edition of [MR1925197]. MR3859905 https://doi.org/10.1515/9783110463453

[23] HELMES, K. and STOCKBRIDGE, R. H. (2010). Construction of the value function and optimal rules in optimal stopping of one-dimensional diffusions. *Adv. Appl. Probab.* **42** 158–182. MR2666923 https://doi.org/10.1239/aap/1269611148

[24] HUANG, W. and HASKELL, W. B. (2017). Risk-aware Q-learning for Markov decision processes. In 2017 *IEEE 56th Annual Conference on Decision and Control (CDC)* 4928–4933. https://doi.org/10.1109/CDC.2017.8264388

[25] JELITO, D., PITERA, M. and STETTNER, Ł. (2021). Risk sensitive optimal stopping. *Stochastic Process. Appl.* **136** 125–144. MR4238105 https://doi.org/10.1016/j.spa.2021.03.005

[26] KAHNEMAN, D. and TVERSKY, A. (1979). Prospect theory: An analysis of decision under risk. *Econometrica* **47** 263–292. MR3618580 https://doi.org/10.2307/1914185

[27] KALLENBERG, O. (2002). *Foundations of Modern Probability*, 2nd ed. *Probability and Its Applications (New York)*. Springer, New York. MR1876169 https://doi.org/10.1007/978-1-4757-4015-8

[28] KARATZAS, I. and ZAMFIRESCU, I.-M. (2008). Martingale approach to stochastic differential games of control and stopping. *Ann. Probab.* **36** 1495–1527. MR2435857 https://doi.org/10.1214/07-AOP367

[29] KOSMALA, T., MARTYR, R. and MORIARTY, J. (2023). Markov risk mappings and risk-sensitive optimal prediction. *Math. Methods Oper. Res.* **97** 91–116. MR4544686 https://doi.org/10.1007/s00186-022-00802-z

[30] KUPPER, M. and SCHACHERMAYER, W. (2009). Representation results for law invariant time consistent functions. *Math. Financ. Econ.* **2** 189–210. MR2540511 https://doi.org/10.1007/s11579-009-0019-9

[31] LAM, T., VERMA, A., KIAN HSIANG LOW, B. and JAILLET, P. (2023). Risk-aware reinforcement learning with coherent risk measures and non-linear function approximation. In *The Eleventh International Conference on Learning Representations*.

[32] NUTZ, M. and SONER, H. M. (2012). Superhedging and dynamic risk measures under volatility uncertainty. *SIAM J. Control Optim.* **50** 2065–2089. MR2974730 https://doi.org/10.1137/100814925

[33] NUTZ, M. and ZHANG, J. (2015). Optimal stopping under adverse nonlinear expectation and related games. *Ann. Appl. Probab.* **25** 2503–2534. MR3375882 https://doi.org/10.1214/14-AAP1054

[34] PENG, S. (1997). Backward SDE and related g-expectation. In *Backward Stochastic Differential Equations (Paris, 1995–1996). Pitman Res. Notes Math. Ser.* **364** 141–160. Longman, Harlow. MR1752680

[35] PENG, S. (2007). G-expectation, G-Brownian motion and related stochastic calculus of Itô type. In *Stochastic Analysis and Applications. Abel Symp.* **2** 541–567. Springer, Berlin. MR2397805 https://doi.org/10.1007/978-3-540-70847-6_25

[36] PENG, S. (2007). G-Brownian motion and dynamic risk measure under volatility uncertainty. arXiv preprint. Available at arXiv:0711.2834.

[37] PESKIR, G. and SHIRYAEV, A. (2006). *Optimal Stopping and Free-Boundary Problems. Lectures in Mathematics ETH Zürich*. Birkhäuser, Basel. MR2256030

[38] PFLUG, G. C. and RÖMISCH, W. (2007). *Modeling, Measuring and Managing Risk*. World Scientific, Hackensack, NJ. MR2424523 https://doi.org/10.1142/9789812708724

[39] PUGH, C. C. (2015). *Real Mathematical Analysis*, 2nd ed. *Undergraduate Texts in Mathematics*. Springer, Cham. MR3380933 https://doi.org/10.1007/978-3-319-17771-7

[40] ROGERS, L. C. G. and WILLIAMS, D. (2000). *Diffusions*, *Markov Processes*, *and Martingales*. *Vol*. 2: *Itô Calculus*, 2nd ed. Cambridge Univ. Press, Cambridge.

[41] ROGERS, L. C. G. and WILLIAMS, D. (2000). *Diffusions*, *Markov Processes*, *and Martingales*. *Vol*. 1: *Foundations*, 2nd ed. Cambridge Univ. Press, Cambridge.

[42] RÖHL, S. (2001). Ein linearer Programmierungsansatz zur Lösung von Stopp- und Steuerungsproblemen. PhD thesis, Humboldt-Universität zu Berlin, Wirtschaftswissenschaftliche Fakultät. https://doi.org/10.18452/14586

[43] SERFOZO, R. (2009). *Basics of Applied Stochastic Processes*. *Probability and Its Applications* (*New York*). Springer, Berlin. MR2484222 https://doi.org/10.1007/978-3-540-89332-5

[44] WANG, W. and XU, H. (2023). Preference robust distortion risk measure and its application. *Math*. *Finance* **33** 389–434. MR4567940 https://doi.org/10.1111/mafi.12379

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A MEAN-FIELD VERSION OF BANK–EL KAROUI'S REPRESENTATION OF STOCHASTIC PROCESSES

BY XIHAO HE[1,a], XIAOLU TAN[2,b] AND JUN ZOU[2,c]

[1]*Department of Mathematics, University of Michigan,* [a]*hexihao@umich.edu*
[2]*Department of Mathematics, The Chinese University of Hong Kong,* [b]*xiaolu.tan@cuhk.edu.hk,* [c]*zou@math.cuhk.edu.hk*

We investigate a mean-field version of Bank–El Karoui's representation theorem of stochastic processes. Under three different technical conditions, we established some existence and uniqueness results, using three different approaches. As motivation and first applications, the results of mean-field representation provide a unified approach for studying various mean-field games (MFGs) in the setting with common noise and multiple populations, including the MFG of timing and the MFG with singular control, etc. As a crucial technical step, a stability result was provided on the classical Bank–El Karoui's representation theorem. It has its own interest and other applications, such as deriving stability results of optimizers (in the strong sense) for a class of optimal stopping and singular control problems.

REFERENCES

[1] BANK, P. (2005). Optimal control under a dynamic fuel constraint. *SIAM J. Control Optim.* **44** 1529–1541. MR2190919 https://doi.org/10.1137/040616966

[2] BANK, P. and BESSLICH, D. (2021). On a stochastic representation theorem for Meyer-measurable processes. *Ann. Inst. Henri Poincaré Probab. Stat.* **57** 1336–1368. MR4291443 https://doi.org/10.1214/20-aihp1113

[3] BANK, P. and EL KAROUI, N. (2004). A stochastic representation theorem with applications to optimization and obstacle problems. *Ann. Probab.* **32** 1030–1067. MR2044673 https://doi.org/10.1214/aop/1079021471

[4] BANK, P. and FÖLLMER, H. (2003). American options, multi-armed bandits, and optimal consumption plans: A unifying view. In *Paris–Princeton Lectures on Mathematical Finance*, 2002. *Lecture Notes in Math.* **1814** 1–42. Springer, Berlin. MR2021789 https://doi.org/10.1007/978-3-540-44859-4_1

[5] BANK, P. and KAUPPILA, H. (2017). Convex duality for stochastic singular control problems. *Ann. Appl. Probab.* **27** 485–516. MR3619793 https://doi.org/10.1214/16-AAP1209

[6] BANK, P. and RIEDEL, F. (2001). Optimal consumption choice with intertemporal substitution. *Ann. Appl. Probab.* **11** 750–788. MR1865023 https://doi.org/10.1214/aoap/1015345348

[7] BERTUCCI, C. (2018). Optimal stopping in mean field games, an obstacle problem approach. *J. Math. Pures Appl.* (9) **120** 165–194. MR3906158 https://doi.org/10.1016/j.matpur.2017.09.016

[8] BERTUCCI, C. (2020). Fokker–Planck equations of jumping particles and mean field games of impulse control. *Ann. Inst. H. Poincaré C, Anal. Non Linéaire* **37** 1211–1244. MR4138232 https://doi.org/10.1016/j.anihpc.2020.04.006

[9] BERTUCCI, C. (2021). Monotone solutions for mean field games master equations: Finite state space and optimal stopping. *J. Éc. Polytech. Math.* **8** 1099–1132. MR4275225 https://doi.org/10.5802/jep.167

[10] BILLINGSLEY, P. (1979). *Probability and Measure. Wiley Series in Probability and Mathematical Statistics*. Wiley, New York. MR0534323

[11] BISMUT, J.-M. and SKALLI, B. (1977). Temps d'arrêt optimal, théorie générale des processus et processus de Markov. *Z. Wahrsch. Verw. Gebiete* **39** 301–313. MR0651550 https://doi.org/10.1007/BF01877497

[12] BONSALL, F. F. (1962). *Lectures on Some Fixed Point Theorems of Functional Analysis*. Tata, Bombay. Notes by K. B. Vedak. MR0198173

[13] BOUVERET, G., DUMITRESCU, R. and TANKOV, P. (2020). Mean-field games of optimal stopping: A relaxed solution approach. *SIAM J. Control Optim.* **58** 1795–1821. MR4119327 https://doi.org/10.1137/18M1233480

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[14] BURZONI, M. and CAMPI, L. (2023). Mean field games with absorption and common noise with a model of bank run. *Stochastic Process*. *Appl*. **164** 206–241. MR4620847 https://doi.org/10.1016/j.spa.2023.07.007

[15] CAMPI, L., DE ANGELIS, T., GHIO, M. and LIVIERI, G. (2022). Mean-field games of finite-fuel capacity expansion with singular controls. *Ann*. *Appl*. *Probab*. **32** 3674–3717. MR4497856 https://doi.org/10.1214/21-aap1771

[16] CAO, H., DIANETTI, J. and FERRARI, G. (2023). Stationary discounted and ergodic mean field games with singular controls. *Math*. *Oper*. *Res*. **48** 1871–2382. MR4675937 https://doi.org/10.1287/moor.2022.1323

[17] CAO, H. and GUO, X. (2022). MFGs for partially reversible investment. *Stochastic Process*. *Appl*. **150** 995–1014. MR4440173 https://doi.org/10.1016/j.spa.2020.09.006

[18] CAO, H., GUO, X. and LEE, J. S. (2023). Approximation of N-player stochastic games with singular controls by mean field games. *Numer*. *Algebra Control Optim*. **13** 604–629. MR4588434 https://doi.org/10.3934/naco.2023001

[19] CARDALIAGUET, P. (2010). Notes on mean field games. Technical report.

[20] CARMONA, R. and DELARUE, F. (2018). *Probabilistic Theory of Mean Field Games with Applications*. *I–II*. *Probability Theory and Stochastic Modelling*. Springer, Cham. MR3753660

[21] CARMONA, R., DELARUE, F. and LACKER, D. (2016). Mean field games with common noise. *Ann*. *Probab*. **44** 3740–3803. MR3572323 https://doi.org/10.1214/15-AOP1060

[22] CARMONA, R., DELARUE, F. and LACKER, D. (2017). Mean field games of timing and models for bank runs. *Appl*. *Math*. *Optim*. **76** 217–260. MR3679343 https://doi.org/10.1007/s00245-017-9435-z

[23] CHIAROLLA, M. B. and FERRARI, G. (2014). Identifying the free boundary of a stochastic, irreversible investment problem via the Bank–El Karoui representation theorem. *SIAM J*. *Control Optim*. **52** 1048–1070. MR3181698 https://doi.org/10.1137/11085195X

[24] DIANETTI, J., FERRARI, G., FISCHER, M. and NENDEL, M. (2023). A unifying framework for submodular mean field games. *Math*. *Oper*. *Res*. **48** 1679–1710. MR4702114 https://doi.org/10.1287/moor.2022.1316

[25] DUMITRESCU, R., LEUTSCHER, M. and TANKOV, P. (2021). Control and optimal stopping mean field games: A linear programming approach. *Electron*. *J*. *Probab*. **26** Paper No. 157, 49. MR4350982 https://doi.org/10.1214/21-ejp713

[26] EL KAROUI, N. (1981). Les aspects probabilistes du contrôle stochastique. In *Ninth Saint Flour Probability Summer School*—1979 (*Saint Flour*, 1979). *Lecture Notes in Math*. **876** 73–238. Springer, Berlin. MR0637471

[27] FU, G. (2023). Extended mean field games with singular controls. *SIAM J*. *Control Optim*. **61** 285–314. MR4550674 https://doi.org/10.1137/20M1384750

[28] FU, G. and HORST, U. (2017). Mean field games with singular controls. *SIAM J*. *Control Optim*. **55** 3833–3868. MR3731028 https://doi.org/10.1137/17M1123742

[29] HE, X., TAN, X. and ZOU, J. (2023). An exit contract optimization problem. *ESAIM Control Optim*. *Calc*. *Var*. **29** Paper No. 82, 33. MR4664845 https://doi.org/10.1051/cocv/2023064

[30] HUANG, M., MALHAMÉ, R. P. and CAINES, P. E. (2006). Large population stochastic dynamic games: Closed-loop McKean–Vlasov systems and the Nash certainty equivalence principle. *Commun*. *Inf*. *Syst*. **6** 221–252. MR2346927 https://doi.org/10.4310/cis.2006.v6.n3.a5

[31] JACOD, J. and MÉMIN, J. (1981). Sur un type de convergence intermédiaire entre la convergence en loi et la convergence en probabilité. In *Seminar on Probability*, *XV* (*Univ*. *Strasbourg*, *Strasbourg*, 1979/1980) (*French*). *Lecture Notes in Math*. **850** 529–546. Springer, Berlin. MR0622586

[32] JACOD, J. and SHIRYAEV, A. N. (1987). *Limit Theorems for Stochastic Processes*. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **288**. Springer, Berlin. MR0959133 https://doi.org/10.1007/978-3-662-02514-7

[33] KAMAE, T. and KRENGEL, U. (1978). Stochastic partial ordering. *Ann*. *Probab*. **6** 1044–1049 (1979). MR0512419

[34] LASRY, J.-M. and LIONS, P.-L. (2007). Mean field games. *Jpn*. *J*. *Math*. **2** 229–260. MR2295621 https://doi.org/10.1007/s11537-007-0657-8

[35] LIONS, P.-L. (2009). Cours Au Coll‘ege de France (Lectures on Nov 27th, Dec 4th–11th, 2009). Available at www.college-de-france.fr.

[36] MA, J. and WANG, Y. (2009). On variant reflected backward SDEs, with applications. *J*. *Appl*. *Math*. *Stoch*. *Anal*. Art. ID 854768, 26. MR2511615 https://doi.org/10.1155/2009/854768

[37] NUTZ, M. (2018). A mean field game of optimal stopping. *SIAM J*. *Control Optim*. **56** 1206–1221. MR3780736 https://doi.org/10.1137/16M1078331

[38] QIAN, Z. and XU, M. (2018). Reflected backward stochastic differential equations with resistance. *Ann*. *Appl*. *Probab*. **28** 888–911. MR3784491 https://doi.org/10.1214/17-AAP1319

[39] REN, Z., TAN, X., TOUZI, N. and YANG, J. (2023). Entropic optimal planning for path-dependent mean field games. *SIAM J. Control Optim.* **61** 1415–1437. MR4599316 https://doi.org/10.1137/22M1484444
[40] TOPKIS, D. M. (1998). *Supermodularity and Complementarity*. *Frontiers of Economic Research*. Princeton Univ. Press, Princeton, NJ. MR1614637

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TWO-DIMENSIONAL SUPERCRITICAL GROWTH DYNAMICS WITH ONE-DIMENSIONAL NUCLEATION

BY DANIEL BLANQUICETT[1,a], JANKO GRAVNER[1,b], DAVID SIVAKOFF[2,c] AND LUKE WILSON[3,d]

[1]*Department of Mathematics, University of California, Davis,* [a]*drblanquicett@ucdavis.edu,* [b]*gravner@math.ucdavis.edu*
[2]*Departments of Statistics and Mathematics, The Ohio State University,* [c]*dsivakoff@stat.osu.edu*
[3]*Departments of Physics and Mathematics, The College of Wooster,* [d]*lwilson24@wooster.edu*

We introduce a class of cellular automata growth models on the two-dimensional integer lattice with finite cross neighborhoods. These dynamics are determined by a Young diagram $\mathcal{Z}$ and the radius ρ of the neighborhood, which we assume to be sufficiently large, so that the resulting dynamics are supercritical. A point becomes occupied if the pair of counts of currently occupied points in the horizontal and vertical parts of the cross neighborhood lies outside $\mathcal{Z}$. Starting with a small density p of occupied points, we focus on the first time T at which the origin is occupied. We show that T scales as a power of $1/p$, and identify that power, when $\mathcal{Z}$ is a triangle, a rectangle, and the union of a finite rectangle with an infinite strip. The case of a triangle corresponds to the classical bootstrap percolation model. We give partial results when $\mathcal{Z}$ is a union of two finite rectangles. The distinguishing feature of these dynamics is nucleation of lines that grow to significant length before most of the space is covered.

REFERENCES

[1] AIZENMAN, M. and LEBOWITZ, J. L. (1988). Metastability effects in bootstrap percolation. *J. Phys. A* **21** 3801–3813. MR0968311

[2] BALISTER, P., BOLLOBÁS, B., LEE, J. and NARAYANAN, B. (2018). Line percolation. *Random Structures Algorithms* **52** 597–616. MR3809690 https://doi.org/10.1002/rsa.20755

[3] BALISTER, P., BOLLOBÁS, B., MORRIS, R. and SMITH, P. (2025). The Critical Length for Growing a Droplet. *Mem. Amer. Math. Soc.* **310** v+180. MR4935836 https://doi.org/10.1090/memo/1571

[4] BALISTER, P. N., BOLLOBÁS, B., MORRIS, R. and SMITH, P. Universality for monotone cellular automata. Available at arXiv:2203.13806.

[5] BALOGH, J., BOLLOBÁS, B., DUMINIL-COPIN, H. and MORRIS, R. (2012). The sharp threshold for bootstrap percolation in all dimensions. *Trans. Amer. Math. Soc.* **364** 2667–2701. MR2888224 https://doi.org/10.1090/S0002-9947-2011-05552-2

[6] BLANQUICETT, D. Three-dimensional 2-critical bootstrap percolation: the stable sets approach. Available at arXiv:2201.11365.

[7] BLANQUICETT, D. (2020). Anisotropic bootstrap percolation in three dimensions. *Ann. Probab.* **48** 2591–2614. MR4152652 https://doi.org/10.1214/20-AOP1434

[8] BLANQUICETT, D. (2024). The d-dimensional bootstrap percolation models with axial neighbourhoods. *Stochastic Process. Appl.* **174** Paper No. 104383. MR4746579 https://doi.org/10.1016/j.spa.2024.104383

[9] BOLLOBÁS, B., DUMINIL-COPIN, H., MORRIS, R. and SMITH, P. (2023). Universality for two-dimensional critical cellular automata. *Proc. Lond. Math. Soc.* (3) **126** 620–703. MR4550150 https://doi.org/10.1112/plms.12497

[10] BOLLOBÁS, B., SMITH, P. and UZZELL, A. (2015). Monotone cellular automata in a random environment. *Combin. Probab. Comput.* **24** 687–722. MR3350030 https://doi.org/10.1017/S0963548315000012

[11] CHALUPA, J., LEATH, P. L. and REICH, G. R. (1979). Bootstrap percolation on a Bethe lattice. *J. Phys. C* **12** L31–L35.

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[12] CHEN, J. et al. (2018). Building two-dimensional materials one row at a time: Avoiding the nucleation barrier. *Science* **362** 1135–1139.

[13] GRAVNER, J. and GRIFFEATH, D. (1996). First passage times for threshold growth dynamics on $\mathbf{Z}^2$. *Ann. Probab.* **24** 1752–1778. MR1415228 https://doi.org/10.1214/aop/1041903205

[14] GRAVNER, J., HOFFMAN, C., PFEIFFER, J. and SIVAKOFF, D. (2015). Bootstrap percolation on the Hamming torus. *Ann. Appl. Probab.* **25** 287–323. MR3297774 https://doi.org/10.1214/13-AAP996

[15] GRAVNER, J., KRNC, M., MILANIČ, M. and RAYMOND, J.-P. Young domination on Hamming rectangles. Available at arXiv:2501.03788.

[16] GRAVNER, J., PAGUYO, J. E. and SLIVKEN, E. (2019). Maximal spanning time for neighborhood growth on the Hamming plane. *SIAM J. Discrete Math.* **33** 976–993. MR3968241 https://doi.org/10.1137/17M1153807

[17] GRAVNER, J. and SIVAKOFF, D. (2017). Bootstrap percolation on products of cycles and complete graphs. *Electron. J. Probab.* **22** Paper No. 29. MR3629873 https://doi.org/10.1214/17-EJP43

[18] GRAVNER, J. and SIVAKOFF, D. (2020). Bootstrap percolation on the product of the two-dimensional lattice with a Hamming square. *Ann. Appl. Probab.* **30** 145–174. MR4068308 https://doi.org/10.1214/19-AAP1497

[19] GRAVNER, J., SIVAKOFF, D. and SLIVKEN, E. (2017). Neighborhood growth dynamics on the Hamming plane. *Electron. J. Combin.* **24** Paper No. 4.29. MR3723568 https://doi.org/10.37236/6400

[20] HARTARSKY, I. and MEZEI, T. R. (2020). Complexity of two-dimensional bootstrap percolation difficulty: Algorithm and NP-hardness. *SIAM J. Discrete Math.* **34** 1444–1459. MR4117299 https://doi.org/10.1137/19M1239933

[21] HARTARSKY, I. and MORRIS, R. (2019). The second term for two-neighbour bootstrap percolation in two dimensions. *Trans. Amer. Math. Soc.* **372** 6465–6505. MR4024528 https://doi.org/10.1090/tran/7828

[22] HOLROYD, A. E. (2003). Sharp metastability threshold for two-dimensional bootstrap percolation. *Probab. Theory Related Fields* **125** 195–224. MR1961342 https://doi.org/10.1007/s00440-002-0239-x

[23] HOLROYD, A. E., LIGGETT, T. M. and ROMIK, D. (2004). Integrals, partitions, and cellular automata. *Trans. Amer. Math. Soc.* **356** 3349–3368. MR2052953 https://doi.org/10.1090/S0002-9947-03-03417-2

[24] JUNGBLUT, S. and DELLAGO, C. (2016). Pathways to self-organization: Crystallization via nucleation and growth. *Eur. Phys. J. E* **77**.

[25] MARIS, H. J. (2006). Introduction to the physics of nucleation. *C. R. Phys.* **7** 946–958.

[26] MORRIS, R. (2017). Bootstrap percolation, and other automata. *European J. Combin.* **66** 250–263. MR3692148 https://doi.org/10.1016/j.ejc.2017.06.024

[27] SLIVKEN, E. Bootstrap percolation on the Hamming torus with threshold 2. Available at arXiv:1407.2317.

[28] VAN ENTER, A. C. D. (1987). Proof of Straley's argument for bootstrap percolation. *J. Stat. Phys.* **48** 943–945. MR0914911 https://doi.org/10.1007/BF01019705

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DIFFUSION LIMITS IN THE QUARTER PLANE AND NONSEMIMARTINGALE REFLECTED BROWNIAN MOTION

BY RAMI ATAR[1,a] AND AMARJIT BUDHIRAJA[2,b]

[1]*Viterbi Faculty of Electrical and Computer Engineering, Technion—Israel Institute of Technology,* [a]*rami@technion.ac.il*
[2]*Department of Statistics and Operations Research, University of North Carolina at Chapel Hill,* [b]*budhiraj@email.unc.edu*

We consider a continuous-time random walk in the quarter plane for which the transition intensities are constant on each of the four faces $(0,\infty)^2$, $F_1 = \{0\} \times (0,\infty)$, $F_2 = (0,\infty) \times \{0\}$ and $\{(0,0)\}$. We show that when rescaled diffusively it converges in law to a Brownian motion with oblique reflection direction $d^{(i)}$ on face F_i, $i = 1, 2$, defined via the Varadhan–Williams submartingale problem (*Comm. Pure Appl. Math.* **38** (1985) 405–443). A parameter denoted by α was introduced in (*Comm. Pure Appl. Math.* **38** (1985) 405–443), measuring the extent to which $d^{(i)}$ are inclined toward the origin. In the case of the quarter plane, α takes values in $(-2, 2)$, and it is known that the reflected Brownian motion is a semimartingale if and only if $\alpha \in (-2, 1)$. Convergence results via both the Skorohod map and the invariance principle for semimartingale reflected Brownian motion are known to hold in various settings in arbitrary dimension. In the case of the quarter plane, the invariance principle was proved for $\alpha \in (-2, 1)$ whereas for tools based on the Skorohod map to be applicable it is necessary (but not sufficient) that $\alpha \in [-1, 1)$. Another tool that has been used to prove convergence in general dimension is the extended Skorohod map, which in the case of the quarter plane provides convergence for $\alpha = 1$. This paper focuses on the range $\alpha \in (1, 2)$, where the Skorohod problem and the extended Skorohod problem do not possess a unique solution, the limit process is not a semimartingale, and convergence to reflected Brownian motion has not been shown before. The result has implications on the asymptotic analysis of two Markovian queueing models: The *generalized processor sharing model with parallelization slowdown*, and the *coupled processor model*. In both cases, the diffusion limit in heavy traffic is characterized by the aforementioned reflected Brownian motion. The restriction of our treatment to dimension two is due to the fact that, for analogous models in higher dimension, the well posedness of the submartingale problem for the candidate limit process is an open problem.

REFERENCES

[1] BOXMA, O. and IVANOVS, J. (2013). Two coupled Lévy queues with independent input. *Stoch. Syst.* **3** 574–590. MR3353211 https://doi.org/10.1214/13-SSY116

[2] BURDZY, K. and CHEN, Z.-Q. (2008). Discrete approximations to reflected Brownian motion. *Ann. Probab.* **36** 698–727. MR2393994 https://doi.org/10.1214/009117907000000240

[3] COHEN, J. W. and BOXMA, O. J. (2000). *Boundary Value Problems in Queueing System Analysis*. *North-Holland Mathematics Studies* **79**. Elsevier, Amsterdam.

[4] COSTANTINI, C. and KURTZ, T. G. (2018). Existence and uniqueness of reflecting diffusions in cusps. *Electron. J. Probab.* **23** Paper No. 84. MR3858912 https://doi.org/10.1214/18-EJP204

[5] COSTANTINI, C. and KURTZ, T. G. (2024). A reverse ergodic theorem for inhomogeneous killed Markov chains and application to a new uniqueness result for reflecting diffusions. *Ann. Appl. Probab.* **34** 3665–3700.

[6] DAĬ, D. G. and VIL'YAMS, R. D. (1995). Existence and uniqueness of semimartingale reflecting Brownian motions in convex polyhedra. *Theory Probab. Appl.* **50** 3–53.

MSC2020 subject classifications. 60J65, 60B10, 60J55, 60J70.
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[7] DEBLASSIE, R. D. and TOBY, E. H. (1993). On the semimartingale representation of reflecting Brownian motion in a cusp. *Probab. Theory Related Fields* **94** 505–524. MR1201557 https://doi.org/10.1007/BF01192561

[8] DUPUIS, P. and ISHII, H. (1991). On Lipschitz continuity of the solution mapping to the Skorokhod problem, with applications. *Stoch. Stoch. Rep.* **35** 31–62. MR1110990 https://doi.org/10.1080/17442509108833688

[9] DUPUIS, P. and RAMANAN, K. (1999). Convex duality and the Skorokhod problem. I, II. *Probab. Theory Related Fields* **115** 153–195, 197–236. MR1720348 https://doi.org/10.1007/s004400050269

[10] FAYOLLE, G. and IASNOGORODSKI, R. (1979). Two coupled processors: The reduction to a Riemann-Hilbert problem. *Z. Wahrsch. Verw. Gebiete* **47** 325–351. MR0525314 https://doi.org/10.1007/BF00535168

[11] FAYOLLE, G., MALYSHEV, V., IASNOGORODSKI, R. and FAYOLLE, G. (1999). *Random Walks in the Quarter Plane*. *Probability Theory and Stochastic Modelling* **40**. Springer, Berlin.

[12] GUPTA, V. (2008). Finding the optimal quantum size: Sensitivity analysis of the M/G/1 round-Robin queue. *ACM SIGMETRICS Perform. Eval. Rev.* **36** 104–106.

[13] HARRISON, J. M. and REIMAN, M. I. (1981). Reflected Brownian motion on an orthant. *Ann. Probab.* **9** 302–308. MR0606992

[14] KANG, W. and RAMANAN, K. (2010). A Dirichlet process characterization of a class of reflected diffusions. *Ann. Probab.* **38** 1062–1105. MR2674994 https://doi.org/10.1214/09-AOP487

[15] KANG, W. and RAMANAN, K. (2017). On the submartingale problem for reflected diffusions in domains with piecewise smooth boundaries. *Ann. Probab.* **45** 404–468. MR3601653 https://doi.org/10.1214/16-AOP1153

[16] KWON, Y. and WILLIAMS, R. J. (1991). Reflected Brownian motion in a cone with radially homogeneous reflection field. *Trans. Amer. Math. Soc.* **327** 739–780. MR1028760 https://doi.org/10.2307/2001821

[17] LAKNER, P., LIU, Z. and REED, J. (2023). Reflected Brownian motion with drift in a wedge. *Queueing Syst.* **105** 233–270. MR4670662 https://doi.org/10.1007/s11134-023-09893-9

[18] LAKNER, P., REED, J. and ZWART, B. (2019). On the roughness of the paths of RBM in a wedge. *Ann. Inst. Henri Poincaré Probab. Stat.* **55** 1566–1598. MR4010945 https://doi.org/10.1214/18-aihp928

[19] YUN, M., ZHOU, Y., ARORA, A. and CHOI, H.-A. (2009). Channel-assignment and scheduling in wireless mesh networks considering switching overhead. In 2009 *IEEE International Conference on Communications* 1–6. IEEE Press, New York.

[20] PAREKH, A. K. and GALLAGER, R. G. (1993). A generalized processor sharing approach to flow control in integrated services networks: The single-node case. *IEEE/ACM Trans. Netw.* **1** 344–357.

[21] PENG, E. (2022). Exact response time analysis of preemptive priority scheduling with switching overhead. *ACM SIGMETRICS Perform. Eval. Rev.* **49** 72–74.

[22] RAMANAN, K. (2006). Reflected diffusions defined via the extended Skorokhod map. *Electron. J. Probab.* **11** 934–992. MR2261058 https://doi.org/10.1214/EJP.v11-360

[23] RAMANAN, K. and REIMAN, M. I. (2003). Fluid and heavy traffic diffusion limits for a generalized processor sharing model. *Ann. Appl. Probab.* **13** 100–139. MR1951995 https://doi.org/10.1214/aoap/1042765664

[24] RAMANAN, K. and REIMAN, M. I. (2008). The heavy traffic limit of an unbalanced generalized processor sharing model. *Ann. Appl. Probab.* **18** 22–58. MR2380890 https://doi.org/10.1214/07-AAP438

[25] REIMAN, M. I. (1984). Open queueing networks in heavy traffic. *Math. Oper. Res.* **9** 441–458. MR0757317 https://doi.org/10.1287/moor.9.3.441

[26] REIMAN, M. I. and WILLIAMS, R. J. (1988). A boundary property of semimartingale reflecting Brownian motions. *Probab. Theory Related Fields* **77** 87–97. MR0921820 https://doi.org/10.1007/BF01848132

[27] STROOCK, D. W. and VARADHAN, S. R. S. (1971). Diffusion processes with boundary conditions. *Comm. Pure Appl. Math.* **24** 147–225. MR0277037 https://doi.org/10.1002/cpa.3160240206

[28] TAYLOR, L. M. and WILLIAMS, R. J. (1993). Existence and uniqueness of semimartingale reflecting Brownian motions in an orthant. *Probab. Theory Related Fields* **96** 283–317. MR1231926 https://doi.org/10.1007/BF01292674

[29] THOMPSON, S., LIPSKY, L., TASNEEM, S. and ZHANG, F. (2009). Analysis of round-Robin implementations of processor sharing, including overhead. In 2009 *Eighth IEEE International Symposium on Network Computing and Applications* 60–65. IEEE Press, New York.

[30] VARADHAN, S. R. S. and WILLIAMS, R. J. (1985). Brownian motion in a wedge with oblique reflection. *Comm. Pure Appl. Math.* **38** 405–443. MR0792398 https://doi.org/10.1002/cpa.3160380405

[31] WHITT, W. (2002). *Stochastic-Process Limits*: *An Introduction to Stochastic-Process Limits and Their Application to Queues*. *Springer Series in Operations Research*. Springer, New York. MR1876437

[32] WILLIAMS, R. J. (1998). An invariance principle for semimartingale reflecting Brownian motions in an orthant. *Queueing Syst. Theory Appl.* **30** 5–25. MR1663755 https://doi.org/10.1023/A:1019156702875

[33] WILLIAMS, R. J. (1998). Diffusion approximations for open multiclass queueing networks: Sufficient conditions involving state space collapse. *Queueing Syst*. *Theory Appl*. **30** 27–88. MR1663759 https://doi.org/10.1023/A:1019108819713

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PATHWISE UNIQUENESS FOR MULTIPLICATIVE YOUNG AND ROUGH DIFFERENTIAL EQUATIONS DRIVEN BY FRACTIONAL BROWNIAN MOTION

BY TOYOMU MATSUDA[1,a] AND AVI MAYORCAS[2,b]

[1] *Independent,* [a] *toyomumatsudawork@gmail.com*
[2] *Department of Mathematical Sciences, University of Bath,* [b] *am2735@bath.ac.uk*

We show *pathwise uniqueness* of multiplicative SDEs, in arbitrary dimensions, driven by fractional Brownian motion with Hurst parameter $H \in (1/3, 1)$ with volatility coefficient σ that is at least γ-Hölder continuous for $\gamma > \frac{1}{2H} \vee \frac{1-H}{H}$. This improves upon the long-standing results of (*Math. Res. Lett.* **1** (1994) 451–464; *Rev. Mat. Iberoam.* **14** (1998) 215–310; *Appl. Math. Res. Express. AMRX* (2008) Art. ID abm009) which cover the same regime but require σ to be at least $\frac{1}{H}$-Hölder continuous. Our central innovation is to combine stochastic averaging estimates with refined versions of the stochastic sewing lemma, due to (*Electron. J. Probab.* **25** (2020) Paper No. 38; *Stoch. Partial Differ. Equ. Anal. Comput.* **11** (2023) 714–729; *Forum Math. Sigma* **12** (2024) Paper No. e52).

REFERENCES

[1] ANZELETTI, L., LÊ, K. and LING, C. (2023). Path-by-path uniqueness for stochastic differential equations under Krylov-Röckner condition. Available at arXiv:2304.06802.

[2] ANZELETTI, L., RICHARD, A. and TANRÉ, E. (2023). Regularisation by fractional noise for one-dimensional differential equations with distributional drift. *Electron. J. Probab.* **28** Paper No. 135. MR4664455 https://doi.org/10.1214/23-ejp1010

[3] APOLINÁRIO, G. B., CHEVILLARD, L. and MOURRAT, J.-C. (2022). Dynamical fractional and multifractal fields. *J. Stat. Phys.* **186** Paper No. 15. MR4357095 https://doi.org/10.1007/s10955-021-02867-2

[4] ATHREYA, S., BUTKOVSKY, O. and MYTNIK, L. (2020). Strong existence and uniqueness for stable stochastic differential equations with distributional drift. *Ann. Probab.* **48** 178–210. MR4079434 https://doi.org/10.1214/19-AOP1358

[5] BAHOURI, H., CHEMIN, J.-Y. and DANCHIN, R. (2011). *Fourier Analysis and Nonlinear Partial Differential Equations*. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **343**. Springer, Heidelberg. MR2768550 https://doi.org/10.1007/978-3-642-16830-7

[6] BAILLEUL, I., CATELLIER, R. and DELARUE, F. (2020). Solving mean field rough differential equations. *Electron. J. Probab.* **25** Paper No. 21. MR4073682 https://doi.org/10.1214/19-ejp409

[7] BENELLI, R. and WEISS, M. (2021). From sub- to superdiffusion: Fractional Brownian motion of membraneless organelles in early C. elegans embryos. *New J. Phys.* **23** 063072.

[8] BERAN, J. (1994). *Statistics for Long-Memory Processes*. *Monographs on Statistics and Applied Probability* **61**. CRC Press, New York. MR1304490

[9] BERMAN, S. M. (1973). Local nondeterminism and local times of Gaussian processes. *Bull. Amer. Math. Soc.* **79** 475–477. MR0317396 https://doi.org/10.1090/S0002-9904-1973-13225-2

[10] BUTKOVSKY, O. and MYTNIK, L. (2019). Regularization by noise and flows of solutions for a stochastic heat equation. *Ann. Probab.* **47** 165–212. MR3909968 https://doi.org/10.1214/18-AOP1259

[11] CASS, T., LITTERER, C. and LYONS, T. (2013). Integrability and tail estimates for Gaussian rough differential equations. *Ann. Probab.* **41** 3026–3050. MR3112937 https://doi.org/10.1214/12-AOP821

[12] CATELLIER, R. and DUBOSCQ, R. (2025). Regularization by noise for rough differential equations driven by Gaussian rough paths. *Ann. Probab.* **53** 79–139. MR4852889 https://doi.org/10.1214/24-aop1701

[13] CATELLIER, R. and GUBINELLI, M. (2016). Averaging along irregular curves and regularisation of ODEs. *Stochastic Process. Appl.* **126** 2323–2366. MR3505229 https://doi.org/10.1016/j.spa.2016.02.002

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[14] COMTE, F., COUTIN, L. and RENAULT, E. (2012). Affine fractional stochastic volatility models. *Ann. Finance* **8** 337–378. MR2922801 https://doi.org/10.1007/s10436-010-0165-3

[15] COMTE, F. and RENAULT, E. (1996). Long memory continuous time models. *J. Econometrics* **73** 101–149. MR1410003 https://doi.org/10.1016/0304-4076(95)01735-6

[16] COUTIN, L. and QIAN, Z. (2002). Stochastic analysis, rough path analysis and fractional Brownian motions. *Probab. Theory Related Fields* **122** 108–140. MR1883719 https://doi.org/10.1007/s004400100158

[17] CRISAN, D., HOLM, D. D., LEAHY, J.-M. and NILSSEN, T. (2022). Variational principles for fluid dynamics on rough paths. *Adv. Math.* **404** Paper No. 108409. MR4413218 https://doi.org/10.1016/j.aim.2022.108409

[18] CRISAN, D., HOLM, D. D., LEAHY, J.-M. and NILSSEN, T. (2022). Solution properties of the incompressible Euler system with rough path advection. *J. Funct. Anal.* **283** Paper No. 109632. MR4462317 https://doi.org/10.1016/j.jfa.2022.109632

[19] DAREIOTIS, K. and GERENCSÉR, M. (2024). Path-by-path regularisation through multiplicative noise in rough, Young, and ordinary differential equations. *Ann. Probab.* **52** 1864–1902. MR4791421 https://doi.org/10.1214/24-aop1686

[20] DAREIOTIS, K., GERENCSÉR, M. and LÊ, K. (2023). Quantifying a convergence theorem of Gyöngy and Krylov. *Ann. Appl. Probab.* **33** 2291–2323. MR4583671 https://doi.org/10.1214/22-aap1867

[21] DAREIOTIS, K., GERENCSÉR, M., LÊ, K. and LING, C. (2024). Regularisation by Gaussian rough path lifts of fractional Brownian motions.

[22] DAVIE, A. M. (2008). Differential equations driven by rough paths: An approach via discrete approximation. *Appl. Math. Res. Express. AMRX* Art. ID abm009. MR2387018

[23] DELARUE, F. and SALKELD, W. (2022). Probabilistic rough paths II Lions–Taylor expansions and random controlled rough paths. Available at arXiv:2203.01185.

[24] DELARUE, F. and SALKELD, W. (2025). An example driven introduction to probabilistic rough paths. In *Séminaire de Probabilités LII. Lecture Notes in Math.* **2363** 27–119. Springer, Cham. MR4923078 https://doi.org/10.1007/978-3-031-86422-3_2

[25] FARANDA, D., PONS, F. M. E., DUBRULLE, B., DAVIAUD, F., SAINT-MICHEL, B., HERBERT, E. and CORTET, P.-P. (2014). Modelling and analysis of turbulent datasets using auto regressive moving average processes. *Phys. Fluids* **26** 105101.

[26] FEYEL, D. and DE LA PRADELLE, A. (2006). Curvilinear integrals along enriched paths. *Electron. J. Probab.* **11** no. 34. MR2261056 https://doi.org/10.1214/EJP.v11-356

[27] FLANDOLI, F. (2011). *Random Perturbation of PDEs and Fluid Dynamic Models. Lecture Notes in Math.* **2015**. Springer, Heidelberg. MR2796837 https://doi.org/10.1007/978-3-642-18231-0

[28] FLANDOLI, F. (2017). Random initial conditions and noise in ordinary and partial differential equations. Avaibale at http://users.dma.unipi.it/flandoli/randomSNS_LN2.pdf.

[29] FLANDOLI, F. and LUO, D. (2021). High mode transport noise improves vorticity blow-up control in 3D Navier–Stokes equations. *Probab. Theory Related Fields* **180** 309–363. MR4265023 https://doi.org/10.1007/s00440-021-01037-5

[30] FLANDOLI, F. and RUSSO, F. (2023). Reduced dissipation effect in stochastic transport by Gaussian noise with regularity greater than 1/2. Available at arXiv:2305.19293.

[31] FRANZKE, C. L. E., O'KANE, T. J., BERNER, J., WILLIAMS, P. D. and LUCARINI, V. (2014). Stochastic climate theory and modeling. *Wiley Interdiscip. Rev.: Clim. Change* **6** 63–78.

[32] FRIZ, P. K. and HAIRER, M. (2020). *A Course on Rough Paths. Universitext*. Springer, Cham. MR4174393 https://doi.org/10.1007/978-3-030-41556-3

[33] FRIZ, P. K., SALKELD, W. and WAGENHOFER, T. (2025). Weak error estimates for rough volatility models. *Ann. Appl. Probab.* **35** 64–98. MR4871704 https://doi.org/10.1214/24-aap2109

[34] FRIZ, P. K. and VICTOIR, N. B. (2010). *Multidimensional Stochastic Processes as Rough Paths: Theory and Applications. Cambridge Studies in Advanced Mathematics* **120**. Cambridge Univ. Press, Cambridge. MR2604669 https://doi.org/10.1017/CBO9780511845079

[35] GALEATI, L. (2022). Pathwise methods in regularisation by noise. PhD thesis, Rheinische Friedrich–Wilhelms-Universität Bonn.

[36] GALEATI, L. (2023). Nonlinear Young differential equations: A review. *J. Dynam. Differential Equations* **35** 985–1046. MR4594437 https://doi.org/10.1007/s10884-021-09952-w

[37] GALEATI, L. and GERENCSÉR, M. (2022). Solution theory of fractional SDEs in complete subcritical regimes.

[38] GALEATI, L. and HARANG, F. A. (2022). Regularization of multiplicative SDEs through additive noise. *Ann. Appl. Probab.* **32** 3930–3963. MR4498199 https://doi.org/10.1214/21-aap1778

[39] GALEATI, L., HARANG, F. A. and MAYORCAS, A. (2023). Distribution dependent SDEs driven by additive fractional Brownian motion. *Probab. Theory Related Fields* **185** 251–309. MR4528970 https://doi.org/10.1007/s00440-022-01145-w

[40] GALEATI, L., LEAHY, J.-M. and NILSSEN, T. (2025). On the well-posedness of (nonlinear) rough continuity equations.

[41] GALEATI, L. and LING, C. (2023). Stability estimates for singular SDEs and applications. *Electron. J. Probab.* **28** Paper No. 24. MR4546633 https://doi.org/10.1214/23-ejp913

[42] GERENCSÉR, M. (2023). Regularisation by regular noise. *Stoch. Partial Differ. Equ. Anal. Comput.* **11** 714–729. MR4588621 https://doi.org/10.1007/s40072-022-00242-0

[43] GESS, B. and TSATSOULIS, P. (2020). Synchronization by noise for the stochastic quantization equation in dimensions 2 and 3. *Stoch. Dyn.* **20** 2040006. MR4161970 https://doi.org/10.1142/S0219493720400067

[44] GUBINELLI, M. (2004). Controlling rough paths. *J. Funct. Anal.* **216** 86–140. MR2091358 https://doi.org/10.1016/j.jfa.2004.01.002

[45] HAIRER, M. and LI, X.-M. (2020). Averaging dynamics driven by fractional Brownian motion. *Ann. Probab.* **48** 1826–1860. MR4124526 https://doi.org/10.1214/19-AOP1408

[46] HAIRER, M. and LI, X.-M. (2022). Generating diffusions with fractional Brownian motion. *Comm. Math. Phys.* **396** 91–141. MR4499013 https://doi.org/10.1007/s00220-022-04462-2

[47] HARANG, F. A. and PERKOWSKI, N. (2021). C^∞-regularization of ODEs perturbed by noise. *Stoch. Dyn.* **21** Paper No. 2140010. MR4342752 https://doi.org/10.1142/S0219493721400104

[48] HINZ, M., TÖLLE, J. M. and VIITASAARI, L. (2023). Variability of paths and differential equations with BV-coefficients. *Ann. Inst. Henri Poincaré Probab. Stat.* **59** 2036–2082. MR4663516 https://doi.org/10.1214/22-aihp1308

[49] KRYLOV, N. V. (2023). On strong solutions of Itô's equations with $D\sigma$ and b in Morrey classes containing L_d. *Ann. Probab.* **51** 1729–1751. MR4642222 https://doi.org/10.1214/23-aop1630

[50] KURTZ, T. G. (2007). The Yamada–Watanabe–Engelbert theorem for general stochastic equations and inequalities. *Electron. J. Probab.* **12** 951–965. MR2336594 https://doi.org/10.1214/EJP.v12-431

[51] LE GALL, J.-F. (2013). *Mouvement Brownien, Martingales et Calcul Stochastique*. *Mathématiques & Applications (Berlin) [Mathematics & Applications]* **71**. Springer, Heidelberg. MR3184878 https://doi.org/10.1007/978-3-642-31898-6

[52] LÊ, K. (2020). A stochastic sewing lemma and applications. *Electron. J. Probab.* **25** Paper No. 38. MR4089788 https://doi.org/10.1214/20-ejp442

[53] LÊ, K. and LING, C. (2021). Taming singular stochastic differential equations: A numerical method.

[54] LI, X.-M., PANLOUP, F. and SIEBER, J. (2023). On the (non)stationary density of fractional-driven stochastic differential equations. *Ann. Probab.* **51** 2056–2085. MR4666290 https://doi.org/10.1214/23-aop1638

[55] LILLY, J. M., SYKULSKI, A. M., EARLY, J. J. and OLHEDE, S. C. (2017). Fractional Brownian motion, the Matérn process, and stochastic modeling of turbulent dispersion. *Nonlinear Process. Geophys.* **24** 481–514.

[56] LYONS, T. (1994). Differential equations driven by rough signals. I. An extension of an inequality of L. C. Young. *Math. Res. Lett.* **1** 451–464. MR1302388 https://doi.org/10.4310/MRL.1994.v1.n4.a5

[57] LYONS, T. J. (1998). Differential equations driven by rough signals. *Rev. Mat. Iberoam.* **14** 215–310. MR1654527 https://doi.org/10.4171/RMI/240

[58] MANDELBROT, B. B. and VAN NESS, J. W. (1968). Fractional Brownian motions, fractional noises and applications. *SIAM Rev.* **10** 422–437. MR0242239 https://doi.org/10.1137/1010093

[59] MATSUDA, T. and PERKOWSKI, N. (2024). An extension of the stochastic sewing lemma and applications to fractional stochastic calculus. *Forum Math. Sigma* **12** Paper No. e52. MR4730255 https://doi.org/10.1017/fms.2024.32

[60] NUALART, D. (2006). *The Malliavin Calculus and Related Topics*, 2nd ed. *Probability and Its Applications (New York)*. Springer, Berlin. MR2200233

[61] ROVERI, L. and TRIGGIANO, F. (2024). Well-posedness of rough 2D Euler equation with bounded vorticity.

[62] SHAPOSHNIKOV, A. and WRESCH, L. (2022). Pathwise vs. path-by-path uniqueness. *Ann. Inst. Henri Poincaré Probab. Stat.* **58** 1640–1649. MR4452646 https://doi.org/10.1214/21-aihp1187

[63] TRIEBEL, H. (1992). *Theory of Function Spaces. II*. *Monographs in Mathematics* **84**. Birkhäuser, Basel. MR1163193 https://doi.org/10.1007/978-3-0346-0419-2

[64] TZVETKOV, N. (2016). Randmon data wave equations. Available at http://php.math.unifi.it/users/cime/Courses/2016/02/201624-Notes.pdf.

[65] YOUNG, L. C. (1936). An inequality of the Hölder type, connected with Stieltjes integration. *Acta Math.* **67** 251–282. MR1555421 https://doi.org/10.1007/BF02401743

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GRAPHICAL MODELS FOR INFINITE MEASURES WITH APPLICATIONS TO EXTREMES

BY SEBASTIAN ENGELKE[1,a], JEVGENIJS IVANOVS[2,b] AND KIRSTIN STROKORB[3,c]

[1]*Research Institute for Statistics and Information Science, Geneva School of Economics and Management, University of Geneva,* [a]*sebastian.engelke@unige.ch*

[2]*Department of Mathematics, Aarhus University,* [b]*jevgenijs.ivanovs@math.au.dk*

[3]*School of Mathematics, Cardiff University,* [c]*strokorbk@cardiff.ac.uk*

Conditional independence and graphical models are well studied for probability distributions on product spaces. We propose a new notion of conditional independence for any measure Λ on the punctured Euclidean space $\mathbb{R}^d \setminus \{0\}$ that explodes at the origin. The importance of such measures stems from their connection to infinitely divisible and max-infinitely divisible distributions, where they appear as Lévy measures and exponent measures, respectively. We characterize independence and conditional independence for Λ in various ways through kernels and factorization of a modified density, including a Hammersley–Clifford type theorem for undirected graphical models. As opposed to the classical conditional independence, our notion is intimately connected to the support of the measure Λ. Our general theory unifies and extends recent approaches to graphical modeling in the fields of extreme value analysis and Lévy processes. Our results for the corresponding undirected and directed graphical models lay the foundation for new statistical methodology in these areas.

REFERENCES

AMÉNDOLA, C., KLÜPPELBERG, C., LAURITZEN, S. and TRAN, N. M. (2022). Conditional independence in max-linear Bayesian networks. *Ann. Appl. Probab.* **32** 1–45. MR4386833 https://doi.org/10.1214/21-AAP1670

ASENOVA, S., MAZO, G. and SEGERS, J. (2021). Inference on extremal dependence in the domain of attraction of a structured Hüsler-Reiss distribution motivated by a Markov tree with latent variables. *Extremes* **24** 461–500. MR4301085 https://doi.org/10.1007/s10687-021-00407-5

ASENOVA, S. and SEGERS, J. (2023). Extremes of Markov random fields on block graphs: Max-stable limits and structured Hüsler-Reiss distributions. *Extremes* **26** 433–468. MR4627351 https://doi.org/10.1007/s10687-023-00467-9

BALKEMA, A. A. and RESNICK, S. I. (1977). Max-infinite divisibility. *J. Appl. Probab.* **14** 309–319. MR0438425 https://doi.org/10.1017/s002190020010498x

BERG, C., CHRISTENSEN, J. P. R. and RESSEL, P. (1984). *Harmonic Analysis on Semigroups: Theory of Positive Definite and Related Functions*. *Graduate Texts in Mathematics* **100**. Springer, New York. MR0747302 https://doi.org/10.1007/978-1-4612-1128-0

BERTOIN, J. (1996). *Lévy Processes*. *Cambridge Tracts in Mathematics* **121**. Cambridge Univ. Press, Cambridge. MR1406564

CHIAPINO, M. and SABOURIN, A. (2017). Feature clustering for extreme events analysis, with application to extreme stream-flow data. In *New Frontiers in Mining Complex Patterns* (A. Appice, M. Ceci, C. Loglisci, E. Masciari and Z. W. Raś, eds.) 132–147. Springer, Cham. https://doi.org/10.1007/978-3-319-61461-8_9

CHIAPINO, M., SABOURIN, A. and SEGERS, J. (2019). Identifying groups of variables with the potential of being large simultaneously. *Extremes* **22** 193–222. MR3949044 https://doi.org/10.1007/s10687-018-0339-3

DAVYDOV, Y., MOLCHANOV, I. and ZUYEV, S. (2008). Strictly stable distributions on convex cones. *Electron. J. Probab.* **13** no. 11. MR2386734 https://doi.org/10.1214/EJP.v13-487

DAWID, A. P. (1979). Conditional independence in statistical theory. *J. Roy. Statist. Soc. Ser. B, Methodol.* **41** 1–31. MR0535541

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DE HAAN, L. (1984). A spectral representation for max-stable processes. *Ann. Probab.* **12** 1194–1204. MR0757776

DE HAAN, L. and FERREIRA, A. (2006). *Extreme Value Theory: An Introduction. Springer Series in Operations Research and Financial Engineering*. Springer, New York. MR2234156 https://doi.org/10.1007/0-387-34471-3

DE HAAN, L. and RESNICK, S. I. (1977). Limit theory for multivariate sample extremes. *Z. Wahrsch. Verw. Gebiete* **40** 317–337. MR0478290 https://doi.org/10.1007/BF00533086

DLMF. NIST Digital Library of Mathematical Functions (F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds.). Available at http://dlmf.nist.gov/, Release 1.1.7 of 2022-10-15.

DOMBRY, C. and RIBATET, M. (2015). Functional regular variations, Pareto processes and peaks over threshold. *Stat. Interface* **8** 9–17. MR3320385 https://doi.org/10.4310/SII.2015.v8.n1.a2

ENGELKE, S. and HITZ, A. S. (2020). Graphical models for extremes. *J. R. Stat. Soc. Ser. B. Stat. Methodol.* **82** 871–932. MR4136498 https://doi.org/10.1111/rssb.12355

ENGELKE, S. and IVANOVS, J. (2021). Sparse structures for multivariate extremes. *Annu. Rev. Stat. Appl.* **8** 241–270. MR4243547 https://doi.org/10.1146/annurev-statistics-040620-041554

ENGELKE, S., LALANCETTE, M. and VOLGUSHEV, S. (2021). Learning extremal graphical structures in high dimensions. https://doi.org/10.48550/arXiv.2111.00840

ENGELKE, S., IVANOVS, J. and THØSTESEN, J. D. (2024). Lévy graphical models. https://doi.org/10.48550/arXiv.2410.19952

ENGELKE, S., MALINOWSKI, A., KABLUCHKO, Z. and SCHLATHER, M. (2015). Estimation of Hüsler-Reiss distributions and Brown-Resnick processes. *J. R. Stat. Soc. Ser. B. Stat. Methodol.* **77** 239–265. MR3299407 https://doi.org/10.1111/rssb.12074

ENGELKE, S., OPITZ, T. and WADSWORTH, J. (2019). Extremal dependence of random scale constructions. *Extremes* **22** 623–666. MR4031852 https://doi.org/10.1007/s10687-019-00353-3

ENGELKE, S. and VOLGUSHEV, S. (2022). Structure learning for extremal tree models. *J. R. Stat. Soc. Ser. B. Stat. Methodol.* **84** 2055–2087. MR4515566 https://doi.org/10.1111/rssb.12556

FELLER, W. (1971). *An Introduction to Probability Theory and Its Applications. Vol. II*, 2nd ed. Wiley, New York. MR0270403

GISSIBL, N. and KLÜPPELBERG, C. (2018). Max-linear models on directed acyclic graphs. *Bernoulli* **24** 2693–2720. MR3779699 https://doi.org/10.3150/17-BEJ941

GISSIBL, N., KLÜPPELBERG, C. and LAURITZEN, S. (2021). Identifiability and estimation of recursive max-linear models. *Scand. J. Stat.* **48** 188–211. MR4233170 https://doi.org/10.1111/sjos.12446

GISSIBL, N., KLÜPPELBERG, C. and OTTO, M. (2018). Tail dependence of recursive max-linear models with regularly varying noise variables. *Econom. Stat.* **6** 149–167. MR3797980 https://doi.org/10.1016/j.ecosta.2018.02.003

HEFFERNAN, J. E. and TAWN, J. A. (2004). A conditional approach for multivariate extreme values. *J. R. Stat. Soc. Ser. B. Stat. Methodol.* **66** 497–546. MR2088289 https://doi.org/10.1111/j.1467-9868.2004.02050.x

HENTSCHEL, M., ENGELKE, S. and SEGERS, J. (2025). Statistical inference for Hüsler-Reiss graphical models through matrix completions. *J. Amer. Statist. Assoc.* **120** 909–921. MR4917266 https://doi.org/10.1080/01621459.2024.2371978

HULT, H. and LINDSKOG, F. (2007). Extremal behavior of stochastic integrals driven by regularly varying Lévy processes. *Ann. Probab.* **35** 309–339. MR2303951 https://doi.org/10.1214/009117906000000548

HUSER, R., OPITZ, T. and THIBAUD, E. (2021). Max-infinitely divisible models and inference for spatial extremes. *Scand. J. Stat.* **48** 321–348. MR4233175 https://doi.org/10.1111/sjos.12491

HÜSLER, J. and REISS, R.-D. (1989). Maxima of normal random vectors: Between independence and complete dependence. *Statist. Probab. Lett.* **7** 283–286. MR0980699 https://doi.org/10.1016/0167-7152(89)90106-5

IVANOVS, J. (2020). Discussion of ‘Graphical models for extremes’ by Engelke and Hitz.

KABLUCHKO, Z. (2009). Spectral representations of sum- and max-stable processes. *Extremes* **12** 401–424. MR2562988 https://doi.org/10.1007/s10687-009-0083-9

KABLUCHKO, Z., SCHLATHER, M. and DE HAAN, L. (2009). Stationary max-stable fields associated to negative definite functions. *Ann. Probab.* **37** 2042–2065. MR2561440 https://doi.org/10.1214/09-AOP455

KALLENBERG, O. (2002). *Foundations of Modern Probability*, 2nd ed. *Probability and Its Applications* (*New York*). Springer, New York. MR1876169 https://doi.org/10.1007/978-1-4757-4015-8

KLÜPPELBERG, C. and LAURITZEN, S. (2019). Bayesian networks for max-linear models. In *Network Science* 79–97. Springer, Cham. MR3971300 https://doi.org/10.1007/978-3-030-26814-5_6

LAURITZEN, S. L. (1996). *Graphical Models. Oxford Statistical Science Series* **17**. The Clarendon Press, New York. MR1419991

LAURITZEN, S. L., DAWID, A. P., LARSEN, B. N. and LEIMER, H. G. (1990). Independence properties of directed Markov fields. In *Networks*, **20** 491–505. Special issue on influence diagrams. MRMR1064735 https://doi.org/10.1002/net.3230200503

LEDFORD, A. W. and TAWN, J. A. (1996). Statistics for near independence in multivariate extreme values. *Biometrika* **83** 169–187. MR1399163 https://doi.org/10.1093/biomet/83.1.169

LEDFORD, A. W. and TAWN, J. A. (1997). Modelling dependence within joint tail regions. *J. Roy. Statist. Soc. Ser. B, Methodol.* **59** 475–499. MR1440592 https://doi.org/10.1111/1467-9868.00080

MAATHUIS, M., DRTON, M., LAURITZEN, S. and WAINWRIGHT, M., eds. (2019). *Handbook of Graphical Models. Chapman & Hall/CRC Handbooks of Modern Statistical Methods*. CRC Press, Boca Raton, FL. MR3889064

MEYER, N. and WINTENBERGER, O. (2021). Sparse regular variation. *Adv. Appl. Probab.* **53** 1115–1148. MR4342579 https://doi.org/10.1017/apr.2021.14

MISRA, N. and KURUOGLU, E. E. (2016). Stable graphical models. *J. Mach. Learn. Res.* **17** Paper No. 168. MR3567436

MOLCHANOV, I. (2009). Convex and star-shaped sets associated with multivariate stable distributions. I. Moments and densities. *J. Multivariate Anal.* **100** 2195–2213. MR2560363 https://doi.org/10.1016/j.jmva.2009.04.003

PAPASTATHOPOULOS, I. and STROKORB, K. (2016). Conditional independence among max-stable laws. *Statist. Probab. Lett.* **108** 9–15. MR3420420 https://doi.org/10.1016/j.spl.2015.08.008

PEARL, J. (1988). *Probabilistic Reasoning in Intelligent Systems: Networks of Plausible Inference. The Morgan Kaufmann Series in Representation and Reasoning*. Morgan Kaufmann, San Mateo, CA. MR0965765

PEARL, J. (2009). *Causality: Models, Reasoning, and Inference*, 2nd ed. Cambridge Univ. Press, Cambridge. MR2548166 https://doi.org/10.1017/CBO9780511803161

PEARL, J. and PAZ, A. (2022). GRAPHOIDS: Graph-based logic for reasoning about relevance relations – or – when would x tell you more about y if you already know z? In *Probabilistic and Causal Inference. The Works of Judea Pearl* 189–200. Association for Computing Machinery (ACM), New York, NY. https://doi.org/10.1145/3501714

RESNICK, S. (2002). Hidden regular variation, second order regular variation and asymptotic independence. *Extremes* **5** 303–336. MR2002121 https://doi.org/10.1023/A:1025148622954

RESNICK, S. I. (2008). *Extreme Values, Regular Variation and Point Processes. Springer Series in Operations Research and Financial Engineering*. Springer, New York. MR2364939 https://doi.org/10.1007/978-0-387-75953-1

ROOTZÉN, H. and TAJVIDI, N. (2006). Multivariate generalized Pareto distributions. *Bernoulli* **12** 917–930. MR2265668 https://doi.org/10.3150/bj/1161614952

RÖTTGER, F., ENGELKE, S. and ZWIERNIK, P. (2023). Total positivity in multivariate extremes. *Ann. Statist.* **51** 962–1004. MR4630937 https://doi.org/10.1214/23-aos2272

SATO, K. (2013). *Lévy Processes and Infinitely Divisible Distributions. Cambridge Studies in Advanced Mathematics* **68**. Cambridge Univ. Press, Cambridge. MR3185174

SEGERS, J. (2020). One- versus multi-component regular variation and extremes of Markov trees. *Adv. Appl. Probab.* **52** 855–878. MR4153591 https://doi.org/10.1017/apr.2020.22

SIMPSON, E. S., WADSWORTH, J. L. and TAWN, J. A. (2020). Determining the dependence structure of multivariate extremes. *Biometrika* **107** 513–532. MR4138974 https://doi.org/10.1093/biomet/asaa018

SKOROHOD, A. V. (1957). Limit theorems for stochastic processes with independent increments. *Teor. Veroyatn. Primen.* **2** 145–177. MR0094842

STROKORB, K. (2020). Extremal independence old and new. Discussion of 'Graphical models for extremes' by Engelke and Hitz. https://doi.org/10.48550/arXiv.2002.07808

SZÉKELY, L. A. and WANG, H. (2005). On subtrees of trees. *Adv. in Appl. Math.* **34** 138–155. MR2102279 https://doi.org/10.1016/j.aam.2004.07.002

WADSWORTH, J. L., TAWN, J. A., DAVISON, A. C. and ELTON, D. M. (2017). Modelling across extremal dependence classes. *J. R. Stat. Soc. Ser. B. Stat. Methodol.* **79** 149–175. MR3597968 https://doi.org/10.1111/rssb.12157

WANG, Y. and STOEV, S. A. (2010). On the association of sum- and max-stable processes. *Statist. Probab. Lett.* **80** 480–488. MR2593589 https://doi.org/10.1016/j.spl.2009.12.001

YUEN, R. and STOEV, S. (2014). CRPS M-estimation for max-stable models. *Extremes* **17** 387–410. MR3252818 https://doi.org/10.1007/s10687-014-0185-x

ZHANG, Y. and WANG, L. (2020). Conditional independence beyond domain separability: Discussion of Engelke and Hitz (2020). https://doi.org/10.48550/arXiv.2006.01036

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FLUCTUATIONS OF THE 2-SPIN SSK MODEL WITH MAGNETIC FIELD

BY BENJAMIN LANDON[1,a] AND PHILIPPE SOSOE[2,b]

[1]*Department of Mathematics, University of Toronto,* [a]*blandon@math.toronto.edu*
[2]*Department of Mathematics, Cornell University,* [b]*ps934@cornell.edu*

We analyze the fluctuations of the free energy, replica overlaps and overlap with the external field in the quadratic spherical SK model with a magnetic field. We identify several different behaviors for these quantities depending on the size of the magnetic field, confirming predictions by Fyodorov-Le Doussal and recent work of Baik, Collins–Woodfin, Le Doussal and Wu.

REFERENCES

[1] AIZENMAN, M., LEBOWITZ, J. L. and RUELLE, D. (1987). Some rigorous results on the Sherrington-Kirkpatrick spin glass model. *Comm. Math. Phys.* **112** 3–20. MR0904135

[2] AUFFINGER, A., BEN AROUS, G. and ČERNÝ, J. (2013). Random matrices and complexity of spin glasses. *Comm. Pure Appl. Math.* **66** 165–201. MR2999295 https://doi.org/10.1002/cpa.21422

[3] BAIK, J., COLLINS-WOODFIN, E., LE DOUSSAL, P. and WU, H. (2021). Spherical spin glass model with external field. *J. Stat. Phys.* **183** Paper No. 31, 79. MR4257832 https://doi.org/10.1007/s10955-021-02757-7

[4] BAIK, J. and LEE, J. O. (2016). Fluctuations of the free energy of the spherical Sherrington-Kirkpatrick model. *J. Stat. Phys.* **165** 185–224. MR3554380 https://doi.org/10.1007/s10955-016-1610-0

[5] BAIK, J. and LEE, J. O. (2017). Fluctuations of the free energy of the spherical Sherrington-Kirkpatrick model with ferromagnetic interaction. *Ann. Henri Poincaré* **18** 1867–1917. MR3649446 https://doi.org/10.1007/s00023-017-0562-5

[6] BAIK, J. and LEE, J. O. (2020). Free energy of bipartite spherical Sherrington–Kirkpatrick model. *Ann. Inst. Henri Poincaré Probab. Stat.* **56** 2897–2934. MR4164860 https://doi.org/10.1214/20-AIHP1062

[7] BAIK, J., LEE, J. O. and WU, H. (2018). Ferromagnetic to paramagnetic transition in spherical spin glass. *J. Stat. Phys.* **173** 1484–1522. MR3878351 https://doi.org/10.1007/s10955-018-2150-6

[8] BENAYCH-GEORGES, F. and KNOWLES, A. (2016). Lectures on the local semicircle law for Wigner matrices. *Panor. Synthèses* **53**.

[9] BLOEMENDAL, A., ERDŐS, L., KNOWLES, A., YAU, H.-T. and YIN, J. (2014). Isotropic local laws for sample covariance and generalized Wigner matrices. *Electron. J. Probab.* **19** no. 33, 53. MR3183577 https://doi.org/10.1214/ejp.v19-3054

[10] CHEN, W.-K., DEY, P. and PANCHENKO, D. (2017). Fluctuations of the free energy in the mixed p-spin models with external field. *Probab. Theory Related Fields* **168** 41–53. MR3651048 https://doi.org/10.1007/s00440-016-0705-5

[11] CHEN, W.-K. and SEN, A. (2017). Parisi formula, disorder chaos and fluctuation for the ground state energy in the spherical mixed p-spin models. *Comm. Math. Phys.* **350** 129–173. MR3606472 https://doi.org/10.1007/s00220-016-2808-3

[12] CIPOLLONI, G., ERDŐS, L. and SCHRÖDER, D. (2023). Functional central limit theorems for Wigner matrices. *Ann. Appl. Probab.* **33** 447–489. MR4551555 https://doi.org/10.1214/22-aap1820

[13] COLLINS-WOODFIN, E. (2021). Overlaps of a spherical spin glass model with microscopic external field. *Electron. J. Probab.* **26** Paper No. 146, 22. MR4346677 https://doi.org/10.1214/21-ejp722

[14] CUGLIANDOLO, L. F., DEAN, D. S. and YOSHINO, H. (2007). Nonlinear susceptibilities of spherical models. *J. Phys. A: Math. Theor.* **40** 4285–4303. MR2316706 https://doi.org/10.1088/1751-8113/40/16/003

[15] DEMBO, A. and ZEITOUNI, O. (2015). Matrix optimization under random external fields. *J. Stat. Phys.* **159** 1306–1326. MR3350373 https://doi.org/10.1007/s10955-015-1228-7

[16] DEY, P. S. and WU, Q. (2023). Mean field spin glass models under weak external field. *Comm. Math. Phys.* **402** 1205–1258. MR4627320 https://doi.org/10.1007/s00220-023-04742-5

MSC2020 subject classifications. 60B10.
Key words and phrases. Spherical Sherrington–Kirkpatrick model.

[17] FYODOROV, Y. V. and LE DOUSSAL, P. (2014). Topology trivialization and large deviations for the minimum in the simplest random optimization. *J. Stat. Phys.* **154** 466–490. MR3162549 https://doi.org/10.1007/s10955-013-0838-1

[18] KIVIMAE, P. (2021). Critical fluctuations for the spherical Sherrington–Kirkpatrick model in an external field. arXiv preprint. Available at arXiv:1908.07512.

[19] KOSTERLITZ, J., THOULESS, D. and JONES, R. C. (1976). Spherical model of a spin-glass. *Phys. Rev. Lett.* **36** 1217.

[20] LANDON, B. and SOSOE, P. (2020). Applications of mesoscopic CLTs in random matrix theory. *Ann. Appl. Probab.* **30** 2769–2795. MR4187127 https://doi.org/10.1214/20-AAP1572

[21] LANDON, B. and SOSOE, P. (2022). Fluctuations of the overlap at low temperature in the 2-spin spherical SK model. *Ann. Inst. Henri Poincaré Probab. Stat.* **58** 1426–1459. MR4452639 https://doi.org/10.1214/21-aihp1205

[22] NGUYEN, V. L. and SOSOE, P. (2019). Central limit theorem near the critical temperature for the overlap in the 2-spin spherical SK model. *J. Math. Phys.* **60** 103302, 13. MR4016884 https://doi.org/10.1063/1.5065525

[23] PANCHENKO, D. and TALAGRAND, M. (2007). On the overlap in the multiple spherical SK models. *Ann. Probab.* **35** 2321–2355. MR2353390 https://doi.org/10.1214/009117907000000015

[24] RAMÍREZ, J. A., RIDER, B. and VIRÁG, B. (2011). Beta ensembles, stochastic Airy spectrum, and a diffusion. *J. Amer. Math. Soc.* **24** 919–944. MR2813333 https://doi.org/10.1090/S0894-0347-2011-00703-0

[25] WATSON, G. N. (1995). *A Treatise on the Theory of Bessel Functions*. *Cambridge Mathematical Library*. Cambridge Univ. Press, Cambridge. Reprint of the second (1944) edition. MR1349110

[26] WU, H. (2019). *New Applications of Random Matrices in Spin Glass and Machine Learning*. ProQuest LLC, Ann Arbor, MI. Thesis (Ph.D.)–University of Michigan. MR4051545

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THE HORTON–STRAHLER NUMBER OF GALTON–WATSON TREES WITH POSSIBLY INFINITE VARIANCE

BY ROBIN KHANFIR[a]

Department of Mathematics and Statistics, McGill University, [a]*robin.khanfir@mcill.ca*

The Horton–Strahler number, also known as the register function, provides a tool for quantifying the branching complexity of a rooted tree. We consider the Horton–Strahler number of critical Galton–Watson trees conditioned to have size n and whose offspring distribution is in the domain of attraction of an α-stable law with $\alpha \in [1, 2]$. We give tail estimates and when $\alpha \neq 1$, we prove that it grows as $\frac{1}{\alpha} \log_{\alpha/(\alpha-1)} n$ in probability. This extends the result of Brandenberger, Devroye and Reddad dealing with the finite variance case for which $\alpha = 2$. We also characterize the cases where $\alpha = 1$, namely the spectrally positive Cauchy regime, which exhibits more complex behaviors. Our proofs are new and probabilistic; they relate the Horton–Strahler number with other shape parameters such as the height or largest degree.

REFERENCES

[1] ABRAHAM, R. and DELMAS, J.-F. (2014). Local limits of conditioned Galton–Watson trees: The infinite spine case. *Electron. J. Probab.* **19** 19. MR3164755 https://doi.org/10.1214/ejp.v19-2747

[2] ABRAHAM, R. and DELMAS, J.-F. (2015). An introduction to Galton-Watson trees and their local limits. Preprint. Available at arXiv:1506.05571.

[3] ADDARIO-BERRY, L., BRANDENBERGER, A., HAMDAN, J. and KERRIOU, C. (2022). Universal height and width bounds for random trees. *Electron. J. Probab.* **27** 118. MR4479914 https://doi.org/10.1214/22-ejp842

[4] ADDARIO-BERRY, L. and DONDERWINKEL, S. (2024). Random trees have height $O(\sqrt{n})$. *Ann. Probab.* **52** 2238–2280. MR4815972 https://doi.org/10.1214/24-aop1694

[5] ALDOUS, D. and PITMAN, J. (1998). Tree-valued Markov chains derived from Galton–Watson processes. *Ann. Inst. Henri Poincaré Probab. Stat.* **34** 637–686. MR1641670 https://doi.org/10.1016/S0246-0203(98)80003-4

[6] BAMUFLEH, S., AL-WAGDANY, A., ELFEKI, A. and CHAABANI, A. (2020). Developing a geomorphological instantaneous unit hydrograph (GIUH) using equivalent Horton–Strahler ratios for flash flood predictions in arid regions. *Geomat. Nat. Hazards Risk* **11** 1697–1723.

[7] BERGER, Q. (2019). Notes on random walks in the Cauchy domain of attraction. *Probab. Theory Related Fields* **175** 1–44. MR4009704 https://doi.org/10.1007/s00440-018-0887-0

[8] BINGHAM, N. H., GOLDIE, C. M. and TEUGELS, J. L. (1987). *Regular Variation. Encyclopedia of Mathematics and Its Applications* **27**. Cambridge Univ. Press, Cambridge. MR0898871 https://doi.org/10.1017/CBO9780511721434

[9] BRANDENBERGER, A., DEVROYE, L. and REDDAD, T. (2021). The Horton-Strahler number of conditioned Galton–Watson trees. *Electron. J. Probab.* **26** 109. MR4290511 https://doi.org/10.1214/21-EJP678

[10] BURD, G. A., WAYMIRE, E. C. and WINN, R. D. (2000). A self-similar invariance of critical binary Galton–Watson trees. *Bernoulli* **6** 1–21. MR1781179 https://doi.org/10.2307/3318630

[11] CHAVAN, S. R. and SRINIVAS, V. V. (2015). Effect of DEM source on equivalent Horton–Strahler ratio based GIUH for catchments in two Indian river basins. *J. Hydrol.* **528** 463–489.

[12] DEVROYE, L. and KRUSZEWSKI, P. (1995). A note on the Horton–Strahler number for random trees. *Inform. Process. Lett.* **56** 95–99. MR1359172 https://doi.org/10.1016/0020-0190(95)00114-R

[13] DONDERWINKEL, S. and KHANFIR, R. (2024). Tight universal bounds on the height times the width of random trees. Preprint. Available at arXiv:2501.00458.

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[14] DRMOTA, M. and PRODINGER, H. (2006). The register function for t-ary trees. *ACM Trans. Algorithms* **2** 318–334. MR2253784 https://doi.org/10.1145/1159892.1159894

[15] DUQUESNE, T. (2003). A limit theorem for the contour process of conditioned Galton–Watson trees. *Ann. Probab.* **31** 996–1027. MR1964956 https://doi.org/10.1214/aop/1048516543

[16] DUQUESNE, T. (2009). An elementary proof of Hawkes's conjecture on Galton–Watson trees. *Electron. Commun. Probab.* **14** 151–164. MR2497323 https://doi.org/10.1214/ECP.v14-1454

[17] DUQUESNE, T. and LE GALL, J.-F. (2002). Random trees, Lévy processes and spatial branching processes. *Astérisque* **281** vi+147. MR1954248

[18] DUQUESNE, T. and WINKEL, M. (2019). Hereditary tree growth and Lévy forests. *Stochastic Process. Appl.* **129** 3690–3747. MR3997659 https://doi.org/10.1016/j.spa.2018.10.007

[19] ESPARZA, J., LUTTENBERGER, M. and SCHLUND, M. (2016). History of strahler numbers — with a preface. International Conference on Language and Automata Theory and Applications, 2014, Available online at https://archive.model.in.tum.de/um/bibdb/esparza/latarevised16.pdf.

[20] FAC-BENEDA, J. (2013). Fractal structure of the Kashubian hydrographic system. *J. Hydrol.* **488** 48–54.

[21] FELLER, W. (1971). *An Introduction to Probability Theory and Its Applications. Vol. II*, 2nd ed. Wiley, New York. MR0270403

[22] FLAJOLET, P., RAOULT, J.-C. and VUILLEMIN, J. (1979). The number of registers required for evaluating arithmetic expressions. *Theoret. Comput. Sci.* **9** 99–125. MR0535127 https://doi.org/10.1016/0304-3975(79)90009-4

[23] GRIMVALL, A. (1974). On the convergence of sequences of branching processes. *Ann. Probab.* **2** 1027–1045. MR0362529 https://doi.org/10.1214/aop/1176996496

[24] HORTON, R. E. (1945). Erosional development of streams and their drainage basins; hydrophysical approach to quantitative morphology. *Geol. Soc. Amer. Bull.* **56** 275–370.

[25] KARAMATA, J. (1930). Sur un mode de croissance régulière des fonctions. *Mathematica* **4** 38–53.

[26] KEMP, R. (1978/79). The average number of registers needed to evaluate a binary tree optimally. *Acta Inform.* **11** 363–372. MR0533482 https://doi.org/10.1007/BF00289094

[27] KEMPERMAN, J. H. B. (1961). *The Passage Problem for a Stationary Markov Chain. Statistical Research Monographs, Vol. I.* Univ. Chicago Press, Chicago, IL. MR0119226

[28] KESTEN, H. (1986). Subdiffusive behavior of random walk on a random cluster. *Ann. Inst. Henri Poincaré Probab. Stat.* **22** 425–487. MR0871905

[29] KHANFIR, R. (2024). Fluctuations of the Horton–Strahler number of stable Galton–Watson trees. Preprint. Available at arXiv:2401.13771.

[30] KORTCHEMSKI, I. (2013). A simple proof of Duquesne's theorem on contour processes of conditioned Galton–Watson trees. In *Séminaire de Probabilités XLV. Lecture Notes in Math.* **2078** 537–558. Springer, Cham. MR3185928 https://doi.org/10.1007/978-3-319-00321-4_20

[31] KORTCHEMSKI, I. and RICHIER, L. (2019). Condensation in critical Cauchy Bienaymé–Galton–Watson trees. *Ann. Appl. Probab.* **29** 1837–1877. MR3914558 https://doi.org/10.1214/18-AAP1447

[32] KOVCHEGOV, Y. and ZALIAPIN, I. (2020). Random self-similar trees: A mathematical theory of Horton laws. *Probab. Surv.* **17** 1–213. MR4067815 https://doi.org/10.1214/19-PS331

[33] KOVCHEGOV, Y. and ZALIAPIN, I. (2021). Invariance and attraction properties of Galton–Watson trees. *Bernoulli* **27** 1789–1823. MR4260509 https://doi.org/10.3150/20-bej1292

[34] LE GALL, J.-F. (2005). Random trees and applications. *Probab. Surv.* **2** 245–311. MR2203728 https://doi.org/10.1214/154957805100000140

[35] LE GALL, J.-F. and LE JAN, Y. (1998). Branching processes in Lévy processes: The exploration process. *Ann. Probab.* **26** 213–252. MR1617047 https://doi.org/10.1214/aop/1022855417

[36] LYONS, R., PEMANTLE, R. and PERES, Y. (1995). Conceptual proofs of $L \log L$ criteria for mean behavior of branching processes. *Ann. Probab.* **23** 1125–1138. MR1349164

[37] MARCHAL, P. (2008). A note on the fragmentation of a stable tree. *Discrete Math. Theor. Comput. Sci.* DMTCS Proceedings vol. AI, Fifth Colloquium on Mathematics and Computer Science.

[38] NEVEU, J. (1986). Arbres et processus de Galton–Watson. *Ann. Inst. Henri Poincaré Probab. Stat.* **22** 199–207. MR0850756

[39] PECKHAM, S. D. (1995). New results for self-similar trees with applications to river networks. *Water Resour. Res.* **31** 1023–1029.

[40] SKOROHOD, A. V. (1957). Limit theorems for stochastic processes with independent increments. *Teor. Veroyatn. Primen.* **2** 145–177. MR0094842

[41] SLACK, R. S. (1968). A branching process with mean one and possibly infinite variance. *Z. Wahrsch. Verw. Gebiete* **9** 139–145. MR0228077 https://doi.org/10.1007/BF01851004

[42] STRAHLER, A. N. (1952). Hypsometric (area-altitude) analysis of erosional topography. *Geol. Soc. Amer. Bull.* **63** 1117–1142.

[43] VERVAAT, W. (1979). A relation between Brownian bridge and Brownian excursion. *Ann*. *Probab*. **7** 143–149. MR0515820

[44] VIENNOT, X. (1990). Trees. In *Mots*, *Mélanges Offert à M.P. Schützenberger* Hermès, Paris. Available online at http://www.xavierviennot.org/xavier/.

[45] WENDEL, J. G. (1975). Left-continuous random walk and the Lagrange expansion. *Amer. Math*. *Monthly* **82** 494–499. MR0381000 https://doi.org/10.2307/2319745

[46] ZOLOTAREV, V. M. (1957). More exact statements of several theorems in the theory of branching processes. *Teor. Veroyatn*. *Primen*. **2** 256–266. MR0096321

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FINITE DIMENSIONAL PROJECTIONS OF HJB EQUATIONS IN THE WASSERSTEIN SPACE

BY ANDRZEJ ŚWIĘCH[a] AND LUKAS WESSELS[b]

School of Mathematics, Georgia Institute of Technology, [a]*swiech@math.gatech.edu,* [b]*wessels@gatech.edu*

This paper continues the study of controlled interacting particle systems with common noise started in (*SIAM J. Math. Anal.* **53** (2021) 1320–1356) and (*SIAM J. Control Optim.* **61** (2023) 820–851). First, we extend the following results of the previously mentioned works to the case of multiplicative noise: (i) We generalize the convergence of the value functions u_n corresponding to control problems of n particles to the value function V corresponding to an appropriately defined infinite dimensional control problem; (ii) we prove, under certain additional assumptions, $C^{1,1}$ regularity of V in the spatial variable. The second main contribution of the present work is the proof that if DV is continuous (which, in particular, includes the previously proven case of $C^{1,1}$ regularity in the spatial variable), the value function V projects precisely onto the value functions u_n. Using this projection property, we show that optimal controls of the finite dimensional problem correspond to optimal controls of the infinite dimensional problem and vice versa. In the case of a linear state equation, we are able to prove that V projects precisely onto the value functions u_n under relaxed assumptions on the coefficients of the cost functional by using approximation techniques in the Wasserstein space, thus covering cases where V may not be differentiable.

REFERENCES

[1] ALIPRANTIS, C. D. and BORDER, K. C. (2006). *Infinite Dimensional Analysis*: *A Hitchhiker's Guide*, 3rd ed. Springer, Berlin. MR2378491

[2] AMBROSIO, L. and FENG, J. (2014). On a class of first order Hamilton–Jacobi equations in metric spaces. *J*. *Differ*. *Equ*. **256** 2194–2245. MR3160441 https://doi.org/10.1016/j.jde.2013.12.018

[3] AMBROSIO, L., GIGLI, N. and SAVARÉ, G. (2005). *Gradient Flows in Metric Spaces and in the Space of Probability Measures*. *Lectures in Mathematics ETH Zürich*. Birkhäuser, Basel. MR2129498

[4] BADREDDINE, Z. and FRANKOWSKA, H. (2022). Solutions to Hamilton–Jacobi equation on a Wasserstein space. *Calc*. *Var*. *Partial Differential Equations* **61** Paper No. 9, 41. MR4342999 https://doi.org/10.1007/s00526-021-02113-3

[5] BANDINI, E., COSSO, A., FUHRMAN, M. and PHAM, H. (2019). Randomized filtering and Bellman equation in Wasserstein space for partial observation control problem. *Stochastic Process*. *Appl*. **129** 674–711. MR3907014 https://doi.org/10.1016/j.spa.2018.03.014

[6] BAYRAKTAR, E., COSSO, A. and PHAM, H. (2018). Randomized dynamic programming principle and Feynman–Kac representation for optimal control of McKean–Vlasov dynamics. *Trans*. *Amer*. *Math*. *Soc*. **370** 2115–2160. MR3739204 https://doi.org/10.1090/tran/7118

[7] BAYRAKTAR, E., EKREN, I. and ZHANG, X. (2025). Comparison of viscosity solutions for a class of second-order PDEs on the Wasserstein space. *Comm*. *Partial Differential Equations* **50** 570–613. MR4885722 https://doi.org/10.1080/03605302.2025.2459855

[8] BELLMAN, R. (1957). *Dynamic Programming*. Princeton Univ. Press, Princeton, NJ. MR0090477

[9] BENSOUSSAN, A., GRABER, P. J. and YAM, S. C. P. (2024). Control on Hilbert spaces and application to some mean field type control problems. *Ann*. *Appl*. *Probab*. **34** 4085–4136. MR4783038 https://doi.org/10.1214/24-aap2060

[10] BENSOUSSAN, A. and YAM, S. C. P. (2019). Control problem on space of random variables and master equation. *ESAIM Control Optim*. *Calc*. *Var*. **25** Paper No. 10, 36. MR3943358 https://doi.org/10.1051/cocv/2018034

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[11] BERTUCCI, C. (2024). Stochastic optimal transport and Hamilton–Jacobi–Bellman equations on the set of probability measures. *Ann. Inst. H. Poincaré Anal. Non Linéaire*. Published online first. https://doi.org/10.4171/aihpc/138

[12] BUDHIRAJA, A., DUPUIS, P. and FISCHER, M. (2012). Large deviation properties of weakly interacting processes via weak convergence methods. *Ann. Probab.* **40** 74–102. MR2917767 https://doi.org/10.1214/10-AOP616

[13] BURZONI, M., IGNAZIO, V., REPPEN, A. M. and SONER, H. M. (2020). Viscosity solutions for controlled McKean–Vlasov jump-diffusions. *SIAM J. Control Optim.* **58** 1676–1699. MR4114830 https://doi.org/10.1137/19M1290061

[14] CARDALIAGUET, P. (2013). Notes on Mean-Field Games (from P-L. Lions' Lectures at Collège de France). Available at https://www.ceremade.dauphine.fr/~cardaliaguet/MFG20130420.pdf.

[15] CARDALIAGUET, P., DAUDIN, S., JACKSON, J. and SOUGANIDIS, P. E. (2023). An algebraic convergence rate for the optimal control of McKean–Vlasov dynamics. *SIAM J. Control Optim.* **61** 3341–3369. MR4665660 https://doi.org/10.1137/22M1486789

[16] CARDALIAGUET, P., DELARUE, F., LASRY, J.-M. and LIONS, P.-L. (2019). *The Master Equation and the Convergence Problem in Mean Field Games*. *Annals of Mathematics Studies* **201**. Princeton Univ. Press, Princeton, NJ. MR3967062 https://doi.org/10.2307/j.ctvckq7qf

[17] CARDALIAGUET, P., JACKSON, J., MIMIKOS-STAMATOPOULOS, N. and SOUGANIDIS, P. E. (2023). Sharp convergence rates for mean field control in the region of strong regularity. Preprint. Available at arXiv:2312.11373.

[18] CARDALIAGUET, P. and SOUGANIDIS, P. E. (2023). Regularity of the value function and quantitative propagation of chaos for mean field control problems. *NoDEA Nonlinear Differential Equations Appl.* **30** Paper No. 25, 37. MR4530369 https://doi.org/10.1007/s00030-022-00823-x

[19] CARMONA, R. and DELARUE, F. (2015). Forward–backward stochastic differential equations and controlled McKean–Vlasov dynamics. *Ann. Probab.* **43** 2647–2700. MR3395471 https://doi.org/10.1214/14-AOP946

[20] CARMONA, R. and DELARUE, F. (2018). *Probabilistic Theory of Mean Field Games with Applications. I: Mean Field FBSDEs, Control, and Games*. *Probability Theory and Stochastic Modelling* **83**. Springer, Cham. MR3752669

[21] CARMONA, R. and DELARUE, F. (2018). *Probabilistic Theory of Mean Field Games with Applications. II: Mean Field Games with Common Noise and Master Equations*. *Probability Theory and Stochastic Modelling* **84**. Springer, Cham. MR3753660

[22] CECCHIN, A., DAUDIN, S., JACKSON, J. and MARTINI, M. (2024). Quantitative convergence for mean field control with common noise and degenerate idiosyncratic noise. Preprint. Available at arXiv:2409.14053.

[23] CONFORTI, G., KRAAIJ, R. C. TAMANINI, L. and TONON, D. (2024). Hamilton–Jacobi equations for Wasserstein controlled gradient flows: existence of viscosity solutions. Preprint. Available at arXiv:2401.02240.

[24] CONFORTI, G., KRAAIJ, R. C. and TONON, D. (2024). Hamilton–Jacobi equations for controlled gradient flows: cylindrical test functions. Preprint. Available at arXiv:2302.06571.

[25] COSSO, A., GOZZI, F., KHARROUBI, I., PHAM, H. and ROSESTOLATO, M. (2023). Optimal control of path-dependent McKean–Vlasov SDEs in infinite-dimension. *Ann. Appl. Probab.* **33** 2863–2918. MR4612657 https://doi.org/10.1214/22-aap1880

[26] COSSO, A., GOZZI, F., KHARROUBI, I., PHAM, H. and ROSESTOLATO, M. (2024). Master Bellman equation in the Wasserstein space: Uniqueness of viscosity solutions. *Trans. Amer. Math. Soc.* **377** 31–83. MR4684588 https://doi.org/10.1090/tran/8986

[27] COSSO, A. and MARTINI, M. (2023). On smooth approximations in the Wasserstein space. *Electron. Commun. Probab.* **28** Paper No. 30, 11. MR4627412 https://doi.org/10.1214/23-ecp538

[28] COSSO, A. and PHAM, H. (2019). Zero-sum stochastic differential games of generalized McKean–Vlasov type. *J. Math. Pures Appl.* (9) **129** 180–212. MR3998794 https://doi.org/10.1016/j.matpur.2018.12.005

[29] CRANDALL, M. G. and LIONS, P.-L. (1985). Hamilton–Jacobi equations in infinite dimensions. I. Uniqueness of viscosity solutions. *J. Funct. Anal.* **62** 379–396. MR0794776 https://doi.org/10.1016/0022-1236(85)90011-4

[30] DAUDIN, S., DELARUE, F. and JACKSON, J. (2024). On the optimal rate for the convergence problem in mean field control. *J. Funct. Anal.* **287** Paper No. 110660, 94. MR4794919 https://doi.org/10.1016/j.jfa.2024.110660

[31] DAUDIN, S., JACKSON, J. and SEEGER, B. (2025). Well-posedness of Hamilton–Jacobi equations in the Wasserstein space: Non-convex Hamiltonians and common noise. *Comm. Partial Differential Equations* **50** 1–52. MR4858218 https://doi.org/10.1080/03605302.2024.2432678

[32] DAUDIN, S. and SEEGER, B. (2024). A comparison principle for semilinear Hamilton–Jacobi–Bellman equations in the Wasserstein space. *Calc. Var. Partial Differential Equations* **63** Paper No. 106, 36. MR4732306 https://doi.org/10.1007/s00526-024-02718-4

[33] DE FEO, F., ŚWIĘCH, A. and WESSELS, L. (2025). Stochastic optimal control in Hilbert spaces: $C^{1,1}$ regularity of the value function and optimal synthesis via viscosity solutions. *Electron. J. Probab.* **30** Paper No. 36, 39. MR4870298 https://doi.org/10.1214/25-ejp1294

[34] DJETE, M. F. (2022). Extended mean field control problem: A propagation of chaos result. *Electron. J. Probab.* **27** Paper No. 20, 53. MR4379197 https://doi.org/10.1214/21-ejp726

[35] DJETE, M. F., POSSAMAÏ, D. and TAN, X. (2022). McKean–Vlasov optimal control: Limit theory and equivalence between different formulations. *Math. Oper. Res.* **47** 2891–2930. MR4515488 https://doi.org/10.1287/moor.2021.1232

[36] FABBRI, G., GOZZI, F. and ŚWIĘCH, A. (2017). *Stochastic Optimal Control in Infinite Dimension: Dynamic Programming and HJB Equations. Probability Theory and Stochastic Modelling* **82**. Springer, Cham. MR3674558 https://doi.org/10.1007/978-3-319-53067-3

[37] FENG, J. and KATSOULAKIS, M. (2009). A comparison principle for Hamilton–Jacobi equations related to controlled gradient flows in infinite dimensions. *Arch. Ration. Mech. Anal.* **192** 275–310. MR2486597 https://doi.org/10.1007/s00205-008-0133-5

[38] FENG, J., MIKAMI, T. and ZIMMER, J. (2021). A Hamilton–Jacobi PDE associated with hydrodynamic fluctuations from a nonlinear diffusion equation. *Comm. Math. Phys.* **385** 1–54. MR4275780 https://doi.org/10.1007/s00220-021-04110-1

[39] FENG, J. and NGUYEN, T. (2012). Hamilton–Jacobi equations in space of measures associated with a system of conservation laws. *J. Math. Pures Appl.* (9) **97** 318–390. MR2899811 https://doi.org/10.1016/j.matpur.2011.11.004

[40] FENG, J. and ŚWIĘCH, A. (2013). Optimal control for a mixed flow of Hamiltonian and gradient type in space of probability measures. *Trans. Amer. Math. Soc.* **365** 3987–4039. With an appendix by Atanas Stefanov. MR3055687 https://doi.org/10.1090/S0002-9947-2013-05634-6

[41] FISCHER, M. and LIVIERI, G. (2016). Continuous time mean-variance portfolio optimization through the mean field approach. *ESAIM Probab. Stat.* **20** 30–44. MR3528616 https://doi.org/10.1051/ps/2016001

[42] FLEMING, W. H. and SONER, H. M. (2006). *Controlled Markov Processes and Viscosity Solutions*, 2nd ed. *Stochastic Modelling and Applied Probability* **25**. Springer, New York. MR2179357

[43] FORNASIER, M. and SOLOMBRINO, F. (2014). Mean-field optimal control. *ESAIM Control Optim. Calc. Var.* **20** 1123–1152. MR3264236 https://doi.org/10.1051/cocv/2014009

[44] GANGBO, W. (2018). On some analytical aspects of Mean Field Games, Lecture notes based on a class taught in Berkeley during Fall 2018. Available at https://www.math.ucla.edu/~wgangbo/publications/MFG-UCB-Final2.pdf.

[45] GANGBO, W., MAYORGA, S. and ŚWIĘCH, A. (2021). Finite dimensional approximations of Hamilton–Jacobi–Bellman equations in spaces of probability measures. *SIAM J. Math. Anal.* **53** 1320–1356. MR4226237 https://doi.org/10.1137/20M1331135

[46] GANGBO, W. and MÉSZÁROS, A. R. (2022). Global well-posedness of master equations for deterministic displacement convex potential mean field games. *Comm. Pure Appl. Math.* **75** 2685–2801. MR4509653 https://doi.org/10.1002/cpa.22069

[47] GANGBO, W., NGUYEN, T. and TUDORASCU, A. (2008). Hamilton–Jacobi equations in the Wasserstein space. *Methods Appl. Anal.* **15** 155–183. MR2481677 https://doi.org/10.4310/MAA.2008.v15.n2.a4

[48] GANGBO, W. and ŚWIĘCH, A. (2015). Metric viscosity solutions of Hamilton–Jacobi equations depending on local slopes. *Calc. Var. Partial Differential Equations* **54** 1183–1218. MR3385197 https://doi.org/10.1007/s00526-015-0822-5

[49] GANGBO, W. and TUDORASCU, A. (2019). On differentiability in the Wasserstein space and well-posedness for Hamilton–Jacobi equations. *J. Math. Pures Appl.* (9) **125** 119–174. MR3944201 https://doi.org/10.1016/j.matpur.2018.09.003

[50] GERMAIN, M., PHAM, H. and WARIN, X. (2022). Rate of convergence for particle approximation of PDEs in Wasserstein space. *J. Appl. Probab.* **59** 992–1008. MR4507678 https://doi.org/10.1017/jpr.2021.102

[51] GIGA, Y., HAMAMUKI, N. and NAKAYASU, A. (2015). Eikonal equations in metric spaces. *Trans. Amer. Math. Soc.* **367** 49–66. MR3271253 https://doi.org/10.1090/S0002-9947-2014-05893-5

[52] HYND, R. and KIM, H. K. (2015). Value functions in the Wasserstein spaces: Finite time horizons. *J. Funct. Anal.* **269** 968–997. MR3352761 https://doi.org/10.1016/j.jfa.2015.05.011

[53] JERHAOUI, O., PROST, A. and ZIDANI, H. (2024). Viscosity solutions of centralized control problems in measure spaces. *ESAIM Control Optim. Calc. Var.* **30** Paper No. 91, 37. MR4837669 https://doi.org/10.1051/cocv/2024081

[54] JIMENEZ, C., MARIGONDA, A. and QUINCAMPOIX, M. (2020). Optimal control of multiagent systems in the Wasserstein space. *Calc. Var. Partial Differential Equations* **59** Paper No. 58, 45. MR4073204 https://doi.org/10.1007/s00526-020-1718-6

[55] JIMENEZ, C., MARIGONDA, A. and QUINCAMPOIX, M. (2023). Dynamical systems and Hamilton–Jacobi–Bellman equations on the Wasserstein space and their L^2 representations. *SIAM J. Math. Anal.* **55** 5919–5966. MR4650916 https://doi.org/10.1137/22M1503877

[56] JIMENEZ, C. and QUINCAMPOIX, M. (2018). Hamilton Jacobi Isaacs equations for differential games with asymmetric information on probabilistic initial condition. *J. Math. Anal. Appl.* **457** 1422–1451. MR3705362 https://doi.org/10.1016/j.jmaa.2017.08.012

[57] KARATZAS, I. and SHREVE, S. E. (1991). *Brownian Motion and Stochastic Calculus*, 2nd ed. *Graduate Texts in Mathematics* **113**. Springer, New York. MR1121940 https://doi.org/10.1007/978-1-4612-0949-2

[58] KRAAIJ, R. C. (2020). The exponential resolvent of a Markov process and large deviations for Markov processes via Hamilton–Jacobi equations. *Electron. J. Probab.* **25** Paper No. 134, 39. MR4169175 https://doi.org/10.1214/20-ejp539

[59] KRAAIJ, R. C. (2022). A general convergence result for viscosity solutions of Hamilton–Jacobi equations and non-linear semigroups. *J. Funct. Anal.* **282** Paper No. 109346, 55. MR4354859 https://doi.org/10.1016/j.jfa.2021.109346

[60] LACKER, D. (2017). Limit theory for controlled McKean–Vlasov dynamics. *SIAM J. Control Optim.* **55** 1641–1672. MR3654119 https://doi.org/10.1137/16M1095895

[61] LASRY, J.-M. and LIONS, P.-L. (1986). A remark on regularization in Hilbert spaces. *Israel J. Math.* **55** 257–266. MR0876394 https://doi.org/10.1007/BF02765025

[62] LIAO, H., MÉSZÁROS, A. R., MOU, C. and ZHOU, C. (2024). Convergence analysis of controlled particle systems arising in deep learning: From finite to infinite sample size. Preprint. Available at arXiv:2404.05185.

[63] LIONS, P.-L. (1988). Viscosity solutions of fully nonlinear second-order equations and optimal stochastic control in infinite dimensions. I. The case of bounded stochastic evolutions. *Acta Math.* **161** 243–278. MR0971797 https://doi.org/10.1007/BF02392299

[64] LIONS, P.-L. (1989). Viscosity solutions of fully nonlinear second-order equations and optimal stochastic control in infinite dimensions. III. Uniqueness of viscosity solutions for general second-order equations. *J. Funct. Anal.* **86** 1–18. MR1013931 https://doi.org/10.1016/0022-1236(89)90062-1

[65] LIONS, P.-L. (2007–2011). Mean-Field Games. Cours au Collège de France.

[66] LIU, Q., SHANMUGALINGAM, N. and ZHOU, X. (2021). Equivalence of solutions of Eikonal equation in metric spaces. *J. Differ. Equ.* **272** 979–1014. MR4166072 https://doi.org/10.1016/j.jde.2020.10.018

[67] MAYORGA, S. and ŚWIĘCH, A. (2023). Finite dimensional approximations of Hamilton–Jacobi–Bellman equations for stochastic particle systems with common noise. *SIAM J. Control Optim.* **61** 820–851. MR4580743 https://doi.org/10.1137/22M1489186

[68] MCSHANE, E. J. (1934). Extension of range of functions. *Bull. Amer. Math. Soc.* **40** 837–842. MR1562984 https://doi.org/10.1090/S0002-9904-1934-05978-0

[69] MIMIKOS-STAMATOPOULOS, N. (2023). A Mean Field Game approach to Large Deviations for empirical measures under common noise. Preprint. Available at arXiv:2310.08033.

[70] PHAM, H. and WEI, X. (2017). Dynamic programming for optimal control of stochastic McKean–Vlasov dynamics. *SIAM J. Control Optim.* **55** 1069–1101. MR3631380 https://doi.org/10.1137/16M1071390

[71] PHAM, H. and WEI, X. (2018). Bellman equation and viscosity solutions for mean-field stochastic control problem. *ESAIM Control Optim. Calc. Var.* **24** 437–461. MR3843191 https://doi.org/10.1051/cocv/2017019

[72] SONER, H. M. and YAN, Q. (2024). Viscosity solutions for McKean–Vlasov control on a torus. *SIAM J. Control Optim.* **62** 903–923. MR4712815 https://doi.org/10.1137/22M1543732

[73] SONER, H. M. and YAN, Q. (2024). Viscosity solutions of the Eikonal equation on the Wasserstein space. *Appl. Math. Optim.* **90** Paper No. 1, 16. MR4751217 https://doi.org/10.1007/s00245-024-10145-2

[74] SZNITMAN, A.-S. (1991). Topics in propagation of chaos. In *École D'Été de Probabilités de Saint-Flour XIX*—1989. *Lecture Notes in Math.* **1464** 165–251. Springer, Berlin. MR1108185 https://doi.org/10.1007/BFb0085169

[75] VILLANI, C. (2009). *Optimal Transport: Old and New*. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **338**. Springer, Berlin. MR2459454 https://doi.org/10.1007/978-3-540-71050-9

[76] YONG, J. and ZHOU, X. Y. (1999). *Stochastic Controls: Hamiltonian Systems and HJB Equations*. *Applications of Mathematics (New York)* **43**. Springer, New York. MR1696772 https://doi.org/10.1007/978-1-4612-1466-3

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INVARIANT POISSON PROCESSES OF ROADS IN THE 3-REGULAR TREE

BY GUILLAUME BLANC[a]

Laboratoire de Mathématiques d'Orsay, Université Paris-Saclay, [a]*guillaume.blanc@epfl.ch*

In this paper, we study invariant Poisson processes of lines (i.e, bi-infinite geodesics) and roads (i.e, lines with speed limits) in the 3-regular tree. More precisely, there exists a unique (up to multiplicative constant) locally finite Borel measure on the space of lines that is invariant under graph automorphisms, and we consider two Poissonian ways of playing with this invariant measure. First, following Benjamini, Jonasson, Schramm and Tykesson, we consider an invariant Poisson process of lines, and show that there is a critical value of the intensity below which a.s. the vacant set of the process percolates, and above which all its connected components are finite. Then, we consider an invariant Poisson process of roads, and show that there is a critical value of the parameter governing the speed limits of the roads below which a.s. one can drive to infinity in finite time using the road network generated by the process, and above which this is impossible.

REFERENCES

[1] ALDOUS, D. (2014). Scale-invariant random spatial networks. *Electron. J. Probab.* **19** 15. MR3164768 https://doi.org/10.1214/EJP.v19-2920

[2] BENJAMINI, I., JONASSON, J., SCHRAMM, O. and TYKESSON, J. (2009). Visibility to infinity in the hyperbolic plane, despite obstacles. *ALEA Lat. Amer. J. Probab. Math. Stat.* **6** 323–342. MR2546629

[3] BLANC, G. (2024). Fractal properties of Aldous-Kendall random metric. *Ann. Inst. Henri Poincaré Probab. Stat.* **60** 1356–1386. MR4757528 https://doi.org/10.1214/22-aihp1354

[4] KAHN, J. (2016). Improper Poisson line process as SIRSN in any dimension. *Ann. Probab.* **44** 2694–2725. MR3531678 https://doi.org/10.1214/15-AOP1032

[5] KENDALL, W. S. (2017). From random lines to metric spaces. *Ann. Probab.* **45** 469–517. MR3601654 https://doi.org/10.1214/14-AOP935

[6] LAST, G. and PENROSE, M. (2018). *Lectures on the Poisson Process*. *Institute of Mathematical Statistics Textbooks* **7**. Cambridge Univ. Press, Cambridge. MR3791470

[7] LYONS, R. and PERES, Y. (2016). *Probability on Trees and Networks*. *Cambridge Series in Statistical and Probabilistic Mathematics* **42**. Cambridge Univ. Press, New York. MR3616205 https://doi.org/10.1017/9781316672815

[8] MATSUMOTO, A. and YANO, K. (2011). On a zero-one law for the norm process of transient random walk. In *Séminaire de Probabilités XLIII*. *Lecture Notes in Math.* **2006** 105–126. Springer, Berlin. MR2790370 https://doi.org/10.1007/978-3-642-15217-7_4

[9] PEMANTLE, R. and PERES, Y. (1994). Domination between trees and application to an explosion problem. *Ann. Probab.* **22** 180–194. MR1258873

[10] SCHNEIDER, R. and WEIL, W. (2008). *Stochastic and Integral Geometry*. *Probability and Its Applications* (*New York*). Springer, Berlin. MR2455326 https://doi.org/10.1007/978-3-540-78859-1

[11] SIDORAVICIUS, V. and SZNITMAN, A.-S. (2009). Percolation for the vacant set of random interlacements. *Comm. Pure Appl. Math.* **62** 831–858. MR2512613 https://doi.org/10.1002/cpa.20267

[12] SZNITMAN, A.-S. (2010). Vacant set of random interlacements and percolation. *Ann. of Math.* (2) **171** 2039–2087. MR2680403 https://doi.org/10.4007/annals.2010.171.2039

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