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SEPARATING TIMES FOR ONE-DIMENSIONAL GENERAL DIFFUSIONS

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The separating time for two probability measures on a filtered space is an extended stopping time which captures the phase transition between equivalence and singularity. More specifically, two probability measures are equivalent before their separating time and singular afterwards. In this paper, we investigate the separating time for two laws of general one-dimensional regular continuous strong Markov processes, so-called general diffusions, which are parameterized via scale functions and speed measures. Our main result is a representation of the corresponding separating time as (loosely speaking) a hitting time of a deterministic set which is characterized via speed and scale. As hitting times are fairly easy to understand, our result gives access to explicit and easy-to-check sufficient and necessary conditions for two laws of general diffusions to be (locally) absolutely continuous and/or singular. Most of the related literature treats the case of stochastic differential equations. In our setting we encounter several novel features, which are due to general speed and scale on the one hand, and to the fact that we do not exclude (instantaneous or sticky) reflection on the other hand. These new features are discussed in a variety of examples (that can be found in the online supplement). As an application of our main theorem, we investigate the no arbitrage concept no free lunch with vanishing risk (NFLVR) for a single asset financial market whose (discounted) asset is modeled as a general diffusion which is bounded from below (e.g., nonnegative). More precisely, we derive deterministic criteria for NFLVR and we identify the (unique) equivalent local martingale measure as the law of a certain general diffusion on natural scale.

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WEAK POINCARÉ INEQUALITIES FOR MARKOV CHAINS: THEORY AND APPLICATIONS

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We investigate the application of weak Poincaré inequalities (WPI) to Markov chains to study their rates of convergence and to derive complexity bounds. At a theoretical level we investigate the necessity of the existence of WPIs to ensure L^2-convergence, in particular by establishing equivalence with the resolvent uniform positivity-improving (RUPI) condition and providing a counterexample. From a more practical perspective, we extend the celebrated Cheeger's inequalities to the subgeometric setting, and further apply these techniques to study random-walk Metropolis algorithms for heavy-tailed target distributions and to obtain lower bounds on pseudo-marginal algorithms.

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PROBABILISTIC APPROACH TO HEAT KERNELS OF SCHRÖDINGER OPERATORS WITH DECAYING POTENTIALS

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We establish global two-sided heat kernel estimates (for full time and space) of the Schrödinger operator $-\frac{1}{2}\Delta + V$ on $\mathbb{R}^d$, where the potential $V(x)$ is locally bounded and behaves like $c|x|^{-\alpha}$ near infinity with $\alpha \in (0, 2)$ and $c > 0$, or with $\alpha > 0$ and $c < 0$. Our results improve all known results in the literature, and it seems that the current paper is the first one where two-sided matching heat kernel bounds for the long range potentials are established. The results of the paper mostly rely on probabilistic approaches.

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THREE CENTRAL LIMIT THEOREMS FOR THE UNBOUNDED EXCURSION COMPONENT OF A GAUSSIAN FIELD

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For a smooth, stationary Gaussian field f on Euclidean space with fast correlation decay, there is a critical level ℓ_c such that the excursion set $\{f \geq \ell\}$ contains a (unique) unbounded component if and only if $\ell < \ell_c$. We prove central limit theorems for the volume, surface area and Euler characteristic of this unbounded component restricted to a growing box. For planar fields, the results hold at all supercritical levels (i.e., all $\ell < \ell_c$). In higher dimensions the results hold at all sufficiently low levels (all $\ell < -\ell_c < \ell_c$) but could be extended to all supercritical levels by proving the decay of truncated connection probabilities. Our proof is based on the martingale central limit theorem.

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MEAN FIELD STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS WITH NONLINEAR KERNELS

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This work focuses on the mean field stochastic partial differential equations with nonlinear kernels. We first prove the existence and uniqueness of strong and weak solutions for mean field stochastic partial differential equations in the variational framework, then establish the convergence (in certain Wasserstein metric) of the empirical laws of interacting systems to the law of solutions of mean field equations, as the number of particles tends to infinity. The main challenge lies in addressing the inherent interplay between the high nonlinearity of operators and the nonlocal effect of coefficients that depend on the measure. In particular, we do not need to assume any exponential moment control condition of solutions, which extends the range of the applicability of our results.

As applications, we first study a class of finite-dimensional interacting particle systems with polynomial kernels, which are commonly encountered in fields such as the data science and the machine learning. Subsequently, we present several illustrative examples of infinite-dimensional interacting systems with nonlinear kernels, such as the stochastic climate models, the stochastic Allen–Cahn equations, and the stochastic Burgers-type equations.

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A GLOBAL STOCHASTIC MAXIMUM PRINCIPLE FOR MEAN-FIELD FORWARD-BACKWARD STOCHASTIC CONTROL SYSTEMS WITH QUADRATIC GENERATORS

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Our paper is devoted to the study of Peng's stochastic maximum principle (SMP) for a stochastic control problem composed of a controlled forward stochastic differential equation (SDE) as dynamics and a controlled backward SDE which defines the cost functional. Our studies combine the difficulties which come, on one hand, from the fact that the coefficients of both the SDE and the backward SDE are of mean-field type (i.e., they do not only depend on the control process and the solution processes but also on their law), and on the other hand, from the fact that the coefficient of the BSDE is of quadratic growth in Z. Our SMP is novel, it extends in a by far nontrivial way existing results on SMP.

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FROM RANK-BASED MODELS WITH COMMON NOISE TO PATHWISE ENTROPY SOLUTIONS OF SPDES

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We study the mean field limit of a rank-based model with common noise, which arises as an extension to models for the market capitalization of firms in stochastic portfolio theory. We show that, under certain conditions on the drift and diffusion coefficients, the empirical cumulative distribution function converges to the solution of a stochastic PDE. A key step in the proof, which is of independent interest, is to show that any solution to an associated martingale problem is also a pathwise entropy solution to the stochastic PDE, a notion introduced in a recent series of papers (*Stoch. Partial Differ. Equ. Anal. Comput.* **1** (2013) 664–686; *Stoch. Partial Differ. Equ. Anal. Comput.* **2** (2014) 517–538; *Stoch. Partial Differ. Equ. Anal. Comput.* **4** (2016) 635–690; *Comm. Pure Appl. Math.* **70** (2017) 1562–1597; *Stochastic Process. Appl.* **127** (2017) 2961–3004).

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STOCHASTIC OPTIMIZATION ON MATRICES AND A GRAPHON MCKEAN–VLASOV LIMIT

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We consider stochastic gradient descent on the space of large symmetric matrices of suitable functions that are invariant under permuting the rows and columns using the same permutation. We establish deterministic limits of these random curves as the dimensions of the matrices go to infinity while the entries remain bounded. Under a "small noise" assumption, the limit is shown to be the gradient flow of functions on graphons whose existence was established in Oh, Somani, Pal and Tripathi (*J. Theor. Probab.* **37** 1469–1522 (2024)). We also consider limits of stochastic gradient descents with added properly scaled reflected Brownian noise. The limiting curve of graphons is characterized by a family of stochastic differential equations with reflections and can be thought of as an extension of the classical McKean–Vlasov limit for interacting diffusions to the graphon setting. The proofs introduce a family of infinite-dimensional exchangeable arrays of reflected diffusions and a novel notion of propagation of chaos for large matrices of diffusions converging to such arrays in a suitable sense.

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MEAN-FIELD POTTS AND RANDOM-CLUSTER DYNAMICS FROM HIGH-ENTROPY INITIALIZATIONS

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A common obstruction to efficient sampling from high-dimensional distributions with Markov chains is the multimodality of the target distribution because they may get trapped far from stationarity. Still, one hopes that this is only a barrier to the mixing of Markov chains from *worst-case* initializations and can be overcome by choosing high-entropy initializations, for example, a product or weakly correlated distribution. Ideally, from such initializations, the dynamics would escape from the saddle points separating modes quickly and spread its mass between the dominant modes with the correct probabilities.

In this paper, we study convergence from high-entropy initializations for the random-cluster and Potts models on the complete graph—two extensively studied high-dimensional landscapes that pose many complexities like discontinuous phase transitions and asymmetric metastable modes. We study the Chayes–Machta and Swendsen–Wang dynamics for the mean-field random-cluster model and the Glauber dynamics for the Potts model. We sharply characterize the set of product measure initializations from which these Markov chains mix rapidly, even though their mixing times from worst-case initializations are exponentially slow. Our proofs require careful approximations of projections of high-dimensional Markov chains (which are not themselves Markovian) by tractable one-dimensional random processes, followed by analysis of the latter's escape from saddle points separating stable modes.

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TAGGED PARTICLE FLUCTUATIONS FOR TASEP WITH DYNAMICS RESTRICTED BY A MOVING WALL

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We consider the totally asymmetric simple exclusion process on $\mathbb{Z}$ with step initial condition and with the presence of a rightward-moving wall that prevents the particles from jumping. This model was first studied in (*Ann. Inst. Henri Poincaré Probab. Stat.* **60** (2024) 692–720). We extend their work by determining the limiting distribution of a tagged particle in the case where the wall has influence on its fluctuations in neighbourhoods of multiple macroscopic times.

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CONDENSATION IN SCALE-FREE GEOMETRIC GRAPHS WITH EXCESS EDGES

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We identify the upper large deviation probability for the number of edges in scale-free geometric random graph models as the space volume goes to infinity. Our result covers the models of scale-free percolation, the Boolean model with heavy-tailed radius distribution, and the age-dependent random connection model. In all these cases the mechanism behind the large deviation is based on a condensation effect. Loosely speaking, the mechanism randomly selects a finite number of vertices and increases their power, so that they connect to a macroscopic number of vertices in the graph, while the other vertices retain a degree close to their expectation and thus make no more than the expected contribution to the large deviation event. We verify this intuition by means of limit theorems for the empirical distributions of degrees and edge lengths under the conditioning. We observe that at large finite volumes, the edge-length distribution splits into a bulk and travelling wave part of asymptotically positive proportions.

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EMPIRICAL MARTINGALE PROJECTIONS VIA THE ADAPTED WASSERSTEIN DISTANCE

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Given a collection of multidimensional pairs $\{(X_i, Y_i)\}_{1\le i\le n}$, we study the problem of projecting the associated suitably smoothed empirical measure onto the space of martingale couplings (i.e., distributions satisfying $\mathbb{E}[Y|X] = X$) using the adapted Wasserstein distance. We call the resulting distance the *smoothed empirical martingale projection distance* (SE-MPD), for which we obtain an explicit characterization. We also show that the space of martingale couplings remains invariant under the smoothing operation. We study the asymptotic limit of the SE-MPD, which converges at a parametric rate as the sample size increases, if the pairs are either i.i.d. or satisfy appropriate mixing assumptions. Additional finite-sample results are also investigated. Using these results, we introduce a novel consistent martingale coupling hypothesis test, which we apply to test the existence of arbitrage opportunities in recently introduced neural network-based generative models for asset pricing calibration.

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DOUBLE DIMERS ON PLANAR HYPERBOLIC GRAPHS VIA CIRCLE PACKINGS

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In this article we study the double dimer model on hyperbolic Temperleyan graphs via circle packings. We prove that on such graphs, the weak limit of the dimer model exists if and only if the removed black vertex from the boundary of an exhaustion converges to a point on the unit circle in the circle packing representation of the graph. One of our main results is that for such measures, the double dimer model has no bi-infinite path almost surely.

Along the way we prove that in the nonamenable setup, the height function of the dimer model has double exponential tail and faces of height larger than k do not percolate for large enough k. All of these results are new, even for hyperbolic lattices. The proof uses the connection between winding of uniform spanning trees and dimer heights, the notion of stationary random graphs, and the boundary theory of random walk on circle packings.

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A QUANTITATIVE ROBBINS-SIEGMUND THEOREM

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The Robbins–Siegmund theorem is one of the most important results in stochastic optimization, where it is widely used to prove the convergence of stochastic algorithms. We provide a quantitative version of the theorem, establishing a bound on how far one needs to look in order to locate a region of *metastability* in the sense of Tao. Our proof involves a metastable analogue of Doob's theorem for L_1-supermartingales along with a series of technical lemmas that make precise how quantitative information propagates through sums and products of stochastic processes. In this way, our paper establishes a general methodology for finding metastable bounds for stochastic processes that can be reduced to supermartingales, and therefore for obtaining quantitative convergence information across a broad class of stochastic algorithms whose convergence proof relies on some variation of the Robbins–Siegmund theorem. We conclude by discussing how our general quantitative result might be used in practice.

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ATTRIBUTE NETWORK MODELS, STOCHASTIC APPROXIMATION AND NETWORK SAMPLING

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Motivated by the central role of social networks in the diffusion of information, the study of network valued data where nodes and/or edges have attributes, which modulate the dynamics of both network evolution, and information flow on the network itself, has witnessed significant research interest across multiple disciplines. A key ingredient of this general area comprises probabilistic network models that incorporate (a) heterogeneity in edge creation across different attribute groups; (b) temporal network evolution and (c) popularity bias. Such models are then used to understand a host of domain specific questions, including bias in network sampling, PageRank and degree centrality scores and their impact in network ranking and recommendation algorithms. Despite significant interest, for these network models, the main network functional amenable to analysis has so far been degree distribution asymptotics.

In this paper, we analyze dynamic random network models where younger vertices connect to older ones with probabilities proportional to their degrees as well as a propensity kernel governed by their attribute types. Using stochastic approximation techniques we show that, in the large network limit, such networks converge in the local weak sense to limiting infinite random trees with an explicit description in terms of randomly stopped multi-type branching processes. This allows for the derivation of asymptotics for a wide class of network functionals implying, for example, that while degree distribution tail exponents depend on the attribute type (already derived by (*Electron. J. Probab.* **18** (2013) 8)), PageRank centrality scores have the *same* tail exponent across attributes. The limit results also give explicit formulae for the performance of various network sampling mechanisms. One surprising consequence is the efficacy of PageRank and walk based network sampling schemes for directed networks in the setting of rare minorities.

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ESTIMATION AND INFERENCE FOR MULTIVARIATE CONTINUOUS-TIME AUTOREGRESSIVE PROCESSES

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The aim of this paper is to develop estimation and inference methods for the drift parameters of multivariate Lévy-driven continuous-time autoregressive processes of order $p \in \mathbb{N}$. Starting from a continuous-time observation of the process, we develop consistent and asymptotically normal maximum likelihood estimators. We then relax the unrealistic assumption of continuous-time observation by considering natural discretizations based on a combination of Riemann-sum, finite difference, and thresholding approximations. The resulting estimators are also proven to be consistent and asymptotically normal under a general set of conditions, allowing for both finite and infinite jump activity in the driving Lévy process. When discretizing the estimators, allowing for irregularly spaced observations is of great practical importance. In this respect, CAR(p) models are not just relevant for "true" continuous-time processes: a CAR(p) specification provides a natural continuous-time interpolation for modeling irregularly spaced data—even if the observed process is inherently discrete. As a practically relevant application, we consider the setting where the multivariate observation is known to possess a graphical structure. We refer to such a process as GrCAR and discuss the corresponding drift estimators and their properties. The finite sample behavior of all theoretical asymptotic results is empirically assessed by extensive simulation experiments.

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CHARACTERIZING THE FOURTH-MOMENT PHENOMENON OF MONOCHROMATIC SUBGRAPH COUNTS VIA INFLUENCES

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We investigate the distribution of monochromatic subgraph counts in random vertex 2-colorings of large graphs. We give sufficient conditions for the asymptotic normality of these counts and demonstrate their essential necessity (particularly for monochromatic triangles). Our approach refines the fourth-moment theorem to establish new, local *influence-based* conditions for asymptotic normality; these findings more generally provide insight into fourth-moment phenomena for a broader class of Rademacher and Gaussian polynomials.

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MINIMUM STATIONARY VALUES OF SPARSE RANDOM DIRECTED GRAPHS

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We consider the stationary distribution of the simple random walk on the directed configuration model with bounded degrees. Provided that the minimum out-degree is at least 2, with high probability (whp) there is a unique stationary distribution (uniqueness regime). We show that the minimum positive stationary value is whp $n^{-(1+C+o(1))}$ for some constant $C \geq 0$ determined by the degree distribution, answering a question raised by Bordenave, Caputo and Salez (*Probab. Theory Related Fields* **170** (2018) 933–960). In particular, C is the competing combination of two factors: (1) the contribution of atypically "thin" in-neighbourhoods, controlled by subcritical branching processes; and (2) the contribution of atypically "light" trajectories, controlled by large deviation rate functions. Additionally, we give estimates for the expected lower tail of the empirical stationary distribution. As a by-product of our proof, we obtain that the hitting and the cover time are both $n^{1+C+o(1)}$ whp. Our results are in sharp contrast to those of Caputo and Quattropani (*Probab. Theory Related Fields* **178** (2020) 1011–1066) who showed that under the additional condition of minimum in-degree at least 2 (ergodicity regime), stationary values only have logarithmic fluctuations around n^{-1}.

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SAMPLING INVERSE SUBORDINATORS AND SUBDIFFUSIONS

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In this paper, a method to exactly sample the trajectories of inverse subordinators (in the sense of the finite-dimensional distributions), jointly with the undershooting or overshooting process, is provided. The method applies to general strictly increasing subordinators. The (random) running times of these algorithms have finite moments and explicit bounds for the expectations are provided. Additionally, the Monte Carlo approximation of functionals of subdiffusive processes (in the form of time-changed Feller processes) is considered where a central limit theorem and the Berry–Esseen bounds are proved. The approximation of time-changed Itô diffusions is also studied. The strong error, as a function of the time step, is explicitly evaluated demonstrating the strong convergence, and the algorithm's complexity is provided. The Monte Carlo approximation of functionals and its properties for the approximate method is studied as well. An application of our algorithms in the context of weak ergodicity breaking of subdiffusion is also discussed.

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ON THE SPECTRAL RADIUS AND THE CHARACTERISTIC POLYNOMIAL OF A RANDOM MATRIX WITH INDEPENDENT ELEMENTS AND A VARIANCE PROFILE

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In this paper, it is shown that with large probability, the spectral radius of a large non-Hermitian random matrix with a general variance profile does not exceed the square root of the spectral radius of the variance profile matrix. A minimal moment assumption is considered and sparse variance profiles are covered. Following an approach developed recently by Bordenave, Chafaï and García-Zelada, the key theorem states the asymptotic equivalence between the reverse characteristic polynomial of the random matrix at hand and a random analytic function which depends on the variance profile matrix. The result is applied to the case of a non-Hermitian random matrix with a variance profile given by a piecewise constant or a continuous non-negative function, the inhomogeneous (centered) directed Erdős–Rényi model, and more.

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CLOSED-LOOP EQUILIBRIA FOR STACKELBERG GAMES: A STORY ABOUT STOCHASTIC TARGETS

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We provide a general approach to reformulating any continuous-time stochastic Stackelberg differential game under *closed-loop strategies* as a single-level optimisation problem with target constraints. More precisely, we consider a Stackelberg game in which the leader and the follower can both control the drift and the volatility of a stochastic output process, in order to maximise their respective expected utility. The aim is to characterise the Stackelberg equilibrium when the players adopt "closed-loop strategies", that is, their decisions are based *solely* on the historical information of the output process, excluding especially any direct dependence on the underlying driving noise, often unobservable in real-world applications. We first show that, by considering the—second-order—backward stochastic differential equation associated with the continuation utility of the follower as a controlled state variable for the leader, the latter's unconventional optimisation problem can be reformulated as a more standard stochastic control problem with target constraints. Thereafter, adapting the methodology developed by (*J. Eur. Math. Soc. (JEMS)* **4** (2002) 201–236) or (*SIAM J. Control Optim.* **48** (2009/10) 3501–3531), the optimal strategies, as well as the corresponding value of the Stackelberg equilibrium, can be characterised through the solution of a well-specified system of Hamilton–Jacobi–Bellman equations. For a more comprehensive insight, we illustrate our approach through a simple example, facilitating both theoretical and numerical detailed comparisons with the solutions under different information structures studied in the literature.

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ERRATA TO "MANY-SERVER ASYMPTOTICS FOR JOIN-THE-SHORTEST-QUEUE: LARGE DEVIATIONS AND RARE EVENTS"

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This note corrects an error in (*Ann. Appl. Probab.* **31** (2021) 2376–2419), Lemma 5.2.

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