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GENETIC COMPOSITION OF SUPERCRITICAL BRANCHING POPULATIONS UNDER POWER LAW MUTATION RATES

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We aim to understand the evolution of the genetic composition of cancer cell populations. To achieve this, we consider an individual-based model representing a cell population where cells divide, die and mutate along the edges of a finite directed graph (V, E). The process starts with only one cell of trait 0. Following typical parameter values in cancer cell populations we study the model under *power law mutation rates*, in the sense that the mutation probabilities are parameterized by negative powers of a scaling parameter n and the typical sizes of the population of interest are positive powers of n. Under a *nonincreasing growth rate condition*, we describe the time evolution of the first-order asymptotics of the size of each subpopulation in the $\log(n)$ time scale, as well as in the random time scale at which the wild-type population, resp. the total population, reaches the size n^t. In particular, such results allow for the perfect characterization of evolutionary pathways. Without imposing any conditions on the growth rates, we describe the time evolution of the order of magnitude of each subpopulation, whose asymptotic limits are positive nondecreasing piecewise linear continuous functions.

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PROPERTIES OF FIRST PASSAGE PERCOLATION ABOVE THE (HYPOTHETICAL) CRITICAL DIMENSION

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It is not known (and even physicists disagree) whether first passage percolation (FPP) on $\mathbb{Z}^d$ has an upper critical dimension d_c, such that the fluctuation exponent $\chi = 0$ in dimensions $d > d_c$. In part to facilitate study of this question, we may nonetheless try to understand properties of FPP in such dimensions should they exist, in particular how they should differ from $d < d_c$. We show that at least one of three fundamental properties of FPP known or believed to hold when $\chi > 0$ must be false if $\chi = 0$. A particular one of the three is most plausible to fail, and we explore the consequences if it is indeed false. These consequences support the idea that when $\chi = 0$, passage times are "local" in the sense that the passage time from x to y is primarily determined by the configuration near x and y. Such locality is manifested by certain "disc-to-disc" passage times, between discs in parallel hyperplanes, being typically much faster than the fastest mean passage time between points in the two discs.

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CONTROLLING MOMENTS WITH KERNEL STEIN DISCREPANCIES

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Kernel Stein discrepancies (KSDs) measure the quality of a distributional approximation and can be computed even when the target density has an intractable normalizing constant. Notable applications include the diagnosis of approximate MCMC samplers and goodness-of-fit tests for unnormalized statistical models. The present work analyzes the convergence control properties of KSDs. We first show that standard KSDs used for weak convergence control fail to control moment convergence. To address this limitation, we next provide sufficient conditions under which alternative diffusion KSDs control both moment and weak convergence. As an immediate consequence we develop, for each $q > 0$, the first KSDs known to exactly characterize q-Wasserstein convergence.

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THE CRITICAL DISORDERED PINNING MEASURE

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In this paper, we study a disordered pinning model induced by a random walk whose increments have a finite $(2+\kappa)$th moment for some $\kappa > 0$. It is known that this model is marginally relevant, and moreover, it undergoes a phase transition in an intermediate disorder regime. We show that, in the critical window, the point-to-point partition functions converge to a unique limiting random measure, which we call the critical disordered pinning measure. We also obtain an analogous result for a continuous counterpart to the pinning model, which is closely related to two other models: one is a critical stochastic Volterra equation that gives rise to a rough volatility model, and the other is a critical stochastic heat equation with multiplicative noise that is white in time and delta in space.

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FLUCTUATIONS OF THE CONNECTIVITY THRESHOLD AND LARGEST NEAREST-NEIGHBOUR LINK

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Consider a random uniform sample $\mathcal{X}_n$ of n points in a compact region A of Euclidean d-space, $d \geq 2$, with a smooth or (when $d = 2$) polygonal boundary. Fix $k \in \mathbb{N}$. Let $M_k(\mathcal{X}_n)$ be the threshold r at which the geometric graph on these n vertices with distance parameter r becomes k-connected. We show that if $d = 2$ then $n(\pi/|A|)M_1(\mathcal{X}_n)^2 - \log n$ is asymptotically standard Gumbel. For $(d, k) \neq (2, 1)$, it is

$$n(\theta_d/|A|)M_k(\mathcal{X}_n)^d - (2 - 2/d)\log n - (4 - 2k - 2/d)\log\log n$$

that converges in distribution to a nondegenerate limit, where θ_d is the volume of the unit ball. The limit is Gumbel with scale parameter 2 except when $(d, k) = (2, 2)$ where the limit is two component extreme value distributed. The different cases reflect the fact that boundary effects are more important in some cases than others. We also give similar results for the largest k-nearest neighbour link $L_k(\mathcal{X}_n)$ in the sample, and show $M_k(\mathcal{X}_n) = L_k(\mathcal{X}_n)$ with high probability. We provide estimates on rates of convergence and give similar results for Poisson samples in A. Finally, we give similar results even for nonuniform samples, with a less explicit sequence of centring constants.

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STOCHASTIC HEAT EQUATIONS DRIVEN BY SPACE-TIME G-WHITE NOISE UNDER SUBLINEAR EXPECTATION

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In this paper, we study the stochastic heat equation driven by a multiplicative space-time G-white noise within the framework of sublinear expectations. The existence and uniqueness of the mild solution are proved. By generalizing the stochastic Fubini theorem under sublinear expectations, we demonstrate that the mild solution also qualifies as a weak solution. Additionally, we derive moment estimates for the solutions.

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RANDOM LÉVY LOOPTREES AND LÉVY MAPS

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What is the analogue of Lévy processes for random surfaces? Motivated by scaling limits of random planar maps in random geometry, we introduce and study Lévy looptrees and Lévy maps. They are defined using excursions of general Lévy processes with no negative jump and extend the known stable looptrees and stable maps, associated with stable processes. We compute in particular their fractal dimensions in terms of the upper and lower Blumenthal–Getoor exponents of the coding Lévy process. The case where the Lévy process is a stable process with a drift naturally appears in the context of stable-Boltzmann planar maps conditioned on having a fixed number of vertices and edges in a near-critical regime.

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OFF-DIAGONAL LONG-RANGE ORDER FOR THE FREE BOSE GAS VIA THE FEYNMAN–KAC FORMULA

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We consider the path-integral representation of the ideal Bose gas under various boundary conditions. We show that Bose–Einstein condensation occurs at the famous critical density threshold, by proving that its 1-particle-reduced density matrix exhibits off-diagonal long-range order above that threshold, but not below. Our proofs are based on the well-known Feynman–Kac formula and a representation in terms of a crucial Poisson point process. Furthermore, in the condensation regime, we derive a law of large numbers with strong concentration for the number of particles in short loops. In contrast to the situation for free boundary conditions, where the entire condensate sits in just one loop, for all other boundary conditions we obtain the limiting Poisson–Dirichlet distribution for the collection of the lengths of all long loops.

Our proofs are new and purely probabilistic (apart from a standard eigenvalue expansion), using elementary tools like Markov's inequality, Poisson point processes, combinatorial formulas for cardinalities of particular partition sets, and asymptotics for random walks with Pareto-distributed steps.

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EXPONENTIAL ASYMPTOTICS FOR p-MULTIPLE SELF-INTERSECTION LOCAL TIME OF 1-D DIFFUSION PROCESS

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The exponential asymptotics of p-multiple self-intersection local time is extended from Brownian motion to a class of 1-d diffusion processes obeying a stochastic differential equation

$$dX_t = b(X_t)\,dt + dB_t, \quad X_0 = 0,$$

where B_t is Brownian motion, and the drift b is globally Lipschitz continuous and differentiable. The exponential asymptotics of self-intersection local time of X_t crucially depends on its symmetrizing measure $\pi(dx) = e^{2\phi(x)}\,dx$ with $\phi(x) = \int_0^x b(y)\,dy$. If the growth rate of $\mathbb{E}e^{2\phi(B_t)}$ is subexponential, the precise exponential asymptotics described by variation with respect to Dirichlet form of X_t under $\pi(dx)$ is derived, which generalizes the results of Chen and Li (*Probab. Theory Related Fields* **128** (2004) 213–254 and Chen (*Electron. J. Probab.* (2023) **28**) ($d = 1$). In terms of proof techniques, the asymptotics of the Feynman–Kac formula of X_t and the comparison inequality on principle eigenvalue of generator play a key role. If the growth rate of $\mathbb{E}e^{\upsilon\phi(B_t)}$ (for some $\upsilon \in (1, 2]$) is exponential, only the finiteness of exponential asymptotics can be obtained, and even the exponential asymptotic result is abnormal in some special cases. In addition, the variational results between Ornstein–Uhlenbeck process and Brownian motion are compared.

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SMOLUCHOWSKI–KRAMERS DIFFUSION APPROXIMATION FOR SYSTEMS OF STOCHASTIC DAMPED WAVE EQUATIONS WITH NONCONSTANT FRICTION

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We consider systems of damped wave equations with a state-dependent damping coefficient and perturbed by a Gaussian multiplicative noise. Initially, we investigate their well-posedness, under quite general conditions on the friction. Subsequently, we study the validity of the so-called Smoluchowski–Kramers diffusion approximation. We show that, under more stringent conditions on the friction, in the small-mass limit the solution of the system of stochastic damped wave equations converge to the solution of a system of stochastic quasi-linear parabolic equations. In this convergence, an additional drift emerges as a result of the interaction between the noise and the state-dependent friction. The identification of this limit is achieved by using a suitable generalization of the classical method of perturbed test functions, tailored to the current infinite dimensional setting.

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CONDITIONING THE LOGISTIC CONTINUOUS-STATE BRANCHING PROCESS ON NON-EXTINCTION VIA ITS TOTAL PROGENY

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The problem of conditioning a continuous-state branching process with quadratic competition (logistic CB process) on nonextinction is investigated. We first establish that nonextinction is equivalent to the total progeny of the population being infinite. The conditioning we propose is then designed by requiring the total progeny to exceed arbitrarily large exponential random variables. This is related to a Doob's h-transform with an explicit excessive function h. The h-transformed process, that is, the conditioned process, is shown to have a finite lifetime almost surely (it is either killed or it explodes continuously). When starting from positive values, the conditioned process is furthermore characterized, up to its lifetime, as the solution to a certain stochastic equation with jumps. The latter superposes the dynamics of the initial logistic CB process with an additional density-dependent immigration term. Last, it is established that the conditioned process can be starting from zero. Key tools employed are a representation of the logistic CB process through a time-changed generalized Ornstein–Uhlenbeck process, as well as Laplace and Siegmund duality relationships with auxiliary diffusion processes.

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FIRST PASSAGE PERCOLATION ON ERDŐS–RÉNYI GRAPHS WITH GENERAL WEIGHTS

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We consider first passage percolation on the Erdős–Rényi graph with n vertices in which each pair of distinct vertices is connected independently by an edge with probability λ/n for some $\lambda > 1$. The edges of the graph are given nonnegative i.i.d. weights with a nondegenerate distribution such that the probability of zero is not too large. We consider the paths with small total weight between two distinct typical vertices and analyse the joint behaviour of the numbers of edges on such paths, the so-called hopcounts, and the total weights of these paths. For $n \to \infty$, we show that, after a suitable transformation, the pairs of hopcounts and total weights of these paths converge in distribution to a Cox process, that is, a Poisson process with a random intensity measure. The random intensity measure is controlled by two independent random variables, whose distribution is the solution of a distributional fixed point equation and is related to branching processes. For nonarithmetic and arithmetic edge-weight distributions we observe different behaviour. In particular, we derive the limiting distribution for the minimal total weight and the corresponding hopcount(s). Our results generalise earlier work of Bhamidi, van der Hofstad and Hooghiemstra, who assume that edge weights have an absolutely continuous distribution. The main tool we employ is the method of moments.

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FLUCTUATION OF THE LARGEST EIGENVALUE OF A KERNEL MATRIX WITH APPLICATION IN GRAPHON-BASED RANDOM GRAPHS

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In this article, we explore the spectral properties of general random kernel matrices $[\mathsf{K}(U_i, U_j)]_{1\leq i\neq j\leq n}$ from a Lipschitz kernel K with n independent random variables $U_1, U_2, \ldots, U_n$ distributed uniformly over [0, 1]. In particular, we identify a dichotomy in the extreme eigenvalue of the kernel matrix, where, if the kernel K is degenerate, the largest eigenvalue of the kernel matrix (after proper normalization) converges weakly to a weighted sum of independent chi-squared random variables. In contrast, for nondegenerate kernels, it converges to a normal distribution extending and reinforcing earlier results from (*Bernoulli* **6** (2000) 113–167). Further, we apply this result to show a dichotomy in the asymptotic behavior of extreme eigenvalues of Graphon-based random graphs, which are pivotal in modeling complex networks and analyzing large-scale graph behavior. These graphs are generated using a kernel W, termed as graphon, by connecting vertices i and j with probability $W(U_i, U_j)$. Our results show that for a Lipschitz graphon W, if the degree function is constant, the fluctuation of the largest eigenvalue (after proper normalization) converges to the weighted sum of independent chi-squared random variables and an independent Gaussian variable. Otherwise, it converges to a normal distribution.

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THE BASS FUNCTIONAL OF MARTINGALE TRANSPORT

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An interesting question in the field of martingale optimal transport, is to determine the martingale with prescribed initial and terminal marginals which is most correlated to Brownian motion. Under a necessary and sufficient irreducibility condition, the answer to this question is given by a *Bass martingale*. At an intuitive level, the latter can be imagined as an order-preserving and martingale-preserving space transformation of an underlying Brownian motion starting with an initial law α which is tuned to ensure the marginal constraints.

In this article we study how to determine the aforementioned initial condition α. This is done by a careful study of what we dub the *Bass functional*. In our main result we show the equivalence between the existence of minimizers of the Bass functional and the existence of a Bass martingale with prescribed marginals. This complements the convex duality approach in a companion paper by the present authors together with M. Beiglböck, with a purely variational perspective. We also establish an infinitesimal version of this result, and furthermore prove the displacement convexity of the Bass functional along certain generalized geodesics in the 2-Wasserstein space.

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THE MAXIMAL DISPLACEMENT OF RADIALLY SYMMETRIC BRANCHING RANDOM WALK IN $\mathbb{R}^d$

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We consider discrete-time branching random walks with a radially symmetric distribution. Independently of each other individuals generate offspring whose relative locations are given by a copy of a radially symmetric point process $\mathcal{L}$. The number of particles at time t form a supercritical Galton–Watson process. We investigate the maximal distance to the origin of such branching random walks. Conditioned on survival, we show that, under some assumptions on $\mathcal{L}$, it grows in the same way as for branching Brownian motion or a broad class of one-dimensional branching random walks: the first term is linear in time and the second logarithmic. The constants in front of these terms are explicit and depend only on the mean measure of $\mathcal{L}$ and dimension. Our main tool in the proof is a ballot theorem with moving barrier which may be of independent interest.

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DEVIATION MATRIX IN DENUMERABLE MARKOV PROCESSES: CHARACTERISATION, RANK-ONE PERTURBATION AND RESTART

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We study the deviation matrix of denumerable state space, multi-chain Markov processes in continuous time. The deviation matrix is a measure of the cumulative deviation from the limiting probabilities and it plays an important role in many application domains. First, we provide several equivalent necessary and sufficient conditions for the existence of the deviation matrix. Next, we study a relation between the deviation matrix and rank-one perturbations of Markov processes. Based on a rank-one perturbation, we derive a versatile formula for the deviation matrix and apply this formula to Markov processes with restart. On the way, we needed to establish several new properties of rank-one perturbed Markov processes. We feel that those properties can be useful in their own right.

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CONTROL OF MCKEAN–VLASOV SDES WITH CONTAGION THROUGH KILLING AT A STATE-DEPENDENT INTENSITY

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We consider a novel McKean–Vlasov control problem with contagion through killing of particles and common noise. Each particle is killed at an exponential rate according to an intensity process that increases whenever the particle is located in a specific region. The removal of a particle pushes others towards the removal region, which can trigger cascades that see particles exiting the system in rapid succession. We study the control of such a system by a central agent who intends to preserve particles at minimal cost. Our theoretical contribution is twofold. First, we rigorously justify the McKean–Vlasov control problem as the limit of a corresponding sequences of controlled finite particle systems. Our proof is based on a controlled martingale problem and tightness arguments. Second, we connect our framework with models in which particles are killed once they hit the boundary of the removal region. We show that these models appear in the limit as the exponential rate tends to infinity. As a corollary, we obtain new existence results for McKean–Vlasov SDEs with singular interaction through hitting times which extend those in the established literature. We conclude the paper with numerical investigations of our model applied to government control of systemic risk in financial systems.

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NONASYMPTOTIC BOUNDS FOR FORWARD PROCESSES IN DENOISING DIFFUSIONS: ORNSTEIN–UHLENBECK IS HARD TO BEAT

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Denoising diffusion probabilistic models (DDPMs) represent a recent advance in generative modelling that has delivered state-of-the-art results across many domains of applications. Despite their success, a rigorous theoretical understanding of the error within DDPMs, particularly the nonasymptotic bounds required for the comparison of their efficiency, remain scarce. Making minimal assumptions on the initial data distribution, allowing, for example, the manifold hypothesis, this paper presents explicit nonasymptotic bounds on the forward diffusion error in total variation (TV), expressed as a function of the terminal time T.

We parametrise multi-modal data distributions in terms of the distance R to their furthest modes and consider forward diffusions with additive and multiplicative noise. Our analysis rigorously proves that, under mild assumptions, the canonical choice of the Ornstein–Uhlenbeck (OU) process cannot be significantly improved in terms of reducing the terminal time T as a function of R and error tolerance $\varepsilon > 0$. Motivated by data distributions arising in generative modelling, we also establish a cut-off like phenomenon (as $R \to \infty$) for the convergence to its invariant measure in TV of an OU process, initialized at a multi-modal distribution with maximal mode distance R.

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ERRATUM: EXACT FORMULAS FOR RANDOM GROWTH WITH HALF-FLAT INITIAL DATA

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In (*Ann. Appl. Probab.* **26** (2016) 507–548), formulas were obtained for moments of the height function of the asymmetric simple exclusion process at one spatial point, starting from half-periodic initial data. As an application, a formula for a certain generating function of this random variable was provided. The derivation in (*Ann. Appl. Probab.* **26** (2016) 507–548) of this generating function has a mistake, which has been corrected, in a certain parameter regime, in a recent paper (*Ann. Appl. Probab.* **34** (2024) 1136–1176) by Dimitrov and Murthy.

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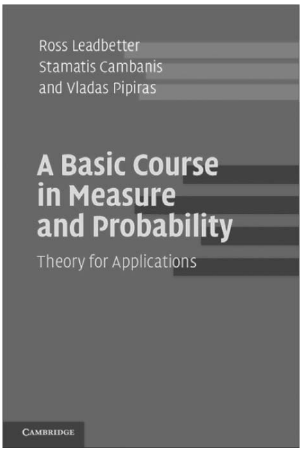

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