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EXPLORATION-EXPLOITATION TRADE-OFF FOR CONTINUOUS-TIME EPISODIC REINFORCEMENT LEARNING WITH LINEAR-CONVEX MODELS

BY LUKASZ SZPRUCH[1,2,a], TANUT TREETANTHIPLOET[2,b] AND YUFEI ZHANG[3,c]

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We develop a probabilistic framework for analysing model-based reinforcement learning in the episodic setting. We then apply it to study finite-time horizon stochastic control problems with linear dynamics but unknown coefficients and convex, but possibly irregular, objective function. Using probabilistic representations, we study regularity of the associated cost functions and establish precise estimates for the performance gap between applying optimal policy derived from estimated and true model parameters. We identify conditions under which this performance gap is quadratic, improving the linear performance gap in recent work [*SIAM J. Control Optim.* **61** (2023) 755–787], which matches the results obtained for stochastic linear-quadratic problems. Next, we propose a phase-based learning algorithm for which we show how to optimise exploration-exploitation trade-off and achieve sublinear regrets in high probability and expectation. When assumptions needed for the quadratic performance gap hold, the algorithm achieves an order $\mathcal{O}(\sqrt{N}\ln N)$ high probability regret over N episodes in the general case, and an order $\mathcal{O}((\ln N)^2)$ expected regret over N episodes, provided that optimal policies for the estimated models are sufficient to explore the parameter space. These regret bounds match the best possible results from the literature. The analysis requires novel concentration inequalities for correlated continuous-time observations, which we derive.

REFERENCES

[1] ABEILLE, M. and LAZARIC, A. (2018). Improved regret bounds for Thompson sampling in linear quadratic control problems. In *International Conference on Machine Learning*, *PMLR* 1–9.

[2] ALLAN, A. L. and COHEN, S. N. (2019). Parameter uncertainty in the Kalman-Bucy filter. *SIAM J. Control Optim.* **57** 1646–1671. MR3950674 https://doi.org/10.1137/18M1167693

[3] ALLAN, A. L. and COHEN, S. N. (2020). Pathwise stochastic control with applications to robust filtering. *Ann. Appl. Probab.* **30** 2274–2310. MR4149529 https://doi.org/10.1214/19-AAP1558

[4] BAIN, A. and CRISAN, D. (2009). *Fundamentals of Stochastic Filtering*. *Stochastic Modelling and Applied Probability* **60**. Springer, New York. MR2454694 https://doi.org/10.1007/978-0-387-76896-0

[5] BASEI, M., GUO, X., HU, A. and ZHANG, Y. (2022). Logarithmic regret for episodic continuous-time linear-quadratic reinforcement learning over a finite-time horizon. *J. Mach. Learn. Res.* **23** Paper No. [178], 34. MR4577131

[6] BENSOUSSAN, A. (1992). *Stochastic Control of Partially Observable Systems*. Cambridge Univ. Press, Cambridge. MR1191160 https://doi.org/10.1017/CBO9780511526503

[7] BERTSEKAS, D. P. and TSITSIKLIS, J. N. (1995). *Neuro-dynamic programming: an overview*, vol. 1.

[8] BESSON, L. and KAUFMANN, E. (2018). What doubling tricks can and can't do for multi-armed bandits. Preprint. Available at arXiv:1803.06971.

[9] COHEN, S. N. and ELLIOTT, R. J. (2015). *Stochastic Calculus and Applications*, 2nd ed. *Probability and Its Applications*. Springer, Cham. MR3443368 https://doi.org/10.1007/978-1-4939-2867-5

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[10] COHEN, S. N. and TREETANTHIPLOET, T. (2020). Asymptotic randomised control with applications to bandits. Preprint. Available at arXiv:2010.07252.

[11] DAVIS, M. H. A. (1977). *Linear Estimation and Stochastic Control*. *Chapman and Hall Mathematics Series*. CRC Press, London. MR0476099

[12] DEAN, S., MANIA, H., MATNI, N., RECHT, B. and TU, S. (2020). On the sample complexity of the linear quadratic regulator. *Found*. *Comput*. *Math*. **20** 633–679. MR4130536 https://doi.org/10.1007/s10208-019-09426-y

[13] DJELLOUT, H., GUILLIN, A. and WU, L. (2004). Transportation cost-information inequalities and applications to random dynamical systems and diffusions. *Ann*. *Probab*. **32** 2702–2732. MR2078555 https://doi.org/10.1214/009117904000000531

[14] DUNCAN, T. E., GUO, L. and PASIK-DUNCAN, B. (1999). Adaptive continuous-time linear quadratic Gaussian control. *IEEE Trans*. *Automat*. *Control* **44** 1653–1662. MR1709871 https://doi.org/10.1109/9.788532

[15] EL KAROUI, N., PENG, S. and QUENEZ, M. C. (1997). Backward stochastic differential equations in finance. *Math*. *Finance* **7** 1–71. MR1434407 https://doi.org/10.1111/1467-9965.00022

[16] FARADONBEH, M. K. S., FARADONBEH, M. S. S. and BAYATI, M. (2022). Thompson sampling efficiently learns to control diffusion processes. Preprint. Available at arXiv:2206.09977.

[17] FLEMING, W. H. and SONER, H. M. (2006). *Controlled Markov Processes and Viscosity Solutions*, 2nd ed. *Stochastic Modelling and Applied Probability* **25**. Springer, New York. MR2179357

[18] GAO, X. and ZHOU, X. Y. (2022). Square-root regret bounds for continuous-time episodic Markov decision processes. Preprint. Available at arXiv:2210.00832.

[19] GAO, X. and ZHOU, X. Y. (2024). Logarithmic regret bounds for continuous-time average-reward Markov decision processes. *SIAM J*. *Control Optim*. **62** 2529–2556. MR4795747 https://doi.org/10.1137/23M1584101

[20] GIEGRICH, M., REISINGER, C. and ZHANG, Y. (2024). Convergence of policy gradient methods for finite-horizon exploratory linear-quadratic control problems. *SIAM J*. *Control Optim*. **62** 1060–1092. MR4721060 https://doi.org/10.1137/22M1533517

[21] GUO, X., HU, A. and ZHANG, Y. (2023). Reinforcement learning for linear-convex models with jumps via stability analysis of feedback controls. *SIAM J*. *Control Optim*. **61** 755–787. MR4580741 https://doi.org/10.1137/21M1414413

[22] GUO, X., XU, R. and ZARIPHOPOULOU, T. (2022). Entropy regularization for mean field games with learning. *Math*. *Oper*. *Res*. **47** 3239–3260. MR4515501 https://doi.org/10.1287/moor.2021.1238

[23] HE, J., ZHOU, D. and GU, Q. (2021). Logarithmic regret for reinforcement learning with linear function approximation. In *International Conference on Machine Learning*, *PMLR* 4171–4180.

[24] JAKSCH, T., ORTNER, R. and AUER, P. (2010). Near-optimal regret bounds for reinforcement learning. *J*. *Mach*. *Learn*. *Res*. **11** 1563–1600. MR2645461

[25] KECHRIS, A. S. (1995). *Classical Descriptive Set Theory*. *Graduate Texts in Mathematics* **156**. Springer, New York. MR1321597 https://doi.org/10.1007/978-1-4612-4190-4

[26] KESKIN, N. B. and ZEEVI, A. (2018). On incomplete learning and certainty-equivalence control. *Oper*. *Res*. **66** 1136–1167. MR3865410 https://doi.org/10.1287/opre.2017.1713

[27] KOZLOWSKI, J. (1992). Optimal allocation of resources to growth and reproduction: Implications for age and size at maturity. *Trends Ecol*. *Evol*. **7** 15–19.

[28] KRYLOV, N. V. (2002). *Introduction to the Theory of Random Processes*. *Graduate Studies in Mathematics* **43**. Amer. Math. Soc., Providence, RI. MR1885884 https://doi.org/10.1090/gsm/043

[29] KRYLOV, N. V. (2009). *Controlled Diffusion Processes*. *Stochastic Modelling and Applied Probability* **14**. Springer, Berlin. MR2723141

[30] LATTIMORE, T. and SZEPESVÁRI, C. (2020). *Bandit Algorithms*. Cambridge Univ. Press, Cambridge.

[31] LIONNET, A., DOS REIS, G. and SZPRUCH, L. (2015). Time discretization of FBSDE with polynomial growth drivers and reaction-diffusion PDEs. *Ann*. *Appl*. *Probab*. **25** 2563–2625. MR3375884 https://doi.org/10.1214/14-AAP1056

[32] LIPTSER, R. S. and SHIRYAYEV, A. N. (1977). *Statistics of Random Processes*. *I*. *General Theory*. *Applications of Mathematics*, *Vol*. 5. Springer, New York. MR0474486

[33] MANIA, H., TU, S. and RECHT, B. (2019). Certainty equivalent control of LQR is efficient. Preprint. Available at arXiv:1902.07826.

[34] MUNOS, R. (2000). A study of reinforcement learning in the continuous case by the means of viscosity solutions. *Mach*. *Learn*. **40** 265–299.

[35] MUNOS, R. (2006). Policy gradient in continuous time. *J*. *Mach*. *Learn*. *Res*. **7** 771–791. MR2274386

[36] MUNOS, R. and BOURGINE, P. (1998). Reinforcement learning for continuous stochastic control problems. In *Advances in Neural Information Processing Systems* 1029–1035.

[37] OSBAND, I., RUSSO, D. and VAN ROY, B. (2013). (More) efficient reinforcement learning via posterior sampling. In *Advances in Neural Information Processing Systems* 3003–3011.

[38] OSBAND, I. and VAN ROY, B. (2014). *Model-based reinforcement learning and the eluder dimension*. Preprint. Available at arXiv:1406.1853.

[39] PARDOUX, É. and PENG, S. (1992). Backward stochastic differential equations and quasilinear parabolic partial differential equations. In *Stochastic Partial Differential Equations and Their Applications* (*Charlotte*, *NC*, 1991). *Lect*. *Notes Control Inf*. *Sci*. **176** 200–217. Springer, Berlin. MR1176785 https://doi.org/10.1007/BFb0007334

[40] REISINGER, C. and ZHANG, Y. (2021). Regularity and stability of feedback relaxed controls. *SIAM J*. *Control Optim*. **59** 3118–3151. MR4310902 https://doi.org/10.1137/20M1312435

[41] RUDIN, C. (2019). Stop explaining black box machine learning models for high stakes decisions and use interpretable models instead. *Nat*. *Mach*. *Intell*. **1** 206–215.

[42] RUSMEVICHIENTONG, P. and TSITSIKLIS, J. N. (2010). Linearly parameterized bandits. *Math*. *Oper*. *Res*. **35** 395–411. MR2674726 https://doi.org/10.1287/moor.1100.0446

[43] SIMCHOWITZ, M. and FOSTER, D. (2020). Naive exploration is optimal for online LQR. In *International Conference on Machine Learning*, *PMLR* 8937–8948.

[44] ŠIŠKA, D. and SZPRUCH, Ł. (2024). Gradient flows for regularized stochastic control problems. *SIAM J*. *Control Optim*. **62** 2036–2070. MR4771046 https://doi.org/10.1137/20M1373645

[45] SUTTON, R. S. and BARTO, A. G. (2018). *Reinforcement Learning*: *An Introduction*, 2nd ed. *Adaptive Computation and Machine Learning*. MIT Press, Cambridge, MA. MR3889951

[46] VERSHYNIN, R. (2018). *High-Dimensional Probability*: *An Introduction with Applications in Data Science*. *Cambridge Series in Statistical and Probabilistic Mathematics* **47**. Cambridge Univ. Press, Cambridge. MR3837109 https://doi.org/10.1017/9781108231596

[47] WAINWRIGHT, M. J. (2019). *High-Dimensional Statistics*: *A Non-asymptotic Viewpoint*. *Cambridge Series in Statistical and Probabilistic Mathematics* **48**. Cambridge Univ. Press, Cambridge. MR3967104 https://doi.org/10.1017/9781108627771

[48] WALIGORA, G. (2009). Tabu search for discrete–continuous scheduling problems with heuristic continuous resource allocation. *European J*. *Oper*. *Res*. **193** 849–856.

[49] WANG, H., ZARIPHOPOULOU, T. and ZHOU, X. Y. (2020). Reinforcement learning in continuous time and space: A stochastic control approach. *J*. *Mach*. *Learn*. *Res*. **21** Paper No. 198, 34. MR4209484

[50] WANG, W., HAN, J., YANG, Z. and WANG, Z. (2021). Global convergence of policy gradient for linear-quadratic mean-field control/game in continuous time. In *International Conference on Machine Learning*, *PMLR* 10772–10782.

[51] YONG, J. and ZHOU, X. Y. (1999). *Stochastic Controls*: *Hamiltonian Systems and HJB Equations*. *Applications of Mathematics* (*New York*) **43**. Springer, New York. MR1696772 https://doi.org/10.1007/978-1-4612-1466-3

[52] ZHANG, J. (2017). *Backward Stochastic Differential Equations*: *From Linear to Fully Nonlinear Theory*. *Probability Theory and Stochastic Modelling* **86**. Springer, New York. MR3699487 https://doi.org/10.1007/978-1-4939-7256-2

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MEAN FIELD OPTIMIZATION PROBLEM REGULARIZED BY FISHER INFORMATION

BY JULIEN CLAISSE[1,a], GIOVANNI CONFORTI[2,b], ZHENJIE REN[3,c] AND SONGBO WANG[4,d]

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Recently there is a rising interest in the research of mean field optimization, in particular because of its role in analyzing the training of neural networks. In this paper by adding the Fisher information as the regularizer, we relate the regularized mean field optimization problem to a so-called mean field Schrödinger (MFS for short) dynamics. We develop an energy-dissipation method to show that the marginal distributions of the MFS dynamics converge exponentially quickly towards the unique minimizer of the regularized optimization problem. Remarkably, the MFS dynamics is proved to be a gradient flow on the probability measure space with respect to the relative entropy. Finally, we propose a Monte Carlo method to sample the marginal distributions of the MFS dynamics.

REFERENCES

[1] BAKRY, D., BARTHE, F., CATTIAUX, P. and GUILLIN, A. (2008). A simple proof of the Poincaré inequality for a large class of probability measures including the log-concave case. *Electron. Commun. Probab.* **13** 60–66. MR2386063 https://doi.org/10.1214/ECP.v13-1352

[2] CARMONA, R. and DELARUE, F. (2018). *Probabilistic Theory of Mean Field Games with Applications. I. Mean Field FBSDEs, Control, and Games. Probability Theory and Stochastic Modelling* **83**. Springer, Cham. MR3752669

[3] CARMONA, R. and DELARUE, F. (2018). *Probabilistic Theory of Mean Field Games with Applications. II. Mean Field Games with Common Noise and Master Equations. Probability Theory and Stochastic Modelling* **84**. Springer, Cham. MR3753660

[4] CHEN, F., REN, Z. and WANG, S. (2025). Uniform-in-time propagation of chaos for mean field Langevin dynamics. *Ann. Inst. Henri Poincaré Probab. Stat.* **61** 2357–2404. MR4981997 https://doi.org/10.1214/24-aihp1499

[5] CHIZAT, L. (2022). Mean-field Langevin dynamics: Exponential convergence and annealing. *Trans. Mach. Learn. Res.*

[6] CHIZAT, L. and BACH, F. (2018). On the global convergence of gradient descent for over-parameterized models using optimal transport. In *Adv. Neural Inf. Process. Syst.* **31** 3036–3046. Curran Associates, Red Hook, NY.

[7] DRONIOU, J. and EYMARD, R. (2016). Uniform-in-time convergence of numerical methods for non-linear degenerate parabolic equations. *Numer. Math.* **132** 721–766. MR3474488 https://doi.org/10.1007/s00211-015-0733-6

[8] EBERLE, A. (2016). Reflection couplings and contraction rates for diffusions. *Probab. Theory Related Fields* **166** 851–886. MR3568041 https://doi.org/10.1007/s00440-015-0673-1

[9] EBERLE, A., GUILLIN, A. and ZIMMER, R. (2019). Quantitative Harris-type theorems for diffusions and McKean–Vlasov processes. *Trans. Amer. Math. Soc.* **371** 7135–7173. MR3939573 https://doi.org/10.1090/tran/7576

[10] ENGEL, E. and DREIZLER, R. M. (2011). *Density Functional Theory: An Advanced Course. Theoretical and Mathematical Physics*. Springer, Heidelberg. MR2743724 https://doi.org/10.1007/978-3-642-14090-7

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[11] ESCHRIG, H. (1996). *The Fundamentals of Density Functional Theory* **1**. Vieweg + Teubner Verlag, Wiesbaden.

[12] HORNIK, K. (1991). Approximation capabilities of multilayer feedforward networks. *Neural Netw.* **4** 251–257.

[13] HU, K., REN, Z., ŠIŠKA, D. and SZPRUCH, Ł. (2021). Mean-field Langevin dynamics and energy landscape of neural networks. *Ann. Inst. Henri Poincaré Probab. Stat.* **57** 2043–2065. MR4328560 https://doi.org/10.1214/20-aihp1140

[14] JORDAN, R., KINDERLEHRER, D. and OTTO, F. (1998). The variational formulation of the Fokker–Planck equation. *SIAM J. Math. Anal.* **29** 1–17. MR1617171 https://doi.org/10.1137/S0036141096303359

[15] KAZEYKINA, A., REN, Z., TAN, X. and YANG, J. (2024). Ergodicity of the underdamped mean-field Langevin dynamics. *Ann. Appl. Probab.* **34** 3181–3226. MR4757500 https://doi.org/10.1214/23-aap2036

[16] LASRY, J.-M. and LIONS, P.-L. (2007). Mean field games. *Jpn. J. Math.* **2** 229–260. MR2295621 https://doi.org/10.1007/s11537-007-0657-8

[17] LEWIN, M. (2022). *Théorie Spectrale et Mécanique Quantique*. *Mathématiques & Applications* (*Berlin*) [*Mathematics & Applications*] **87**. Springer, Cham. MR4454766 https://doi.org/10.1007/978-3-030-93436-1

[18] MA, J. and YONG, J. (1999). *Forward–Backward Stochastic Differential Equations and Their Applications*. *Lecture Notes in Math.* **1702**. Springer, Berlin. MR1704232

[19] MEI, S., MISIAKIEWICZ, T. and MONTANARI, A. (2019). Mean-field theory of two-layers neural networks: Dimension-free bounds and kernel limit. In *Proc.* 32*rd Conf. Learn. Theory*. *Proc. Mach. Learn. Res.* **99** 2388–2464. PMLR.

[20] MONMARCHÉ, P., REN, Z. and WANG, S. (2024). Time-uniform log-Sobolev inequalities and applications to propagation of chaos. *Electron. J. Probab.* **29** Paper No. 154, 38 pp. MR4816013 https://doi.org/10.1214/24-ejp1217

[21] NITANDA, A., WU, D. and SUZUKI, T. (2022). Convex analysis of the mean field Langevin dynamics. In *Proc.* 25*th Int. Conf. Artif. Intell. Stat. Proc. Mach. Learn. Res.* **151** 9741–9757. PMLR.

[22] REED, M. and SIMON, B. (1980). *Methods of Modern Mathematical Physics*. Academic Press, New York.

[23] ROTSKOFF, G., JELASSI, S., BRUNA, J. and VANDEN-EIJNDEN, E. (2019). Neuron birth-death dynamics accelerates gradient descent and converges asymptotically. In *Proc.* 36*th Int. Conf. Mach. Learn. Proc. Mach. Learn. Res.* **97** 5508–5517. PMLR.

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THE MULTICOLOUR EAST MODEL

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We consider the *multicolour East model*, a model of glass forming liquids that is best described as having up to 2^d rotated versions of the East model evolving on the same $\mathbb{Z}^d$ graph while only sharing their nonfacilitating state. The facilitation mechanisms are independent and thus there is a novel dynamic blocking phenomenon between the facilitation for the various rotations. We find sufficient conditions on the model geometry to have a positive spectral gap and prove that with 2^d rotated versions the model is not ergodic. For $d = 2$ we find sufficient conditions on the transition rates so that the dominating term in the spectral gap is given by the East model on $\mathbb{Z}^2$ as the lowest transition rate tends to 0. In particular, we prove this when intuitively the blocking mechanisms should be dominating, that is, when the other transition rates are high. We do this by showing that the various rotated versions cooperate to facilitate the East movement of the slowest version.

REFERENCES

[1] CANCRINI, N., MARTINELLI, F., ROBERTO, C. and TONINELLI, C. (2008). Kinetically constrained spin models. *Probab. Theory Related Fields* **140** 459–504. MR2365481 https://doi.org/10.1007/s00440-007-0072-3

[2] CHLEBOUN, P., FAGGIONATO, A. and MARTINELLI, F. (2016). Relaxation to equilibrium of generalized East processes on $\mathbb{Z}^d$: Renormalization group analysis and energy-entropy competition. *Ann. Probab.* **44** 1817–1863. MR3502595 https://doi.org/10.1214/15-AOP1011

[3] COUZINIÉ, Y. (2022). The multidimensional East model: a multicolour model and a front evolution problem, Ph.D. thesis, Roma Tre University. Available at https://yannick-couzinie.github.io/theses/phd-thesis.

[4] COUZINIÉ, Y. and MARTINELLI, F. (2022). On a front evolution problem for the multidimensional East model. *Electron. J. Probab.* **27** Paper No. 143, 30. MR4505380 https://doi.org/10.1214/22-ejp870

[5] FAGGIONATO, A., MARTINELLI, F., ROBERTO, C. and TONINELLI, C. (2013). The East model: Recent results and new progresses. *Markov Process. Related Fields* **19** 407–452. MR3156959

[6] GARRAHAN, J. P. and CHANDLER, D. (2003). Coarse-grained microscopic model of glass formers. *Proc. Natl. Acad. Sci. USA* **100** 9710–9714.

[7] GARRAHAN, J. P., SOLLICH, P. and TONINELLI, C. (2011). Kinetically constrained models. *Dynamical heterogeneities in glasses, colloids, and Granular Media* **150** 111–137.

[8] GINÉ, E., GRIMMETT, G. R. and SALOFF-COSTE, L. (1997). *Lectures on Probability Theory and Statistics. Lecture Notes in Math.* **1665**. Springer, Berlin. MR1490042 https://doi.org/10.1007/BFb0092617

[9] HARTARSKY, I., MARTINELLI, F. and TONINELLI, C. (2021). Universality for critical KCM: Finite number of stable directions. *Ann. Probab.* **49** 2141–2174. MR4317702 https://doi.org/10.1214/20-aop1500

[10] KORDZAKHIA, G. and LALLEY, S. P. (2006). Ergodicity and mixing properties of the northeast model. *J. Appl. Probab.* **43** 782–792. MR2274800 https://doi.org/10.1239/jap/1158784946

[11] LIGGETT, T. M. (1985). *Interacting Particle Systems. Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **276**. Springer, New York. MR0776231 https://doi.org/10.1007/978-1-4613-8542-4

[12] LIGGETT, T. M. (2010). *Continuous Time Markov Processes: An Introduction. Graduate Studies in Mathematics* **113**. Amer. Math. Soc., Providence, RI. MR2574430 https://doi.org/10.1090/gsm/113

[13] MARÊCHÉ, L. (2019). Exponential convergence to equilibrium for the d-dimensional East model. *Electron. Commun. Probab.* **24** Paper No. 55, 10. MR4003129 https://doi.org/10.1214/19-ecp261

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[14] MARTINELLI, F., SHAPIRA, A. and TONINELLI, C. (2020). Diffusive scaling of the Kob-Andersen model in $\mathbb{Z}^d$. *Ann. Inst. Henri Poincaré Probab. Stat.* **56** 2189–2210. MR4116722 https://doi.org/10.1214/19-AIHP1035

[15] MARTINELLI, F. and TONINELLI, C. (2019). Towards a universality picture for the relaxation to equilibrium of kinetically constrained models. *Ann. Probab.* **47** 324–361. MR3909971 https://doi.org/10.1214/18-AOP1262

[16] SHAPIRA, A. (2020). Kinetically constrained models with random constraints. *Ann. Appl. Probab.* **30** 987–1006. MR4108128 https://doi.org/10.1214/19-AAP1527

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FREE PROBABILITY, PATH DEVELOPMENTS AND SIGNATURE KERNELS AS UNIVERSAL SCALING LIMITS

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Random developments of a path into a matrix Lie group G_N have recently been used to construct signature-based kernels on path space. Two examples include developments into $\mathrm{GL}(N;\mathbb{R})$ and $U(N;\mathbb{C})$, the general linear and unitary groups of dimension N. For the former, Muça Cirone et al. showed that the signature kernel is obtained via a scaling limit of developments with Gaussian vector fields. The second instance was used by Lou et al. to construct a metric between probability measures on path space. We present a unified treatment to obtaining large N limits by leveraging the tools of free probability theory. An important conclusion is that the limiting kernels, while dependent on the choice of Lie group, are nonetheless universal limits with respect to how the development map is randomised. For unitary developments, the limiting kernel is given by the contraction of a signature against the monomials of freely independent semicircular random variables. Using the Schwinger–Dyson equations, we show that this kernel can be obtained by solving a novel quadratic functional equation. We provide a convergent numerical scheme for this equation, together with rates, which does not require computation of signatures themselves.

REFERENCES

[1] ADLAM, B. and PENNINGTON, J. (2020). The neural tangent kernel in high dimensions: Triple descent and a multi-scale theory of generalization. In *International Conference on Machine Learning* 74–84. PMLR.

[2] ANDERSON, G. W., GUIONNET, A. and ZEITOUNI, O. (2010). *An Introduction to Random Matrices*. *Cambridge Studies in Advanced Mathematics* **118**. Cambridge Univ. Press, Cambridge. MR2760897

[3] BENIGNI, L. and PÉCHÉ, S. (2021). Eigenvalue distribution of some nonlinear models of random matrices. *Electron*. *J*. *Probab*. **26** Paper No. 150. MR4346666 https://doi.org/10.1214/21-ejp699

[4] CASS, T., LYONS, T. and XU, X. (2024). Weighted signature kernels. *Ann*. *Appl*. *Probab*. **34** 585–626. MR4696286 https://doi.org/10.1214/23-aap1973

[5] CHEN, K.-T. (1954). Iterated integrals and exponential homomorphisms. *Proc*. *Lond*. *Math*. *Soc*. (3) **4** 502–512. MR0073174 https://doi.org/10.1112/plms/s3-4.1.502

[6] CHEVYREV, I. and LYONS, T. (2016). Characteristic functions of measures on geometric rough paths. *Ann*. *Probab*. **44** 4049–4082. MR3572331 https://doi.org/10.1214/15-AOP1068

[7] CHEVYREV, I. and OBERHAUSER, H. (2022). Signature moments to characterize laws of stochastic processes. *J*. *Mach*. *Learn*. *Res*. **23** Paper No. [176]. MR4577129

[8] DIEHL, J., EBRAHIMI-FARD, K. and TAPIA, N. (2020). Time-warping invariants of multidimensional time series. *Acta Appl*. *Math*. **170** 265–290. MR4163237 https://doi.org/10.1007/s10440-020-00333-x

[9] DIEHL, J., EBRAHIMI-FARD, K. and TAPIA, N. (2020). Iterated-sums signature, quasisymmetric functions and time series analysis. *Sém*. *Lothar*. *Combin*. **84B** Art. 86. MR4138713

[10] DUBACH, G. and PELED, Y. (2021). On words of non-Hermitian random matrices. *Ann*. *Probab*. **49** 1886–1916. MR4260470 https://doi.org/10.1214/20-aop1496

[11] FATIR ANSARI, A., SCARLETT, J. and SOH, H. (2020). A characteristic function approach to deep implicit generative modeling. In 2020 *IEEE/CVF Conference on Computer Vision and Pattern Recognition* (*CVPR*).

[12] GE, J., LIANG, Y.-C., BAI, Z. and PAN, G. (2021). Large-dimensional random matrix theory and its applications in deep learning and wireless communications. *Random Matrices Theory Appl*. **10** Paper No. 2230001. MR4379548 https://doi.org/10.1142/S2010326322300017

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[13] GUIONNET, A. (2019). *Asymptotics of Random Matrices and Related Models*. *CBMS Regional Conference Series in Mathematics* **130**. Amer. Math. Soc., Providence, RI. MR3931304 https://doi.org/10.1090/cbms/130

[14] GUIONNET, A. and MAUREL-SEGALA, E. (2006). Combinatorial aspects of matrix models. *ALEA Lat*. *Amer*. *J*. *Probab*. *Math*. *Stat*. **1** 241–279. MR2249657

[15] GUIONNET, A. and MAUREL-SEGALA, E. (2007). Second order asymptotics for matrix models. *Ann*. *Probab*. **35** 2160–2212. MR2353386 https://doi.org/10.1214/009117907000000141

[16] GUIONNET, A. and SHLYAKHTENKO, D. (2009). Free diffusions and matrix models with strictly convex interaction. *Geom*. *Funct*. *Anal*. **18** 1875–1916. MR2491694 https://doi.org/10.1007/s00039-009-0704-0

[17] GUIONNET, A. and SHLYAKHTENKO, D. (2014). Free monotone transport. *Invent*. *Math*. **197** 613–661. MR3251831 https://doi.org/10.1007/s00222-013-0493-9

[18] IBRAHIM, M. R. and LYONS, T. (2022). ImageSig: A signature transform for ultra-lightweight image recognition. In *Proceedings of the IEEE/CVF Conference on Computer Vision and Pattern Recognition* 3649–3659.

[19] ISSA, Z., HORVATH, B., LEMERCIER, M. and SALVI, C. (2023). Nonadversarial training of neural SDEs with signature kernel scores.

[20] KIDGER, P., KIDGER, P., ARRIBAS, I. P., SALVI, C. and LYONS, T. (2019). Deep signature transforms. In *Neural Information Processing Systems*.

[21] KIRÁLY, F. J. and OBERHAUSER, H. (2019). Kernels for sequentially ordered data. *J*. *Mach*. *Learn*. *Res*. **20** Paper No. 31. MR3948071

[22] LEMERCIER, M., LYONS, T. and SALVI, C. (2025). Log-PDE methods for rough signature kernels.

[23] LEVIN, D., LYONS, T. and NI, H. (2016). Learning from the past, predicting the statistics for the future, learning an evolving system.

[24] LI, S., YU, Z., XIANG, M. and MANDIC, D. (2020). Reciprocal adversarial learning via characteristic functions. In *Advances in Neural Information Processing Systems* **33** 217–228.

[25] LOU, H., LI, S. and NI, H. (2022). Path development network with finite-dimensional Lie group representation.

[26] LOU, H., LI, S. and NI, H. (2023). PCF-GAN: Generating sequential data via the characteristic function of measures on the path space. In *Thirty-Seventh Conference on Neural Information Processing Systems*.

[27] LYONS, T. J., CARUANA, M. and LÉVY, T. (2007). *Differential Equations Driven by Rough Paths*. *Lecture Notes in Math*. **1908**. Springer, Berlin. MR2314753

[28] MUCA CIRONE, N., LEMERCIER, M. and SALVI, C. (2023). Neural signature kernels as infinite-width-depth-limits of controlled ResNets. In *Proceedings of the* 40*th International Conference on Machine Learning*. *Proceedings of Machine Learning Research* **202** 25358–25425. PMLR.

[29] NI, H., SZPRUCH, L., SABATE-VIDALES, M., XIAO, B., WIESE, M. and LIAO, S. (2021). Sig-Wasserstein GANs for time series generation. In *Proceedings of the Second ACM International Conference on AI in Finance* 1–8.

[30] PENNINGTON, J., SCHOENHOLZ, S. and GANGULI, S. (2017). Resurrecting the sigmoid in deep learning through dynamical isometry: Theory and practice. *Adv. Neural Inf. Process*. *Syst*. **30**.

[31] SALVI, C., CASS, T., FOSTER, J., LYONS, T. and YANG, W. (2021). The signature kernel is the solution of a Goursat PDE. *SIAM J*. *Math*. *Data Sci*. **3** 873–899. MR4309861 https://doi.org/10.1137/20M1366794

[32] TAO, T. (2012). *Topics in Random Matrix Theory*. *Graduate Studies in Mathematics* **132**. Amer. Math. Soc., Providence, RI. MR2906465 https://doi.org/10.1090/gsm/132

[33] VERSHYNIN, R. (2018). *High-Dimensional Probability*. *Cambridge Series in Statistical and Probabilistic Mathematics* **47**. Cambridge Univ. Press, Cambridge. MR3837109 https://doi.org/10.1017/9781108231596

[34] VOICULESCU, D. (1985). Symmetries of some reduced free product C^*-algebras. In *Operator Algebras and Their Connections with Topology and Ergodic Theory* (*Buşteni*, 1983). *Lecture Notes in Math*. **1132** 556–588. Springer, Berlin. MR0799593 https://doi.org/10.1007/BFb0074909

[35] VOICULESCU, D. (1986). Addition of certain noncommuting random variables. *J*. *Funct*. *Anal*. **66** 323–346. MR0839105 https://doi.org/10.1016/0022-1236(86)90062-5

[36] VOICULESCU, D. (1987). Multiplication of certain noncommuting random variables. *J*. *Operator Theory* **18** 223–235. MR0915507

[37] VOICULESCU, D. (1991). Limit laws for random matrices and free products. *Invent*. *Math*. **104** 201–220. MR1094052 https://doi.org/10.1007/BF01245072

[38] WANG, Z. and ZHU, Y. (2024). Deformed semicircle law and concentration of nonlinear random matrices for ultra-wide neural networks. *Ann*. *Appl*. *Probab*. **34** 1896–1947. MR4728160 https://doi.org/10.1214/23-aap2010

[39] YANG, W., JIN, L., NI, H. and LYONS, T. (2016). Rotation-free online handwritten character recognition using dyadic path signature features, hanging normalization, and deep neural network. In 2016 23*rd International Conference on Pattern Recognition* (*ICPR*) 4083–4088.

[40] YANG, W., LYONS, T., NI, H., SCHMID, C. and JIN, L. (2022). Developing the path signature methodology and its application to landmark-based human action recognition. In *Stochastic Analysis*, *Filtering*, *and Stochastic Optimization* 431–464. Springer, Cham. MR4433824 https://doi.org/10.1007/978-3-030-98519-6_18

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STATIONARY STATES FOR STABLE PROCESSES WITH PARTIAL RESETTING

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We study a d-dimensional stochastic process $\mathbf{X}$ which arises from a Lévy process $\mathbf{Y}$ by partial resetting, that is, the position of the process $\mathbf{X}$ at a Poisson moment equals $c \in (0, 1)$ times its position right before the moment, and it develops as $\mathbf{Y}$ between these two consecutive moments.

We focus on $\mathbf{Y}$ being a strictly α-stable process with $\alpha \in (0, 2]$ having a transition density: We analyze properties of the transition density p of the process $\mathbf{X}$. We establish a series representation of p. We prove its convergence as time goes to infinity (ergodicity), and we show that the limit $\rho_{\mathbf{Y}}$ (density of the ergodic measure) can be expressed by means of the transition density of the process $\mathbf{Y}$ starting from zero, which results in closed concise formulae for its moments. We show that the process $\mathbf{X}$ reaches a nonequilibrium stationary state. Furthermore, we check that p satisfies the Fokker–Planck equation, and we confirm the harmonicity of $\rho_{\mathbf{Y}}$ with respect to the adjoint generator.

The following cases are discussed in details: Brownian motion, isotropic and d-cylindrical α-stable processes for $\alpha \in (0, 2)$, and α-stable subordinator for $\alpha \in (0, 1)$. We find the asymptotic behavior of $p(t; x, y)$ as $t \to +\infty$ while (t, y) stays in a certain space-time region. For Brownian motion, we discover a phase transition, that is, a change of the asymptotic behavior of $p(t; 0, y)$ with respect to $\rho_{\mathbf{Y}}(y)$.

REFERENCES

[1] AKAHORI, J., CONSTANTINESCU, C., IMAMURA, Y. and PHAM, H. H. (2022). An application of risk theory to mortgage lending. *Scand. Actuar. J.* **5** 447–469. MR4423735 https://doi.org/10.1080/03461238.2021.1995781

[2] ALSMEYER, G. (2003). On the Harris recurrence of iterated random Lipschitz functions and related convergence rate results. *J. Theoret. Probab.* **16** 217–247. MR1956829 https://doi.org/10.1023/A:1022290807360

[3] ARTALEJO, J. R., ECONOMOU, A. and LOPEZ-HERRERO, M. J. (2007). Evaluating growth measures in an immigration process subject to binomial and geometric catastrophes. *Math. Biosci. Eng.* **4** 573–594. MR2388812 https://doi.org/10.3934/mbe.2007.4.573

[4] ASSAF, M., KAMENEV, A. and MEERSON, B. (2008). Population extinction in a time-modulated environment. *Phys. Rev. E* **78** 041123, 13 pp. MR2529574 https://doi.org/10.1103/PhysRevE.78.041123

[5] AVRACHENKOV, K., PIUNOVSKIY, A. and ZHANG, Y. (2013). Markov processes with restart. *J. Appl. Probab.* **50** 960–968. MR3161367 https://doi.org/10.1239/jap/1389370093

[6] BASRAK, B., DAVIS, R. A. and MIKOSCH, T. (2002). Regular variation of GARCH processes. *Stoch. Process. Appl.* **99** 95–115. MR1894253 https://doi.org/10.1016/S0304-4149(01)00156-9

[7] BEHME, A. (2011). Generalized Ornstein–Uhlenbeck processes and extensions. Ph.D. thesis, Braunschweig Universite.

[8] BELL, W. J. (1991). *Searching Behaviour*: *The Behavioural Ecology of Finding Resources*. CRC Press, Boca Raton.

[9] BÉNICHOU, O., COPPEY, M., MOREAU, M., SUET, P.-H. and VOITURIEZ, R. (2005). Optimal search strategies for hidden targets. *Phys. Rev. Lett.* **94** 198101.

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[10] BÉNICHOU, O., LOVERDO, C., MOREAU, M. and VOITURIEZ, R. (2011). Intermittent search strategies. *Rev. Modern Phys*. **83** 81.

[11] BEREZHKOVSKII, A., SZABO, A., URBAKH, R. and KOLOMEISKY, A. (2016). Dependence of the enzymatic velocity on the substrate dissociation rate. *J. Phys. Chem., B* **121** 3437.

[12] BILLINGSLEY, P. (1995). *Probability and Measure*, 3rd ed. *Wiley Series in Probability and Mathematical Statistics*. Wiley, New York. MR1324786

[13] BLUMENTHAL, R. M. and GETOOR, R. K. (1960). Some theorems on stable processes. *Trans. Amer. Math. Soc*. **95** 263–273. MR0119247 https://doi.org/10.2307/1993291

[14] BOGDAN, K., STÓS, A. and SZTONYK, P. (2003). Harnack inequality for stable processes on d-sets. *Studia Math*. **158** 163–198. MR2013738 https://doi.org/10.4064/sm158-2-5

[15] BOROVKOV, A. A. and FOSS, S. G. (1992). Stochastically recursive sequences and their generalizations. *Siberian Adv. Math*. **2** 16–81. MR1157423

[16] BOROVKOV, K. and VERE-JONES, D. (2000). Explicit formulae for stationary distributions of stress release processes. *J. Appl. Probab*. **37** 315–321. MR1780992 https://doi.org/10.1017/s0021900200015539

[17] BÖTTCHER, B., SCHILLING, R. and WANG, J. (2013). *Lévy Matters. III. Lévy-Type Processes: Construction, Approximation and Sample Path Properties*. *Lecture Notes in Math*. **2099**. Springer, Cham. MR3156646 https://doi.org/10.1007/978-3-319-02684-8

[18] BOXMA, O., PERRY, D., STADJE, W. and ZACKS, S. (2006). A Markovian growth-collapse model. *Adv. Appl. Probab*. **38** 221–243. MR2213972 https://doi.org/10.1239/aap/1143936148

[19] BRILLIANTOV, N. V. and PÖSCHEL, T. (2004). *Kinetic Theory of Granular Gases*. *Oxford Graduate Texts*. Oxford Univ. Press, Oxford. MR2101911 https://doi.org/10.1093/acprof:oso/9780198530381.001.0001

[20] CALABRESE, S., PORPORATO, A., LAIO, F., D'ODORICO, P. and RIDOLFI, L. (2017). Age distribution dynamics with stochastic jumps in mortality. *Proc. A* **473** 20170451, 15 pp. MR3741842 https://doi.org/10.1098/rspa.2017.0451

[21] CLARK, J. S. (1989). Ecological disturbance as a renewal process: Theory and application to fire history. *Oikos* **56** 17.

[22] COFRÉ, R., VIDELA, L. and ROSAS, F. (2019). An introduction to the non-equilibrium steady states of maximum entropy spike trains. *Entropy* **21** Paper No. 884, 28 pp. MR4016421 https://doi.org/10.3390/e21090884

[23] DAHLENBURG, M., CHECHKIN, A. V., SCHUMER, R. and METZLER, R. (2021). Stochastic resetting by a random amplitude. *Phys. Rev. E* **103** Paper No. 052123, 22 pp. MR4277661 https://doi.org/10.1103/physreve.103.052123

[24] DERRIDA, B. (2007). Non-equilibrium steady states: Fluctuations and large deviations of the density and of the current. *J. Stat. Mech. Theory Exp*. **2007** P07023, 45 pp. MR2335699 https://doi.org/10.1088/1742-5468/2007/07/p07023

[25] DI BELLO, C., CHECHKIN, A. V., HARTMANN, A. K., PALMOWSKI, Z. and METZLER, R. (2023). Time-dependent probability density function for partial resetting dynamics. *New J. Phys*. **25** Paper No. 082002, 15 pp. MR4642350 https://doi.org/10.1088/1367-2630/aced1d

[26] DOETSCH, G. (1974). *Introduction to the Theory and Application of the Laplace Transformation*. Springer, New York. MR0344810

[27] DUMAS, V., GUILLEMIN, F. and ROBERT, P. (2002). A Markovian analysis of additive-increase multiplicative-decrease algorithms. *Adv. Appl. Probab*. **34** 85–111. MR1895332 https://doi.org/10.1239/aap/1019160951

[28] ELIAZAR, I. and KLAFTER, J. (2004). A growth-collapse model: Lévy inflow, geometric crashes, and generalized Ornstein–Uhlenbeck dynamics. *Phys. A* **334** 1–21. MR2044928 https://doi.org/10.1016/j.physa.2003.11.007

[29] EULE, S. and METZGER, J. (2016). Non-equilibrium steady states of stochastic processes with intermittent resetting. *New J. Phys*. **18** 033006.

[30] EVANS, M. R. and MAJUMDAR, S. N. (2011). Diffusion with stochastic resetting. *Phys. Rev. Lett*. **106** 160601.

[31] EVANS, M. R. and MAJUMDAR, S. N. (2014). Diffusion with resetting in arbitrary spatial dimension. *J. Phys. A* **47** 285001, 19 pp. MR3225982 https://doi.org/10.1088/1751-8113/47/28/285001

[32] EVANS, M. R., MAJUMDAR, S. N. and MALLICK, K. (2013). Optimal diffusive search: Nonequilibrium resetting versus equilibrium dynamics. *J. Phys. A* **46** 185001, 13 pp. MR3055665 https://doi.org/10.1088/1751-8113/46/18/185001

[33] EVANS, M. R., MAJUMDAR, S. N. and SCHEHR, G. (2020). Stochastic resetting and applications. *J. Phys. A* **53** 193001, 67 pp. MR4093464 https://doi.org/10.1088/1751-8121/ab7cfe

[34] FLOREANI, S. and CASANOVA, A. G. (2023). Non-equilibrium steady state of the symmetric exclusion process with reservoirs. Preprint. Available at arXiv:2307.02481.

[35] GERBER, L. R. and HILBORN, R. (2001). Catastrophic events and recovery from low densities in populations of otariids: Implications for risk of extinction. *Mammal Rev.* **31** 131.

[36] GORENFLO, R., KILBAS, A. A., MAINARDI, F. and ROGOSIN, S. (2020). *Mittag-Leffler Functions*, *Related Topics and Applications*. *Springer Monographs in Mathematics*. Springer, Berlin. MR4179587 https://doi.org/10.1007/978-3-662-61550-8

[37] GRIER, D. G. and ROICHMAN, Y. (2006). Holographic optical trapping. *Appl*. *Opt*. **45** 880–887.

[38] GRINCEVIČIUS, A. R. (1975). Random difference equations and renewal theory for products of random matrices. *Lith*. *Math*. *J*. **15** 580–589.

[39] GRÜNWALD, D. and SINGER, R. H. (2010). In vivo imaging of labelled endogenous β-actin mRNA during nucleocytoplasmic transport. *Nature* **467** 604.

[40] GRZYWNY, T. and SZCZYPKOWSKI, K. (2019). Heat kernels of non-symmetric Lévy-type operators. *J. Differ. Equ*. **267** 6004–6064. MR3996792 https://doi.org/10.1016/j.jde.2019.06.013

[41] GRZYWNY, T. and SZCZYPKOWSKI, K. (2020). Lévy processes: Concentration function and heat kernel bounds. *Bernoulli* **26** 3191–3223. MR4140542 https://doi.org/10.3150/20-BEJ1220

[42] GUILLEMIN, F., ROBERT, P. and ZWART, B. (2004). AIMD algorithms and exponential functionals. *Ann*. *Appl*. *Probab*. **14** 90–117. MR2023017 https://doi.org/10.1214/aoap/1075828048

[43] GUPTA, S. and JAYANNAVAR, A. M. (2022). Stochastic resetting: A (very) brief review. *Front*. *Phys*. **10** 789097.

[44] HARDIN, D. (1984). Skewed stable variables and processes. Technical report, North Carolina Univ.

[45] HARTMAN, P. and WINTNER, A. (1942). On the infinitesimal generators of integral convolutions. *Amer. J. Math*. **64** 273–298. MR0006635 https://doi.org/10.2307/2371683

[46] ISERLES, A. (1993). On the generalized pantograph functional-differential equation. *European J*. *Appl*. *Math*. **4** 1–38. MR1208418 https://doi.org/10.1017/S0956792500000966

[47] KELLA, O., PERRY, D. and STADJE, W. (2003). A stochastic clearing model with a Brownian and a compound Poisson component. *Probab*. *Engrg*. *Inform*. *Sci*. **17** 1–22. MR1959382 https://doi.org/10.1017/S026996480317101X

[48] KNOPOVA, V. and SCHILLING, R. L. (2013). A note on the existence of transition probability densities of Lévy processes. *Forum Math*. **25** 125–149. MR3010850 https://doi.org/10.1515/form.2011.108

[49] KOU, S. (2002). A jump-diffusion model for option pricing. *Manag*. *Sci*. **48** 1086.

[50] KUSOLITSCH, N. (2010). Why the theorem of Scheffé should be rather called a theorem of Riesz. *Period*. *Math*. *Hungar.* **61** 225–229. MR2728440 https://doi.org/10.1007/s10998-010-3225-6

[51] LÖPKER, A. and STADJE, W. (2011). Hitting times and the running maximum of Markovian growth-collapse processes. *J*. *Appl*. *Probab*. **48** 295–312. MR2840300 https://doi.org/10.1239/jap/1308662628

[52] MAJUMDAR, S. N., SABHAPANDIT, S. and SCHEHR, G. (2015). Dynamical transition in the temporal relaxation of stochastic processes under resetting. *Phys*. *Rev*. *E* **91** 052131, 8 pp. MR3476293 https://doi.org/10.1103/PhysRevE.91.052131

[53] MALAKAR, K., JEMSEENA, V., KUNDU, A., VIJAY KUMAR, K., SABHAPANDIT, S., MAJUMDAR, S. N., REDNER, S. and DHAR, A. (2018). Steady state, relaxation and first-passage properties of a run-and-tumble particle in one-dimension. *J*. *Stat*. *Mech*. *Theory Exp*. **2018** 043215, 26 pp. MR3801967 https://doi.org/10.1088/1742-5468/aab84f

[54] MAU, X. Y., FENG, X. and PORPORATO, A. (2014). Multiplicative jump processes and applications to leaching of salt and contaminants in the soil. *Phys*. *Rev*. *E* **90** 1.

[55] MENDOZA, E. G. (2010). Sudden stops, financial crises, and leverage. *Amer. Econ*. *Rev*. **100** 1941.

[56] MERTON, R. C. (1976). Option pricing when underlying stock returns are discontinuous. *J*. *Financ*. *Econ*. **3** 125.

[57] MEYN, S. P. and TWEEDIE, R. L. (1993). *Markov Chains and Stochastic Stability*. *Communications and Control Engineering Series*. Springer, London. MR1287609 https://doi.org/10.1007/978-1-4471-3267-7

[58] MIKOSCH, T. and SAMORODNITSKY, G. (2000). The supremum of a negative drift random walk with dependent heavy-tailed steps. *Ann*. *Appl*. *Probab*. **10** 1025–1064. MR1789987 https://doi.org/10.1214/aoap/1019487517

[59] MUKHERJEE, B., SENGUPTA, K. and MAJUMDAR, S. N. (2018). Quantum dynamics with stochastic reset. *Phys*. *Rev. B* **98** 104309.

[60] ODORICO, P. D., LAIO, F., RIDOLFI, L., ODORICO, P. D., LAIO, F. and RIDOLFI, L. (2006). A probabilistic analysis of fire-induced tree-grass coexistence in savannas. *Amer. Nat*. **167** 78.

[61] OTT, T. J. and KEMPERMAN, J. H. B. (2008). Transient behavior of processes in the TCP paradigm. *Probab. Engrg*. *Inform*. *Sci*. **22** 431–471. MR2426601 https://doi.org/10.1017/S0269964808000259

[62] OTT, T. J., KEMPERMAN, J. H. B. and MATHIS, M. (1996). The stationary behavior of ideal TCP congestion avoidance. Available at http://www.teunisott.com/Papers/TCP_Paradigm/TCPwindow.pdf.

[63] PAL, A. (2015). Diffusion in a potential landscape with stochastic resetting. *Phys. Rev. E* **91** 012113.
[64] PLATA, C. A., GUPTA, D. and AZAELE, S. (2020). Asymmetric stochastic resetting: Modeling catastrophic events. *Phys. Rev. E* **102** 052116, 9 pp. MR4189343
[65] REUVENI, S., URBAKH, M. and KLAFTER, J. (2014). Role of substrate unbinding in Michaelis–Menten enzymatic reactions. *Proc. Natl. Acad. Sci. USA* **111** 4391.
[66] ROBIN, T., HADANY, L. and URBAKH, M. (2019). Random search with resetting as a strategy for optimal pollination. *Phys. Rev. E* **99** 052119.
[67] ROBIN, T., REUVENI, S. and URBAKH, M. (2018). Single-molecule theory of enzymatic inhibition. *Nat. Commun.* **9** 779.
[68] ROLDAN, E., LISICA, A., SANCHEZ-TALTAVULL, D. and GRILL, S. W. (2016). Stochastic resetting in backtrack recovery by RNA polymerases. *Phys. Rev. E* **93** 062411.
[69] SATO, K. (1999). *Lévy Processes and Infinitely Divisible Distributions*. *Cambridge Studies in Advanced Mathematics* **68**. Cambridge Univ. Press, Cambridge. MR1739520
[70] FUCHS, J., GOLDT, S. and SEIFERT, U. (2016). Stochastic thermodynamics of resetting. *Europhys. Lett.* **113** 60009.
[71] SHANBHAG, D. N. and SREEHARI, M. (1977). On certain self-decomposable distributions. *Z. Wahrsch. Verw. Gebiete* **38** 217–222. MR0436267 https://doi.org/10.1007/BF00537265
[72] SORNETTE, D. (2017). *Why Stock Markets Crash: Critical Events in Complex Financial Systems*. Princeton Univ. Press, Princeton.
[73] SUWEIS, S., PORPORATO, A., RINALDO, A. and MARITAN, A. (2011). Prescription-induced jump distributions in multiplicative Poisson processes. *Phys. Rev. E* **83** 1.
[74] SUWEIS, S., RINALDO, A., VAN DER ZEE, S. E., DALY, E., MARITAN, A. and PORPORATO, A. (2010). Stochastic modeling of soil salinity. *Geophys. Res. Lett.* **37** 1.
[75] TAL-FRIEDMAN, O., PAL, A., SEKHON, A., REUVENI, S. and ROICHMAN, Y. (2020). Experimental realization of diffusion with stochastic resetting. *J. Phys. Chem. Lett.* **11** 7350–7355.
[76] TAL-FRIEDMAN, O., ROICHMAN, Y. and REUVENI, S. (2022). Diffusion with partial resetting. *Phys. Rev. E* **106** Paper No. 054116, 13 pp. MR4525953 https://doi.org/10.1103/physreve.106.054116
[77] VAN DER HOFSTAD, R., KAPODISTRIA, S., PALMOWSKI, Z. and SHNEER, S. (2023). Unified approach for solving exit problems for additive-increase and multiplicative-decrease processes. *J. Appl. Probab.* **60** 85–105. MR4546112 https://doi.org/10.1017/jpr.2022.27
[78] VAN LEEUWAARDEN, J. S. H., LÖPKER, A. H. and OTT, T. J. (2009). TCP and iso-stationary transformations. *Queueing Syst.* **63** 459–475. MR2576022 https://doi.org/10.1007/s11134-009-9145-6
[79] VAN LEEUWAARDEN, J. S. H., LÖPKER, A. H. and OTT, T. J. (2009). TCP and iso-stationary transformations. *Queueing Syst.* **63** 459–475. MR2576022 https://doi.org/10.1007/s11134-009-9145-6
[80] WHITEHOUSE, J., EVANS, M. R. and MAJUMDAR, S. N. (2013). Effect of partial absorption on diffusion with resetting. *Phys. Rev. E* **87** 022118.
[81] WIŚNIEWOLSKI, M. (2024). On the probabilistic representations of solutions of pantograph equations and triangle coefficients. *J. Differ. Equ.* **379** 600–625. MR4661159 https://doi.org/10.1016/j.jde.2023.10.041
[82] WU, Y. and ZHU, W. Q. (2008). Stochastic analysis of a pulse-type prey-predator model. *Phys. Rev. E* **77** 041911, 9 pp. MR2495461 https://doi.org/10.1103/PhysRevE.77.041911
[83] ZHENG, X. G. (1991). Ergodic theorems for stress release processes. *Stoch. Process. Appl.* **37** 239–258. MR1102872 https://doi.org/10.1016/0304-4149(91)90045-E
[84] ZOLOTAREV, V. M. (1986). *One-Dimensional Stable Distributions*. *Translations of Mathematical Monographs* **65**. Amer. Math. Soc., Providence, RI. MR0854867 https://doi.org/10.1090/mmono/065

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ASYMPTOTICS FOR FIRST-PASSAGE PERCOLATION ON LOGARITHMIC SUBGRAPHS OF $\mathbb{Z}^2$

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For $a > 0$ and $b \geq 0$, let $\mathbb{G}_{a,b}$ be the subgraph of $\mathbb{Z}^2$ induced by the vertices between the first coordinate axis and the graph of the function $f = f_{a,b}(u) = a\log(1+u) + b\log(1+\log(1+u))$, $u \geq 0$. It is known that for $a > 0$, the critical value for Bernoulli percolation on $\mathbb{G}_f = \mathbb{G}_{a,b}$ is strictly between $1/2$ and 1, and that if $b > 2a$ then the percolation phase transition is discontinuous. We study first-passage percolation (FPP) on $\mathbb{G}_{a,b}$ with i.i.d. edge-weights (τ_e) satisfying $p = \mathbb{P}(\tau_e = 0) \in [1/2, 1)$ and the "gap condition" $\mathbb{P}(\tau_e \leq \delta) = p$ for some $\delta > 0$. We find the rate of growth of the expected passage time in $\mathbb{G}_f$ from the origin to the line $x = n$, and show that, while when $p = 1/2$ it is of order $n/(a\log n)$, when $p > 1/2$ it can be of order (a) $n^{c_1}/(\log n)^{c_2}$, (b) $(\log n)^{c_3}$, (c) $\log\log n$, or (d) constant, depending on the relationship between a, b, and p. For more general functions f, we prove a central limit theorem for the passage time and show that its variance grows at the same rate as the mean. As a consequence of our methods, we improve the percolation transition result by showing that the phase transition on $\mathbb{G}_{a,b}$ is discontinuous if and only if $b > a$, and improve "sponge crossing dimensions" asymptotics from the 1980s on subcritical percolation crossing probabilities for tall thin rectangles.

REFERENCES

[1] AHLBERG, D. (2015). Convergence towards an asymptotic shape in first-passage percolation on cone-like subgraphs of the integer lattice. *J. Theoret. Probab.* **28** 198–222. MR3320965 https://doi.org/10.1007/s10959-013-0521-0

[2] AUFFINGER, A., DAMRON, M. and HANSON, J. (2017). 50 *Years of First-Passage Percolation*. *University Lecture Series* **68**. Amer. Math. Soc., Providence, RI. MR3729447 https://doi.org/10.1090/ulect/068

[3] BOLLOBÁS, B. and RIORDAN, O. (2006). *Percolation*. Cambridge Univ. Press, New York. MR2283880 https://doi.org/10.1017/CBO9781139167383

[4] CAMPANINO, M., CHAYES, J. T. and CHAYES, L. (1991). Gaussian fluctuations of connectivities in the subcritical regime of percolation. *Probab. Theory Related Fields* **88** 269–341. MR1100895 https://doi.org/10.1007/BF01418864

[5] CAMPANINO, M. and IOFFE, D. (2002). Ornstein-Zernike theory for the Bernoulli bond percolation on $\mathbb{Z}^d$. *Ann. Probab.* **30** 652–682. MR1905854 https://doi.org/10.1214/aop/1023481005

[6] CHAYES, J. T. and CHAYES, L. (1986). Critical points and intermediate phases on wedges of $\mathbb{Z}^d$. *J. Phys. A* **19** 3033–3048. MR0861221

[7] DAMRON, M., HANSON, J. and LAM, W.-K. (2023). Universality of the time constant for $2D$ critical first-passage percolation. *Ann. Appl. Probab.* **33** 1701–1731. MR4583656 https://doi.org/10.1214/22-aap1808

[8] DAMRON, M., LAM, W.-K. and WANG, X. (2017). Asymptotics for $2D$ critical first passage percolation. *Ann. Probab.* **45** 2941–2970. MR3706736 https://doi.org/10.1214/16-AOP1129

[9] DURRETT, R. (2019). *Probability—Theory and Examples*. 5th ed. *Cambridge Series in Statistical and Probabilistic Mathematics* **49**. Cambridge Univ. Press, Cambridge. MR3930614 https://doi.org/10.1017/9781108591034

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[10] DURRETT, R. and SCHONMANN, R. H. (1988). The contact process on a finite set. II. *Ann. Probab.* **16** 1570–1583. MR0958203

[11] FITZNER, R. and VAN DER HOFSTAD, R. (2017). Mean-field behavior for nearest-neighbor percolation in $d > 10$. *Electron. J. Probab.* **22** 43. MR3646069 https://doi.org/10.1214/17-EJP56

[12] GRIMMETT, G. (1999). *Percolation*, 2nd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **321**. Springer, Berlin. MR1707339 https://doi.org/10.1007/978-3-662-03981-6

[13] GRIMMETT, G. R. (1981). Critical sponge dimensions in percolation theory. *Adv. Appl. Probab.* **13** 314–324. MR0612206 https://doi.org/10.2307/1426686

[14] GRIMMETT, G. R. (1983). Bond percolation on subsets of the square lattice, and the threshold between one-dimensional and two-dimensional behaviour. *J. Phys. A* **16** 599–604. MR0701479

[15] GUT, A. (1988). *Stopped Random Walks. Limit Theorems and Applications*. *Applied Probability. A Series of the Applied Probability Trust* **5**. Springer, New York. MR0916870 https://doi.org/10.1007/978-1-4757-1992-5

[16] JIANG, J. and YAO, C.-L. (2019). Critical first-passage percolation starting on the boundary. *Stoch. Process. Appl.* **129** 2049–2065. MR3958423 https://doi.org/10.1016/j.spa.2018.06.008

[17] KESTEN, H. (1980). The critical probability of bond percolation on the square lattice equals $\frac{1}{2}$. *Comm. Math. Phys.* **74** 41–59. MR0575895

[18] KESTEN, H. and ZHANG, Y. (1997). A central limit theorem for "critical" first-passage percolation in two dimensions. *Probab. Theory Related Fields* **107** 137–160. MR1431216 https://doi.org/10.1007/s004400050080

[19] MCLEISH, D. L. (1974). Dependent central limit theorems and invariance principles. *Ann. Probab.* **2** 620–628. MR0358933 https://doi.org/10.1214/aop/1176996608

[20] YAO, C.-L. (2013). A CLT for winding angles of the arms for critical planar percolation. *Electron. J. Probab.* **18** 85. MR3109624 https://doi.org/10.1214/EJP.v18-2285

[21] YAO, C.-L. (2014). Law of large numbers for critical first-passage percolation on the triangular lattice. *Electron. Commun. Probab.* **19** 18. MR3183571 https://doi.org/10.1214/ECP.v19-3268

[22] YAO, C.-L. (2018). Limit theorems for critical first-passage percolation on the triangular lattice. *Stoch. Process. Appl.* **128** 445–460. MR3739504 https://doi.org/10.1016/j.spa.2017.05.002

[23] YAO, C.-L. (2019). Asymptotics for 2D critical and near-critical first-passage percolation. *Probab. Theory Related Fields* **175** 975–1019. MR4026611 https://doi.org/10.1007/s00440-019-00908-2

[24] ZHANG, Y. (1999). Double behavior of critical first-passage percolation. In *Perplexing Problems in Probability*. *Progress in Probability* **44** 143–158. Birkhäuser, Boston, MA. MR1703129

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QUANTITATIVE INSTABILITY FOR STOCHASTIC SCALAR REACTION-DIFFUSION EQUATIONS

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This work studies the instability of stochastic scalar reaction diffusion equations, driven by a multiplicative noise that is white in time and smooth in space, near to zero, which is assumed to be a fixed point for the equation. We prove that if the Lyapunov exponent at zero is positive, then the flow of nonzero solutions admits uniform bounds on small negative moments. The proof builds on ideas from stochastic homogenisation. We require suitable corrector estimates for the solution to a Poisson problem involving an infinite-dimensional projective process, together with a linearisation step that hinges on quantitative parametrix-like arguments. Overall, we are able to construct an appropriate Lyapunov functional for the nonlinear dynamics and address some problems left open in the literature.

REFERENCES

[1] BEDROSSIAN, J., BLUMENTHAL, A. and PUNSHON-SMITH, S. (2021). Almost-sure enhanced dissipation and uniform-in-diffusivity exponential mixing for advection-diffusion by stochastic Navier–Stokes. *Probab. Theory Related Fields* **179** 777–834. MR4242626 https://doi.org/10.1007/s00440-020-01010-8

[2] BEDROSSIAN, J., BLUMENTHAL, A. and PUNSHON-SMITH, S. (2022). Lagrangian chaos and scalar advection in stochastic fluid mechanics. *J. Eur. Math. Soc.* (*JEMS*) **24** 1893–1990. MR4404792 https://doi.org/10.4171/jems/1140

[3] BEDROSSIAN, J., BLUMENTHAL, A. and PUNSHON-SMITH, S. (2022). A regularity method for lower bounds on the Lyapunov exponent for stochastic differential equations. *Invent. Math.* **227** 429–516. MR4372219 https://doi.org/10.1007/s00222-021-01069-7

[4] BERGLUND, N. and GENTZ, B. (2013). Sharp estimates for metastable lifetimes in parabolic SPDEs: Kramers' law and beyond. *Electron. J. Probab.* **18** no. 24. MR3035752 https://doi.org/10.1214/EJP.v18-1802

[5] BLÖMKER, D. and NEAMŢU, A. (2023). Bifurcation theory for SPDEs: Finite-time Lyapunov exponents and amplitude equations. *SIAM J. Appl. Dyn. Syst.* **22** 2150–2179. MR4624849 https://doi.org/10.1137/23M1549638

[6] BLUMENTHAL, A., COTI ZELATI, M. and GVALANI, R. S. (2023). Exponential mixing for random dynamical systems and an example of Pierrehumbert. *Ann. Probab.* **51** 1559–1601. MR4597327 https://doi.org/10.1214/23-aop1627

[7] BLUMENTHAL, A., ENGEL, M. and NEAMŢU, A. (2023). On the pitchfork bifurcation for the Chafee–Infante equation with additive noise. *Probab. Theory Related Fields* **187** 603–627. MR4664581 https://doi.org/10.1007/s00440-023-01235-3

[8] BUSHELL, P. J. (1973). Hilbert's metric and positive contraction mappings in a Banach space. *Arch. Ration. Mech. Anal.* **52** 330–338. MR0336473 https://doi.org/10.1007/BF00247467

[9] BUTKOVSKY, O. and SCHEUTZOW, M. (2020). Couplings via comparison principle and exponential ergodicity of SPDEs in the hypoelliptic setting. *Comm. Math. Phys.* **379** 1001–1034. MR4163359 https://doi.org/10.1007/s00220-020-03834-w

[10] COTI ZELATI, M. and HAIRER, M. (2021). A noise-induced transition in the Lorenz system. *Comm. Math. Phys.* **383** 2243–2274. MR4244269 https://doi.org/10.1007/s00220-021-04000-6

[11] DA PRATO, G. and ZABCZYK, J. (1992). A note on stochastic convolution. *Stoch. Anal. Appl.* **10** 143–153. MR1154532 https://doi.org/10.1080/07362999208809260

MSC2020 subject classifications. 60H15.
Key words and phrases. SPDEs, Instability, Lyapunov exponents, reaction-diffusion.

[12] DUNLAP, A., GU, Y. and KOMOROWSKI, T. (2023). Fluctuation exponents of the KPZ equation on a large torus. *Comm. Pure Appl. Math.* **76** 3104–3149. MR4642816 https://doi.org/10.1002/cpa.22110

[13] ETHERIDGE, A., FREEMAN, N. and PENINGTON, S. (2017). Branching Brownian motion, mean curvature flow and the motion of hybrid zones. *Electron. J. Probab.* **22** Paper No. 103. MR3733661 https://doi.org/10.1214/17-EJP127

[14] FLANDOLI, F., GESS, B. and SCHEUTZOW, M. (2017). Synchronization by noise for order-preserving random dynamical systems. *Ann. Probab.* **45** 1325–1350. MR3630300 https://doi.org/10.1214/16-AOP1088

[15] GESS, B. and TSATSOULIS, P. (2024). Lyapunov exponents and synchronisation by noise for systems of SPDEs. *Ann. Probab.* **52** 1903–1953. MR4791422 https://doi.org/10.1214/24-aop1690

[16] GESS, B. and YAROSLAVTSEV, I. (2021). Stabilization by transport noise and enhanced dissipation in the Kraichnan model.

[17] GU, Y. and KOMOROWSKI, T. (2024). KPZ on torus: Gaussian fluctuations. *Ann. Inst. Henri Poincaré Probab. Stat.* **60** 1570–1618. MR4780499 https://doi.org/10.1214/23-aihp1392

[18] HAIRER, M., MATTINGLY, J. C. and SCHEUTZOW, M. (2011). Asymptotic coupling and a general form of Harris' theorem with applications to stochastic delay equations. *Probab. Theory Related Fields* **149** 223–259. MR2773030 https://doi.org/10.1007/s00440-009-0250-6

[19] HAIRER, M. and ROSATI, T. (2023). Spectral gap for projective processes of linear SPDEs. arXiv preprint. Available at arXiv:2307.07472.

[20] HENING, A. and NGUYEN, D. H. (2018). Coexistence and extinction for stochastic Kolmogorov systems. *Ann. Appl. Probab.* **28** 1893–1942. MR3809480 https://doi.org/10.1214/17-AAP1347

[21] KHOSHNEVISAN, D., KIM, K., MUELLER, C. and SHIU, S.-Y. (2023). Phase analysis for a family of stochastic reaction-diffusion equations. *Electron. J. Probab.* **28** Paper No. 10. MR4620551 https://doi.org/10.1214/23-ejp983

[22] PERKOWSKI, N. and VAN ZUIJLEN, W. (2023). Quantitative heat-kernel estimates for diffusions with distributional drift. *Potential Anal.* **59** 731–752. MR4625000 https://doi.org/10.1007/s11118-021-09984-3

[23] ROSATI, T. C. (2022). Synchronization for KPZ. *Stoch. Dyn.* **22** Paper No. 2250010. MR4439607 https://doi.org/10.1142/S0219493722500101

[24] SCHREIBER, S. J., BENAÏM, M. and ATCHADÉ, K. A. S. (2011). Persistence in fluctuating environments. *J. Math. Biol.* **62** 655–683. MR2786721 https://doi.org/10.1007/s00285-010-0349-5

[25] SINAĬ, Y. G. (1991). Two results concerning asymptotic behavior of solutions of the Burgers equation with force. *J. Stat. Phys.* **64** 1–12. MR1117645 https://doi.org/10.1007/BF01057866

[26] VAN NEERVEN, J. (2007/2008). Stochastic evolution equations. *ISEM Lecture Notes*.

[27] WALSH, J. B. (1986). An introduction to stochastic partial differential equations. In *École D'été de Probabilités de Saint-Flour, XIV*—1984. *Lecture Notes in Math.* **1180** 265–439. Springer, Berlin. MR0876085 https://doi.org/10.1007/BFb0074920

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THE METRIC REMOVABILITY OF INTERFACES IN THE DIRECTED LANDSCAPE

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The directed landscape is a prominent model of random geometry which is believed to be the universal scaling limit of all planar random geometries in the Kardar–Parisi–Zhang universality class. It comes equipped with a few different natural simple curves associated to it, such as geodesics and interfaces. Given such a curve, one might wonder whether the geometry of this curve determines the entire landscape, or if in fact, there is nontrivial extra information actually present "on" the curve. In this paper, we show that the former is true for an interface in the directed landscape, while the latter is true for a geodesic instead. Further, as is used in the proof of the first assertion above, we precisely identify the correct Hausdorff dimension of the set of times where any geodesic intersects an interface—we show that this set a.s. has dimension zero.

REFERENCES

ANG, M., HOLDEN, N. and SUN, X. (2024). Integrability of SLE via conformal welding of random surfaces. *Comm. Pure Appl. Math.* **77** 2651–2707. MR4720219 https://doi.org/10.1002/cpa.22180

BASU, R. and BHATIA, M. (2023). A Peano curve from mated geodesic trees in the directed landscape.

BASU, R., GANGULY, S. and HAMMOND, A. (2021). Fractal geometry of Airy_2 processes coupled via the Airy sheet. *Ann. Probab.* **49** 485–505. MR4203343 https://doi.org/10.1214/20-AOP1444

BASU, R., SARKAR, S. and SLY, A. (2019). Coalescence of geodesics in exactly solvable models of last passage percolation. *J. Math. Phys.* **60** 093301, 22 pp. MR4002528 https://doi.org/10.1063/1.5093799

BATES, E., GANGULY, S. and HAMMOND, A. (2022). Hausdorff dimensions for shared endpoints of disjoint geodesics in the directed landscape. *Electron. J. Probab.* **27** Paper No. 1, 44 pp. MR4361743 https://doi.org/10.1214/21-ejp706

BHATIA, M. (2022). Atypical stars on a directed landscape geodesic.

BHATIA, M. (2024). Duality in the directed landscape and its applications to fractal geometry. *Int. Math. Res. Not. IMRN* **2024** 9633–9662. MR4761771 https://doi.org/10.1093/imrn/rnae051

BHATIA, M. (2026a). Conditional independence for Peano bubbles in the directed landscape. Upcoming.

BHATIA, M. (2026b). The directed landscape is determined by any parametrised geodesic tree. Upcoming.

BUSANI, O. (2024). Diffusive scaling limit of the Busemann process in last passage percolation. *Ann. Probab.* **52** 1650–1712. https://doi.org/10.1214/24-AOP1681

BUSANI, O. (2025). Non-existence of three non-coalescing infinite geodesics with the same direction in the directed landscape. *Probab. Theory Related Fields* 1–47. https://doi.org/10.1007/s00440-025-01451-z

BUSANI, O., SEPPÄLÄINEN, T. and SORENSEN, E. (2024). The stationary horizon and semi-infinite geodesics in the directed landscape. *Ann. Probab.* **52** 1–66. MR4698024 https://doi.org/10.1214/23-aop1655

BUSEMANN, H. (2012). *The Geometry of Geodesics*. Academic Press, New York. MR0075623

CORWIN, I., HAMMOND, A., HEGDE, M. and MATETSKI, K. (2023). Exceptional times when the KPZ fixed point violates Johansson's conjecture on maximizer uniqueness. *Electron. J. Probab.* **28** Paper No. 11, 81 pp. MR4536117 https://doi.org/10.1214/22-ejp898

DAUVERGNE, D. (2024). Non-uniqueness times for the maximizer of the KPZ fixed point. *Adv. Math.* **442** Paper No. 109550, 41 pp. MR4708162 https://doi.org/10.1016/j.aim.2024.109550

DAUVERGNE, D. (2025). The 27 geodesic networks in the directed landscape. *Invent. Math.* **242** 123–220. MR4953176 https://doi.org/10.1007/s00222-025-01355-8

DAUVERGNE, D., ORTMANN, J. and VIRÁG, B. (2022). The directed landscape. *Acta Math.* **229** 201–285. MR4554223 https://doi.org/10.4310/acta.2022.v229.n2.a1

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DAUVERGNE, D., SARKAR, S. and VIRÁG, B. (2022). Three-halves variation of geodesics in the directed landscape. *Ann. Probab.* **50** 1947–1985. MR4474505 https://doi.org/10.1214/22-aop1574

DAUVERGNE, D. and VIRÁG, B. (2021). The scaling limit of the longest increasing subsequence. Preprint. Available at arXiv:2104.08210.

DAUVERGNE, D. and ZHANG, L. (2024). Disjoint optimizers and the directed landscape. *Mem. Amer. Math. Soc.* **303** v+103. MR4837616 https://doi.org/10.1090/memo/1524

DING, J., DUBÉDAT, J. and GWYNNE, E. (2023). Introduction to the Liouville quantum gravity metric. In *ICM—International Congress of Mathematicians. Vol.* 6. *Sections* 12–14 4212–4244. EMS Press, Berlin. MR4680401

DUPLANTIER, B., MILLER, J. and SHEFFIELD, S. (2021). Liouville quantum gravity as a mating of trees. *Astérisque* **427** viii+257. MR4340069 https://doi.org/10.24033/ast

FERRARI, P., MARTIN, J. and PIMENTEL, L. (2006). Roughening and inclination of competition interfaces. *Phys. Rev. E, Stat. Nonlin. Soft Matter Phys.* **73** 031602. https://doi.org/10.1103/PhysRevE.73.031602

GANGULY, S. and HAMMOND, A. (2023). The geometry of near ground states in Gaussian polymer models. *Electron. J. Probab.* **28** Paper No. 60, 80 pp. MR4583067 https://doi.org/10.1214/22-ejp895

GANGULY, S. and ZHANG, L. (2022). Fractal geometry of the space-time difference profile in the directed landscape via construction of geodesic local times.

HAMMOND, A. (2020). Exponents governing the rarity of disjoint polymers in Brownian last passage percolation. *Proc. Lond. Math. Soc.* (3) **120** 370–433. MR4008374 https://doi.org/10.1112/plms.12292

HOFFMAN, C. (2005). Coexistence for Richardson type competing spatial growth models. *Ann. Appl. Probab.* **15** 739–747. MR2114988 https://doi.org/10.1214/105051604000000729

HUGHES, L. and MILLER, J. (2022). Equivalence of metric gluing and conformal welding in γ-Liouville quantum gravity for $\gamma \in (0, 2)$. *Ann. Inst. Henri Poincaré Probab. Stat.* **61** 1961–2003. https://doi.org/10.1214/24-AIHP1480

JONES, P. W. and SMIRNOV, S. K. (2000). Removability theorems for Sobolev functions and quasiconformal maps. *Ark. Mat.* **38** 263–279. MR1785402 https://doi.org/10.1007/BF02384320

KALMYKOV, S., KOVALEV, L. V. and RAJALA, T. (2019). Removable sets for intrinsic metric and for holomorphic functions. *J. Anal. Math.* **139** 751–772. MR4041119 https://doi.org/10.1007/s11854-024-0076-2

KARDAR, M., PARISI, G. and ZHANG, Y.-C. (1986). Dynamic scaling of growing interfaces. *Phys. Rev. Lett.* **56** 889–892. https://doi.org/10.1103/PhysRevLett.56.889

KAVVADIAS, K., MILLER, J. and SCHOUG, L. (2022). Conformal removability of SLE_4.

MARTIN, J. B., SLY, A. and ZHANG, L. (2025). Convergence of the environment seen from geodesics in exponential last-passage percolation. *J. Eur. Math. Soc.* (*JEMS*) **27** 877–970. https://doi.org/10.4171/JEMS/1594

MILLER, J. and QIAN, W. (2020). The geodesics in Liouville quantum gravity are not Schramm-Loewner evolutions. *Probab. Theory Related Fields* **177** 677–709. MR4126929 https://doi.org/10.1007/s00440-019-00949-7

NEWMAN, C. M. (1995). A surface view of first-passage percolation. In *Proceedings of the International Congress of Mathematicians, Vol.* 1, 2 (*Zürich*, 1994) 1017–1023. Birkhäuser, Basel. MR1404001

PIMENTEL, L. P. R. (2016). Duality between coalescence times and exit points in last-passage percolation models. *Ann. Probab.* **44** 3187–3206. MR3551194 https://doi.org/10.1214/15-AOP1044

RAHMAN, M. and VIRÁG, B. (2025). Infinite geodesics, competition interfaces and the second class particle in the scaling limit. *Ann. Inst. Henri Poincaré Probab. Stat.* **61** 1075–1126. MR4901636 https://doi.org/10.1214/23-aihp1454

ROHDE, S. and SCHRAMM, O. (2005). Basic properties of SLE. *Ann. of Math.* (2) **161** 883–924. MR2153402 https://doi.org/10.4007/annals.2005.161.883

SHEFFIELD, S. (2016). Conformal weldings of random surfaces: SLE and the quantum gravity zipper. *Ann. Probab.* **44** 3474–3545. MR3551203 https://doi.org/10.1214/15-AOP1055

SHEFFIELD, S. (2023). What is a random surface? In *ICM—International Congress of Mathematicians. Vol.* 2. *Plenary Lectures* 1202–1258. EMS Press, Berlin. MR4680280

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GENERAL MARKOVIAN RANDOMIZED EQUILIBRIUM EXISTENCE AND CONSTRUCTION IN ZERO-SUM DYNKIN GAMES FOR DIFFUSIONS

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One of the most classical games for stochastic processes is the zero-sum Dynkin (stopping) game. We present a complete equilibrium solution to a general formulation of this game with an underlying one-dimensional diffusion. A key result is the construction of a characterizable global ϵ-Nash equilibrium in Markovian randomized stopping times for every $\epsilon > 0$. This is achieved by leveraging the well-known equilibrium structure under a restrictive ordering condition on the payoff functions, leading to a novel approach based on an appropriate notion of randomization that allows for solving the general game without any ordering condition. Additionally, we provide conditions for the existence of pure and randomized Nash equilibria (with $\epsilon = 0$). Our results enable explicit identification of equilibrium stopping times and their corresponding values in many cases, illustrated by several examples.

REFERENCES

[1] BAURDOUX, E. J. and KYPRIANOU, A. E. (2004). Further calculations for Israeli options. *Stoch. Stoch. Rep.* **76** 549–569. MR2100021 https://doi.org/10.1080/10451120412331313438

[2] BAXTER, J. R. and CHACON, R. V. (1977). Compactness of stopping times. *Z. Wahrsch. Verw. Gebiete* **40** 169–181. MR0517871 https://doi.org/10.1007/BF00736045

[3] BAYRAKTAR, E. and YAO, S. (2017). On the robust Dynkin game. *Ann. Appl. Probab.* **27** 1702–1755. MR3678483 https://doi.org/10.1214/16-AAP1243

[4] BODNARIU, A., CHRISTENSEN, S. and LINDENSJÖ, K. (2024). Local time pushed mixed equilibrium strategies for time-inconsistent stopping problems. *SIAM J. Control Optim.* **62** 1261–1290. MR4729445 https://doi.org/10.1137/22M1506651

[5] BORODIN, A. N. and SALMINEN, P. (1996). *Handbook of Brownian Motion—Facts and Formulae*. *Probability and Its Applications*. Birkhäuser, Basel. MR1477407 https://doi.org/10.1007/978-3-0348-7652-0

[6] BUCKDAHN, R. and ENGELBERT, H. J. (1984). Randomized stopping times: Doob's optional sampling theorem and optimal stopping. *Math. Nachr.* **115** 237–247. MR0755279 https://doi.org/10.1002/mana.19841150117

[7] CHRISTENSEN, S. and LINDENSJÖ, K. (2020). Time-inconsistent stopping, myopic adjustment and equilibrium stability: With a mean-variance application. In *Stochastic Modeling and Control* (J. Jakubowski, M. Nieweglowski, M. Rasonyi and L. Stettner, eds.) *Banach Center Publ.* **122** 53–76. Polish Acad. Sci. Inst. Math., Warsaw. MR4276004 https://doi.org/10.4064/bc122-4

[8] CHRISTENSEN, S. and LINDENSJÖ, K. (2020). On time-inconsistent stopping problems and mixed strategy stopping times. *Stoch. Process. Appl.* **130** 2886–2917. MR4080735 https://doi.org/10.1016/j.spa.2019.08.010

[9] CHRISTENSEN, S., LINDENSJÖ, K. and NEUMANN, B. A. (2023). Markovian randomized equilibria for general Markovian Dynkin games in discrete time. arXiv preprint. Available at arXiv:2307.13413.

[10] CHRISTENSEN, S. and SCHULTZ, B. (2024). On the existence of Markovian randomized equilibria in Dynkin games of war-of-attrition-type. arXiv preprint. Available at arXiv:2406.09820.

[11] CVITANIĆ, J. and KARATZAS, I. (1996). Backward stochastic differential equations with reflection and Dynkin games. *Ann. Probab.* **24** 2024–2056. MR1415239 https://doi.org/10.1214/aop/1041903216

[12] DE ANGELIS, T., FERRARI, G. and MORIARTY, J. (2018). Nash equilibria of threshold type for two-player nonzero-sum games of stopping. *Ann. Appl. Probab.* **28** 112–147. MR3770874 https://doi.org/10.1214/17-AAP1301

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[13] DE ANGELIS, T., MERKULOV, N. and PALCZEWSKI, J. (2022). On the value of non-Markovian Dynkin games with partial and asymmetric information. *Ann. Appl. Probab.* **32** 1774–1813. MR4430001 https://doi.org/10.1214/21-aap1721

[14] DECAMPS, J.-P., GENSBITTEL, F. and MARIOTTI, T. (2023). The war of attrition under uncertainty: Theory and robust testable implications. Available at SSRN 4642426.

[15] DÉCAMPS, J.-P., GENSBITTEL, F. and MARIOTTI, T. (2024). Mixed Markov-perfect equilibria in the continuous-time war of attrition. arXiv preprint. Available at arXiv:2407.04878.

[16] DYNKIN, E. B. (1969). A game-theoretic version of an optimal stopping problem. *Dokl. Akad. Nauk SSSR* **185** 16–19. MR0241121

[17] EKSTRÖM, E., GLOVER, K. and LENIEC, M. (2017). Dynkin games with heterogeneous beliefs. *J. Appl. Probab.* **54** 236–251. MR3632616 https://doi.org/10.1017/jpr.2016.97

[18] EKSTRÖM, E., LINDENSJÖ, K. and OLOFSSON, M. (2022). How to detect a salami slicer: A stochastic controller-and-stopper game with unknown competition. *SIAM J. Control Optim.* **60** 545–574. MR4385647 https://doi.org/10.1137/21M139044X

[19] EKSTRÖM, E. and PESKIR, G. (2008). Optimal stopping games for Markov processes. *SIAM J. Control Optim.* **47** 684–702. MR2385859 https://doi.org/10.1137/060673916

[20] EKSTRÖM, E. and VILLENEUVE, S. (2006). On the value of optimal stopping games. *Ann. Appl. Probab.* **16** 1576–1596. MR2260074 https://doi.org/10.1214/105051606000000204

[21] FERENSTEIN, E. Z. (2007). Randomized stopping games and Markov market games. *Math. Methods Oper. Res.* **66** 531–544. MR2357664 https://doi.org/10.1007/s00186-006-0143-8

[22] GRÜN, C. (2013). On Dynkin games with incomplete information. *SIAM J. Control Optim.* **51** 4039–4065. MR3120752 https://doi.org/10.1137/120891800

[23] HOBSON, D. and ZENG, M. (2020). Randomised rules for stopping problems. *J. Appl. Probab.* **57** 981–1004. MR4148067 https://doi.org/10.1017/jpr.2020.43

[24] KARATZAS, I. and SHREVE, S. E. (1991). *Brownian Motion and Stochastic Calculus*, 2nd ed. *Graduate Texts in Mathematics* **113**. Springer, New York. MR1121940 https://doi.org/10.1007/978-1-4612-0949-2

[25] KIFER, Y. (2000). Game options. *Finance Stoch.* **4** 443–463. MR1779588 https://doi.org/10.1007/PL00013527

[26] KOBYLANSKI, M., QUENEZ, M.-C. and DE CAMPAGNOLLE, M. R. (2014). Dynkin games in a general framework. *Stochastics* **86** 304–329. MR3180039 https://doi.org/10.1080/17442508.2013.778860

[27] LANDO, D. (1998). On Cox processes and credit risky securities. *Rev. Deriv. Res.* **2** 99–120.

[28] LARAKI, R. and SOLAN, E. (2005). The value of zero-sum stopping games in continuous time. *SIAM J. Control Optim.* **43** 1913–1922. MR2137507 https://doi.org/10.1137/S0363012903429025

[29] MARTYR, R. and MORIARTY, J. (2021). Nonzero-sum games of optimal stopping and generalized Nash equilibrium problems. *SIAM J. Control Optim.* **59** 1443–1465. MR4243661 https://doi.org/10.1137/18M119803X

[30] PENNANEN, T. and PERKKIÖ, A.-P. (2025). Optimal stopping without Snell envelopes. *Proc. Amer. Math. Soc.* **153** 1045–1059. MR4862142 https://doi.org/10.1090/proc/15806

[31] PESKIR, G. (2009). Optimal stopping games and Nash equilibrium. *Teor. Veroyatn. Primen.* **53** 558–571. MR2759714 https://doi.org/10.1137/S0040585X97983821

[32] PESKIR, G. and SHIRYAEV, A. (2006). *Optimal Stopping and Free-Boundary Problems*. *Lectures in Mathematics ETH Zürich*. Birkhäuser, Basel. MR2256030

[33] REVUZ, D. and YOR, M. (1999). *Continuous Martingales and Brownian Motion*, 3rd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **293**. Springer, Berlin. MR1725357 https://doi.org/10.1007/978-3-662-06400-9

[34] RIEDEL, F. and STEG, J.-H. (2017). Subgame-perfect equilibria in stochastic timing games. *J. Math. Econom.* **72** 36–50. MR3697074 https://doi.org/10.1016/j.jmateco.2017.06.006

[35] ROSENBERG, D., SOLAN, E. and VIEILLE, N. (2001). Stopping games with randomized strategies. *Probab. Theory Related Fields* **119** 433–451. MR1821142 https://doi.org/10.1007/PL00008766

[36] SCHULTZ, B. (2024). On differentiability of reward functionals corresponding to Markovian randomized stopping times. arXiv preprint. Available at arXiv:2411.11393.

[37] SHIRYAYEV, A. N. (1978). *Optimal Stopping Rules*. *Applications of Mathematics* **8**. Springer, New York. Translated from the Russian by A. B. Aries. MR0468067

[38] SHMAYA, E. and SOLAN, E. (2014). Equivalence between random stopping times in continuous time. arXiv preprint. Available at arXiv:1403.7886.

[39] SOLAN, E., TSIRELSON, B. and VIEILLE, N. (2012). Random stopping times in stopping problems and stopping games. arXiv preprint. Available at arXiv:1211.5802.

[40] TOUZI, N. and VIEILLE, N. (2002). Continuous-time Dynkin games with mixed strategies. *SIAM J. Control Optim.* **41** 1073–1088. MR1972503 https://doi.org/10.1137/S0363012900369812

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UNIQUENESS OF THE SOLUTION OF THE FILTERING EQUATIONS IN SPACES OF MEASURES FOR GENERAL SIGNAL AND OBSERVATION PROCESSES

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This paper addresses a generalized filtering framework in which the signal process X and observation process Y are both driven by correlated Brownian motions, and the coefficients of their governing stochastic differential equations depend jointly on (X, Y), with the exception of the diffusion coefficient of the observation process, which does not depend upon the signal. Unlike many prior works, the observation equation may have a degenerate (noninvertible or even zero) diffusion coefficient. In this framework, we derive filtering equations and prove of equivalence between the uniqueness of the nonlinear Kushner–Stratonovich equation and the linear Zakai equation. Finally we give a novel proof of uniqueness for the Zakai equation using a backward stochastic partial differential equation (BSPDE), overcoming the limitations of classical duality arguments. This approach successfully handles the randomness and anticipation introduced by the observation-dependent coefficients, which are not tractable under traditional deterministic PDE methods.

REFERENCES

[1] BAIN, A. and CRISAN, D. (2009). *Fundamentals of Stochastic Filtering*. *Stochastic Modelling and Applied Probability* **60**. Springer, New York. MR2454694 https://doi.org/10.1007/978-0-387-76896-0

[2] BEN-ISRAEL, A. and GREVILLE, T. N. E. (2003). *Generalized Inverses*: *Theory and Applications*, 2nd ed. *CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC* **15**. Springer, New York. MR1987382

[3] BENSOUSSAN, A. (1992). *Stochastic Control of Partially Observable Systems*. Cambridge Univ. Press, Cambridge. MR1191160 https://doi.org/10.1017/CBO9780511526503

[4] CRISAN, D., KOURITZIN, M. and XIONG, J. (2009). Nonlinear filtering with signal dependent observation noise. *Electron*. *J*. *Probab*. **14** no. 63, 1863–1883. MR2540851 https://doi.org/10.1214/EJP.v14-687

[5] CRISAN, D. and ROZOVSKIĬ, B., eds. (2011). *The Oxford Handbook of Nonlinear Filtering*. Oxford Univ. Press, Oxford. MR2882749

[6] DU, K. and MENG, Q. (2010). A revisit to W_2^n-theory of super-parabolic backward stochastic partial differential equations in $\mathbb{R}^d$. *Stoch*. *Process*. *Appl*. **120** 1996–2015. MR2673985 https://doi.org/10.1016/j.spa.2010.06.001

[7] DU, K., TANG, S. and ZHANG, Q. (2013). $W^{m,p}$-solution ($p \geq 2$) of linear degenerate backward stochastic partial differential equations in the whole space. *J*. *Differ*. *Equ*. **254** 2877–2904. MR3017034 https://doi.org/10.1016/j.jde.2013.01.013

[8] FOTSA-MBOGNE, D. J. and PARDOUX, E. (2017). Nonlinear filtering with degenerate noise. *Electron*. *Commun*. *Probab*. **22** Paper No. 44, 14. MR3693770 https://doi.org/10.1214/17-ECP74

[9] JOANNIDES, M. and LEGLAND, F. (1997). Nonlinear filtering with continuous time perfect observations and noninformative quadratic variation. In *Proceedings of the* 36*th IEEE Conference on Decision and Control*, *San Diego*, *December* 10–12, 1997, 1645–1650.

[10] KALLIANPUR, G. (1980). *Stochastic Filtering Theory*. *Applications of Mathematics* **13**. Springer, New York. MR0583435

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[11] KARATZAS, I. and SHREVE, S. E. (1991). *Brownian Motion and Stochastic Calculus*, 2nd ed. *Graduate Texts in Mathematics* **113**. Springer, New York. MR1121940 https://doi.org/10.1007/978-1-4612-0949-2

[12] KRYLOV, N. V. and ROZOVSKIĬ, B. L. (1981). On the first integrals and Liouville equations for diffusion processes. In *Stochastic Differential Systems* (*Visegrád*, 1980). *Lect*. *Notes Control Inf*. *Sci*. **36** 117–125. Springer, Berlin. MR0653654

[13] LIPTSER, R. S. and SHIRYAEV, A. N. (2001). *Statistics of Random Processes*. *I*: *General Theory*, expanded ed. *Applications of Mathematics* (*New York*) **5**. Springer, Berlin. MR1800857

[14] PARDOUX, E. (1975). Equations aux dérivées partielles stochastiques non linéaries monotones Ph.D. thesis, Univ Paris XI, Orsay.

[15] PARDOUX, E. (1979). Stochastic partial differential equations and filtering of diffusion processes. *Stochastics* **3** 127–167. MR0553909 https://doi.org/10.1080/17442507908833142

[16] PARDOUX, É. (1991). Filtrage non linéaire et équations aux dérivées partielles stochastiques associées. In *École D'Été de Probabilités de Saint-Flour XIX*—1989. *Lecture Notes in Math*. **1464** 67–163. Springer, Berlin. MR1108184

[17] PARDOUX, E. and RĂŞCANU, A. (2014). *Stochastic Differential Equations*, *Backward SDEs*, *Partial Differential Equations*. *Stochastic Modelling and Applied Probability* **69**. Springer, Cham. MR3308895 https://doi.org/10.1007/978-3-319-05714-9

[18] PROTTER, P. E. (2005). *Stochastic Integration and Differential Equations*. *Stochastic Modelling and Applied Probability* **21**. Springer, Berlin. MR2273672 https://doi.org/10.1007/978-3-662-10061-5

[19] ROZOVSKY, B. L. and LOTOTSKY, S. V. (2018). *Stochastic Evolution Systems*: *Linear Theory and Applications to Non-linear Filtering*. *Probability Theory and Stochastic Modelling* **89**. Springer, Cham. MR3839316 https://doi.org/10.1007/978-3-319-94893-5

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ASYMPTOTICALLY UNBIASED APPROXIMATION OF THE QSD OF DIFFUSION PROCESSES WITH A DECREASING TIME STEP EULER SCHEME

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We build and study a recursive algorithm based on the occupation measure of an Euler scheme with decreasing step for the numerical approximation of the quasistationary distribution (QSD) of an elliptic diffusion in a bounded domain. We prove the almost sure convergence of the procedure for a family of redistributions and show that we can also recover the approximation of the rate of survival and the convergence in distribution of the algorithm. This last point follows from some new bounds on the weak error related to diffusion dynamics with renewal.

REFERENCES

[1] ALDOUS, D., FLANNERY, B. and PALACIOS, J.-L. (1988). Two applications of urn processes: The fringe analysis of search trees and the simulation of quasi-stationary distributions of Markov chains. *Probab. Engrg. Inform. Sci.* **2** 293–307.

[2] BARTLETT, M. S. (1960). *Stochastic Population Models in Ecology and Epidemiology. Methuen's Monographs on Applied Probability and Statistics*. Methuen & Co., Ltd., London; Wiley, New York. MR0118550

[3] BENAÏM, M., CHAMPAGNAT, N. and VILLEMONAIS, D. (2021). Stochastic approximation of quasi-stationary distributions for diffusion processes in a bounded domain. *Ann. Inst. Henri Poincaré Probab. Stat.* **57** 726–739. MR4260481 https://doi.org/10.1214/20-aihp1093

[4] BENAIM, M., CLOEZ, B. and PANLOUP, F. (2018). Stochastic approximation of quasi-stationary distributions on compact spaces and applications. *Ann. Appl. Probab.* **28** 2370–2416. MR3843832 https://doi.org/10.1214/17-AAP1360

[5] BURDZY, K., HOŁYST, R. and MARCH, P. (2000). A Fleming–Viot particle representation of the Dirichlet Laplacian. *Comm. Math. Phys.* **214** 679–703. MR1800866 https://doi.org/10.1007/s002200000294

[6] CHAMPAGNAT, N., COULIBALY-PASQUIER, K. A. and VILLEMONAIS, D. (2018). Criteria for exponential convergence to quasi-stationary distributions and applications to multi-dimensional diffusions. In *Séminaire de Probabilités XLIX. Lecture Notes in Math.* **2215** 165–182. Springer, Cham. MR3837104

[7] COLLET, P., MARTÍNEZ, S. and SAN MARTÍN, J. (2013). *Quasi-Stationary Distributions: Markov Chains, Diffusions and Dynamical Systems. Probability and Its Applications (New York)*. Springer, Heidelberg. MR2986807 https://doi.org/10.1007/978-3-642-33131-2

[8] DARROCH, J. N. and SENETA, E. (1965). On quasi-stationary distributions in absorbing discrete-time finite Markov chains. *J. Appl. Probab.* **2** 88–100. MR0179842 https://doi.org/10.2307/3211876

[9] DI GESÙ, G., LELIÈVRE, T., LE PEUTREC, D. and NECTOUX, B. (2016). Jump Markov models and transition state theory: The quasi-stationary distribution approach. *Faraday Discuss.* **195** 469–495.

[10] DUDLEY, R. M. (2002). *Real Analysis and Probability. Cambridge Studies in Advanced Mathematics* **74**. Cambridge Univ. Press, Cambridge. MR1932358 https://doi.org/10.1017/CBO9780511755347

[11] FERRARI, P. A., KESTEN, H., MARTINEZ, S. and PICCO, P. (1995). Existence of quasi-stationary distributions. A renewal dynamical approach. *Ann. Probab.* **23** 501–521. MR1334159

[12] GILBARG, D. and TRUDINGER, N. S. (2001). *Elliptic Partial Differential Equations of Second Order*, 1998 ed. *Classics in Mathematics*. Springer, Berlin. MR1814364

[13] GRIGORESCU, I. and KANG, M. (2004). Hydrodynamic limit for a Fleming-Viot type system. *Stoch. Process. Appl.* **110** 111–143. MR2052139 https://doi.org/10.1016/j.spa.2003.10.010

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[14] GROISMAN, P. and JONCKHEERE, M. (2013). Simulation of quasi-stationary distributions on countable spaces. *Markov Process*. *Related Fields* **19** 521–542. MR3156964

[15] LAMBERTON, D. and PAGÈS, G. (2002). Recursive computation of the invariant distribution of a diffusion. *Bernoulli* **8** 367–405. MR1913112 https://doi.org/10.1142/S0219493703000838

[16] LELIÈVRE, T., PILLAUD-VIVIEN, L. and REYGNER, J. (2018). Central limit theorem for stationary Fleming-Viot particle systems in finite spaces. *ALEA Lat*. *Amer. J. Probab. Math*. *Stat*. **15** 1163–1182. MR3860819 https://doi.org/10.30757/alea.v15-43

[17] LEMAIRE, V. and MENOZZI, S. (2010). On some non asymptotic bounds for the Euler scheme. *Electron*. *J. Probab.* **15** no. 53, 1645–1681. MR2735377 https://doi.org/10.1214/ejp.v15-814

[18] MÉLÉARD, S. and VILLEMONAIS, D. (2012). Quasi-stationary distributions and population processes. *Probab. Surv.* **9** 340–410. MR2994898 https://doi.org/10.1214/11-PS191

[19] PANLOUP, F. and REYGNER, J. (2026). Supplement to "Asymptotically unbiased approximation of the QSD of diffusion processes with a decreasing time step Euler scheme." https://doi.org/10.1214/25-AAP2259SUPP

[20] POLLOCK, M., FEARNHEAD, P., JOHANSEN, A. M. and ROBERTS, G. O. (2020). Quasi-stationary Monte Carlo and the ScaLE algorithm. *J. R. Stat*. *Soc*. *Ser. B*. *Stat*. *Methodol*. **82** 1167–1221. MR4176340

[21] REVUZ, D. and YOR, M. (1999). *Continuous Martingales and Brownian Motion*, 3rd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **293**. Springer, Berlin. MR1725357 https://doi.org/10.1007/978-3-662-06400-9

[22] SCHILLING, R. L. and PARTZSCH, L. (2014). *Brownian Motion*: *An Introduction to Stochastic Processes*, 2nd ed. *De Gruyter Graduate*. de Gruyter, Berlin. MR3234570 https://doi.org/10.1515/9783110307306

[23] VILLANI, C. (2009). *Optimal Transport*: *Old and New*. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **338**. Springer, Berlin. MR2459454 https://doi.org/10.1007/978-3-540-71050-9

[24] VILLEMONAIS, D. (2014). General approximation method for the distribution of Markov processes conditioned not to be killed. *ESAIM Probab. Stat*. **18** 441–467. MR3333998 https://doi.org/10.1051/ps/2013045

[25] WANG, A. Q., ROBERTS, G. O. and STEINSALTZ, D. (2020). An approximation scheme for quasi-stationary distributions of killed diffusions. *Stoch*. *Process*. *Appl*. **130** 3193–3219. MR4080743 https://doi.org/10.1016/j.spa.2019.09.010

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ASYMPTOTIC EXPANSION OF THE HARD-TO-SOFT EDGE TRANSITION

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By showing that the symmetrically transformed Bessel kernel admits a full asymptotic expansion for large parameter, we establish a hard-to-soft edge transition expansion. This resolves a conjecture recently proposed by Bornemann.

REFERENCES

[1] ALDOUS, D. and DIACONIS, P. (1999). Longest increasing subsequences: From patience sorting to the Baik–Deift–Johansson theorem. *Bull. Amer. Math. Soc.* (*N.S.*) **36** 413–432. MR1694204 https://doi.org/10.1090/S0273-0979-99-00796-X

[2] BAIK, J., DEIFT, P. and JOHANSSON, K. (1999). On the distribution of the length of the longest increasing subsequence of random permutations. *J. Amer. Math. Soc.* **12** 1119–1178. MR1682248 https://doi.org/10.1090/S0894-0347-99-00307-0

[3] BAIK, J., DEIFT, P. and SUIDAN, T. (2016). *Combinatorics and Random Matrix Theory. Graduate Studies in Mathematics* **172**. Amer. Math. Soc., Providence, RI. MR3468920 https://doi.org/10.1090/gsm/172

[4] BAIK, J. and JENKINS, R. (2013). Limiting distribution of maximal crossing and nesting of Poissonized random matchings. *Ann. Probab.* **41** 4359–4406. MR3161478 https://doi.org/10.1214/12-AOP781

[5] BORNEMANN, F. (2016). A note on the expansion of the smallest eigenvalue distribution of the LUE at the hard edge. *Ann. Appl. Probab.* **26** 1942–1946. MR3513610 https://doi.org/10.1214/15-AAP1121

[6] BORNEMANN, F. (2024). Asymptotic expansions relating to the distribution of the length of longest increasing subsequences. *Forum Math. Sigma* **12** Paper No. e36, 56. MR4717646 https://doi.org/10.1017/fms.2024.13

[7] BORNEMANN, F. (2024). A Stirling-type formula for the distribution of the length of longest increasing subsequences. *Found. Comput. Math.* **24** 915–953. MR4760357 https://doi.org/10.1007/s10208-023-09604-z

[8] BORODIN, A. and FORRESTER, P. J. (2003). Increasing subsequences and the hard-to-soft edge transition in matrix ensembles *J. Phys. A* **36** 2963–2981. MR1986402 https://doi.org/10.1088/0305-4470/36/12/307

[9] CHARLIER, C. and LENELLS, J. (2023). The hard-to-soft edge transition: Exponential moments, central limit theorems and rigidity. *J. Approx. Theory* **285** Paper No. 105833, 50. MR4507555 https://doi.org/10.1016/j.jat.2022.105833

[10] CHOUP, L. N. (2006). Edgeworth expansion of the largest eigenvalue distribution function of GUE and LUE. *Int. Math. Res. Not.* Art. ID 61049, 32. MR2233711 https://doi.org/10.1155/IMRN/2006/61049

[11] DEIFT, P. and ZHOU, X. (1993). A steepest descent method for oscillatory Riemann–Hilbert problems. Asymptotics for the MKdV equation. *Ann. of Math.* (2) **137** 295–368. MR1207209 https://doi.org/10.2307/2946540

[12] DEIFT, P. A. (1999). *Orthogonal Polynomials and Random Matrices: A Riemann–Hilbert Approach. Courant Lecture Notes in Mathematics* **3**. New York Univ., New York. MR1677884

[13] EDELMAN, A., GUIONNET, A. and PÉCHÉ, S. (2016). Beyond universality in random matrix theory. *Ann. Appl. Probab.* **26** 1659–1697. MR3513602 https://doi.org/10.1214/15-AAP1129

[14] EL KAROUI, N. (2006). A rate of convergence result for the largest eigenvalue of complex white Wishart matrices. *Ann. Probab.* **34** 2077–2117. MR2294977 https://doi.org/10.1214/009117906000000502

[15] FORRESTER, P. J. (1993). The spectrum edge of random matrix ensembles. *Nuclear Phys. B* **402** 709–728. MR1236195 https://doi.org/10.1016/0550-3213(93)90126-A

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[16] FORRESTER, P. J. and HUGHES, T. D. (1994). Complex Wishart matrices and conductance in mesoscopic systems: Exact results. *J. Math. Phys.* **35** 6736–6747. MR1303076 https://doi.org/10.1063/1.530639

[17] FORRESTER, P. J. and MAYS, A. (2023). Finite size corrections relating to distributions of the length of longest increasing subsequences. *Adv. in Appl. Math.* **145** Paper No. 102482, 33. MR4534654 https://doi.org/10.1016/j.aam.2022.102482

[18] FORRESTER, P. J. and TRINH, A. K. (2019). Finite-size corrections at the hard edge for the Laguerre β ensemble. *Stud. Appl. Math.* **143** 315–336. MR4019586 https://doi.org/10.1111/sapm.12279

[19] JOHANSSON, K. (1998). The longest increasing subsequence in a random permutation and a unitary random matrix model. *Math. Res. Lett.* **5** 63–82. MR1618351 https://doi.org/10.4310/MRL.1998.v5.n1.a6

[20] KUIJLAARS, A. B. J., MCLAUGHLIN, K. T.-R., VAN ASSCHE, W. and VANLESSEN, M. (2004). The Riemann–Hilbert approach to strong asymptotics for orthogonal polynomials on $[-1, 1]$. *Adv. Math.* **188** 337–398. MR2087231 https://doi.org/10.1016/j.aim.2003.08.015

[21] LAURENZI, B. J. Polynomials associated with the gigher derivatives of the Airy functions $\mathrm{Ai}(z)$ and $\mathrm{Ai}'(z)$. Preprint. Available at arXiv:1110.2025.

[22] NAGAO, T. and FORRESTER, P. J. (1995). Asymptotic correlations at the spectrum edge of random matrices. *Nuclear Phys. B* **435** 401–420. MR1313625 https://doi.org/10.1016/0550-3213(94)00545-P

[23] OLVER, F. W. J. (1952). Some new asymptotic expansions for Bessel functions of large orders. *Proc. Camb. Philos. Soc.* **48** 414–427. MR0048638

[24] OLVER, F. W. J., OLDE DAALHUIS, A. B., LOZIER, D. W., SCHNEIDER, B. I., BOISVERT, R. F., CLARK, C. W., MILLER, B. R., SAUNDERS, B. V., COHL, H. S. et al., eds. NIST Digital Library of Mathematical Functions. Available at http://dlmf.nist.gov/. Release 1.1.10 of 2023-06-15.

[25] PERRET, A. and SCHEHR, G. (2016). Finite N corrections to the limiting distribution of the smallest eigenvalue of Wishart complex matrices. *Random Matrices Theory Appl.* **5** 1650001, 27. MR3455282 https://doi.org/10.1142/S2010326316500015

[26] ROMIK, D. (2015). *The Surprising Mathematics of Longest Increasing Subsequences*. *Institute of Mathematical Statistics Textbooks* **4**. Cambridge Univ. Press, New York. MR3468738

[27] SILVERSTEIN, J. W. and COMBETTES, P. L. (1992). Signal detection via spectral theory of large dimensional random matrices. *IEEE Trans. Signal Process.* **40** 2100–2105.

[28] TRACY, C. A. and WIDOM, H. (1994). Level-spacing distributions and the Airy kernel. *Comm. Math. Phys.* **159** 151–174. MR1257246

[29] ULAM, S. M. (1961). Monte Carlo calculations in problems of mathematical physics. In *Modern Mathematics for the Engineer: Second Series*. *Univ. California Engineering Extension Series* 261–281. McGraw-Hill, New York. MR0129165

[30] WISHART, J. (1928). The generalized product moment distribution in samples from a normal multivariate population. *Biometrika A* **20** 32–52.

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ERGODICITY AND MIXING OF INVARIANT CAPACITIES AND APPLICATIONS

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We introduce the notion of common conditional expectation to investigate Birkhoff's ergodic theorem and subadditive ergodic theorem for invariant upper probabilities. If, in addition, the upper probability is ergodic, we construct an invariant probability to characterize the limit of the ergodic mean. Moreover, this skeleton probability is the unique ergodic probability in the core of the upper probability, that is equal to all probabilities in the core on all invariant sets. We have the following applications of these two theorems:

- provide a strong law of large numbers for ergodic stationary sequence on upper probability spaces;
- prove the multiplicative ergodic theorem on upper probability spaces;
- establish a criterion for the ergodicity of upper probabilities in terms of independence.

Furthermore, we introduce and study weak mixing for capacity preserving systems. Using the skeleton idea, we also provide several characterizations of weak mixing for invariant upper probabilities.

Finally, we provide examples of ergodic and weakly mixing capacity preserving systems. As applications, we obtain new results in the classical ergodic theory, for example, in characterizing dynamical properties on probability preserving systems, such as weak mixing, periodicity. Moreover, we use our results in the nonlinear theory to deduce the asymptotic independence, Birkhoff's type ergodic theorem, subadditive ergodic theorem, and multiplicative ergodic theorem for noninvariant probabilities.

REFERENCES

[1] ALIPRANTIS, C. D. and BORDER, K. C. (1999). *Infinite-Dimensional Analysis*: *A Hitchhiker's Guide*, 2nd ed. Springer, Berlin. MR1717083 https://doi.org/10.1007/978-3-662-03961-8
[2] ARNOLD, L. (1998). *Random Dynamical Systems*. *Springer Monographs in Mathematics*. Springer, Berlin. MR1723992 https://doi.org/10.1007/978-3-662-12878-7
[3] BARTL, D. (2020). Conditional nonlinear expectations. *Stoch*. *Process*. *Appl*. **130** 785–805. MR4046520 https://doi.org/10.1016/j.spa.2019.03.014
[4] BILLOT, A. (1992). From fuzzy set theory to nonadditive probabilities: How have economists reacted? *Fuzzy Sets Syst*. **49** 75–90. MR1177948 https://doi.org/10.1016/0165-0114(92)90111-G
[5] BIRKHOFF, G. D. (1931). Proof of the ergodic theorem. *Proc*. *Natl*. *Acad*. *Sci*. *USA* **17** 656–660.
[6] BOURGAIN, J. (1989). Pointwise ergodic theorems for arithmetic sets. *Publ*. *Math*. *Inst*. *Hautes Études Sci*. **69** 5–45. MR1019960
[7] BUCZOLICH, Z. and MAULDIN, R. D. (2010). Divergent square averages. *Ann*. *of Math*. (2) **171** 1479–1530. MR2680392 https://doi.org/10.4007/annals.2010.171.1479
[8] CARLIER, G. and DANA, R. A. (2003). Core of convex distortions of a probability. *J*. *Econom*. *Theory* **113** 199–222. MR2021300 https://doi.org/10.1016/S0022-0531(03)00122-4

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[9] CERREIA-VIOGLIO, S., MACCHERONI, F. and MARINACCI, M. (2016). Ergodic theorems for lower probabilities. *Proc. Amer. Math. Soc.* **144** 3381–3396. MR3503706 https://doi.org/10.1090/proc/13086

[10] CHEN, Z. (2016). Strong laws of large numbers for sub-linear expectations. *Sci. China Math.* **59** 945–954. MR3484493 https://doi.org/10.1007/s11425-015-5095-0

[11] CHEN, Z., WU, P. and LI, B. (2013). A strong law of large numbers for non-additive probabilities. *Internat. J. Approx. Reason.* **54** 365–377. MR3021837 https://doi.org/10.1016/j.ijar.2012.06.002

[12] CHOQUET, G. (1953/54). Theory of capacities. *Ann. Inst. Fourier* (*Grenoble*) **5** 131–295 (1955). MR0080760

[13] DA PRATO, G. and ZABCZYK, J. (1996). *Ergodicity for Infinite-Dimensional Systems*. *London Mathematical Society Lecture Note Series* **229**. Cambridge Univ. Press, Cambridge. MR1417491 https://doi.org/10.1017/CBO9780511662829

[14] DE COOMAN, G. and MIRANDA, E. (2008). Weak and strong laws of large numbers for coherent lower previsions. *J. Statist. Plann. Inference* **138** 2409–2432. MR2414259 https://doi.org/10.1016/j.jspi.2007.10.020

[15] DENNEBERG, D. (2000). Non-additive measure and integral, basic concepts and their role for applications. In *Fuzzy Measures and Integrals*. *Stud. Fuzziness Soft Comput.* **40** 42–69. Physica, Heidelberg. MR1767745

[16] DIESTEL, J. (1984). *Sequences and Series in Banach Spaces*. *Graduate Texts in Mathematics* **92**. Springer, New York. MR0737004 https://doi.org/10.1007/978-1-4612-5200-9

[17] DOOB, J. L. (1953). *Stochastic Processes*. Wiley, New York; CRC Press, London. MR0058896

[18] DUNFORD, N. and SCHWARTZ, J. T. (1988). *Linear Operators*. *Part I*. *General Theory*. *Wiley Classics Library*. Wiley, New York. MR1009162

[19] EINSIEDLER, M. and WARD, T. (2011). *Ergodic Theory with a View Towards Number Theory*. *Graduate Texts in Mathematics* **259**. Springer, London. MR2723325 https://doi.org/10.1007/978-0-85729-021-2

[20] FENG, C., QU, B. and ZHAO, H. (2020). A sufficient and necessary condition of PS-ergodicity of periodic measures and generated ergodic upper expectations. *Nonlinearity* **33** 5324–5354. MR4157861 https://doi.org/10.1088/1361-6544/ab9584

[21] FENG, C., WU, P. and ZHAO, H. (2020). Ergodicity of invariant capacities. *Stoch. Process. Appl.* **130** 5037–5059. MR4108481 https://doi.org/10.1016/j.spa.2020.02.010

[22] FENG, C. and ZHAO, H. (2021). Ergodicity of sublinear Markovian semigroups. *SIAM J. Math. Anal.* **53** 5646–5681. MR4320888 https://doi.org/10.1137/20M1356518

[23] FOLLAND, G. B. (1999). *Real Analysis*: *Modern Techniques and Their Applications*, 2nd ed. *Pure and Applied Mathematics* (*New York*). Wiley, New York. MR1681462

[24] FURSTENBERG, H. (1981). *Recurrence in Ergodic Theory and Combinatorial Number Theory*. Princeton Univ. Press, Princeton, NJ. MR0603625

[25] FURSTENBERG, H. and KESTEN, H. (1960). Products of random matrices. *Ann. Math. Statist.* **31** 457–469. MR0121828 https://doi.org/10.1214/aoms/1177705909

[26] GLASNER, E. (2003). *Ergodic Theory via Joinings*. *Mathematical Surveys and Monographs* **101**. Amer. Math. Soc., Providence, RI. MR1958753 https://doi.org/10.1090/surv/101

[27] HAMMERSLEY, J. M. and WELSH, D. J. A. (1965). First-passage percolation, subadditive processes, stochastic networks, and generalized renewal theory. In *Proc. Internat. Res. Semin.*, *Statist. Lab.*, *Univ. California*, *Berkeley*, *Calif.*, 1963 61–110. Springer, New York. MR0198576

[28] HUREWICZ, W. (1944). Ergodic theorem without invariant measure. *Ann. of Math.* (2) **45** 192–206. MR0009427 https://doi.org/10.2307/1969081

[29] KINGMAN, J. F. C. (1968). The ergodic theory of subadditive stochastic processes. *J. Roy. Statist. Soc. Ser. B*, *Methodol.* **30** 499–510. MR0254907

[30] KINGMAN, J. F. C. (1973). Subadditive ergodic theory. *Ann. Probab.* **1** 883–909. MR0356192 https://doi.org/10.1214/aop/1176996798

[31] KRENGEL, U. (1985). *Ergodic Theorems*. *De Gruyter Studies in Mathematics* **6**. de Gruyter, Berlin. MR0797411 https://doi.org/10.1515/9783110844641

[32] MACCHERONI, F. and MARINACCI, M. (2005). A strong law of large numbers for capacities. *Ann. Probab.* **33** 1171–1178. MR2135316 https://doi.org/10.1214/009117904000001062

[33] MARINACCI, M. and MONTRUCCHIO, L. (2004). Introduction to the mathematics of ambiguity. In *Uncertainty in Economic Theory*: *Essays in Honor of David Schmeidler's* 65*th Birthday* (I. Gilboa, ed.) Routledge, New York.

[34] OSELEDEC, V. I. (1968). A multiplicative ergodic theorem. Characteristic Ljapunov, exponents of dynamical systems. *Tr. Mosk. Mat. Obŝ.* **19** 179–210. MR0240280

[35] OXTOBY, J. C. (1952). Ergodic sets. *Bull. Amer. Math. Soc.* **58** 116–136. MR0047262 https://doi.org/10.1090/S0002-9904-1952-09580-X

[36] PENG, S. (2007). G-expectation, G-Brownian motion and related stochastic calculus of Itô type. In *Stochastic Analysis and Applications*. *Abel Symp*. **2** 541–567. Springer, Berlin. MR2397805 https://doi.org/10.1007/978-3-540-70847-6_25
[37] PENG, S. (2009). Survey on normal distributions, central limit theorem, Brownian motion and the related stochastic calculus under sublinear expectations. *Sci*. *China Ser. A* **52** 1391–1411. MR2520583 https://doi.org/10.1007/s11425-009-0121-8
[38] QUAS, A. (2012). Ergodicity and mixing properties. In *Mathematics of Complexity and Dynamical Systems*. *Vols*. 1–3 225–240. Springer, New York. MR3220673 https://doi.org/10.1007/978-1-4614-1806-1_15
[39] ROSENBLATT, J. M. and WIERDL, M. (1995). Pointwise ergodic theorems via harmonic analysis. In *Ergodic Theory and Its Connections with Harmonic Analysis* (*Alexandria*, 1993). *London Mathematical Society Lecture Note Series* **205** 3–151. Cambridge Univ. Press, Cambridge. MR1325697 https://doi.org/10.1017/CBO9780511574818.002
[40] RUDIN, W. (1991). *Functional analysis*, 2nd ed. *International Series in Pure and Applied Mathematics* McGraw-Hill, Inc., New York. Cambridge Univ. Press, Cambridge, 1995, pp. 3–151.
[41] SCHMEIDLER, D. (1986). Integral representation without additivity. *Proc*. *Amer. Math*. *Soc*. **97** 255–261. MR0835875 https://doi.org/10.2307/2046508
[42] SCHMEIDLER, D. (1989). Subjective probability and expected utility without additivity. *Econometrica* **57** 571–587. MR0999273 https://doi.org/10.2307/1911053
[43] TEMAM, R. (1988). *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*. *Applied Mathematical Sciences* **68**. Springer, New York. MR0953967 https://doi.org/10.1007/978-1-4684-0313-8
[44] VOLNÝ, D. (1987). Martingale decompositions of stationary processes. *Yokohama Math*. *J*. **35** 113–121. MR0928378
[45] WALLEY, P. (1991). *Statistical Reasoning with Imprecise Probabilities*. *Monographs on Statistics and Applied Probability* **42**. CRC Press, London. MR1145491 https://doi.org/10.1007/978-1-4899-3472-7
[46] WALTERS, P. (1982). *An Introduction to Ergodic Theory*. *Graduate Texts in Mathematics* **79**. Springer, New York. MR0648108
[47] WANG, Z. and KLIR, G. J. (2009). *Generalized Measure Theory*. *IFSR International Series on Systems Science and Engineering* **25**. Springer, New York. MR2453907 https://doi.org/10.1007/978-0-387-76852-6

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BERRY–ESSEEN INEQUALITY FOR RANDOM WALKS CONDITIONED TO STAY POSITIVE

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We consider random walks conditioned to stay positive. When the mean of increments is zero and variance is finite it is known that their distribution converges to the Rayleigh law. In the present paper we derive a Berry–Esseen type estimate and show that if the third absolute moment is finite then the rate of convergence is of order $n^{-1/2}$.

REFERENCES

[1] ALESHKYAVICHENE, A. K. (1973). Nonuniform estimate of speed of convergence of the distribution of the maxima of sequences of sums of independent random variables. *Lith. Math. J.* **13** 356–378. https://doi.org/10.1007/BF01563425

[2] ALEŠKEVIČIENĖ, A. K. (2006). On calculation of moments of ladder heights. *Lith. Math. J.* **46** 129–145. https://doi.org/10.1007/s10986-006-0016-0

[3] BERRY, A. C. (1941). The accuracy of the Gaussian approximation to the sum of independent variates. *Trans. Amer. Math. Soc.* **49** 122–136. MR0003498 https://doi.org/10.2307/1990053

[4] BOLTHAUSEN, E. (1976). On a functional central limit theorem for random walks conditioned to stay positive. *Ann. Probab.* **4** 480–485. MR0415702 https://doi.org/10.1214/aop/1176996098

[5] BRYN-JONES, A. and DONEY, R. A. (2006). A functional limit theorem for random walk conditioned to stay non-negative. *J. Lond. Math. Soc.* (2) **74** 244–258. MR2254563 https://doi.org/10.1112/S0024610706022964

[6] DENISOV, D., SAKHANENKO, A. and WACHTEL, V. (2018). First-passage times for random walks with nonidentically distributed increments. *Ann. Probab.* **46** 3313–3350. MR3857857 https://doi.org/10.1214/17-AOP1248

[7] DENISOV, D., TARASOV, A. and WACHTEL, V. (2024). Asymptotic expansions for normal deviations of random walks conditioned to stay positive. Preprint. Available at arXiv:2412.09145.

[8] DENISOV, D., TARASOV, A. and WACHTEL, V. (2026). Asymptotic expansions for random walks conditioned to stay positive. *Electron. J. Probab.* **31** 1–51. MR5022991 https://doi.org/10.1214/25-EJP1467

[9] GRAMA, I. and XIAO, H. (2024). Gaussian heat kernel asymptotics for conditioned random walks. Preprint. Available at arXiv:2412.08932.

[10] GRAMA, I. and XIAO, H. (2025). Conditioned local limit theorems for random walks on the real line. *Ann. Inst. Henri Poincaré Probab. Stat.* **61** 403–456. MR4863050 https://doi.org/10.1214/23-aihp1426

[11] GRIFFIN, P. S., JAIN, N. C. and PRUITT, W. E. (1984). Approximate local limit theorems for laws outside domains of attraction. *Ann. Probab.* **12** 45–63. MR0723729

[12] HONG, W. and SUN, M. (2024). Berry-Esseen theorem for random walks conditioned to stay positive. *Electron. Commun. Probab.* **29** Paper No. 32, 8. MR4781867 https://doi.org/10.1214/24-ECP605

[13] IGLEHART, D. L. (1974). Functional central limit theorems for random walks conditioned to stay positive. *Ann. Probab.* **2** 608–619. MR0362499 https://doi.org/10.1214/aop/1176996607

[14] MOGUL'SKII, A. A. (1974). Absolute estimates for moments of certain boundary functionals. *Theory Probab. Appl.* **18** 340–347.

[15] NAGAEV, S. V. (1969). An estimate of the rate of convergence of the distribution of the maximum of the sums of independent random variables. *Sibirsk. Mat. Zh.* **10** 614–633. MR0245075

[16] PETROV, V. V. (1975). *Sums of Independent Random Variables. Ergebnisse der Mathematik und Ihrer Grenzgebiete* [*Results in Mathematics and Related Areas*], Band **82**. Springer, New York. MR0388499

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[17] TYURIN, I. S. (2010). Sharpening the upper bounds for constants in Lyapunov's theorem. *Uspekhi Mat. Nauk* **65** 201–201. MR2682728 https://doi.org/10.1070/RM2010v065n03ABEH004688

[18] VATUTIN, V. A. and WACHTEL, V. (2009). Local probabilities for random walks conditioned to stay positive. *Probab. Theory Related Fields* **143** 177–217. MR2449127 https://doi.org/10.1007/s00440-007-0124-8

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FORWARD-BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS ON TENSOR FIELDS AND A STOCHASTIC REPRESENTATION OF NAVIER–STOKES EQUATIONS ON RIEMANNIAN MANIFOLDS

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In this paper we will introduce a class of forward-backward stochastic differential equations on tensor fields of Riemannian manifolds, which are related to semilinear parabolic partial differential equations on tensor fields. Moreover, we will use these forward-backward stochastic differential equations to give a stochastic representation of incompressible Navier–Stokes equations on Riemannian manifolds, where some extra conditions used in (*Potential Anal.* **48** (2018) 181–206) are not required.

REFERENCES

[1] ARNAUDON, M. and CRUZEIRO, A. B. (2012). Lagrangian Navier–Stokes diffusions on manifolds: Variational principle and stability. *Bull. Sci. Math.* **136** 857–881. MR2995006 https://doi.org/10.1016/j.bulsci.2012.06.007

[2] ARNAUDON, M., CRUZEIRO, A. B. and FANG, S. (2018). Generalized stochastic Lagrangian paths for the Navier–Stokes equation. *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) **18** 1033–1060. MR3807594

[3] BECK, C., BECKER, S., CHERIDITO, P., JENTZEN, A. and NEUFELD, A. (2021). Deep splitting method for parabolic PDEs. *SIAM J. Sci. Comput.* **43** A3135–A3154. MR4310910 https://doi.org/10.1137/19M1297919

[4] BISMUT, J.-M. (1976). Linear quadratic optimal stochastic control with random coefficients. *SIAM J. Control Optim.* **14** 419–444. MR0406663 https://doi.org/10.1137/0314028

[5] BLACHE, F. (2005). Backward stochastic differential equations on manifolds. *Probab. Theory Related Fields* **132** 391–437. MR2197107 https://doi.org/10.1007/s00440-004-0400-9

[6] BLACHE, F. (2006). Backward stochastic differential equations on manifolds. II. *Probab. Theory Related Fields* **136** 234–262. MR2240788 https://doi.org/10.1007/s00440-005-0482-z

[7] BUSNELLO, B. (1999). A probabilistic approach to the two-dimensional Navier–Stokes equations. *Ann. Probab.* **27** 1750–1780. MR1742887 https://doi.org/10.1214/aop/1022677547

[8] BUSNELLO, B., FLANDOLI, F. and ROMITO, M. (2005). A probabilistic representation for the vorticity of a three-dimensional viscous fluid and for general systems of parabolic equations. *Proc. Edinb. Math. Soc.* (2) **48** 295–336. MR2157249 https://doi.org/10.1017/S0013091503000506

[9] CHEN, X. and CRUZEIRO, A. B. (2013). Stochastic geodesics and forward-backward stochastic differential equations on Lie groups. In *Discrete Contin. Dyn. Syst. Dynamical Systems, Differential Equations and Applications, 9th AIMS Conference* **Suppl.**, 115–121.

[10] CHEN, X., CRUZEIRO, A. B. and QIAN, Z. M. Navier–Stokes equation and forward-backward stochastic differential system in the Besov spaces. Available at arXiv:1305.0647.

[11] CHEN, X. and YE, W. (2021). A probabilistic representation for heat flow of harmonic map on manifolds with time-dependent Riemannian metric. *Statist. Probab. Lett.* **177** Paper No. 109165. MR4275670 https://doi.org/10.1016/j.spl.2021.109165

[12] CHEN, X. and YE, W. (2021). A study of backward stochastic differential equation on a Riemannian manifold. *Electron. J. Probab.* **26** Paper No. 85. MR4278596 https://doi.org/10.1214/21-ejp649

[13] CONSTANTIN, P. and IYER, G. (2008). A stochastic Lagrangian representation of the three-dimensional incompressible Navier–Stokes equations. *Comm. Pure Appl. Math.* **61** 330–345. MR2376844 https://doi.org/10.1002/cpa.20192

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[14] Cruzeiro, A. B. and Qian, Z. M. (2014). Backward stochastic differential equations associated with the vorticity equations. *J. Funct. Anal.* **267** 660–677. MR3212719 https://doi.org/10.1016/j.jfa.2014.05.011

[15] Cruzeiro, A. B. and Shamarova, E. (2009). Navier–Stokes equations and forward-backward SDEs on the group of diffeomorphisms of a torus. *Stoch. Process. Appl.* **119** 4034–4060. MR2565558 https://doi.org/10.1016/j.spa.2009.09.011

[16] Darling, R. W. R. (1995). Constructing gamma-martingales with prescribed limit, using backwards SDE. *Ann. Probab.* **23** 1234–1261. MR1349170

[17] Delarue, F. and Menozzi, S. (2006). A forward-backward stochastic algorithm for quasi-linear PDEs. *Ann. Appl. Probab.* **16** 140–184. MR2209339 https://doi.org/10.1214/105051605000000674

[18] Delbaen, F., Qiu, J. and Tang, S. (2015). Forward-backward stochastic differential systems associated to Navier–Stokes equations in the whole space. *Stoch. Process. Appl.* **125** 2516–2561. MR3332847 https://doi.org/10.1016/j.spa.2015.02.014

[19] Down, D., Meyn, S. P. and Tweedie, R. L. (1995). Exponential and uniform ergodicity of Markov processes. *Ann. Probab.* **23** 1671–1691. MR1379163

[20] E, W., Han, J. and Jentzen, A. (2022). Algorithms for solving high dimensional PDEs: From nonlinear Monte Carlo to machine learning. *Nonlinearity* **35** 278–310. MR4356985 https://doi.org/10.1088/1361-6544/ac337f

[21] Elworthy, K. D. (1982). *Stochastic Differential Equations on Manifolds*. *London Mathematical Society Lecture Note Series* **70**. Cambridge Univ. Press, Cambridge. MR0675100

[22] Elworthy, K. D., Le Jan, Y. and Li, X.-M. (1997). Concerning the geometry of stochastic differential equations and stochastic flows. In *New Trends in Stochastic Analysis* (*Charingworth*, 1994) 107–131. World Scientific, River Edge, NJ. MR1654511

[23] Elworthy, K. D., Le Jan, Y. and Li, X.-M. (1999). *On the Geometry of Diffusion Operators and Stochastic Flows*. *Lecture Notes in Math.* **1720**. Springer, Berlin. MR1735806 https://doi.org/10.1007/BFb0103064

[24] Émery, M. (1989). *Stochastic Calculus in Manifolds*. *Universitext*. Springer, Berlin. MR1030543 https://doi.org/10.1007/978-3-642-75051-9

[25] Estrade, A. and Pontier, M. (2001). Backward stochastic differential equations in a Lie group. In *Séminaire de Probabilités*, *XXXV*. *Lecture Notes in Math.* **1755** 241–259. Springer, Berlin. MR1837291 https://doi.org/10.1007/978-3-540-44671-2_18

[26] Fang, S. (2020). Nash embedding, shape operator and Navier–Stokes equation on a Riemannian manifold. *Acta Math. Appl. Sin. Engl. Ser.* **36** 237–252. MR4083442 https://doi.org/10.1007/s10255-020-0928-1

[27] Fang, S. and Luo, D. (2018). Constantin and Iyer's representation formula for the Navier–Stokes equations on manifolds. *Potential Anal.* **48** 181–206. MR3748390 https://doi.org/10.1007/s11118-017-9631-0

[28] Fang, S. and Malliavin, P. (1993). Stochastic analysis on the path space of a Riemannian manifold. I. Markovian stochastic calculus. *J. Funct. Anal.* **118** 249–274. MR1245604 https://doi.org/10.1006/jfan.1993.1145

[29] Feng, C., Wang, X. and Zhao, H. (2018). Quasi-linear PDEs and forward–backward stochastic differential equations: Weak solutions. *J. Differ. Equ.* **264** 959–1018. MR3720835 https://doi.org/10.1016/j.jde.2017.09.030

[30] Güneysu, B. and Pigola, S. (2015). The Calderón-Zygmund inequality and Sobolev spaces on noncompact Riemannian manifolds. *Adv. Math.* **281** 353–393. MR3366843 https://doi.org/10.1016/j.aim.2015.03.027

[31] Hsu, E. P. (2002). *Stochastic Analysis on Manifolds*. *Graduate Studies in Mathematics* **38**. Amer. Math. Soc., Providence, RI. MR1882015 https://doi.org/10.1090/gsm/038

[32] Ikeda, N. and Watanabe, S. (1981). *Stochastic Differential Equations and Diffusion Processes*. *North-Holland Mathematical Library* **24**. North-Holland, Amsterdam. MR0637061

[33] Le Jan, Y. and Sznitman, A. S. (1997). Stochastic cascades and 3-dimensional Navier–Stokes equations. *Probab. Theory Related Fields* **109** 343–366. MR1481125 https://doi.org/10.1007/s004400050135

[34] Ledesma, D. S. (2020). A local solution to the Navier–Stokes equations on manifolds via stochastic representation. *Nonlinear Anal.* **198** 111927. MR4091311 https://doi.org/10.1016/j.na.2020.111927

[35] Pardoux, É. and Peng, S. (1992). Backward stochastic differential equations and quasilinear parabolic partial differential equations. In *Stochastic Partial Differential Equations and Their Applications* (*Charlotte*, *NC*, 1991). *Lect. Notes Control Inf. Sci.* **176** 200–217. Springer, Berlin. MR1176785 https://doi.org/10.1007/BFb0007334

[36] Pardoux, É. and Peng, S. (1990). Adapted solution of a backward stochastic differential equation. *Syst. Control Lett.* **14** 55–61. MR1037747 https://doi.org/10.1016/0167-6911(90)90082-6

[37] PARDOUX, É. and TANG, S. (1999). Forward-backward stochastic differential equations and quasilinear parabolic PDEs. *Probab. Theory Related Fields* **114** 123–150. MR1701517 https://doi.org/10.1007/s004409970001

[38] PENG, S. (1991). Probabilistic interpretation for systems of quasilinear parabolic partial differential equations. *Stoch. Stoch. Rep.* **37** 61–74. MR1149116

[39] PIERFELICE, V. (2017). The incompressible Navier–Stokes equations on non-compact manifolds. *J. Geom. Anal.* **27** 577–617. MR3606562 https://doi.org/10.1007/s12220-016-9691-1

[40] VILLANI, C. (2009). *Optimal Transport: Old and New. Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **338**. Springer, Berlin. MR2459454 https://doi.org/10.1007/978-3-540-71050-9

[41] WANG, F.-Y. (2005). *Functional Inequalities, Markov Processes and Spectral Theory*. Science Press, Beijing.

[42] WU, Z. and YU, Z. (2014). Probabilistic interpretation for a system of quasilinear parabolic partial differential equation combined with algebra equations. *Stoch. Process. Appl.* **124** 3921–3947. MR3264433 https://doi.org/10.1016/j.spa.2014.07.013

[43] XING, H. and ŽITKOVIĆ, G. (2018). A class of globally solvable Markovian quadratic BSDE systems and applications. *Ann. Probab.* **46** 491–550. MR3758736 https://doi.org/10.1214/17-AOP1190

[44] ZHANG, Q. and ZHAO, H. (2012). Probabilistic representation of weak solutions of partial differential equations with polynomial growth coefficients. *J. Theoret. Probab.* **25** 396–423. MR2914434 https://doi.org/10.1007/s10959-011-0350-y

[45] ZHANG, X. (2010). A stochastic representation for backward incompressible Navier–Stokes equations. *Probab. Theory Related Fields* **148** 305–332. MR2653231 https://doi.org/10.1007/s00440-009-0234-6

[46] ZHANG, X. (2011). Quasi-invariant stochastic flows of SDEs with non-smooth drifts on compact manifolds. *Stoch. Process. Appl.* **121** 1373–1388. MR2794981 https://doi.org/10.1016/j.spa.2011.03.001

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ERGODIC AND MIXING PROPERTIES OF THE 2D NAVIER–STOKES EQUATIONS WITH A DEGENERATE MULTIPLICATIVE GAUSSIAN NOISE

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In this paper, we establish the ergodic and mixing properties of stochastic 2D Navier–Stokes equations driven by a highly degenerate multiplicative Gaussian noise. The noise can appear in as few as four directions, and its intensity depends on the solution. The case of additive Gaussian noise was previously treated by Hairer and Mattingly (*Ann. of Math. (2)* **164** (2006) 993–1032). To derive the ergodic and mixing properties in the present setting, we employ Malliavin calculus to establish the asymptotically strong Feller property. The primary challenge lies in proving the "invertibility" of the Malliavin matrix, which differs fundamentally from the additive case.

REFERENCES

[1] BRZEŹNIAK, Z., FERRARIO, B., MAURELLI, M. and ZANELLA, M. (2025). Global well posedness and ergodic results in regular Sobolev spaces for the nonlinear Schrödinger equation with multiplicative noise and arbitrary power of the nonlinearity. *Discrete Contin. Dyn. Syst.* **45** 3217–3257. MR4893139 https://doi.org/10.3934/dcds.2025018

[2] CONSTANTIN, P. and FOIAS, C. (1988). *Navier–Stokes Equations. Chicago Lectures in Mathematics*. Univ. Chicago Press, Chicago, IL. MR0972259

[3] DAREIOTIS, K., GESS, B. and TSATSOULIS, P. (2020). Ergodicity for stochastic porous media equations with multiplicative noise. *SIAM J. Math. Anal.* **52** 4524–4564. MR4154312 https://doi.org/10.1137/19M1278521

[4] DONG, Z., ZHANG, R. and ZHANG, T. (2023). Ergodicity for stochastic conservation laws with multiplicative noise. *Comm. Math. Phys.* **400** 1739–1789. MR4595608 https://doi.org/10.1007/s00220-022-04629-x

[5] E, W. and MATTINGLY, J. C. (2001). Ergodicity for the Navier–Stokes equation with degenerate random forcing: Finite-dimensional approximation. *Comm. Pure Appl. Math.* **54** 1386–1402. MR1846802 https://doi.org/10.1002/cpa.10007

[6] EYINK, G. L. (1996). Exact results on stationary turbulence in 2D: Consequences of vorticity conservation. *Phys. D* **91** 97–142. MR1377188 https://doi.org/10.1016/0167-2789(95)00250-2

[7] FERRARIO, B. and ZANELLA, M. (2024). Uniqueness of the invariant measure and asymptotic stability for the 2D Navier–Stokes equations with multiplicative noise. *Discrete Contin. Dyn. Syst.* **44** 228–262. MR4671522 https://doi.org/10.3934/dcds.2023102

[8] FÖLDES, J., GLATT-HOLTZ, N., RICHARDS, G. and THOMANN, E. (2015). Ergodic and mixing properties of the Boussinesq equations with a degenerate random forcing. *J. Funct. Anal.* **269** 2427–2504. MR3390008 https://doi.org/10.1016/j.jfa.2015.05.014

[9] GAWARECKI, L. and MANDREKAR, V. (2011). *Stochastic Differential Equations in Infinite Dimensions with Applications to Stochastic Partial Differential Equations. Probability and Its Applications (New York)*. Springer, Heidelberg. MR2560625 https://doi.org/10.1007/978-3-642-16194-0

[10] HAIRER, M. and MATTINGLY, J. C. (2006). Ergodicity of the 2D Navier–Stokes equations with degenerate stochastic forcing. *Ann. of Math.* (2) **164** 993–1032. MR2259251 https://doi.org/10.4007/annals.2006.164.993

[11] HAIRER, M. and MATTINGLY, J. C. (2008). Spectral gaps in Wasserstein distances and the 2D stochastic Navier–Stokes equations. *Ann. Probab.* **36** 2050–2091. MR2478676 https://doi.org/10.1214/08-AOP392

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Key words and phrases. Stochastic Navier–Stokes equations, Malliavin calculus, ergodicity.

[12] HAIRER, M. and MATTINGLY, J. C. (2011). A theory of hypoellipticity and unique ergodicity for semilinear stochastic PDEs. *Electron. J. Probab.* **16** 658–738. MR2786645 https://doi.org/10.1214/EJP.v16-875

[13] HUANG, J., PENG, X., WANG, X. and ZHANG, J. (2025). Ergodicity for stochastic 2D Boussinesq equations with a highly degenerate pure jump Levy noise. Preprint. Available at arXiv:2503.18045.

[14] KALLENBERG, O. (1997). *Foundations of Modern Probability*. *Probability and Its Applications* (*New York*). Springer, New York. MR1464694

[15] KARATZAS, I. and SHREVE, S. E. (1991). *Brownian Motion and Stochastic Calculus*, 2nd ed. *Graduate Texts in Mathematics* **113**. Springer, New York. MR1121940 https://doi.org/10.1007/978-1-4612-0949-2

[16] KUKSIN, S., NERSESYAN, V. and SHIRIKYAN, A. (2020). Exponential mixing for a class of dissipative PDEs with bounded degenerate noise. *Geom. Funct. Anal.* **30** 126–187. MR4081057 https://doi.org/10.1007/s00039-020-00525-5

[17] KUKSIN, S. and SHIRIKYAN, A. (2002). Coupling approach to white-forced nonlinear PDEs. *J. Math. Pures Appl.* (9) **81** 567–602. MR1912412 https://doi.org/10.1016/S0021-7824(02)01259-X

[18] KUKSIN, S. and SHIRIKYAN, A. (2012). *Mathematics of Two-Dimensional Turbulence*. *Cambridge Tracts in Mathematics* **194**. Cambridge Univ. Press, Cambridge. MR3443633 https://doi.org/10.1017/CBO9781139137119

[19] MATTINGLY, J. C. (2002). Exponential convergence for the stochastically forced Navier–Stokes equations and other partially dissipative dynamics. *Comm. Math. Phys.* **230** 421–462. MR1937652 https://doi.org/10.1007/s00220-002-0688-1

[20] MATTINGLY, J. C. and PARDOUX, É. (2006). Malliavin calculus for the stochastic 2D Navier–Stokes equation. *Comm. Pure Appl. Math.* **59** 1742–1790. MR2257860 https://doi.org/10.1002/cpa.20136

[21] NOVIKOV, E. A. (1965). Functionals and the random-force method in turbulence theory. *Sov. Phys. JETP* **20** 1290–1294.

[22] NUALART, D. (2006). *The Malliavin Calculus and Related Topics*, 2nd ed. *Probability and Its Applications* (*New York*). Springer, Berlin. MR2200233

[23] ODASSO, C. (2007). Exponential mixing for the 3D stochastic Navier–Stokes equations. *Comm. Math. Phys.* **270** 109–139. MR2276442 https://doi.org/10.1007/s00220-006-0156-4

[24] ODASSO, C. (2008). Exponential mixing for stochastic PDEs: The non-additive case. *Probab. Theory Related Fields* **140** 41–82. MR2357670 https://doi.org/10.1007/s00440-007-0057-2

[25] PENG, X., ZHAI, J. and ZHANG, T. (2024). Ergodicity for 2D Navier–Stokes equations with a degenerate pure jump noise. Preprint. Available at arXiv:2405.00414.

[26] REVUZ, D. and YOR, M. (1999). *Continuous Martingales and Brownian Motion*, 3rd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **293**. Springer, Berlin. MR1725357 https://doi.org/10.1007/978-3-662-06400-9

[27] RÖCKNER, M., ZHU, R. and ZHU, X. (2015). Sub and supercritical stochastic quasi-geostrophic equation. *Ann. Probab.* **43** 1202–1273. MR3342662 https://doi.org/10.1214/13-AOP887

[28] SHIRIKYAN, A. (2015). Control and mixing for 2D Navier–Stokes equations with space-time localised noise. *Ann. Sci. Éc. Norm. Supér.* (4) **48** 253–280. MR3346171 https://doi.org/10.24033/asens.2244

[29] SHIRIKYAN, A. (2021). Controllability implies mixing II. Convergence in the dual-Lipschitz metric. *J. Eur. Math. Soc.* (*JEMS*) **23** 1381–1422. MR4228283 https://doi.org/10.4171/jems/1036

[30] STANIŠIĆ, M. M. (1988). *The Mathematical Theory of Turbulence*, 2nd ed. *Universitext*. Springer, New York. MR0918774 https://doi.org/10.1007/978-1-4612-3840-9

[31] VISHIK, M. I., KOMECH, A. I. and FURSIKOV, A. V. (1979). Some mathematical problems of statistical hydromechanics. *Uspekhi Mat. Nauk* **34** 135–210. MR0562801

[32] WANG, R., ZHAI, J. and ZHANG, T. (2016). Exponential mixing for stochastic model of two-dimensional second grade fluids. *Nonlinear Anal.* **132** 196–213. MR3433962 https://doi.org/10.1016/j.na.2015.11.009

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A PHASE TRANSITION IN SAMPLING FROM RESTRICTED BOLTZMANN MACHINES

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Restricted Boltzmann machines are a class of undirected graphical models that play a key role in deep learning and unsupervised learning. In this study, we prove a phase transition phenomenon in the mixing time of the Gibbs sampler for a one-parameter restricted Boltzmann machine. Specifically, the mixing time varies logarithmically, polynomially, and exponentially with the number of vertices depending on whether the parameter c is above, equal to, or below a critical value $c_\star \approx -5.87$. A key insight from our analysis is the link between the Gibbs sampler and a dynamic system, which we use to quantify the former based on the behavior of the latter. To study the critical case $c = c_\star$, we develop a new isoperimetric inequality for the sampler's stationary distribution by showing that the distribution is nearly log-concave. Conditions for rapid and torpid convergence of a class of generic binomial chain are also provided.

REFERENCES

[1] AIZENMAN, M. and HOLLEY, R. (1987). Rapid convergence to equilibrium of stochastic Ising models in the Dobrushin Shlosman regime. In *Percolation Theory and Ergodic Theory of Infinite Particle Systems* (*Minneapolis*, *Minn*., 1984–1985). *IMA Vol. Math. Appl.* **8** 1–11. Springer, New York. MR0894538 https://doi.org/10.1007/978-1-4613-8734-3_1

[2] ANDRIEU, C., LEE, A., POWER, S. and WANG, A. Q. (2024). Explicit convergence bounds for Metropolis Markov chains: Isoperimetry, spectral gaps and profiles. *Ann. Appl. Probab.* **34** 4022–4071. MR4783036 https://doi.org/10.1214/24-aap2058

[3] BALASUBRAMANIAN, K., CHEWI, S., ERDOGDU, M. A., SALIM, A. and ZHANG, S. (2022). Towards a theory of non-log-concave sampling: First-order stationarity guarantees for Langevin Monte Carlo. In *Conference on Learning Theory* 2896–2923. PMLR.

[4] BELLONI, A. and CHERNOZHUKOV, V. (2009). On the computational complexity of MCMC-based estimators in large samples. *Ann. Statist.* **37** 2011–2055. MR2533478 https://doi.org/10.1214/08-AOS634

[5] BLUME, L. E. (1993). The statistical mechanics of strategic interaction. *Games Econom. Behav.* **5** 387–424. MR1227917 https://doi.org/10.1006/game.1993.1023

[6] BUTKOVSKY, O. (2014). Subgeometric rates of convergence of Markov processes in the Wasserstein metric. *Ann. Appl. Probab.* **24** 526–552. MR3178490 https://doi.org/10.1214/13-AAP922

[7] CATTIAUX, P. and GUILLIN, A. (2022). Functional inequalities for perturbed measures with applications to log-concave measures and to some Bayesian problems. *Bernoulli* **28** 2294–2321. MR4474544 https://doi.org/10.3150/21-bej1419

[8] CHANG, H. and ZHOU, Q. (2024). Dimension-free relaxation times of informed MCMC samplers on discrete spaces. arXiv preprint. Available at arXiv:2404.03867.

[9] COATES, A., NG, A. and LEE, H. (2011). An analysis of single-layer networks in unsupervised feature learning. In *Proceedings of the Fourteenth International Conference on Artificial Intelligence and Statistics* 215–223. JMLR Workshop and Conference Proceedings.

[10] DIACONIS, P. and FREEDMAN, D. (1999). Iterated random functions. *SIAM Rev.* **41** 45–76. MR1669737 https://doi.org/10.1137/S0036144598338446

[11] DING, J., LUBETZKY, E. and PERES, Y. (2009). The mixing time evolution of Glauber dynamics for the mean-field Ising model. *Comm. Math. Phys.* **289** 725–764. MR2506768 https://doi.org/10.1007/s00220-009-0781-9

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Key words and phrases. Restricted Boltzmann machine, Gibbs sampler, mixing time, phase transition.

[12] DOUC, R., MOULINES, E., PRIOURET, P. and SOULIER, P. (2018). *Markov Chains*. *Springer Series in Operations Research and Financial Engineering*. Springer, Cham. MR3889011 https://doi.org/10.1007/978-3-319-97704-1

[13] DUVERNEY, D. (2024). *An Introduction to Hypergeometric Functions*, French ed. Birkhäuser/Springer, Cham. MR4824080 https://doi.org/10.1007/978-3-031-65144-1

[14] DWIVEDI, R., CHEN, Y., WAINWRIGHT, M. J. and YU, B. (2019). Log-concave sampling: Metropolis-Hastings algorithms are fast. *J. Mach. Learn. Res.* **20** 183. MR4048994

[15] ETHIER, S. N. and KURTZ, T. G. (1986). *Markov Processes – Characterization and Convergence*. *Wiley Series in Probability and Mathematical Statistics*: *Probability and Mathematical Statistics*. Wiley, New York. MR0838085 https://doi.org/10.1002/9780470316658

[16] FORT, G., MOULINES, E., ROBERTS, G. O. and ROSENTHAL, J. S. (2003). On the geometric ergodicity of hybrid samplers. *J. Appl. Probab.* **40** 123–146. MR1953771 https://doi.org/10.1239/jap/1044476831

[17] FREIDLIN, M. I. and WENTZELL, A. D. (1984). *Random Perturbations of Dynamical Systems*. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **260**. Springer, New York. Translated from the Russian by Joseph Szücs. MR0722136 https://doi.org/10.1007/978-1-4684-0176-9

[18] GEMAN, S. and GEMAN, D. (1984). Stochastic relaxation, Gibbs distributions, and the Bayesian restoration of images. *IEEE Trans. Pattern Anal. Mach. Intell.* **PAMI-6(6)** 721–741.

[19] HINTON, G. E. (2012). A practical guide to training restricted Boltzmann machines. In *Neural Networks*: *Tricks of the Trade* 599–619. Springer, Berlin.

[20] HINTON, G. E., OSINDERO, S. and TEH, Y.-W. (2006). A fast learning algorithm for deep belief nets. *Neural Comput.* **18** 1527–1554. MR2224485 https://doi.org/10.1162/neco.2006.18.7.1527

[21] HINTON, G. E. and SALAKHUTDINOV, R. R. (2006). Reducing the dimensionality of data with neural networks. *Science* **313** 504–507. MR2242509 https://doi.org/10.1126/science.1127647

[22] HWANG, S.-H. and REY-BELLET, L. (2021). Positive feedback in coordination games: Stochastic evolutionary dynamics and the logit choice rule. *Games Econom. Behav.* **126** 355–373. MR4205727 https://doi.org/10.1016/j.geb.2021.01.003

[23] IMHOF, L. A. and NOWAK, M. A. (2006). Evolutionary game dynamics in a Wright-Fisher process. *J. Math. Biol.* **52** 667–681. MR2235522 https://doi.org/10.1007/s00285-005-0369-8

[24] JONES, G. L. and HOBERT, J. P. (2001). Honest exploration of intractable probability distributions via Markov chain Monte Carlo. *Statist. Sci.* **16** 312–334. MR1888447 https://doi.org/10.1214/ss/1015346317

[25] KIM, J. and ROČKOVÁ, V. (2025). On mixing rates for Bayesian CART. *Electron. J. Stat.* **19** 3041–3067. MR4930535 https://doi.org/10.1214/25-EJS2397

[26] KUSHNER, H. J. and YIN, G. G. (2003). *Stochastic Approximation and Recursive Algorithms and Applications*, 2nd ed. *Applications of Mathematics* (*New York*) **35**. Springer, New York. Stochastic Modelling and Applied Probability. MR1993642

[27] LAROCHELLE, H. and BENGIO, Y. (2008). Classification using discriminative restricted Boltzmann machines. In *Proceedings of the* 25*th International Conference on Machine Learning* 536–543.

[28] LAWLER, G. F. and SOKAL, A. D. (1988). Bounds on the L^2 spectrum for Markov chains and Markov processes: A generalization of Cheeger's inequality. *Trans. Amer. Math. Soc.* **309** 557–580. MR0930082 https://doi.org/10.2307/2000925

[29] LEVIN, D. A., LUCZAK, M. J. and PERES, Y. (2010). Glauber dynamics for the mean-field Ising model: Cut-off, critical power law, and metastability. *Probab. Theory Related Fields* **146** 223–265. MR2550363 https://doi.org/10.1007/s00440-008-0189-z

[30] LEVIN, D. A. and PERES, Y. (2017). *Markov Chains and Mixing Times*, 2nd ed. American Mathematical Soc., Providence.

[31] LOVÁSZ, L. (1999). Hit-and-run mixes fast. *Math. Program.* **86** 443–461. MR1733749 https://doi.org/10.1007/s101070050099

[32] MARTELLI, M. (2011). *Introduction to Discrete Dynamical Systems and Chaos*. Wiley, New York.

[33] MEYN, S. P. and TWEEDIE, R. L. (1993). *Markov Chains and Stochastic Stability*. *Communications and Control Engineering Series*. Springer London, Ltd., London. MR1287609 https://doi.org/10.1007/978-1-4471-3267-7

[34] MEYN, S. P. and TWEEDIE, R. L. (1994). Computable bounds for geometric convergence rates of Markov chains. *Ann. Appl. Probab.* **4** 981–1011. MR1304770

[35] THE ROYAL SWEDISH ACADEMY OF SCIENCES (2024). Press release for the Nobel prize in physics. https://www.nobelprize.org/prizes/physics/2024/press-release/.

[36] OLIVIERI, E. and VARES, M. E. (2005). *Large Deviations and Metastability*. *Encyclopedia of Mathematics and Its Applications* **100**. Cambridge Univ. Press, Cambridge. MR2123364 https://doi.org/10.1017/CBO9780511543272

[37] QIN, Q. (2024). Convergence bounds for Monte Carlo Markov chains. arXiv preprint. Available at arXiv: 2409.14656.

[38] QIN, Q. and HOBERT, J. P. (2021). On the limitations of single-step drift and minorization in Markov chain convergence analysis. *Ann. Appl. Probab.* **31** 1633–1659. MR4312841 https://doi.org/10.1214/20-aap1628

[39] QIN, Q. and HOBERT, J. P. (2022). Geometric convergence bounds for Markov chains in Wasserstein distance based on generalized drift and contraction conditions. *Ann. Inst. Henri Poincaré Probab. Stat.* **58** 872–889. MR4421611 https://doi.org/10.1214/21-aihp1195

[40] ROBERTS, G. O. and ROSENTHAL, J. S. (1997). Geometric ergodicity and hybrid Markov chains. *Electron. Commun. Probab.* **2** 13–25. MR1448322 https://doi.org/10.1214/ECP.v2-981

[41] ROBERTS, G. O. and ROSENTHAL, J. S. (2001). Markov chains and de-initializing processes. *Scand. J. Stat.* **28** 489–504. MR1858413 https://doi.org/10.1111/1467-9469.00250

[42] ROSENTHAL, J. S. (1995). Minorization conditions and convergence rates for Markov chain Monte Carlo. *J. Amer. Statist. Assoc.* **90** 558–566. MR1340509

[43] SALAKHUTDINOV, R., MNIH, A. and HINTON, G. (2007). Restricted Boltzmann machines for collaborative filtering. In *Proceedings of the 24th International Conference on Machine Learning* 791–798.

[44] TATARU, P., SIMONSEN, M., BATAILLON, T. and HOBOLTH, A. (2017). Statistical inference in the Wright–Fisher model using allele frequency data. *Syst. Biol.* **66** e30–e46.

[45] TOSH, C. (2016). Mixing rates for the alternating Gibbs sampler over restricted Boltzmann machines and friends. In *International Conference on Machine Learning* 840–849. PMLR.

[46] ZHOU, Q., YANG, J., VATS, D., ROBERTS, G. O. and ROSENTHAL, J. S. (2022). Dimension-free mixing for high-dimensional Bayesian variable selection. *J. R. Stat. Soc. Ser. B. Stat. Methodol.* **84** 1751–1784. MR4515557 https://doi.org/10.1111/rssb.12546

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KERNEL MEAN EMBEDDING TOPOLOGY: WEAK AND STRONG FORMS FOR STOCHASTIC KERNELS AND IMPLICATIONS FOR MODEL LEARNING

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We introduce a novel topology, called Kernel Mean Embedding Topology, for stochastic kernels, in a weak and strong form. This topology, defined on the spaces of Bochner integrable functions from a signal space to a space of probability measures endowed with a Hilbert space structure, allows for a versatile formulation. This construction allows one to obtain both a strong and weak formulation. (i) For its weak formulation, we highlight the utility on relaxed policy spaces, and investigate connections with the Young narrow topology and Borkar (or w^*)-topology, and establish equivalence properties. We report that, while both the w^*-topology and kernel mean embedding topology are relatively compact, they are not closed. Conversely, while the Young narrow topology is closed, it lacks relative compactness. (ii) We show that the strong form provides an appropriate formulation for placing topologies on spaces of models characterized by stochastic kernels with explicit robustness and learning theoretic implications on optimal stochastic control under discounted or average cost criteria. (iii) We thus show that this topology possesses several properties making it ideal to study optimality and approximations (under the weak formulation) and robustness (under the strong formulation) for many applications.

REFERENCES

[1] ALDOUS, D. J. (1981). *Weak Convergence and the General Theory of Processes*. Editeur inconnu.

[2] ARAPOSTATHIS, A. and BORKAR, V. S. (2010). Uniform recurrence properties of controlled diffusions and applications to optimal control. *SIAM J. Control Optim.* **48** 4181–4223. MR2665463 https://doi.org/10.1137/090762464

[3] ARAPOSTATHIS, A., BORKAR, V. S. and GHOSH, M. K. (2012). *Ergodic Control of Diffusion Processes*. *Encyclopedia of Mathematics and Its Applications* **143**. Cambridge Univ. Press, Cambridge. MR2884272

[4] ARAPOSTATHIS, A., BORKAR, V. S. and GHOSH, M. K. (2012). *Ergodic Control of Diffusion Processes*. *Encyclopedia of Mathematics and Its Applications* **143**. Cambridge Univ. Press, Cambridge. MR2884272

[5] AUMANN, R. J. (1961). Mixed and behavior strategies in infinite extensive games. Technical report, Princeton Univ., Princeton, NJ.

[6] BACKHOFF-VERAGUAS, J., BARTL, D., BEIGLBÖCK, M. and EDER, M. (2020). All adapted topologies are equal. *Probab. Theory Related Fields* **178** 1125–1172. MR4168395 https://doi.org/10.1007/s00440-020-00993-8

[7] BACKHOFF-VERAGUAS, J., BARTL, D., BEIGLBÖCK, M. and EDER, M. (2020). Adapted Wasserstein distances and stability in mathematical finance. *Finance Stoch.* **24** 601–632. MR4118990 https://doi.org/10.1007/s00780-020-00426-3

[8] BALDER, E. J. (1988). Generalized equilibrium results for games with incomplete information. *Math. Oper. Res.* **13** 265–276. MR0942618 https://doi.org/10.1287/moor.13.2.265

[9] BALDER, E. J. (1997). Consequences of denseness of Dirac Young measures. *J. Math. Anal. Appl.* **207** 536–540. MR1438929 https://doi.org/10.1006/jmaa.1997.5306

[10] BARTL, D., BEIGLBÖCK, M. and PAMMER, G. (2024). The Wasserstein space of stochastic processes. *J. Eur. Math. Soc.*.

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[11] BARTL, D. and WIESEL, J. (2023). Sensitivity of multiperiod optimization problems with respect to the adapted Wasserstein distance. *SIAM J. Financial Math.* **14** 704–720. MR4599319 https://doi.org/10.1137/22M1537746

[12] BÄUERLE, N. and LANGE, D. (2018). Optimal control of partially observable piecewise deterministic Markov processes. *SIAM J. Control Optim.* **56** 1441–1462. MR3784901 https://doi.org/10.1137/17M1134731

[13] BÄUERLE, N. and RIEDER, U. (2010). Optimal control of piecewise deterministic Markov processes with finite time horizon. In *Modern Trends in Controlled Stochastic Processes*: *Theory and Applications* 123–143.

[14] BAYRAKTAR, E., BÄUERLE, N. and KARA, A. D. (2022). Finite approximations and Q learning for mean field type multi agent control. Preprint. Available at arXiv:2211.09633.

[15] BAYRAKTAR, E., DOLINSKY, Y. and GUO, J. (2020). Continuity of utility maximization under weak convergence. *Math. Financ. Econ.* **14** 725–757. MR4135703 https://doi.org/10.1007/s11579-020-00274-x

[16] BEIGLBÖCK, M., JOURDAIN, B., MARGHERITI, W. and PAMMER, G. (2022). Approximation of martingale couplings on the line in the adapted weak topology. *Probab. Theory Related Fields* **183** 359–413. MR4421177 https://doi.org/10.1007/s00440-021-01103-y

[17] BERTSEKAS, D. P. and SHREVE, S. E. (1978). *Stochastic Optimal Control*: *The Discrete Time Case*. *Mathematics in Science and Engineering* **139**. Academic Press, New York. MR0511544

[18] BILLINGSLEY, P. (1995). *Probability and Measure*, 3rd ed. *Wiley Series in Probability and Mathematical Statistics*. Wiley, New York. MR1324786

[19] BISMUT, J. M. (1973). *Théorie Probabiliste du Contrôle des Diffusions* **181**.

[20] BORKAR, V. S. (1989). A topology for Markov controls. *Appl. Math. Optim.* **20** 55–62. MR0989430 https://doi.org/10.1007/BF01447645

[21] BORKAR, V. S. (1993). White-noise representations in stochastic realization theory. *SIAM J. Control Optim.* **31** 1093–1102. MR1233993 https://doi.org/10.1137/0331050

[22] CARMELI, C., DE VITO, E., TOIGO, A. and UMANITÀ, V. (2010). Vector valued reproducing kernel Hilbert spaces and universality. *Anal. Appl.* (*Singap.*) **8** 19–61. MR2603770 https://doi.org/10.1142/S0219530510001503

[23] CASTAING, C., FITTE, P. R. D. and VALADIER, M. (2004). *Young Measures on Topological Spaces*: *With Applications in Control Theory and Probability Theory* **571**. Springer Science & Business Media, Berlin.

[24] CEMBRANOS, P. and MENDOZA, J. (1997). *Banach Spaces of Vector-Valued Functions*. *Lecture Notes in Math.* **1676**. Springer, Berlin. MR1489231 https://doi.org/10.1007/BFb0096765

[25] DIAZ, S. and MAYORAL, F. (1999). On compactness in spaces of Bochner integrable functions. *Acta Math. Hungar.* **83** 231–239. MR1692302 https://doi.org/10.1023/A:1006721122987

[26] DIESTEL, J. and UHL, J. J. JR. (1977). *Vector Measures*. *Mathematical Surveys* **15**. Amer. Math. Soc., Providence, RI. MR0453964

[27] DIEUDONNÉ, J. (1951). Sur le théorème de Lebesgue-Nikodym. V. *Canad. J. Math.* **3** 129–139. MR0044611 https://doi.org/10.4153/cjm-1951-015-9

[28] DUDLEY, R. M. (2004). *Real Analysis and Probability*. *Cambridge Studies in Advanced Mathematics* **74**. Cambridge Univ. Press, Cambridge. MR1932358 https://doi.org/10.1017/CBO9780511755347

[29] FATTORINI, H. O. (1994). Existence theory and the maximum principle for relaxed infinite-dimensional optimal control problems. *SIAM J. Control Optim.* **32** 311–331. MR1261142 https://doi.org/10.1137/S0363012991220244

[30] FEINBERG, E. A., KASYANOV, P. O. and ZADOIANCHUK, N. V. (2012). Average cost Markov decision processes with weakly continuous transition probabilities. *Math. Oper. Res.* **37** 591–607. MR2997893 https://doi.org/10.1287/moor.1120.0555

[31] FLEMING, W. H. (1982). Generalized solutions in optimal stochastic control. In *Proc. U.R.I. Conference on Control*, *University of Rhode Island*.

[32] FLORESCU, L. C. and GODET-THOBIE, C. (2012). *Young Measures and Compactness in Measure Spaces*. de Gruyter, Berlin. MR2953092 https://doi.org/10.1515/9783110280517

[33] FOLLAND, G. B. (1999). *Real Analysis*: *Modern Techniques and Their Applications*, 2nd ed. *Pure and Applied Mathematics* (*New York*). Wiley, New York. MR1681462

[34] GHOJOGH, B., GHODSI, A., KARRAY, F. and CROWLEY, M. (2021). Reproducing kernel Hilbert space, Mercer's theorem, eigenfunctions, Nyström method, and use of kernels in machine learning: Tutorial and survey. Preprint. Available at arXiv:2106.08443.

[35] GĪHMAN, Ĭ. Ī. and SKOROHOD, A. V. (2012). *Controlled Stochastic Processes*. Springer, New York. MR0544839

[36] GYÖRFI, L. and KOHLER, M. (2007). Nonparametric estimation of conditional distributions. *IEEE Trans. Inf. Theory* **53** 1872–1879. MR2317148 https://doi.org/10.1109/TIT.2007.894631

[37] HELLWIG, M. F. (1996). Sequential decisions under uncertainty and the maximum theorem. *J. Math. Econom.* **25** 443–464. MR1394333 https://doi.org/10.1016/0304-4068(95)00739-3
[38] HERNÁNDEZ-LERMA, O. and LASSERRE, J. B. (1996). *Discrete-Time Markov Control Processes: Basic Optimality Criteria. Applications of Mathematics (New York)* **30**. Springer, New York. MR1363487 https://doi.org/10.1007/978-1-4612-0729-0
[39] HERNÁNDEZ-LERMA, O. and LASSERRE, J. B. (2003). *Markov Chains and Invariant Probabilities. Progress in Mathematics* **211**. Birkhäuser, Basel. MR1974383 https://doi.org/10.1007/978-3-0348-8024-4
[40] KARA, A. D., RAGINSKY, M. and YÜKSEL, S. (2022). Robustness to incorrect models and data-driven learning in average-cost optimal stochastic control. *Automatica J. IFAC* **139** Paper No. 110179, 13 pp. MR4383037 https://doi.org/10.1016/j.automatica.2022.110179
[41] KARA, A. D. and YÜKSEL, S. (2020). Robustness to incorrect system models in stochastic control. *SIAM J. Control Optim.* **58** 1144–1182. MR4091166 https://doi.org/10.1137/18M1208058
[42] KLEBANOV, I., SCHUSTER, I. and SULLIVAN, T. J. (2020). A rigorous theory of conditional mean embeddings. *SIAM J. Math. Data Sci.* **2** 583–606. MR4121886 https://doi.org/10.1137/19M1305069
[43] KLEBANOV, I., SPRUNGK, B. and SULLIVAN, T. J. (2021). The linear conditional expectation in Hilbert space. *Bernoulli* **27** 2267–2299. MR4303883 https://doi.org/10.3150/20-BEJ1308
[44] KUSHNER, H. J. (2014). A partial history of the early development of continuous-time nonlinear stochastic systems theory. *Automatica J. IFAC* **50** 303–334. MR3163780 https://doi.org/10.1016/j.automatica.2013.10.013
[45] KUSHNER, H. J. and DUPUIS, P. (2001). *Numerical Methods for Stochastic Control Problems in Continuous Time: Stochastic Modelling and Applied Probability*, 2nd ed. *Applications of Mathematics (New York)* **24**. Springer, New York. MR1800098 https://doi.org/10.1007/978-1-4613-0007-6
[46] LI, Z., MEUNIER, D., MOLLENHAUER, M. and GRETTON, A. (2022). Optimal rates for regularized conditional mean embedding learning. *Adv. Neural Inf. Process. Syst.* **35** 4433–4445.
[47] MASCOLO, E. and MIGLIACCIO, L. (1989). Relaxation methods in control theory. *Appl. Math. Optim.* **20** 97–103. MR0989434 https://doi.org/10.1007/BF01447649
[48] MCSHANE, E. J. (1967). Relaxed controls and variational problems. *SIAM J. Control* **5** 438–485. MR0218949
[49] MERTENS, J.-F., SORIN, S. and ZAMIR, S. (2015). *Repeated Games. Econometric Society Monographs* **55**. Cambridge Univ. Press, New York. MR3560551 https://doi.org/10.1017/CBO9781139343275
[50] MILGROM, P. R. and WEBER, R. J. (1985). Distributional strategies for games with incomplete information. *Math. Oper. Res.* **10** 619–632. MR0812820 https://doi.org/10.1287/moor.10.4.619
[51] MOLLENHAUER, M. and KOLTAI, P. (2023). Nonparametric approximation of conditional expectation operators.
[52] MOLLENHAUER, M., MUCKE, N. and SULLIVAN, T. J. (2024). Learning linear operators: Infinite-dimensional regression as a well-behaved noncompact inverse problem.
[53] MRANI-ZENTAR, O. and YÜKSEL, S. (2025). Data-driven nonparametric model learning and adaptive control of MDPs with Borel spaces: Identifiability and near optimal design. Preprint.
[54] MUANDET, K., FUKUMIZU, K., SRIPERUMBUDUR, B. and SCHOLKOPF, B. (2017). Kernel mean embedding of distributions: A review and beyond. *Found. Trends Mach. Learn.* **10** 1–141.
[55] PAMMER, G. (2024). A note on the adapted weak topology in discrete time. *Electron. Commun. Probab.* **29** Paper No. 1, 13 pp. MR4696436 https://doi.org/10.1214/23-ECP572
[56] PAPAGEORGIOU, N. S. and WINKERT, P. (2018). *Applied Nonlinear Functional Analysis: An Introduction. De Gruyter Graduate*. de Gruyter, Berlin. MR3823796
[57] PARK, J. and MUANDET, K. (2020). A measure-theoretic approach to kernel conditional mean embeddings. In *Advances in Neural Information Processing Systems* (H. Larochelle, M. Ranzato, R. Hadsell, M. F. Balcan and H. Lin, eds.) **33** 21247–21259.
[58] PAULSEN, V. I. and RAGHUPATHI, M. (2016). *An Introduction to the Theory of Reproducing Kernel Hilbert Spaces. Cambridge Studies in Advanced Mathematics* **152**. Cambridge Univ. Press, Cambridge. MR3526117 https://doi.org/10.1017/CBO9781316219232
[59] PRADHAN, S. and YÜKSEL, S. (2024). Continuity of cost in Borkar control topology and implications on discrete space and time approximations for controlled diffusions under several criteria. *Electron. J. Probab.* **29** Paper No. 37, 32 pp. MR4718442 https://doi.org/10.1214/24-ejp1093
[60] PRADHAN, S. and YÜKSEL, S. (2024). Near optimality of Lipschitz and smooth policies in controlled diffusions. *Syst. Control Lett.* **193** Paper No. 105943, 6 pp. MR4813588 https://doi.org/10.1016/j.sysconle.2024.105943
[61] PRADHAN, S. and YÜKSEL, S. (2025). Controlled diffusions under full, partial, and decentralized information: Existence of optimal policies and discrete-time approximations. *SIAM J. Control Optim.* **63** 3674–3702. MR4970190 https://doi.org/10.1137/23M1625196

[62] SALDI, N. (2020). A topology for team policies and existence of optimal team policies in stochastic team theory. *IEEE Trans. Automat. Control* **65** 310–317. MR4052878 https://doi.org/10.1109/tac.2019.2911798

[63] SALDI, N., LINDER, T. and YÜKSEL, S. (2018). *Finite Approximations in Discrete-Time Stochastic Control: Quantized Models and Asymptotic Optimality. Systems & Control: Foundations & Applications*. Birkhäuser/Springer, Cham. MR3791811 https://doi.org/10.1007/978-3-319-79033-6

[64] SALDI, N. and YÜKSEL, S. (2022). Geometry of information structures, strategic measures and associated stochastic control topologies. *Probab. Surv.* **19** 450–532. MR4456780 https://doi.org/10.1214/20-ps356

[65] SALDI, N., YÜKSEL, S. and LINDER, T. (2016). Near optimality of quantized policies in stochastic control under weak continuity conditions. *J. Math. Anal. Appl.* **435** 321–337. MR3423399 https://doi.org/10.1016/j.jmaa.2015.10.008

[66] SIMON-GABRIEL, C.-J., BARP, A., SCHÖLKOPF, B. and MACKEY, L. (2023). Metrizing weak convergence with maximum mean discrepancies. *J. Mach. Learn. Res.* **24** Paper No. [184], 20 pp. MR4633573

[67] SONG, L., BOOTS, B., SIDDIQI, S. M., GORDON, G. J. and SMOLA, A. (2010). Hilbert space embeddings of hidden Markov models. In *International Conference on Machine Learning*.

[68] SONG, L., FUKUMIZU, K. and GRETTON, A. (2013). Kernel embeddings of conditional distributions: A unified kernel framework for nonparametric inference in graphical models. *IEEE Signal Process. Mag.* **30** 98–111.

[69] SONG, L., HUANG, J., SMOLA, A. and FUKUMIZU, K. (2009). Hilbert space embeddings of conditional distributions with applications to dynamical systems. In *Proceedings of the 26th Annual International Conference on Machine Learning. ICML '09* 961–968. Association for Computing Machinery, New York, NY, USA.

[70] SRIPERUMBUDUR, B. K., FUKUMIZU, K. and LANCKRIET, G. R. G. (2011). Universality, characteristic kernels and RKHS embedding of measures. *J. Mach. Learn. Res.* **12** 2389–2410. MR2825431

[71] SRIPERUMBUDUR, B. K., GRETTON, A., FUKUMIZU, K., SCHÖLKOPF, B. and LANCKRIET, G. R. G. (2010). Hilbert space embeddings and metrics on probability measures. *J. Mach. Learn. Res.* **11** 1517–1561. MR2645460

[72] TAMÁS, A. and CSÁJI, B. C. (2024). Recursive estimation of conditional kernel mean embeddings. *J. Mach. Learn. Res.* **25** Paper No. 264, 35 pp. MR4810970

[73] VALDIER, M. (1994). A course on Young measures.

[74] WARGA, J. (2014). *Optimal Control of Differential and Functional Equations*. Academic Press, New York. MR0372708

[75] WHEEDEN, R. L. and ZYGMUND, A. (1977). *Measure and Integral: An Introduction to Real Analysis. Pure and Applied Mathematics* **43**. Dekker, New York–Basel. MR0492146

[76] YOUNG, L. C. (1937). Generalized curves and the existence of an attained absolute minimum in the calculus of variations. *C. R. Soc. Sci. Lett. Varsovie* **30** 212–234.

[77] YÜKSEL, S. (2020). A universal dynamic program and refined existence results for decentralized stochastic control. *SIAM J. Control Optim.* **58** 2711–2739. MR4146376 https://doi.org/10.1137/18M1221382

[78] YÜKSEL, S. (2024). On Borkar and Young relaxed control topologies and continuous dependence of invariant measures on control policy. *SIAM J. Control Optim.* **62** 2367–2386. MR4784080 https://doi.org/10.1137/23M1571940

[79] YÜKSEL, S. and LINDER, T. (2012). Optimization and convergence of observation channels in stochastic control. *SIAM J. Control Optim.* **50** 864–887. MR2914232 https://doi.org/10.1137/100808976

[80] ZHOU, Y., SONG, Y. and YÜKSEL, S. (2024). Robustness to model approximation, learning, and sample complexity in Wasserstein regular MDPs. Preprint. Available at arXiv:2410.14116.

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STRONG EXISTENCE AND UNIQUENESS FOR SINGULAR SDES DRIVEN BY STABLE PROCESSES

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We consider the one-dimensional stochastic differential equation

$$X_t = x_0 + L_t + \int_0^t \mu(X_s)\,ds, \quad t \ge 0,$$

where μ is a finite measure of Kato class K_η with $\eta \in (0, \alpha - 1]$ and $(L_t)_{t\ge0}$ is a symmetric α-stable process with $\alpha \in (1, 2)$. We derive weak and strong well-posedness for this equation when $\eta \le \alpha - 1$ and $\eta < \alpha - 1$, respectively and show that the condition $\eta \le \alpha - 1$ is sharp for weak existence. We furthermore reformulate the equation in terms of the local time of the solution $(X_t)_{t\ge0}$ and prove its well-posedness. To this end, we also derive a Tanaka-type formula for a symmetric, α-stable processes with $\alpha \in (1, 2)$ that is perturbed by an adapted, right-continuous process of finite variation.

REFERENCES

[1] ADAMS, D. R. and HEDBERG, L. I. (1996). *Function Spaces and Potential Theory*. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **314**. Springer, Berlin. MR1411441 https://doi.org/10.1007/978-3-662-03282-4

[2] AINSWORTH, M. and GLUSA, C. (2018). Towards an efficient finite element method for the integral fractional Laplacian on polygonal domains. In *Contemporary Computational Mathematics—a Celebration of the* 80*th Birthday of Ian Sloan*, *Vol.* 1, 2 17–57. Springer, Cham. MR3822227

[3] ALBEVERIO, S. and MA, Z. M. (1992). Additive functionals, nowhere Radon and Kato class smooth measures associated with Dirichlet forms. *Osaka J. Math.* **29** 247–265. MR1173989

[4] ANTUNOVIĆ, T., BURDZY, K., PERES, Y. and RUSCHER, J. (2011). Isolated zeros for Brownian motion with variable drift. *Electron. J. Probab.* **16** no. 65, 1793–1814. MR2842087 https://doi.org/10.1214/EJP.v16-927

[5] APPLEBAUM, D. (2009). *Lévy Processes and Stochastic Calculus*, 2nd ed. *Cambridge Studies in Advanced Mathematics* **116**. Cambridge Univ. Press, Cambridge. MR2512800 https://doi.org/10.1017/CBO9780511809781

[6] ATHREYA, S., BUTKOVSKY, O. and MYTNIK, L. (2020). Strong existence and uniqueness for stable stochastic differential equations with distributional drift. *Ann. Probab.* **48** 178–210. MR4079434 https://doi.org/10.1214/19-AOP1358

[7] AUGELLO, G., VALENTI, D. and SPAGNOLO, B. (2010). Non-Gaussian noise effects in the dynamics of a short overdamped Josephson junction. *Eur. Phys. J. B* **78** 225–234.

[8] BAHOURI, H., CHEMIN, J.-Y. and DANCHIN, R. (2011). *Fourier Analysis and Nonlinear Partial Differential Equations*. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **343**. Springer, Heidelberg. MR2768550 https://doi.org/10.1007/978-3-642-16830-7

[9] BASS, R. F. and CHEN, Z.-Q. (2001). Stochastic differential equations for Dirichlet processes. *Probab. Theory Related Fields* **121** 422–446. MR1867429 https://doi.org/10.1007/s004400100151

[10] BASS, R. F. and CHEN, Z.-Q. (2003). Brownian motion with singular drift. *Ann. Probab.* **31** 791–817. MR1964949 https://doi.org/10.1214/aop/1048516536

[11] BASS, R. F. and CHEN, Z.-Q. (2005). One-dimensional stochastic differential equations with singular and degenerate coefficients. *Sankhyā* **67** 19–45. MR2203887

[12] BERTOIN, J. (1996). *Lévy Processes*. *Cambridge Tracts in Mathematics* **121**. Cambridge Univ. Press, Cambridge. MR1406564

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[13] BLEI, S. and ENGELBERT, H.-J. (2013). One-dimensional stochastic differential equations with generalized and singular drift. *Stoch. Process. Appl.* **123** 4337–4372. MR3096356 https://doi.org/10.1016/j.spa.2013.06.014

[14] BLEI, S. and ENGELBERT, H.-J. (2014). One-dimensional stochastic differential equations with generalized drift. *Theory Probab. Appl.* **58** 345–357. MR3403001 https://doi.org/10.1137/S0040585X97986655

[15] BLUMENTHAL, R. M. and GETOOR, R. K. (1968). *Markov Processes and Potential Theory*. *Pure and Applied Mathematics* **29**. Academic Press, New York. MR0264757

[16] BOGDAN, K. and JAKUBOWSKI, T. (2007). Estimates of heat kernel of fractional Laplacian perturbed by gradient operators. *Comm. Math. Phys.* **271** 179–198. MR2283957 https://doi.org/10.1007/s00220-006-0178-y

[17] BOYLAN, E. S. (1964). Local times for a class of Markoff processes. *Illinois J. Math.* **8** 19–39. MR0158434

[18] BROX, T. (1986). A one-dimensional diffusion process in a Wiener medium. *Ann. Probab.* **14** 1206–1218. MR0866343

[19] CHARBONNEAU, P., KURCHAN, J., PARISI, G., URBANI, P. and ZAMPONI, F. (2014). Fractal free energy landscapes in structural glasses. *Nat. Commun.* **5**. https://doi.org/10.1038/ncomms4725

[20] CHAUDRU DE RAYNAL, P. and MENOZZI, S. (2022). On multidimensional stable-driven stochastic differential equations with Besov drift. *Electron. J. Probab.* **27** Paper No. 163, 52. MR4525442 https://doi.org/10.1214/22-ejp864

[21] CHEN, Z.-Q., KIM, P. and SONG, R. (2012). Dirichlet heat kernel estimates for fractional Laplacian with gradient perturbation. *Ann. Probab.* **40** 2483–2538. MR3050510 https://doi.org/10.1214/11-AOP682

[22] CHEN, Z.-Q. and WANG, L. (2016). Uniqueness of stable processes with drift. *Proc. Amer. Math. Soc.* **144** 2661–2675. MR3477084 https://doi.org/10.1090/proc/12909

[23] DEL CASTILLO-NEGRETE, D., GONCHAR, V. Y. and CHECHKIN, A. (2008). Fluctuation-driven directed transport in the presence of Lévy flights. *Phys. A, Stat. Mech. Appl.* **387** 6693–6704.

[24] DITLEVSEN, P. D. (1999). Observation of α-stable noise induced millennial climate changes from an ice-core record. *Geophys. Res. Lett.* **26** 1441–1444.

[25] DYBIEC, B. and GUDOWSKA-NOWAK, E. (2009). Lévy stable noise-induced transitions: Stochastic resonance, resonant activation and dynamic hysteresis. *J. Stat. Mech. Theory Exp.* **2009** P05004.

[26] ENGELBERT, H.-J. and KURENOK, V. P. (2019). The Tanaka formula for symmetric stable processes with index α, $0 < \alpha < 2$. *Theory Probab. Appl.* **64** 264–289. MR3943123 https://doi.org/10.1137/S0040585X97T989489

[27] ENGELBERT, H. J. and SCHMIDT, W. (1985). On one-dimensional stochastic differential equations with generalized drift. In *Stochastic Differential Systems* (*Marseille-Luminy*, 1984). *Lect. Notes Control Inf. Sci.* **69** 143–155. Springer, Berlin. MR0798317 https://doi.org/10.1007/BFb0005069

[28] ÉTORÉ, P. and MARTINEZ, M. (2018). Time inhomogeneous stochastic differential equations involving the local time of the unknown process, and associated parabolic operators. *Stoch. Process. Appl.* **128** 2642–2687. MR3811700 https://doi.org/10.1016/j.spa.2017.09.018

[29] FRAUENFELDER, H., SLIGAR, S. G. and WOLYNES, P. G. (1991). The energy landscapes and motions of proteins. *Science* **254** 1598–1603.

[30] GROH, J. (1986). On Brownian motion with irregular drift. *Illinois J. Math.* **30** 417–428. MR0850340

[31] GYÖNGY, I. and MARTÍNEZ, T. (2001). On stochastic differential equations with locally unbounded drift. *Czechoslovak Math. J.* **51** 763–783. MR1864041 https://doi.org/10.1023/A:1013764929351

[32] HU, Y., LÊ, K. and MYTNIK, L. (2017). Stochastic differential equation for Brox diffusion. *Stoch. Process. Appl.* **127** 2281–2315. MR3652414 https://doi.org/10.1016/j.spa.2016.10.010

[33] KIM, P. and SONG, R. (2006). Two-sided estimates on the density of Brownian motion with singular drift. *Illinois J. Math.* **50** 635–688. MR2247841

[34] KIM, P. and SONG, R. (2014). Stable process with singular drift. *Stoch. Process. Appl.* **124** 2479–2516. MR3192504 https://doi.org/10.1016/j.spa.2014.03.006

[35] KNOPOVA, V. and KULIK, A. (2017). Intrinsic compound kernel estimates for the transition probability density of Lévy-type processes and their applications. *Probab. Math. Statist.* **37** 53–100. MR3652202 https://doi.org/10.19195/0208-4147.37.1.3

[36] KREMP, H. and PERKOWSKI, N. (2022). Multidimensional SDE with distributional drift and Lévy noise. *Bernoulli* **28** 1757–1783. MR4411510 https://doi.org/10.3150/21-bej1394

[37] KREMP, H. K. (2022). *Topics in Particle Systems and Singular SDEs*, *Dissertation*. Free Univ., Berlin.

[38] KRYLOV, N. V. and RÖCKNER, M. (2005). Strong solutions of stochastic equations with singular time dependent drift. *Probab. Theory Related Fields* **131** 154–196. MR2117951 https://doi.org/10.1007/s00440-004-0361-z

[39] KURTZ, T. G. (2011). Equivalence of stochastic equations and martingale problems. In *Stochastic Analysis* 2010 113–130. Springer, Heidelberg. MR2789081 https://doi.org/10.1007/978-3-642-15358-7_6

[40] KURTZ, T. G. (2014). Weak and strong solutions of general stochastic models. *Electron. Commun. Probab.* **19** no. 58, 16. MR3254737 https://doi.org/10.1214/ECP.v19-2833

[41] KYPRIANOU, A. E. (2014). *Fluctuations of Lévy Processes with Applications: Introductory Lectures*, 2nd ed. *Universitext*. Springer, Heidelberg. MR3155252 https://doi.org/10.1007/978-3-642-37632-0

[42] LE GALL, J.-F. (1984). One-dimensional stochastic differential equations involving the local times of the unknown process. In *Stochastic Analysis and Applications* (*Swansea*, 1983). *Lecture Notes in Math.* **1095** 51–82. Springer, Berlin. MR0777514 https://doi.org/10.1007/BFb0099122

[43] LI, Y., XU, Y. and KURTHS, J. (2017). Roughness-enhanced transport in a tilted ratchet driven by Lévy noise. *Phys. Rev. E* **96** 052121.

[44] MYTNIK, L. and WEINBERGER, J. (2026). Supplement to "Strong existence and uniqueness for singular SDEs driven by stable processes." https://doi.org/10.1214/25-AAP2267SUPP

[45] PRIOLA, E. (2012). Pathwise uniqueness for singular SDEs driven by stable processes. *Osaka J. Math.* **49** 421–447. MR2945756

[46] PROTTER, P. E. (2004). *Stochastic Integration and Differential Equations*, 2nd ed. *Stochastic Modelling and Applied Probability* **21**. Springer, Berlin. MR2020294

[47] SALMINEN, P. and YOR, M. (2007). Tanaka formula for symmetric Lévy processes. In *Séminaire de Probabilités XL. Lecture Notes in Math.* **1899** 265–285. Springer, Berlin. MR2409011 https://doi.org/10.1007/978-3-540-71189-6_14

[48] SHEN, X., ZHANG, H., XU, Y. and MENG, S. (2015). Observation of alpha-stable noise in the laser gyroscope data. *IEEE Sens. J.* **16** 1998–2003.

[49] SOURYAL, M. R., LARRSON, E. G., PERIC, B. M. and VOJCIC, B. R. (2005). Soft-decision metrics for coded orthogonal signaling in symmetric alpha-stable noise. In *Proceedings*. (*ICASSP* '05). *IEEE International Conference on Acoustics, Speech, and Signal Processing*, 2005 **3** iii/697-iii/700. https://doi.org/10.1109/ICASSP.2005.1415805

[50] STEIN, E. M. (1970). *Singular Integrals and Differentiability Properties of Functions*. *Princeton Mathematical Series* **30**. Princeton Univ. Press, Princeton, NJ. MR0290095

[51] STROOCK, D. W. and YOR, M. (1981). Some remarkable martingales. In *Seminar on Probability, XV* (*Univ. Strasbourg, Strasbourg*, 1979/1980) (*French*). *Lecture Notes in Math.* **850** 590–603. Springer, Berlin. MR0622590

[52] TANAKA, H., TSUCHIYA, M. and WATANABE, S. (1974). Perturbation of drift-type for Lévy processes. *J. Math. Kyoto Univ.* **14** 73–92. MR0368146 https://doi.org/10.1215/kjm/1250523280

[53] TRIEBEL, H. (1988). Characterizations of Besov–Hardy–Sobolev spaces: A unified approach. *J. Approx. Theory* **52** 162–203. MR0929302 https://doi.org/10.1016/0021-9045(88)90055-X

[54] TRIEBEL, H. (2010). *Theory of Function Spaces*. *Modern Birkhäuser Classics*. Birkhäuser, Basel. Reprint of 1983 edition [MR0730762]. Also published in 1983 by Birkhäuser Verlag [MR0781540]. MR3024598

[55] TSUKADA, H. (2018). A potential theoretic approach to Tanaka formula for asymmetric Lévy processes. In *Séminaire de Probabilités XLIX*. *Lecture Notes in Math.* **2215** 521–542. Springer, Cham. MR3838874

[56] TSUKADA, H. (2019). Tanaka formula for strictly stable processes. *Probab. Math. Statist.* **39** 39–60. MR3964382 https://doi.org/10.19195/0208-4147.39.1.3

[57] VERETENNIKOV, A. Y. (1981). On strong solutions and explicit formulas for solutions of stochastic integral equations. *Math. USSR, Sb.* **39** 387–403. https://doi.org/10.1070/SM1981v039n03ABEH001522

[58] WOYCZYŃSKI, W. A. (2001). Lévy processes in the physical sciences. In *Lévy Processes* 241–266. Birkhäuser, Boston, MA. MR1833700

[59] XIONG, J., ZHENG, J. and ZHOU, X. (2019). Unique strong solutions of Lévy processes driven stochastic differential equations with discontinuous coefficients. *Stochastics* **91** 592–612. MR3939347 https://doi.org/10.1080/17442508.2019.1567729

[60] ZHANG, X. and ZHAO, G. (2018). Singular Brownian diffusion processes. *Commun. Math. Stat.* **6** 533–581. MR3877717 https://doi.org/10.1007/s40304-018-0164-7

[61] ZVONKIN, A. K. (1975). A transformation of the phase space of a diffusion process that will remove the drift. *Mat. Sb.* (*N.S.*) **22** 129–149. MR0336813

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A LOCALISATION PHASE TRANSITION FOR THE CATALYTIC BRANCHING RANDOM WALK

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We show the existence of a phase transition between a localisation and a nonlocalisation regime for a branching random walk with a catalyst at the origin. More precisely, we consider a continuous-time branching random walk that jumps at rate one, with simple random walk jumps on $\mathbb{Z}^d$, and that branches (with binary branching) at rate $\lambda > 0$ everywhere, except at the origin, where it branches at rate $\lambda_0 > \lambda$. We show that, if λ_0 is large enough, then the occupation measure of the branching random walk localises (i.e., when normalised by the total number of particles, it converges almost surely without spatial renormalisation), whereas, if λ_0 is close enough to λ, then the occupation measure delocalises, in the sense that the proportion of particles in any finite given set converges almost surely to zero. The case $\lambda = 0$ (when branching only occurs at the origin) has been extensively studied in the literature and a transition between localisation and nonlocalisation was also exhibited in this case. Interestingly, the transition that we observe, conjecture, and partially prove in this paper occurs at the same threshold as in the case $\lambda = 0$. One of the strengths of our result is that, in the localisation regime, we are able to prove convergence of the occupation measure, while existing results in the case $\lambda = 0$ give convergence of moments instead.

REFERENCES

[1] ALBEVERIO, S. and BOGACHEV, L. V. (2000). Branching random walk in a catalytic medium. I. Basic equations. *Positivity* **4** 41–100. MR1740207 https://doi.org/10.1023/A:1009818620550

[2] ALBEVERIO, S., BOGACHEV, L. V. and YAROVAYA, E. B. (1998). Asymptotics of branching symmetric random walk on the lattice with a single source. *C. R. Acad. Sci. Paris Sér. I Math.* **326** 975–980. MR1649878 https://doi.org/10.1016/S0764-4442(98)80125-0

[3] ANDRÉ, M. and DUCHAMPS, J.-J. (2025). Sharp $L \log L$ condition for supercritical Galton–Watson processes with countable types. arXiv preprint. Available at arXiv:2503.05575.

[4] ATHREYA, K. B. and NEY, P. E. (1972). *Branching Processes*. *Die Grundlehren der Mathematischen Wissenschaften* **196**. Springer, New York. MR0373040

[5] BANDYOPADHYAY, A. and THACKER, D. (2017). Pólya urn schemes with infinitely many colors. *Bernoulli* **23** 3243–3267. MR3654806 https://doi.org/10.3150/16-BEJ844

[6] BANDYOPADHYAY, A. and THACKER, D. (2022). A new approach to Pólya urn schemes and its infinite color generalization. *Ann. Appl. Probab.* **32** 46–79. MR4386521 https://doi.org/10.1214/21-aap1671

[7] BANSAYE, V., BERAH, T. and CLOEZ, B. (2025). On the strong law of large numbers and $L \log L$ condition for supercritical general branching processes. arXiv preprint. Available at arXiv:2503.03324.

[8] BENAÏM, M. (2006). Dynamics of stochastic approximation algorithms. In *Séminaire de Probabilités, XXXIII*. *Lecture Notes in Math.* **1709** 1–68. Springer, Berlin. MR1767993 https://doi.org/10.1007/BFb0096509

[9] BIGGINS, J. D. (1992). Uniform convergence of martingales in the branching random walk. *Ann. Probab.* **20** 137–151. MR1143415

[10] BOCHAROV, S. and HARRIS, S. C. (2014). Branching random walk in an inhomogeneous breeding potential. In *Séminaire de Probabilités XLVI*. *Lecture Notes in Math.* **2123** 1–32. Springer, Cham. MR3330812 https://doi.org/10.1007/978-3-319-11970-0_1

[11] BOCHAROV, S. and HARRIS, S. C. (2014). Branching Brownian motion with catalytic branching at the origin. *Acta Appl. Math.* **134** 201–228. MR3273694 https://doi.org/10.1007/s10440-014-9879-y

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[12] BOCHAROV, S. and HARRIS, S. C. (2016). Limiting distribution of the rightmost particle in catalytic branching Brownian motion. *Electron. Commun. Probab.* **21** Paper No. 70, 12. MR3564217 https://doi.org/10.1214/16-ECP22

[13] BULINSKAYA, E. V. (2010). Catalytic branching random walk on three-dimensional lattice. *Theory Stoch. Process.* **16** 23–32. MR2777898

[14] BULINSKAYA, E. V. (2011). Limit distributions arising in branching random walks on integer lattices. *Lith. Math. J.* **51** 310–321. MR2832227 https://doi.org/10.1007/s10986-011-9128-2

[15] BULINSKAYA, E. V. (2018). Spread of a catalytic branching random walk on a multidimensional lattice. *Stoch. Process. Appl.* **128** 2325–2340. MR3804795 https://doi.org/10.1016/j.spa.2017.09.007

[16] BULINSKAYA, E. V. (2019). Fluctuations of the propagation front of a catalytic branching walk. *Teor. Veroyatn. Primen.* **64** 642–670. MR4030815 https://doi.org/10.4213/tvp5325

[17] CARMONA, P. and HU, Y. (2014). The spread of a catalytic branching random walk. *Ann. Inst. Henri Poincaré Probab. Stat.* **50** 327–351. MR3189074 https://doi.org/10.1214/12-AIHP529

[18] CASTELL, F., GÜN, O. and MAILLARD, G. (2011). Parabolic Anderson model with a finite number of moving catalysts. In *Probability in Complex Physical Systems. Springer Proc. Math.* **11** 91–117. Springer, Heidelberg. MR3372846 https://doi.org/10.1007/978-3-642-23811-6_5

[19] CHAMPAGNAT, N. and VILLEMONAIS, D. (2023). General criteria for the study of quasi-stationarity. *Electron. J. Probab.* **28** Paper No. 22, 84. MR4546021 https://doi.org/10.1214/22-ejp880

[20] COX, J. T., GROAH, J., DAWSON, D. A. and GREVEN, A. (2004). *Mutually Catalytic Super Branching Random Walks: Large Finite Systems and Renormalization Analysis* **809**. Amer. Math. Soc., Providence.

[21] DAWSON, D. A. and FLEISCHMANN, K. (1994). A super-Brownian motion with a single point catalyst. *Stoch. Process. Appl.* **49** 3–40. MR1258279 https://doi.org/10.1016/0304-4149(94)90110-4

[22] DÖRING, L. and ROBERTS, M. I. (2013). Catalytic branching processes via spine techniques and renewal theory. In *Séminaire de Probabilités XLV. Lecture Notes in Math.* **2078** 305–322. Springer, Cham. MR3185920 https://doi.org/10.1007/978-3-319-00321-4_12

[23] DUFLO, M. (1997). *Random Iterative Models. Applications of Mathematics (New York)* **34**. Springer, Berlin. Translated from the 1990 French original by Stephen S. Wilson and revised by the author. MR1485774 https://doi.org/10.1007/978-3-662-12880-0

[24] FIODOROV, A. and MUIRHEAD, S. (2014). Complete localisation and exponential shape of the parabolic Anderson model with Weibull potential field. *Electron. J. Probab.* **19** no. 58, 27. MR3238778 https://doi.org/10.1214/EJP.v19-3203

[25] FLEISCHMANN, K. and LE GALL, J.-F. (1995). A new approach to the single point catalytic super-Brownian motion. *Probab. Theory Related Fields* **102** 63–82. MR1351711 https://doi.org/10.1007/BF01295222

[26] GÄRTNER, J. and DEN HOLLANDER, F. (2006). Intermittency in a catalytic random medium. *Ann. Probab.* **34** 2219–2287. MR2294981 https://doi.org/10.1214/009117906000000467

[27] GÄRTNER, J. and HEYDENREICH, M. (2006). Annealed asymptotics for the parabolic Anderson model with a moving catalyst. *Stoch. Process. Appl.* **116** 1511–1529. MR2269214 https://doi.org/10.1016/j.spa.2006.04.002

[28] GÄRTNER, J. and MOLCHANOV, S. A. (1990). Parabolic problems for the Anderson model. I. Intermittency and related topics. *Comm. Math. Phys.* **132** 613–655. MR1069840

[29] GREVEN, A., KLENKE, A. and WAKOLBINGER, A. (1999). The longtime behavior of branching random walk in a catalytic medium. *Electron. J. Probab.* **4** no. 12, 80. MR1690316 https://doi.org/10.1214/EJP.v4-49

[30] JANSON, S. (2019). Random replacements in Pólya urns with infinitely many colours. *Electron. Commun. Probab.* **24** Paper No. 23, 11. MR3953824 https://doi.org/10.1214/19-ECP226

[31] KESTEN, H. and SIDORAVICIUS, V. (2003). Branching random walk with catalysts. *Electron. J. Probab.* **8** no. 5, 51. MR1961167 https://doi.org/10.1214/EJP.v8-127

[32] KÖNIG, W., LACOIN, H., MÖRTERS, P. and SIDOROVA, N. (2009). A two cities theorem for the parabolic Anderson model. *Ann. Probab.* **37** 347–392. MR2489168 https://doi.org/10.1214/08-AOP405

[33] LALLEY, S. and SELLKE, T. (1988). Traveling waves in inhomogeneous branching Brownian motions. I. *Ann. Probab.* **16** 1051–1062. MR0942755

[34] LALLEY, S. and SELLKE, T. (1989). Travelling waves in inhomogeneous branching Brownian motions. II. *Ann. Probab.* **17** 116–127. MR0972775

[35] LAWLER, G. F. and LIMIC, V. (2010). *Random Walk: A Modern Introduction. Cambridge Studies in Advanced Mathematics* **123**. Cambridge Univ. Press, Cambridge. MR2677157 https://doi.org/10.1017/CBO9780511750854

[36] MAILLER, C. and MARCKERT, J.-F. (2017). Measure-valued Pólya urn processes. *Electron. J. Probab.* **22** Paper No. 26, 33. MR3629870 https://doi.org/10.1214/17-EJP47

[37] MAILLER, C. and VILLEMONAIS, D. (2020). Stochastic approximation on noncompact measure spaces and application to measure-valued Pólya processes. *Ann. Appl. Probab.* **30** 2393–2438. MR4149532 https://doi.org/10.1214/20-AAP1561

[38] MOLCHANOV, S. A. and YAROVAYA, E. B. (2012). Branching processes with lattice spatial dynamics and a finite set of particle generation centers. *Dokl. Akad. Nauk* **86** 638–641. MR3052247 https://doi.org/10.1134/S1064562412040278

[39] ORTGIESE, M. and ROBERTS, M. I. (2017). One-point localization for branching random walk in Pareto environment. *Electron. J. Probab.* **22** Paper No. 6, 20. MR3613699 https://doi.org/10.1214/16-EJP22

[40] RÉVÉSZ, P. (1994). *Random Walks of Infinitely Many Particles*. World Scientific, River Edge, NJ. MR1645302 https://doi.org/10.1142/2376

[41] TOPCHII, V. and VATUTIN, V. (2003). Individuals at the origin in the critical catalytic branching random walk. In *Discrete Random Walks* (*Paris*, 2003). *Discrete Math. Theor. Comput. Sci. Proc.*, *AC* 325–332. Assoc. Discrete Math. Theor. Comput. Sci., Nancy. MR2042398

[42] VATUTIN, V. A. and TOPCHIĬ, V. A. (2005). A limit theorem for critical catalytic branching random walks. *Teor. Veroyatn. Primen.* **49** 498–518. MR2144864 https://doi.org/10.1137/S0040585X97981214

[43] VATUTIN, V. A. and TOPCHIĬ, V. A. (2011). Catalytic branching random walks on $\mathbb{Z}^d$ with branching at the origin. *Mat. Tr.* **14** 28–72. MR2961768 https://doi.org/10.3103/s1055134413020065

[44] VATUTIN, V. A., TOPCHIĬ, V. A. and KHU, Y. (2012). A branching random walk in the lattice Z^4 with branching at only the origin. *Teor. Veroyatn. Primen.* **56** 193–212. MR3136471 https://doi.org/10.1137/s0040585x97985352

[45] WATANABE, S. (1967). Limit theorem for a class of branching processes. In *Markov Processes and Potential Theory* (*Proc. Sympos. Math. Res. Center, Madison, Wis.*, 1967). Wiley, New York. MR0237007

[46] YAROVAYA, E. B. (2010). Monotonicity of the probability of returning to the source in models of a branching random walk. *Vestnik Moskov. Univ. Ser. I Mat. Mekh.* **65** 78–80. MR2682356 https://doi.org/10.3103/S0027132210020063

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MIXED MARKOV-PERFECT EQUILIBRIA IN THE CONTINUOUS-TIME WAR OF ATTRITION

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We prove the existence of a Markov-perfect equilibrium in randomized stopping times for a model of the war of attrition in which the underlying state variable follows a homogenous linear diffusion. We first prove that the space of Markovian randomized stopping times can be topologized as a compact absolute retract. This in turn enables us to use a powerful fixed-point theorem by Eilenberg and Montgomery (*Amer. J. Math.* **68** (1946) 214–222) to prove our existence theorem. We illustrate our results with an example of a war of attrition that admits a mixed-strategy Markov-perfect equilibrium but no pure-strategy Markov-perfect equilibrium.

REFERENCES

[1] ALBEVERIO, S. and MA, Z. M. (1992). Additive functionals, nowhere Radon and Kato class smooth measures associated with Dirichlet forms. *Osaka J. Math.* **29** 247–265. MR1173989
[2] ATTARD, N. (2017). Nash equilibrium in nonzero-sum games of optimal stopping for Brownian motion. *Adv. Appl. Probab.* **49** 430–445. MR3668384 https://doi.org/10.1017/apr.2017.8
[3] ATTARD, N. (2018). Nonzero-sum games of optimal stopping for Markov processes. *Appl. Math. Optim.* **77** 567–597. MR3787841 https://doi.org/10.1007/s00245-016-9388-7
[4] BAXTER, J. R. and CHACON, R. V. (1977). Compactness of stopping times. *Z. Wahrsch. Verw. Gebiete* **40** 169–181. MR0517871 https://doi.org/10.1007/BF00736045
[5] BEER, G. (1993). *Topologies on Closed and Closed Convex Sets. Mathematics and Its Applications* **268**. Kluwer Academic, Dordrecht. MR1269778 https://doi.org/10.1007/978-94-015-8149-3
[6] BENSOUSSAN, A. and FRIEDMAN, A. (1977). Nonzero-sum stochastic differential games with stopping times and free boundary problems. *Trans. Amer. Math. Soc.* **231** 275–327. MR0453082 https://doi.org/10.2307/1997905
[7] BILLINGSLEY, P. (2013). *Convergence of Probability Measures*. Wiley, New York.
[8] BLISS, C. and NALEBUFF B, B. (1984). Dragon-slaying and ballroom dancing: The private supply of a public good. *J. Public Econ.* **25** 1–12.
[9] BLUMENTHAL, R. M. and GETOOR, R. K. (1968). *Markov Processes and Potential Theory*. Academic Press, New York.
[10] BORODIN, A. N. and SALMINEN, P. (2002). *Handbook of Brownian Motion—Facts and Formulae*, 2nd ed. *Probability and Its Applications*. Birkhäuser, Basel. MR1912205 https://doi.org/10.1007/978-3-0348-8163-0
[11] CATTIAUX, P. and LEPELTIER, J.-P. (1990). Existence of a quasi-Markov Nash equilibrium for nonzero sum Markov stopping games. *Stoch. Stoch. Rep.* **30** 85–103. MR1054193 https://doi.org/10.1080/17442509008833635
[12] CHRISTENSEN, S. and SCHULTZ, B. (2024). On the Existence of Markovian Randomized Equilibria in Dynkin Games of War-of-Attrition-Type. https://arxiv.org/abs/2406.09820.
[13] CHRONOPOULOS, M. and SIDDIQUI, A. (2015). When is it better to wait for a new version? Optimal replacement of an emerging technology under uncertainty. *Ann. Oper. Res.* **235** 177–201. MR3428590 https://doi.org/10.1007/s10479-015-2010-6
[14] DE ANGELIS, T., FERRARI, G. and MORIARTY, J. (2018). Nash equilibria of threshold type for two-player nonzero-sum games of stopping. *Ann. Appl. Probab.* **28** 112–147. MR3770874 https://doi.org/10.1214/17-AAP1301

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[15] DE ANGELIS, T., MERKULOV, N. and PALCZEWSKI, J. (2022). On the value of non-Markovian Dynkin games with partial and asymmetric information. *Ann. Appl. Probab.* **32** 1774–1813. MR4430001 https://doi.org/10.1214/21-aap1721

[16] DEBREU, G. (1952). A social equilibrium existence theorem. *Proc. Natl. Acad. Sci. USA* **38** 886–893. MR0050251 https://doi.org/10.1073/pnas.38.10.886

[17] DÉCAMPS, J.-P., GENSBITTEL, F. and MARIOTTI, T. (2023). The War of Attrition under Uncertainty: Theory and Robust Testable Implications. https://hal.science/hal-03816389v3.

[18] DÉCAMPS, J.-P., GENSBITTEL, F. and MARIOTTI, T. (2025). Investment timing and technological breakthroughs. *Math. Oper. Res.* **50** 1478–1513. MR4902798 https://doi.org/10.1287/moor.2022.0022

[19] DELLACHERIE, C. and MEYER, P.-A. (1982). *Probabilities and Potential B*, *Theory of Martingales*. North-Holland, Amsterdam.

[20] EILENBERG, S. and MONTGOMERY, D. (1946). Fixed point theorems for multi-valued transformations. *Amer. J. Math.* **68** 214–222. MR0016676 https://doi.org/10.2307/2371832

[21] EKSTRÖM, E. and PESKIR, G. (2008). Optimal stopping games for Markov processes. *SIAM J. Control Optim.* **47** 684–702. MR2385859 https://doi.org/10.1137/060673916

[22] EKSTRÖM, E. and VILLENEUVE, S. (2006). On the value of optimal stopping games. *Ann. Appl. Probab.* **16** 1576–1596. MR2260074 https://doi.org/10.1214/105051606000000204

[23] EL KAROUI, N. (1981). Les aspects probabilistes du contrôle stochastique. In *Ninth Saint Flour Probability Summer School—1979* (*Saint Flour*, 1979). *Lecture Notes in Math.* **876** 73–238. Springer, Berlin. MR0637471

[24] EL KAROUI, N., LEPELTIER, J.-P. and MILLET, A. (1992). A probabilistic approach to the reduite in optimal stopping. *Probab. Math. Statist.* **13** 97–121. MR1199792

[25] FOLLAND, G. B. (1999). *Real Analysis*: *Modern Techniques and Their Applications*. Wiley, New York.

[26] FUDENBERG, D. and TIROLE, J. (1985). Preemption and rent equalization in the adoption of new technology. *Rev. Econ. Stud.* **52** 383–401. MR0797373 https://doi.org/10.2307/2297660

[27] FUKUSHIMA, M., OSHIMA, Y. and TAKEDA, M. (2011). *Dirichlet Forms and Symmetric Markov Processes*, extended ed. *De Gruyter Studies in Mathematics* **19**. de Gruyter, Berlin. MR2778606

[28] GEORGIADIS, G., KIM, Y. and KWON, H. D. (2022). The absence of attrition in a war of attrition under complete information. *Games Econom. Behav.* **131** 171–185. MR4347449 https://doi.org/10.1016/j.geb.2021.11.004

[29] GIECZEWSKI, G. (2025). Evolving wars of attrition. *J. Econom. Theory* **224** 105967. MR4849877 https://doi.org/10.1016/j.jet.2025.105967

[30] GLICKSBERG, I. L. (1952). A further generalization of the Kakutani fixed theorem, with application to Nash equilibrium points. *Proc. Amer. Math. Soc.* **3** 170–174. MR0046638 https://doi.org/10.2307/2032478

[31] HAMADENE, S. and ZHANG, J. (2010). The continuous time nonzero-sum Dynkin game problem and application in game options. *SIAM J. Control Optim.* **48** 3659–3669.

[32] HUISMAN, K. J. M. and KORT, P. M. (2004). Strategic technology adoption taking into account future technological improvements: A real options approach. *European J. Oper. Res.* **159** 705–728. MR2078864 https://doi.org/10.1016/S0377-2217(03)00421-1

[33] JACOD, J. and SHIRYAEV, A. N. (2003). *Limit Theorems for Stochastic Processes*. Springer, Berlin, New York.

[34] KAKUTANI, S. (1941). A generalization of Brouwer's fixed point theorem. *Duke Math. J.* **8** 457–459. MR0004776

[35] KALLENBERG, O. (2017). *Random Measures*, *Theory and Applications*. *Probability Theory and Stochastic Modelling* **77**. Springer, Cham. MR3642325 https://doi.org/10.1007/978-3-319-41598-7

[36] KWON, H. D. and PALCZEWSKI, J. (2025). Exit Game with Private Information. *Math. Oper. Res.* **50** 2433–2469. MR4989835 https://doi.org/10.1287/moor.2022.0285

[37] LAMBRECHT, B. M. (2001). The impact of debt financing on entry and exit in a duopoly. *Rev. Financ. Stud.* **14** 765–804.

[38] LARAKI, R. and SOLAN, E. (2005). The value of zero-sum stopping games in continuous time. *SIAM J. Control Optim.* **43** 1913–1922. MR2137507 https://doi.org/10.1137/S0363012903429025

[39] LARAKI, R. and SOLAN, E. (2013). Equilibrium in two-player non-zero-sum Dynkin games in continuous time. *Stochastics* **85** 997–1014. MR3176497 https://doi.org/10.1080/17442508.2012.726222

[40] LEPELTIER, J.-P. and ETOURNEAU, E. (1987). A problem of non-zero sum stopping game. In *Stochastic Differential Systems*: *Proceedings of the IFIP-WG* 7/1 *Working Conference* (H. J. Engelbert and W. Schmidt, eds.). *Lecture Notes in Control and Information Sciences* **96** 310–315. Springer, Berlin.

[41] LEPELTIER, J.-P. and MAINGUENEAU, M. A. (1984). Le jeu de Dynkin en théorie générale sans l'hypothèse de Mokobodski. *Stochastics* **13** 25–44. MR0752475 https://doi.org/10.1080/17442508408833309

[42] LIPTSER, R. S. and SHIRYAEV, A. N. (1977). *Statistics of Random Processes*: *I. General Theory*. Springer, New York.

[43] MARTYR, R. and MORIARTY, J. (2021). Nonzero-sum games of optimal stopping and generalized Nash equilibrium problems. *SIAM J. Control Optim.* **59** 1443–1465. MR4243661 https://doi.org/10.1137/18M119803X

[44] MASKIN, E. and TIROLE, J. (2001). Markov perfect equilibrium. I. Observable actions. *J. Econom. Theory* **100** 191–219. MR1860033 https://doi.org/10.1006/jeth.2000.2785

[45] MCLENNAN, A. (2018). *Advanced Fixed Point Theory for Economics*. Springer, Singapore. MR3822724 https://doi.org/10.1007/978-981-13-0710-2

[46] MEYER, P.-A. (1978). Convergence faible et compacité des temps d'arrêt d'après Baxter et Chacon. In *Séminaire de Probabilités, XII* (*Univ. Strasbourg, Strasbourg*, 1976/1977). *Lecture Notes in Math.* **649** 411–423. Springer, Berlin. MR0520015

[47] MURTO, P. (2004). Exit in duopoly under uncertainty. *Rand J. Econ.* **35** 111–127.

[48] RENY, P. J. (2011). On the existence of monotone pure-strategy equilibria in Bayesian games. *Econometrica* **79** 499–553. MR2809378 https://doi.org/10.3982/ECTA8934

[49] REVUZ, D. (1970). Mesures associées aux fonctionnelles additives de Markov. I. *Trans. Amer. Math. Soc.* **148** 501–531. MR0279890 https://doi.org/10.2307/1995386

[50] REVUZ, D. and YOR, M. (1999). *Continuous Martingales and Brownian Motion*. Springer, Berlin, New York.

[51] RIEDEL, F. and STEG, J.-H. (2017). Subgame-perfect equilibria in stochastic timing games. *J. Math. Econom.* **72** 36–50. MR3697074 https://doi.org/10.1016/j.jmateco.2017.06.006

[52] SAKAI, K. (2013). *Geometric Aspects of General Topology*. *Springer Monographs in Mathematics*. Springer, Tokyo. MR3099433 https://doi.org/10.1007/978-4-431-54397-8

[53] SELTEN, R. (1970). *Preispolitik der Mehrproduktenunternehmung in der Statischen Theorie*. Springer, Berlin, New York.

[54] SELTEN, R. (1975). Reexamination of the perfectness concept for equilibrium points in extensive games. *Internat. J. Game Theory* **4** 25–55. MR0395896 https://doi.org/10.1007/BF01766400

[55] SHARPE, M. J. (1971). Exact multiplicative functionals in duality. *Indiana Univ. Math. J.* **21** 27–60.

[56] TOUZI, N. and VIEILLE, N. (2002). Continuous-time Dynkin games with mixed strategies. *SIAM J. Control Optim.* **41** 1073–1088. MR1972503 https://doi.org/10.1137/S0363012900369812

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A GENERAL PARTIAL CRAMÉR'S CONDITION FOR EDGEWORTH EXPANSION OF A FUNCTION OF SAMPLE MEANS WITH APPLICATIONS

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A large class of statistics can be formulated as smooth functions of sample means of random vectors. In this paper, we propose a general partial Cramér's condition (GPCC) and apply it to establish the validity of the Edgeworth expansion for the distribution function of these functions of sample means. Additionally, we apply the proposed theorems to several specific statistics. In particular, by verifying the GPCC, we demonstrate for the first time the validity of the formal Edgeworth expansion of Pearson's correlation coefficient between random variables with absolutely continuous and discrete components. Furthermore, we conduct a series of simulation studies that show the Edgeworth expansion has higher accuracy.

REFERENCES

ANGST, J. and POLY, G. (2017). A weak Cramér condition and application to Edgeworth expansions. *Electron. J. Probab.* **22** 1–24. MR3683368 https://doi.org/10.1214/17-EJP77

BABU, G. J. and BAI, Z. D. (1993). Edgeworth expansions of a function of sample means under minimal moment conditions and partial Cramér's condition. *Sankhyā Ser. A* **55** 244–258. MR1319127

BABU, G. J. and SINGH, K. (1989). On Edgeworth expansions in the mixture cases. *Ann. Statist.* **17** 443–447. MR0981463 https://doi.org/10.1214/aos/1176347029

BAI, Z. D. and RAO, C. R. (1991). Edgeworth expansion of a function of sample means. *Ann. Statist.* **19** 1295–1315. MR1126326 https://doi.org/10.1214/aos/1176348250

BAI, Z. D. and RAO, C. R. (1992). A note on Edgeworth expansion for ratio of sample means. *Sankhyā Ser. A* **54** 309–322. MR1216289

BHATTACHARYA, R. N. and GHOSH, J. K. (1978). On the validity of the formal Edgeworth expansion. *Ann. Statist.* **6** 434–451. MR0471142

BHATTACHARYA, R. N. and GHOSH, J. K. (1988). On moment conditions for valid formal Edgeworth expansions. *J. Multivariate Anal.* **27** 68–79. MR0971173 https://doi.org/10.1016/0047-259X(88)90116-9

BHATTACHARYA, R. N. and RAO, R. R. (2010). *Normal Approximation and Asymptotic Expansions*. Wiley, New York.

BLOZNELIS, M. and GÖTZE, F. (2022). Edgeworth approximations for distributions of symmetric statistics. *Probab. Theory Related Fields* **183** 1153–1235. MR4453324 https://doi.org/10.1007/s00440-022-01144-x

BOBKOV, S. G., CHISTYAKOV, G. P. and GÖTZE, F. (2013). Rate of convergence and Edgeworth-type expansion in the entropic central limit theorem. *Ann. Probab.* **41** 2479–2512. MR3112923 https://doi.org/10.1214/12-AOP780

CALONICO, S., CATTANEO, M. D. and FARRELL, M. H. (2022). Coverage error optimal confidence intervals for local polynomial regression. *Bernoulli* **28** 2998–3022. MR4474570 https://doi.org/10.3150/21-bej1445

CHATTERJEE, A. and LAHIRI, S. N. (2018). Edgeworth expansions for a class of spectral density estimators and their applications to interval estimation. *Statist. Sinica* **28** 2591–2608. MR3839875

CHUNG, K.-L. (1946). The approximate distribution of Student's statistic. *Ann. Math. Stat.* **17** 447–465. MR0018390 https://doi.org/10.1214/aoms/1177730884

CRAMÉR, H. (1928). On the composition of elementary errors: First paper: Mathematical deductions. *Scand. Actuar. J.* **1928** 13–74.

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DECROUEZ, G. and HALL, P. (2013). Normal approximation and smoothness for sums of means of lattice-valued random variables. *Bernoulli* **19** 1268–1293. MR3102551 https://doi.org/10.3150/12-BEJSP02

FELLER, W. (1971). *An Introduction to Probability Theory and Its Applications*. *Vol. II*, 2nd ed. Wiley, New York. MR0270403

GÖTZE, F. and HIPP, C. (1978). Asymptotic expansions in the central limit theorem under moment conditions. *Z. Wahrsch. Verw. Gebiete* **42** 67–87. MR0467882 https://doi.org/10.1007/BF00534208

HALL, P. (1987). Edgeworth expansion for Student's t statistic under minimal moment conditions. *Ann. Probab.* **15** 920–931. MR0893906

HALL, P. (2013). *The Bootstrap and Edgeworth Expansion*. Springer, Berlin.

HE, S. and LAM, H. (2024). Higher-order coverage errors of batching methods via Edgeworth expansions on t-statistics. *Ann. Statist.* **52** 1360–1383. MR4804812 https://doi.org/10.1214/24-aos2377

HSU, P. L. (1945). The approximate distributions of the mean and variance of a sample of independent variables. *Ann. Math. Stat.* **16** 1–29. MR0011911 https://doi.org/10.1214/aoms/1177731168

JIANG, C., LA VECCHIA, D., RONCHETTI, E. and SCAILLET, O. (2023). Saddlepoint approximations for spatial panel data models. *J. Amer. Statist. Assoc.* **118** 1164–1175. MR4595485 https://doi.org/10.1080/01621459.2021.1981913

JING, B.-Y. and WANG, Q. (2010). A unified approach to Edgeworth expansions for a general class of statistics. *Statist. Sinica* **20** 613–636. MR2682633

KABLUCHKO, Z., MARYNYCH, A. and SULZBACH, H. (2017). General Edgeworth expansions with applications to profiles of random trees. *Ann. Appl. Probab.* **27** 3478–3524. MR3737930 https://doi.org/10.1214/17-AAP1285

KOLASSA, J. E. and KUFFNER, T. A. (2020). On the validity of the formal Edgeworth expansion for posterior densities. *Ann. Statist.* **48** 1940–1958. MR4134781 https://doi.org/10.1214/19-AOS1871

OGASAWARA, H. (2006). Asymptotic expansion of the sample correlation coefficient under nonnormality. *Comput. Statist. Data Anal.* **50** 891–910. MR2210737 https://doi.org/10.1016/j.csda.2004.11.001

PETROV, V. V. (1975). *Sums of Independent Random Variables*. Springer, New York.

PODOLSKIJ, M. and YOSHIDA, N. (2016). Edgeworth expansion for functionals of continuous diffusion processes. *Ann. Appl. Probab.* **26** 3415–3455. MR3582807 https://doi.org/10.1214/16-AAP1179

WEI, Y., HU, J. and BAI, Z. (2026). Supplement to "A general partial Cramér's condition for Edgeworth expansion of a function of sample means with applications." https://doi.org/10.1214/25-AAP2271SUPP

ZHANG, Y. and XIA, D. (2022). Edgeworth expansions for network moments. *Ann. Statist.* **50** 726–753. MR4404918 https://doi.org/10.1214/21-aos2125

ZHILOVA, M. (2022). New Edgeworth-type expansions with finite sample guarantees. *Ann. Statist.* **50** 2545–2561. MR4500618 https://doi.org/10.1214/22-aos2192

ZHOU, X.-H., GAO, S. and HUI, S. L. (1997). Methods for comparing the means of two independent log-normal samples. *Biometrics* **53** 1129–1135.

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CENTRAL LIMIT THEOREM FOR SUPERDIFFUSIVE REFLECTED BROWNIAN MOTION

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We study the second-order asymptotics around the superdiffusive strong law (*Ann. Inst. Henri Poincaré Probab. Stat.* **59** (2023) 1813–1843) of a multidimensional driftless diffusion with oblique reflection from the boundary in a generalised parabolic domain. In the unbounded direction we prove the limit is Gaussian with the usual diffusive scaling, while in the appropriately scaled cross-sectional slice we establish convergence to the invariant law of a reflecting diffusion in a unit ball. Using the separation of time scales, we also show asymptotic independence between these two components. The parameters of the limit laws are explicit in the growth rate of the boundary and the asymptotic diffusion matrix and reflection vector field. A phase transition occurs when the domain becomes too narrow, in which case we prove that the central limit theorem for the unbounded component fails.

REFERENCES

[1] ANDERSON, D. F., HIGHAM, D. J., LEITE, S. C. and WILLIAMS, R. J. (2019). On constrained Langevin equations and (bio)chemical reaction networks. *Multiscale Model. Simul.* **17** 1–30. MR3895328 https://doi.org/10.1137/18M1190999

[2] BABOVSKY, H., BARDOS, C. and PŁATKOWSKI, T. (1991). Diffusion approximation for a Knudsen gas in a thin domain with accommodation on the boundary. *Asymptot. Anal.* **3** 265–289. MR1094677

[3] BENAÏM, M. and HURTH, T. (2022). *Markov Chains on Metric Spaces—a Short Course*. *Universitext*. Springer, Cham. MR4559704 https://doi.org/10.1007/978-3-031-11822-7

[4] BENSOUSSAN, A., LIONS, J.-L. and PAPANICOLAOU, G. C. (1979). Boundary layers and homogenization of transport processes. *Publ. Res. Inst. Math. Sci.* **15** 53–157. MR0533346 https://doi.org/10.2977/prims/1195188427

[5] BOUCHAUD, J.-P. and GEORGES, A. (1990). Anomalous diffusion in disordered media: Statistical mechanisms, models and physical applications. *Phys. Rep.* **195** 127–293. MR1081295 https://doi.org/10.1016/0370-1573(90)90099-N

[6] BREŠAR, M., MIJATOVIĆ, A. and WADE, A. (2025). Brownian motion with asymptotically normal reflection in unbounded domains: From transience to stability. *Ann. Probab.* **53** 175–222. MR4852005 https://doi.org/10.1214/24-aop1703

[7] BURDZY, K., CHEN, Z.-Q., MARSHALL, D. and RAMANAN, K. (2017). Obliquely reflected Brownian motion in nonsmooth planar domains. *Ann. Probab.* **45** 2971–3037. MR3706737 https://doi.org/10.1214/16-AOP1130

[8] BURDZY, K., CHEN, Z.-Q. and MARSHALL, D. E. (2006). Traps for reflected Brownian motion. *Math. Z.* **252** 103–132. MR2209154 https://doi.org/10.1007/s00209-005-0849-y

[9] BURDZY, K. and SALISBURY, T. S. (1999). On minimal parabolic functions and time-homogeneous parabolic h-transforms. *Trans. Amer. Math. Soc.* **351** 3499–3531. MR1661309 https://doi.org/10.1090/S0002-9947-99-02471-X

[10] CHOLAQUIDIS, A., FRAIMAN, R., LUGOSI, G. and PATEIRO-LÓPEZ, B. (2016). Set estimation from reflected Brownian motion. *J. R. Stat. Soc. Ser. B. Stat. Methodol.* **78** 1057–1078. MR3557189 https://doi.org/10.1111/rssb.12149

[11] COSTANTINI, C. and KURTZ, T. G. (2006). Diffusion approximation for transport processes with general reflection boundary conditions. *Math. Models Methods Appl. Sci.* **16** 717–762. MR2226124 https://doi.org/10.1142/S0218202506001339

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[12] COSTANTINI, C. and KURTZ, T. G. (2018). Existence and uniqueness of reflecting diffusions in cusps. *Electron. J. Probab.* **23** Paper No. 84, 21. MR3858912 https://doi.org/10.1214/18-EJP204

[13] COSTANTINI, C. and KURTZ, T. G. (2024). A reverse ergodic theorem for inhomogeneous killed Markov chains and application to a new uniqueness result for reflecting diffusions. *Ann. Appl. Probab.* **34** 3665–3700. MR4783027 https://doi.org/10.1214/23-aap2047

[14] DUPUIS, P. and ISHII, H. (1993). SDEs with oblique reflection on nonsmooth domains. *Ann. Probab.* **21** 554–580. MR1207237

[15] EBERLE, A. (2009). *Markov Processes. Lecture Notes at Univ. Bonn.*

[16] ERNST, P. A. and FRANCESCHI, S. (2021). Asymptotic behavior of the occupancy density for obliquely reflected Brownian motion in a half-plane and Martin boundary. *Ann. Appl. Probab.* **31** 2991–3016. MR4350980 https://doi.org/10.1214/21-aap1681

[17] FOSCHINI, G. J. (1982). Equilibria for diffusion models of pairs of communicating computers—symmetric case. *IEEE Trans. Inf. Theory* **28** 273–284. MR0651627 https://doi.org/10.1109/TIT.1982.1056473

[18] FRANCESCHI, S. and RASCHEL, K. (2019). Integral expression for the stationary distribution of reflected Brownian motion in a wedge. *Bernoulli* **25** 3673–3713. MR4010969 https://doi.org/10.3150/19-bej1107

[19] GRADINARU, M. and OFFRET, Y. (2013). Existence and asymptotic behaviour of some time-inhomogeneous diffusions. *Ann. Inst. Henri Poincaré Probab. Stat.* **49** 182–207. MR3060153 https://doi.org/10.1214/11-AIHP469

[20] HAN, Z., HU, Y. and LEE, C. (2016). Optimal pricing barriers in a regulated market using reflected diffusion processes. *Quant. Finance* **16** 639–647. MR3473978 https://doi.org/10.1080/14697688.2015.1034163

[21] HARRISON, J. M. (1978). The diffusion approximation for tandem queues in heavy traffic. *Adv. Appl. Probab.* **10** 886–905. MR0509222 https://doi.org/10.2307/1426665

[22] HARRISON, J. M. and WILLIAMS, R. J. (1987). Brownian models of open queueing networks with homogeneous customer populations. *Stochastics* **22** 77–115. MR0912049 https://doi.org/10.1080/17442508708833469

[23] HOBSON, D. G. and ROGERS, L. C. G. (1993). Recurrence and transience of reflecting Brownian motion in the quadrant. *Math. Proc. Cambridge Philos. Soc.* **113** 387–399. MR1198420 https://doi.org/10.1017/S0305004100076040

[24] ICHIBA, T., PAPATHANAKOS, V., BANNER, A., KARATZAS, I. and FERNHOLZ, R. (2011). Hybrid atlas models. *Ann. Appl. Probab.* **21** 609–644. MR2807968 https://doi.org/10.1214/10-AAP706

[25] KALLENBERG, O. (2021). *Foundations of Modern Probability*, 2nd ed. *Probability Theory and Stochastic Modelling* **99**. Springer, Cham. MR4226142 https://doi.org/10.1007/978-3-030-61871-1

[26] KANG, W. and RAMANAN, K. (2014). Characterization of stationary distributions of reflected diffusions. *Ann. Appl. Probab.* **24** 1329–1374. MR3210998 https://doi.org/10.1214/13-AAP947

[27] KANG, W. and RAMANAN, K. (2017). On the submartingale problem for reflected diffusions in domains with piecewise smooth boundaries. *Ann. Probab.* **45** 404–468. MR3601653 https://doi.org/10.1214/16-AOP1153

[28] KULIK, A. and SCHEUTZOW, M. (2015). A coupling approach to Doob's theorem. *Atti Accad. Naz. Lincei, Rend. Lincei, Mat. Appl.* **26** 83–92. MR3345324 https://doi.org/10.4171/RLM/694

[29] KULYK, O., PILIPENKO, A. and RŒLLY, S. (2024). On reflected diffusions in cones and cylinders. *Ukrainian Math. J.* **75** 1693–1721. Reprint of (2023). Ukraïn. Mat. Zh. **75** no. 11, 1497–1521. MR4747119

[30] KWON, Y. and WILLIAMS, R. J. (1991). Reflected Brownian motion in a cone with radially homogeneous reflection field. *Trans. Amer. Math. Soc.* **327** 739–780. MR1028760 https://doi.org/10.2307/2001821

[31] LEIMKUHLER, B., SHARMA, A. and TRETYAKOV, M. V. (2023). Simplest random walk for approximating Robin boundary value problems and ergodic limits of reflected diffusions. *Ann. Appl. Probab.* **33** 1904–1960. MR4583661 https://doi.org/10.1214/22-aap1856

[32] LIONS, P.-L. and SZNITMAN, A.-S. (1984). Stochastic differential equations with reflecting boundary conditions. *Comm. Pure Appl. Math.* **37** 511–537. MR0745330 https://doi.org/10.1002/cpa.3160370408

[33] LIPSHUTZ, D. and RAMANAN, K. (2019). Pathwise differentiability of reflected diffusions in convex polyhedral domains. *Ann. Inst. Henri Poincaré Probab. Stat.* **55** 1439–1476. MR4010941 https://doi.org/10.1214/18-aihp924

[34] LOPER, J. (2020). Uniform ergodicity for Brownian motion in a bounded convex set. *J. Theoret. Probab.* **33** 22–35. MR4064292 https://doi.org/10.1007/s10959-018-0848-7

[35] MENSHIKOV, M., POPOV, S. and WADE, A. (2017). *Non-homogeneous Random Walks: Lyapunov Function Methods for Near-Critical Stochastic Systems. Cambridge Tracts in Mathematics* **209**. Cambridge Univ. Press, Cambridge. MR3587911 https://doi.org/10.1017/9781139208468

[36] MENSHIKOV, M. V., MIJATOVIĆ, A. and WADE, A. R. (2023). Reflecting Brownian motion in generalized parabolic domains: Explosion and superdiffusivity. *Ann. Inst. Henri Poincaré Probab. Stat.* **59** 1813–1843. MR4663508 https://doi.org/10.1214/22-aihp1309

[37] MENSHIKOV, M. V., MIJATOVIĆ, A. and WADE, A. R. (2021). Reflecting random walks in curvilinear wedges. In *In and Out of Equilibrium* 3. *Celebrating Vladas Sidoravicius*. *Progress in Probability* **77** 637–675. Birkhäuser, Cham. MR4237286

[38] MENSHIKOV, M. V. and WADE, A. R. (2010). Rate of escape and central limit theorem for the supercritical Lamperti problem. *Stoch. Process. Appl.* **120** 2078–2099. MR2673989 https://doi.org/10.1016/j.spa.2010.06.004

[39] METZLER, R., JEON, J. H., CHERSTVYA, A. G. and BARKAID, E. (2014). Anomalous diffusion models and their properties: Non-stationarity, non-ergodicity, and ageing at the centenary of single particle tracking. *Phys. Chem. Chem. Phys.* **16** 24128.

[40] MIJATOVIĆ, A., SAUZEDDE, I. and WADE, A. (2024). Presentation entitled "Central limit theorem for superdiffusive reflected Brownian motion". Available at https://youtu.be/SVcVtZafrVY?si=0DzYzgZHqJveE_Y—YouTube presentation on the YouTube channel https://www.youtube.com/@prob-am7844–Prob-AM.

[41] OLIVEIRA, F. A., FERREIRA, R. M. S., LAPAS, L. C. and VAINSTEIN, M. H. (2019). Anomalous diffusion: A basic mechanism for the evolution of inhomogeneous systems. *Front. Phys.* **7**. https://doi.org/10.3389/fphy.2019.00018

[42] PINSKY, R. G. (2009). Transcience/recurrence for normally reflected Brownian motion in unbounded domains. *Ann. Probab.* **37** 676–686. MR2510020 https://doi.org/10.1214/08-AOP416

[43] RAMANAN, K. and REIMAN, M. I. (2008). The heavy traffic limit of an unbalanced generalized processor sharing model. *Ann. Appl. Probab.* **18** 22–58. MR2380890 https://doi.org/10.1214/07-AAP438

[44] SERAFIN, G. (2022). Laplace Dirichlet heat kernels in convex domains. *J. Differ. Equ.* **314** 700–732. MR4369183 https://doi.org/10.1016/j.jde.2022.01.026

[45] THORISSON, H. (2000). *Coupling*, *Stationarity*, *and Regeneration*. *Probability and Its Applications* (*New York*). Springer, New York. MR1741181 https://doi.org/10.1007/978-1-4612-1236-2

[46] VARADHAN, S. R. S. and WILLIAMS, R. J. (1985). Brownian motion in a wedge with oblique reflection. *Comm. Pure Appl. Math.* **38** 405–443. MR0792398 https://doi.org/10.1002/cpa.3160380405

[47] WILLIAMS, R. J. (1985). Recurrence classification and invariant measure for reflected Brownian motion in a wedge. *Ann. Probab.* **13** 758–778. MR0799421

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CONSISTENCY OF MLE IN PARTIALLY OBSERVED DIFFUSION MODELS ON A TORUS

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In this paper, we consider a general partially observed diffusion model with periodic coefficients and with nondegenerate diffusion component. The coefficients of such a model depend on an unknown (static and deterministic) parameter which needs to be estimated based on the observed component of the diffusion process. We show that, given enough regularity of the diffusion coefficients, a maximum likelihood estimator of the unknown parameter converges to the true parameter value as the sample size grows to infinity.

REFERENCES

[1] ALEXANDROVICH, G., HOLZMANN, H. and LEISTER, A. (2016). Nonparametric identification and maximum likelihood estimation for hidden Markov models. *Biometrika* **103** 423–434. MR3509896 https://doi.org/10.1093/biomet/asw001

[2] ALIPRANTIS, C. D. and BORDER, K. C. (2006). *Infinite Dimensional Analysis*: *A Hitchhiker's Guide*, 3rd ed. Springer, Berlin. MR2378491

[3] ALLMAN, E. S., MATIAS, C. and RHODES, J. A. (2009). Identifiability of parameters in latent structure models with many observed variables. *Ann*. *Statist*. **37** 3099–3132. MR2549554 https://doi.org/10.1214/09-AOS689

[4] ARONSON, D. G. (1967). Bounds for the fundamental solution of a parabolic equation. *Bull*. *Amer*. *Math*. *Soc*. **73** 890–896. MR0217444 https://doi.org/10.1090/S0002-9904-1967-11830-5

[5] BUDHIRAJA, A. and KUSHNER, H. J. (1999). Approximation and limit results for nonlinear filters over an infinite time interval. *SIAM J*. *Control Optim*. **37** 1946–1979. MR1720146 https://doi.org/10.1137/S0363012997328178

[6] CAPPÉ, O., MOULINES, E. and RYDÉN, T. (2005). *Inference in Hidden Markov Models*. *Springer Series in Statistics*. Springer, New York. MR2159833

[7] CHIGANSKY, P. (2006). Stability of nonlinear filters: A survey. In *Lecture Notes* Petropolis, Brazil.

[8] CHIGANSKY, P. (2009). Maximum likelihood estimator for hidden Markov models in continuous time. *Stat*. *Inference Stoch*. *Process*. **12** 139–163. MR2511676 https://doi.org/10.1007/s11203-008-9025-4

[9] CHIGANSKY, P., LIPTSER, R. and VAN HANDEL, R. (2011). Intrinsic methods in filter stability. In *The Oxford Handbook of Nonlinear Filtering* 319–351. Oxford Univ. Press, Oxford. MR2884601

[10] DA PRATO, G., FUHRMAN, M. and MALLIAVIN, P. (1999). Asymptotic ergodicity of the process of conditional law in some problem of non-linear filtering. *J*. *Funct*. *Anal*. **164** 356–377. MR1695555 https://doi.org/10.1006/jfan.1999.3399

[11] DANNEMANN, J., HOLZMANN, H. and LEISTER, A. (2014). Semiparametric hidden Markov models: Identifiability and estimation. *Wiley Interdiscip*. *Rev*.: *Comput*. *Stat*. **6** 418–425.

[12] BAIN, A. and CRISAN, D. (2009). *Fundamentals of Stochastic Filtering*. Springer, Berlin.

[13] DOUC, R., MOULINES, E., OLSSON, J. and VAN HANDEL, R. (2011). Consistency of the maximum likelihood estimator for general hidden Markov models. *Ann*. *Statist*. **39** 474–513. MR2797854 https://doi.org/10.1214/10-AOS834

[14] DOUC, R., MOULINES, É. and RYDÉN, T. (2004). Asymptotic properties of the maximum likelihood estimator in autoregressive models with Markov regime. *Ann*. *Statist*. **32** 2254–2304. MR2102510 https://doi.org/10.1214/009053604000000021

[15] ELLIOTT, R. J., AGGOUN, L. and MOORE, J. B. (2008). *Hidden Markov Models*: *Estimation and Control*. Springer, New York.

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[16] FRIEDMAN, A. (1983). *Partial Differential Equations of Parabolic Type*. Krieger, Huntington, NY.

[17] GENON-CATALOT, V. and LAREDO, C. (2006). Leroux's method for general hidden Markov models. *Stoch. Process. Appl.* **116** 222–243. MR2197975 https://doi.org/10.1016/j.spa.2005.10.005

[18] IBRAGIMOV, I. A. and HAS' MINSKII, R. Z. (2013). *Statistical Estimation: Asymptotic Theory* 16. Springer, Berlin.

[19] KHASMINSKII, R. Z. and KUTOYANTS, Y. A. (2018). On parameter estimation of hidden telegraph process. *Bernoulli* **24** 2064–2090. MR3757523 https://doi.org/10.3150/16-BEJ920

[20] KRYLOV, N. V. (1987). *Nonlinear Elliptic and Parabolic Equations of the Second Order*. *Mathematics and Its Applications (Soviet Series)* **7**. Reidel, Dordrecht. MR0901759 https://doi.org/10.1007/978-94-010-9557-0

[21] KUTOYANTS, Y. A. (2013). *Statistical Inference for Ergodic Diffusion Processes*. Springer, London.

[22] KUTOYANTS, Y. A. (2019). On parameter estimation of hidden ergodic Ornstein-Uhlenbeck process. *Electron. J. Stat.* **13** 4508–4526. MR4029160 https://doi.org/10.1214/19-EJS1631

[23] LE GLAND, F. and OUDJANE, N. (2004). Stability and uniform approximation of nonlinear filters using the Hilbert metric and application to particle filters. *Ann. Appl. Probab.* **14** 144–187. MR2023019 https://doi.org/10.1214/aoap/1075828050

[24] LEROUX, B. G. (1992). Maximum-likelihood estimation for hidden Markov models. *Stoch. Process. Appl.* **40** 127–143. MR1145463 https://doi.org/10.1016/0304-4149(92)90141-C

[25] LIGONNIÈRE, M. (2023). On the contraction properties of a pseudo-Hilbert projective metric. Available at arXiv:2312.11147.

[26] LINDVALL, T. and ROGERS, L. C. G. (1986). Coupling of multidimensional diffusions by reflection. *Ann. Probab.* **14** 860–872. MR0841588

[27] LIPTSER, R. S. and SHIRYAEV, A. N. (2001). *Statistics of Random Processes. I: General Theory*, expanded ed. *Applications of Mathematics (New York)* **5**. Springer, Berlin. MR1800857

[28] NADTOCHIY, S. and YIN, Y. (2022). Consistency of MLE for partially observed diffusions, with application in market microstructure modeling. Available at arXiv:2201.07656.

[29] OCONE, D. and PARDOUX, E. (1996). Asymptotic stability of the optimal filter with respect to its initial condition. *SIAM J. Control Optim.* **34** 226–243. MR1372912 https://doi.org/10.1137/S0363012993256617

[30] PARDOUX, E. (1981/82). Équations du filtrage non linéaire, de la prédiction et du lissage. *Stochastics* **6** 193–231. MR0665400 https://doi.org/10.1080/17442508208833204

[31] STANNAT, W. (2005). Stability of the filter equation for a time-dependent signal on $\mathbb{R}^d$. *Appl. Math. Optim.* **52** 39–71. MR2141176 https://doi.org/10.1007/s00245-005-0820-7

[32] STROOCK, D. W. and VARADHAN, S. R. S. (1979). *Multidimensional Diffusion Processes*. *Grundlehren der Mathematischen Wissenschaften* **233**. Springer, Berlin. MR0532498

[33] VAN HANDEL, R. (2007). *Filtering, Stability, and Robustness*. ProQuest LLC, Ann Arbor, MI. Thesis (Ph.D.)—California Institute of Technology. MR3192970

[34] VAN HANDEL, R. (2009). The stability of conditional Markov processes and Markov chains in random environments. *Ann. Probab.* **37** 1876–1925. MR2561436 https://doi.org/10.1214/08-AOP448