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LARGEST ENTRIES OF HIGH-DIMENSIONAL SAMPLE COVARIANCE MATRIX UNDER AUTOREGRESSIVE STRUCTURE: ASYMPTOTIC THEORY AND PHASE TRANSITION

BY TIEFENG JIANG[1,a] AND TUAN PHAM[2,b]

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We investigate the asymptotic distribution of the largest off-diagonal entry of the sample covariance matrix under the autoregressive structure driven by a parameter $r = r_n \in (0, 1)$. Our analysis reveals a phase transition at the order of $\sqrt{\log p}/\sqrt{n}$ in the asymptotic distribution of the largest entries when r is bounded away from 1. Additionally, as r converges to 1 with logarithmic rate, we show that there is a subtle clustering effect among the entries, leading to an asymptotic distribution characterized by a Gumbel law with an extremal index strictly between 0 and 1. This unexpected behavior is counterintuitive and represents a significant departure from the standard Gumbel law observed in (*Ann. Statist.* **53** (2025) 907–928) for sample correlation matrices: the seemingly simpler sample covariance matrices can exhibit more complex behavior than the sample correlation matrices. Our proof methodology relies on high-dimensional Gaussian approximation techniques and employs a blocking argument to create independence.

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TIME-UNIFORM SELF-NORMALIZED CONCENTRATION FOR VECTOR-VALUED PROCESSES

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Self-normalized processes arise naturally in many learning-related tasks. While self-normalized concentration has been extensively studied for scalar-valued processes, there are few results for multidimensional processes outside of the sub-Gaussian setting. In this work, we construct a general, self-normalized inequality for $\mathbb{R}^d$-valued processes that satisfy a simple yet broad "sub-ψ" tail condition, which generalizes assumptions based on cumulant generating functions. From this general inequality, we derive an upper law of the iterated logarithm for sub-ψ vector-valued processes, which is tight up to small constants. We show how our inequality can be leveraged to derive a variety of novel, self-normalized concentration inequalities under both light and heavy-tailed observations. Further, we provide applications in prototypical statistical tasks, such as parameter estimation in online linear regression, autoregressive modeling, and bounded mean estimation via a new (multivariate) empirical Bernstein concentration inequality.

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KERNEL LIMIT FOR A CLASS OF RECURRENT NEURAL NETWORKS TRAINED ON ERGODIC DATA SEQUENCES

BY SAMUEL CHUN HEI LAM[1,a], JUSTIN SIRIGNANO[1,b] AND KONSTANTINOS SPILIOPOULOS[2,c]

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Mathematical methods are developed to characterize the asymptotics of recurrent neural networks (RNN) as the number of hidden units, data samples in the sequence, hidden state updates, and training steps simultaneously grow to infinity. In the case of an RNN with a simplified weight matrix, we prove the convergence of the RNN to the solution of an infinite-dimensional ODE coupled with the fixed point of a random algebraic equation. The analysis requires addressing several challenges which are unique to RNNs. In typical mean-field applications (e.g., feedforward neural networks), discrete updates are of magnitude $\mathcal{O}(1/N)$ and the number of updates is $\mathcal{O}(N)$. Therefore, the system can be represented as an Euler approximation of an appropriate ODE/PDE, which it will converge to as $N \to \infty$. However, the RNN hidden layer updates are $\mathcal{O}(1)$. Therefore, RNNs cannot be represented as a discretization of an ODE/PDE and standard mean-field techniques cannot be applied. Instead, we develop a fixed point analysis for the evolution of the RNN memory states, with convergence estimates in terms of the number of update steps and the number of hidden units. The RNN hidden layer is studied as a function in a Sobolev space, whose evolution is governed by the data sequence (a Markov chain), the parameter updates, and its dependence on the RNN hidden layer at the previous time step. Due to the strong correlation between updates, a Poisson equation must be used to bound the fluctuations of the RNN around its limit equation. These mathematical methods give rise to the neural tangent kernel (NTK) limits for RNNs trained on data sequences as the number of data samples and size of the neural network grow to infinity.

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ON THE CONVERGENCE OF DYNAMIC IMPLEMENTATIONS OF HAMILTONIAN MONTE CARLO AND NO U-TURN SAMPLERS

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There is substantial empirical evidence about the success of dynamic implementations of Hamiltonian Monte Carlo (HMC), such as the no U-turn sampler (NUTS), in many challenging inference problems but theoretical results about their behavior are scarce. The aim of this paper is to fill this gap. We consider a general class of MCMC algorithms we call dynamic HMC. First, we show that this general framework encompasses NUTS as a particular case, implying the invariance of the target distribution as a by-product. Second and most importantly, we present the first ergodicity result for NUTS and prove that the NUTS variant currently implemented in major software packages is ergodic. Under conditions similar to the ones existing for HMC, we also show that NUTS is geometrically ergodic. Finally, we improve existing convergence results for HMC and show that the method is ergodic without any boundedness condition on the stepsize or the number of leapfrog steps in the case where the target is a perturbation of a Gaussian distribution.

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SHARPER ANALYSIS OF THE RANDOM GRAPH d-PROCESS VIA A BALLS-IN-BINS MODEL

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A random graph d-process starts with an empty graph on n vertices, and adds one edge at each time step, chosen uniformly at random from those pairs of vertices which are not yet edges and have current degrees less than d. If, in the final graph, at most one vertex has degree $d-1$ and all others have degree d, we call the process saturated. We present a new approach to analysing this process based on random allocation of balls in bins. This allows us to get improved results on the degree distribution throughout the process and, consequently, to determine the asymptotic probability of nonsaturation of the process.

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THE ASYMPTOTIC BEHAVIOR OF FRAUDULENT ALGORITHMS

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Let U be a Morse function on a compact connected m-dimensional Riemannian manifold, $m \geq 2$, satisfying $\min U = 0$ and let $\mathcal{U} = \{x \in M : U(x) = 0\}$ be the set of global minimizers. Consider the stochastic algorithm $X^{(\beta)} := (X^{(\beta)}(t))_{t\geq 0}$ taking values in M, whose generator is $U\triangle[\cdot] - \beta\langle\nabla U, \nabla[\cdot]\rangle$, where $\beta \in \mathbb{R}$ is a real parameter. We show that for $\beta > \frac{m}{2} - 1$, $X^{(\beta)}(t)$ converges a.s. as $t \to \infty$, toward a point $p \in \mathcal{U}$ and that each $p \in \mathcal{U}$ has a positive probability to be selected when $X^{(\beta)}(0) \notin \mathcal{U}$. On the other hand, for $\beta < \frac{m}{2} - 1$ and when the initial law does not charge $\mathcal{U}$, the law of $X^{(\beta)}(t)$ converges in total variation (at an exponential rate) toward the probability measure π_β having density proportional to $U(x)^{-1-\beta}$ with respect to the Riemannian measure.

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LOEWNER THEORY FOR BERNSTEIN FUNCTIONS II: APPLICATIONS TO INHOMOGENEOUS CONTINUOUS-STATE BRANCHING PROCESSES

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This paper continues the research project launched in (*Constr. Approx.* (2025) **61** 379–412) and aimed at studying time-inhomogeneous one-dimensional branching processes (mainly on a continuous but also on a discrete state space) with the help of recent achievements in Loewner theory dealing with evolution families of holomorphic self-maps in simply connected domains of the complex plane. Under a suitable stochastic continuity condition, we show that the families of the Laplace exponents of branching processes on $[0,\infty]$ can be characterized as topological (i.e., depending continuously on the time parameters) reverse evolution families whose elements are Bernstein functions. For the case of a stronger regularity w.r.t. time, we establish a Loewner–Kufarev type ODE for the Laplace exponents and characterize branching processes with finite mean in terms of the vector field driving this ODE. Similar results are obtained for families of probability generating functions of branching processes on the discrete state space $\{0,1,2,\dots\}\cup\{\infty\}$. In addition, we find a necessary and sufficient condition for "spatial" embeddability of such branching processes into branching processes on $[0,\infty]$. Finally, we give some probabilistic interpretations of the Denjoy–Wolff point at 0 and at ∞.

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EMPIRICAL BERNSTEIN IN SMOOTH BANACH SPACES

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Existing concentration bounds for bounded vector-valued random variables include extensions of the scalar Hoeffding and Bernstein inequalities. While the latter is typically tighter, it requires knowing a bound on the variance of the random variables. We derive a new vector-valued empirical Bernstein inequality, which makes use of an empirical estimator of the variance instead of the true variance. The bound holds in 2-smooth separable Banach spaces, which include finite-dimensional Euclidean spaces and separable Hilbert spaces. The resulting confidence sets are instantiated for both the batch setting (where the sample size is fixed) and the sequential setting (where the sample size is a stopping time). The confidence set width asymptotically exactly matches that achieved by Bernstein in the leading term.

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FOUNDATIONS OF BARE-SIMULATION OPTIMIZATION OF DISTANCES

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The constrained minimization—respectively maximization—of dissimilarity-modeling divergences (i.e., generally nonsymmetric, directed distances) and of related generalized entropies is a fundamental task in many areas of quantitative sciences. On $\mathbb{R}^K$ of arbitrary dimension K, we derive a method which tackles such kind of constrained optimization problems—and beyond—by limits of sequences of appropriately constructed, dimension-free, typically comfortably simulable random vectors; almost no assumptions (like convexity) on the set of constraints are needed. This very largely extends our recent results on f-divergences which can be connected to light-tailed probability distributions in a certain manner (cf. *IEEE Trans. Inform. Theory* (2023) **69** 3062–3120). For instance, in the current paper we cover constrained optimizations of *arbitrary* f-divergences, Bregman distances, scaled Bregman distances and weighted ℓ_r-distances.

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UP-DOWN ORDERED CHINESE RESTAURANT PROCESSES WITH TWO-SIDED IMMIGRATION, EMIGRATION AND DIFFUSION LIMITS

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We establish scaling limit theorems for the up-down ordered Chinese restaurant processes (oCRPs) of Rogers and Winkel as processes in a space of interval partitions. As previously conjectured, the limits are self-similar diffusions previously constructed directly in the continuum. We extend the oCRP model and the results to a three-parameter family $\mathrm{oCRP}^{(\alpha)}(\theta_1,\theta_2)$, $\alpha \in (0,1)$, $\theta_1,\theta_2 \geq 0$. We use the scaling limit approach to extend existing stationarity results to the full three-parameter family, identifying an extended family of Poisson–Dirichlet interval partitions. Their ranked sequence of interval lengths has Poisson–Dirichlet distribution with parameters $\alpha \in (0,1)$ and $\theta := \theta_1 + \theta_2 - \alpha \geq -\alpha$, including for the first time the usual range of $\theta > -\alpha$ rather than being restricted to $\theta \geq 0$. This has applications to Fleming–Viot processes, nested interval partition evolutions and tree-valued Markov processes, notably relying on the extended parameter range.

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DOWNWARD CONDITIONAL MONOTONICITY GIVES SURVIVAL AND EXTINCTION FOR CONTACT PROCESSES IN RANDOM ENVIRONMENTS

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The concept of downward conditional monotonicity for the Markov-modulated Poisson process (MMPP) is introduced and used to derive the optimal stochastic domination of a standard Poisson point process. The maximum arrival rate for the Poisson process which allows this domination to exist is shown to be related to an eigenvalue extracted from the generator matrix of the quasi-birth–death (QBD) formulation of the MMPP. This allows derivation of survival and extinction regimes for a large family of contact processes whose infection and recovery rates vary over time according to an underlying random environment with a finite number of states. Direct comparison with standard contact processes which dominate from above and below accomplishes this.

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UNIFORM-IN-TIME WEAK PROPAGATION OF CHAOS FOR CONSENSUS-BASED OPTIMIZATION

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We study the uniform-in-time weak propagation of chaos for the consensus-based optimization (CBO) method on a bounded searching domain. We apply the methodology for studying long-time behaviors of interacting particle systems developed in the work of (*Ann. Inst. Henri Poincaré Probab. Stat.* **61** (2025) 1021–1074). Our work shows that the weak error has order $O(N^{-1})$ uniformly in time, where N denotes the number of particles. The main strategy behind the proofs are the decomposition of the weak errors using the linearized Fokker–Planck equations and the exponential decay of their Sobolev norms. Consequently, our result leads to the joint convergence of the empirical distribution of the CBO particle system to the Dirac-delta distribution at the global minimizer in population size and running time in Wasserstein-type metrics.

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INTERSECTIONS OF BRANCHING RANDOM WALKS ON $\mathbb{Z}^8$

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We consider random walks on $\mathbb{Z}^8$ indexed by the infinite invariant tree, which consists of an infinite spine and finite random trees attached to it on both sides. We establish the precise order of the nonintersection probability between one walk indexed by one side of the tree, and an independent one indexed by both sides of an independent tree. This is analogous to the result by Lawler from the '90s for two independent simple random walks on $\mathbb{Z}^4$. We also prove a weak law of large numbers for the branching capacity of the range of a branching random walk.

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FUNCTIONAL CENTRAL LIMIT THEOREM FOR THE PRINCIPAL EIGENVALUE OF DYNAMIC ERDŐS–RÉNYI RANDOM GRAPHS

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In this paper we consider a dynamic version of the Erdős–Rényi random graph, in which edges independently appear and disappear in time, with the on- and off times being exponentially distributed. The focus lies on the evolution of the principle eigenvalue of the adjacency matrix in the regime that the number of vertices grows large. The main result is a functional central limit theorem, which displays that the principal eigenvalue essentially inherits the characteristics of the dynamics of the individual edges.

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QUANTITATIVE EQUILIBRIUM FLUCTUATIONS FOR INTERACTING PARTICLE SYSTEMS

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We consider a class of interacting particle systems in continuous space of nongradient type, which are reversible with respect to Poisson point processes with constant density. For these models, a rate of convergence was recently obtained in (*Ann. Probab.* **50** (2022) 1885–1946) for certain finite-volume approximations of the bulk diffusion matrix. Here, we show how to leverage this to obtain quantitative versions of a number of results capturing the large-scale fluctuations of these systems, such as the convergence of two-point correlation functions and the Green–Kubo formula.

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CONVERGENCE OF COORDINATE ASCENT VARIATIONAL INFERENCE FOR LOG-CONCAVE MEASURES VIA OPTIMAL TRANSPORT

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Mean field variational inference (VI) is the problem of finding the closest product (factorized) measure, in the sense of relative entropy, to a given high-dimensional probability measure ρ. The well-known coordinate ascent variational inference (CAVI) algorithm aims to approximate this product measure by iteratively optimizing over one coordinate (factor) at a time, which can be done explicitly. Despite its popularity, the convergence of CAVI remains poorly understood. In this paper, we prove the convergence of CAVI for log-concave densities ρ. If additionally $\log \rho$ has Lipschitz gradient, we find a linear rate of convergence, and if also ρ is strongly log-concave, we find an exponential rate. Our analysis starts from the observation that mean field VI, while notoriously nonconvex in the usual sense, is in fact displacement convex in the sense of optimal transport when ρ is log-concave. This allows us to adapt techniques from the optimization literature on coordinate descent algorithms in Euclidean space.

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TIME CORRELATIONS IN KPZ MODELS WITH DIFFUSIVE INITIAL CONDITIONS

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Temporal correlation for randomly growing interfaces in the KPZ universality class is a topic of recent interest. Following the conjectures in (*SIGMA Symmetry Integrability Geom. Methods Appl.* **12** (2016) Paper No. 074, 23), most of the works so far have been concentrated on the three special initial conditions, namely droplet, flat, and stationary. We focus on studying the time correlation problem for generic random initial conditions with diffusive growth. While these generalize the stationary initial conditions, the transition from stationary to diffusive initial conditions fundamentally disrupts the integrability of the model, necessitating the development of new analytical techniques. We formulate our results in terms of the positive temperature exactly solvable model of the inverse-gamma polymer and obtain upper and lower bounds, up to a constant, for the correlation between the free energy of two polymers whose endpoints are close together or far apart. Although the exponents governing the time correlation decay that we obtain are the same as in the droplet initial condition recently studied in (*Comm. Math. Phys.* **405** (2024) Paper No. 163, 72), the mechanism producing the correlations is different when the endpoints of the polymers are far apart in time. We obtain new results that have not been covered previously in either zero- or positive-temperature KPZ models, complementing the results obtained in (*Math. Phys. Anal. Geom.* **22** (2019) Paper No. 1, 33), where the exponent was only obtained in the case of close endpoints and for a narrower class of initial conditions in the zero temperature exponential LPP model. Our arguments rely on the one-point moderate deviation estimates, which have recently been obtained using stationary polymer techniques and thus do not depend on complicated exact formulae.

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DEEP OPERATOR BSDE: A NUMERICAL SCHEME TO APPROXIMATE SOLUTION OPERATORS

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Motivated by dynamic risk measures and conditional g-expectations, in this work we propose a numerical method to approximate the solution operator given by a backward stochastic differential equation (BSDE). The main ingredients for this are the Wiener chaos decomposition and the classical Euler scheme for BSDEs. We show convergence of this scheme under very mild assumptions, and provide a rate of convergence in more restrictive cases. We then implement it using neural networks, and we present several numerical examples where we can check the accuracy of the method.

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ON THE MCMC PERFORMANCE IN BERNOULLI GROUP TESTING AND THE RANDOM MAX-SET COVER PROBLEM

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The group testing problem is a canonical inference task where one seeks to identify k infected individuals out of a population of n people, based on the outcomes of N group tests. Of particular interest is the case of Bernoulli group testing (BGT), where each individual participates in each test independently and with a fixed probability. BGT is known to be "information-theoretically" optimal, as there exists a decoder that can approximately recover the set of infected individuals with high probability as n grows using $N^* = \log_2 \binom{n}{k}$ BGT tests, which is the minimum required number of tests among *all* group testing designs.

An important open question in the field is if a polynomial-time decoder exists for BGT which succeeds also with N^* samples. In a recent paper (IZ'21) some evidence was presented (but no proof) that a simple low-temperature MCMC method could succeed. The evidence was based on a first-moment (or "annealed") analysis of the landscape and simulations showing the MCMC success for $n \approx 1{,}000s$.

In this work, we prove that, despite the intriguing success in simulations for small n, the proposed class of MCMC methods for BGT with N^* samples takes super-polynomial-in-n time to identify the infected individuals. We show that the suggested first-moment picture by the previous work has been an artifact of "rare bad" events, and via a delicate conditional second-moment method we conclude that an overlap gap property takes place in BGT leading to bottlenecks for the MCMC methods. Towards obtaining our results, we establish the tight max-satisfiability thresholds of random k-set cover, a result of potentially independent interest in the study of random constraint satisfaction problems.

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ENTROPIC CONDITIONAL CENTRAL LIMIT THEOREM AND HADAMARD COMPRESSION

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The Hadamard compression has been proved to achieve the compression limit for mixed-distributed input signals, by making the uncertainty of the discrete part vanish. The limit of the continuous part is still unknown. We find that the Hadamard transform is closely related to the conditional central limit theorem since they both satisfy a core entropic property. We first establish an entropic *conditional central limit theorem* (CCLT), which is stronger than the classical CCLT. Second, through building a suitable probability space and extending the entropic CCLT, we show that for the continuous input under the iterated Hadamard transform, almost every distribution of the output conditional on the values of the previous signals will tend to be Gaussian, and the conditional distribution is in fact insensitive to the condition. The results enable us to make a theoretical study concerning the Hadamard compression, which provides a solid theoretical analysis supporting the simulation results in a previous work.

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SHARP CONNECTIVITY BOUNDS FOR THE VACANT SET OF RANDOM INTERLACEMENTS

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We consider percolation of the vacant set of random interlacements at intensity u in dimensions three and higher, and derive lower bounds on the truncated two-point function for all values of $u > 0$. These bounds are sharp up to principal exponential order for all u in dimension three and all $u \neq u_*$ in higher dimensions, where u_* refers to the critical parameter of the model, and they match the upper bounds derived in the article (Goswami, Rodriguez, Shulzhenko (2025)). In dimension three, our results further imply that the truncated two-point function grows at large distances x at a rate that depends on x only through its Euclidean norm, which offers a glimpse of the expected (Euclidean) invariance of the scaling limit at criticality. The decay rate is atypical, it incurs a logarithmic correction and comes with an explicit prefactor that converges to 0 as the parameter u approaches the critical point u_* from either side. A particular challenge stems from the combined effects of lack of monotonicity due to the truncation in the super-critical phase, and the precise (rotationally invariant) controls we seek, that measure the effects of a certain "harmonic humpback" function. Among others, their derivation relies on rather fine estimates for hitting probabilities of the random walk in arbitrary direction e, which witness this invariance at the discrete level, and preclude straightforward applications of projection arguments.

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ON THE MAXIMAL CORRELATION OF SOME STOCHASTIC PROCESSES

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We study the maximal correlation coefficient $R(X, Y)$ between two stochastic processes X and Y. In the case when (X, Y) is a random walk, we find $R(X, Y)$ using the Csáki–Fischer identity and the lower semicontinuity of the map $\mathrm{Law}(X, Y) \to R(X, Y)$. When (X, Y) is a two-dimensional Lévy process, we express $R(X, Y)$ in terms of the Lévy measure of the process and the covariance matrix of the diffusion part of the process. Consequently, for a two-dimensional α-stable random vector (X, Y) with $0 < \alpha < 2$, we express $R(X, Y)$ in terms of α and the spectral measure τ of the α-stable distribution. We also establish analogs and extensions of the Dembo–Kagan–Shepp–Yu inequality and the Madiman–Barron inequality.

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EXISTENCE AND NONUNIQUENESS OF ERGODIC LERAY–HOPF SOLUTIONS TO THE STOCHASTIC POWER-LAW FLOWS

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We study long time behavior of shear-thinning fluid flows in $d \geq 3$ dimensions, driven by additive stochastic forcing of trace class, with power-law indices ranging from 1 to $\frac{2d}{d+2}$. We particularly focus on Leray–Hopf solutions, that is, on analytically weak solutions satisfying energy inequality.

Introducing a new kind of energy related functional into the technique of convex integration enables the construction of infinitely many such solutions that are probabilistically strong for a certain initial value. Furthermore, we provide global in time estimates which lead to the existence of infinitely many stationary and even ergodic Leray–Hopf solutions.

These results represent the first construction of Leray–Hopf solutions in the framework of stochastic shear-thinning fluids within this range of power-law indices.

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LIMIT PROFILE FOR THE BERNOULLI–LAPLACE URN VIA DIFFUSIONS

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We introduce a technique to establish *limit profiles* for Markov chains by leveraging tools from stochastic analysis—in particular, approximations via SDEs. We demonstrate our arguments by analysing the *Bernoulli–Laplace urn* model:

- initially, one urn contains k red balls and a second $n-k$ blue balls;
- in each step, a pair of balls is chosen uniform and their locations are switched.

Cutoff is known to occur at $\frac{1}{2}n\log\min\{k,\sqrt{n}\}$ with window order n whenever $1 \ll k \le \frac{1}{2}n$. We refine this by determining the *limit profile*: a function Φ such that

$$d_{\mathsf{TV}}\left(\frac{1}{2}n\log\min\{k,\sqrt{n}\}+\theta n\right) \to \Phi(\theta) \quad \text{as } n\to\infty \text{ for all } \theta\in\mathbb{R}.$$

Its birth–death representation approximates an Ornstein–Uhlenbeck diffusion, once appropriately rescaled, started from the bulk (according to its equilibrium distribution), when $k \gg \sqrt{n}$. Precise concentration results as the chain enters the bulk are needed.

We also establish the limit profile when $1 \ll k \lesssim \sqrt{n}$, via a different method.

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